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I.—**On Generalised Functions of Legendre and Bessel.** By the **Rev. F. H. Jackson**, H.M.S. "Irresistible." *Communicated by* Dr W. PEDDIE.

(Part I. received July 27 ; Part II. August 6, 1903. Read November 16, 1903. Issued separately February 5, 1904.)

PART I.

1.

In this paper some properties of the functions $J_{[n]}(x, \lambda)$, $P_{[n]}(x, \lambda)$, $Q_{[n]}(x, \lambda)$ will be investigated. These functions are generalised forms of BESSEL'S and LEGENDRE'S Functions. Two interesting expressions are obtained for the sum of the coefficients of x in the series $P_{[n]}(x)$ and $\frac{d^{(\nu)}P_{[n]}}{dx^{(\nu)}}$. Throughout the paper $[n]$ denotes $\frac{p^n-1}{p-1}$.

$J_{[n]}(x, \lambda)$ denotes the convergent series

$$\sum_{r=0}^{r=\infty} \frac{\lambda^{n+2r} x^{[n+2r]}}{[r]!\,[n+r]!\,(2)_r (2)_{n+r}} \quad . \quad . \quad . \quad (1)$$

in which

$[n]!$ represents $[1][2][3] \ldots . [n]$

$(2)_n$,, $p+1 \cdot p^2+1 \cdot p^3+1 \ldots . p^n+1$

if $p=1$ we see that

$[n]!$ reduces to $n!$

$(2)_n$,, 2^n

$P_{[n]}(x\lambda)$ denotes

$$\sum_{r=0}^{r=\infty} (-1)^r \frac{[2n-2r]!}{[r]!\,[n-r]!\,[n-2r]!\,(2)_r (2)_{n-r}} p^{r.r+2} \lambda^{n-2r} x^{[n-2r]} \quad . \quad . \quad (2)$$

which when $p=1$ reduces to LEGENDRE'S function P_n

$Q_{[n]}(x, \lambda)$ denotes

$$\sum_{r=0}^{r=\infty} \frac{[n+2r]!\,[n+r]!\,(2)_{n+r}}{[r]!\,[2n+2r+1]!\,(2)_r} p^{3r} \lambda^{-n-2r-1} x^{[-n-2r-1]} \quad . \quad . \quad (3)$$

If n be not integral then $[n]!$ must be replaced by the function $\Pi_p([n])$ which is defined as

$$\Pi_p([n]) = \underset{\kappa=\infty}{\mathrm{L}} \frac{[1][2][3] \ldots . [\kappa]}{[n+1][n+2][n+3] \ldots . [n+\kappa]} [\kappa]^n p^{\frac{n.n+1}{2}} \quad . \quad (4)$$

reducing when $p=1$ to GAUSS's expression for $\Pi(n)$ or $\Gamma(n+1)$

$$\Gamma(n+1) = \prod_{\kappa=\infty} \frac{1\cdot 2\cdot 3\cdot 4 \ldots\ldots \kappa}{n+1\cdot n+2\cdot n+3 \ldots\ldots n+\kappa}\kappa^n$$

The infinite product (4) is convergent for all values of n except negative integers:

$(2)_n$ will be in general

$$\frac{\Pi_{p^2}([n])}{\Pi_p([n])} \times (p+1)^n \quad . \quad . \quad . \quad . \quad (5)$$

for the infinite product $\Pi_{p^2}([n])$ is

$$\frac{\dfrac{p^2-1}{p^2-1}\cdot\dfrac{p^4-1}{p^2-1}\cdot\dfrac{p^6-1}{p^2-1}\cdots\cdots\dfrac{p^{2\kappa}-1}{p^2-1}}{\dfrac{p^{2n+2}-1}{p^2-1}\cdot\dfrac{p^{2n+1}-1}{p^2-1}\cdots\cdots\dfrac{p^{2n+2\kappa}-1}{p^2-1}}\left\{\frac{p^{2\kappa}-1}{p^2-1}\right\}^n p^{n.n+1}$$

which is

$$\Pi_p([n]) \times \frac{p+1\cdot p^2+1\ p^3+1\ldots\ldots p^\kappa+1}{p^{n+1}+1\ p^{n+2}+1\ldots\ldots p^{n+\kappa}+1}\left\{\frac{p^\kappa+1}{p+1}\right\}^n p^{\frac{n.n+1}{2}}$$

$$\Pi_p([n]) \times \frac{(2)_n}{(p+1)^n}$$

$(2)_n$ denoting a convergent infinite product reducing for integral values of n to $(p+1)(p^2+1)\ldots(p^n+1)$.

The difference theorem for $\Pi_p([x])$ is $\Pi_p([x])=[x]\Pi_p([x-1])$. The multiplication theorem which is the generalisation of

$$\Gamma(x)\Gamma\left(x+\frac{1}{n}\right)\cdots\Gamma\left(x+\frac{n-1}{n}\right)=\Gamma(nx)\,.\,(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nx}$$

is investigated in another paper.

As obtained from the differential equations the series are perfectly general with regard to n, and the question of $[n]!$ or $\Pi([n])$ only arises in connection with the arbitrary constant multiplier of each series.

2.

In a paper on generalised forms of the series of BESSEL and LEGENDRE (*Proc. Edin. Math. Soc.*, vol. xxi.), I have shown that if $J_{[n]}(x)$ denote the series

$$A\left\{x^{[n]}+\frac{x^{[n+2]}}{[2][2n+2]}+\frac{x^{[n+4]}}{[2][4][2n+2][2n+4]}+\ldots\ldots\right\} \quad . \quad . \quad (6)$$

then $y=J_{[n]}(x)$ satisfies a differential equation

$$px^{[2]}\frac{d^{(2)}y}{dx^{(2)}}+\{1-[n]-[-n]\}x\frac{dy}{dx}+[n][-n]y=x^{[2]}J_{[n]}(x^p) \quad . \quad . \quad (7)$$

$\frac{d^{(2)}}{dx^{(2)}}$ denoting $\frac{d}{d(x^p)}\cdot\frac{d}{dx}$ . . . . .

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Introducing a parameter λ we see that (6) and (7) may be stated more generally as

$$y = A\left\{ x^{[n]} + \frac{\lambda^2 x^{[n+2]}}{[2][2n+2]} + \frac{\lambda^4 x^{[n+4]}}{[2][4][2n+2][2n+4]} + \cdots \right\} \quad . \qquad (8)$$

and

$$px^{[2]}\frac{d^{(2)}y}{dx^{(2)}} + \left\{ 1 - [n] - [-n] \right\} x\frac{dy}{dx} + [n][-n]y = \lambda^2 x^{[2]} J_{[n]}(x^{p/2}\lambda) \quad . \qquad (9)$$

In the series (8) give to the arbitrary constant A the value $\frac{\lambda^n}{(2)_n[n]!}$ or generally $\frac{\lambda^n}{(2)_n \Pi([n])}$, then throughout the paper $J_{[n]}(x,\lambda)$ will denote the series

$$\frac{\lambda^n}{(2)_n[n]!} \sum_{r=0}^{r=\infty} \frac{\lambda^{2r} x^{[n+2r]}}{[2][4] \ldots [2r] \cdot [2n+2] \ldots [2n+2r]} \quad . \qquad (10)$$

which can be written in the form

$$\sum \frac{\lambda^{n+2r} x^{[n+2r]}}{(2)_r (2)_{n+r} [r]! [n+r]!} \quad . \qquad (11)$$

Since

$$\begin{aligned} [2][4] \ldots [2r] &= \frac{p^2-1}{p-1} \cdot \frac{p^4-1}{p-1} \cdot \ldots \cdot \frac{p^{2r}-1}{p-1} \\ &= p+1 \cdot p^2+1 \cdots p^r+1 \cdot \frac{p-1}{p-1} \cdot \frac{p^2-1}{p-1} \cdot \ldots \cdot \frac{p^r-1}{p-1} \\ &= (2)_r [r]! \end{aligned}$$

and in general (r not integral)

$$(p+1)^r \Pi_{p^2}([r]) = (2)_r \Pi_p([r])$$

3.

It may be verified at once that

$$\frac{d}{dx}\left\{ J_{[0]}(x\lambda) \right\} = \lambda J_{[1]}(x^p \cdot \lambda) \quad . \qquad (12)$$

and in general that

$$\frac{d}{d(x^{p^n})} \left\{ x^{-[n]} J_{[n]}(x,\lambda) \right\} = \lambda J_{[n+1]}(x^p \cdot \lambda) \quad . \qquad (13)$$

for

$$\begin{aligned} & x^{-[n]} J_{[n]}(x\lambda) \\ &= \frac{\lambda^{n+2r} x^{[n+2r]-[n]}}{(2)_r (2)_{n+r} [r]! [n+r]!} \\ &= \sum \frac{\lambda^{n+2r} x^{p^n [2r]}}{(2)_r (2)_{n+r} [r]! [n+r]!} \quad . \qquad (14) \end{aligned}$$

since

$$[n+2r] - [n] = \frac{p^{n+2r}-1}{p-1} - \frac{p^n-1}{p-1} = p^n \frac{p^{2r}-1}{p-1} = p^n [2r]$$

Differentiating the series (14) with regard to x^{p^n} we see that the differential coefficient of the first term is zero and we obtain the series

$$\begin{aligned} & \sum_{r=1}^{r=\infty} \frac{\lambda^{n+2r}[2r]x^{p^{n+1}[2r-1]}}{(2)_r(2)_{n+r}[r]![n+r]!} \quad . \quad . \quad . \quad . \quad (15) \\ = & \sum_{r=1}^{r=\infty} \frac{\lambda^{n+2r}x^{-p[n]}x^{p[n+2r-1]}}{(2)_r(2)_{n+r}[r-1]![n+r]!} \\ = & \lambda x^{-p[n]} \sum_{r=0}^{r=\infty} \frac{\lambda^{n+2r+1}x^{p[n+1+2r]}}{(2)_r(2)_{n+1+r}[r]![n+1+r]!} \\ = & \lambda x^{-p[n]}\mathrm{J}_{[n+1]}(x^p\lambda) \end{aligned}$$

which establishes

$$\frac{d}{d(x^{p^n})}\left\{ x^{-[n]}\mathrm{J}_{[n]}(x\lambda) \right\} = \lambda x^{-p[n]}\mathrm{J}_{[n+1]}(x^p,\lambda) \quad . \quad . \quad . \quad (16)$$

By a change of the independent variable this may be written in the form

$$\mathrm{J}_{[n+1]}(x^{p^{n+1}}\lambda) = \frac{1}{\lambda}x^{p^{n+1}[n]}\frac{d}{d(x^{p^{2n}})}\left\{ x^{-p^n[n]}\mathrm{J}_{[n]}(x^{p^n}\lambda) \right\} \quad . \quad . \quad (17)$$

from which we have by repeating the operations

$$\begin{aligned} & \lambda^{n+1}x^{-p^{n+1}[n+1]}\mathrm{J}_{[n+1]}(x^{p^{n+1}}\lambda) = \\ & \frac{1}{x^{p^{2n+1}}}\frac{d}{d(x^{p^{2n}})}\left\{ \frac{1}{x^{p^{2n-1}}}\frac{d}{d(x^{p^{2n-2}})}\left\{ \cdots\cdots \right\}\frac{1}{x^{p^3}}\frac{d}{d(x^{p^2})}\left\{ \frac{1}{x^p}\frac{d\mathrm{J}_0(x\lambda)}{dx} \right\}\right\}\cdots\Big\}\Big\}\Big\} \quad (18) \end{aligned}$$

a theorem analogous to

$$\frac{\lambda^{n+1}}{x^{n+1}}\mathrm{J}_{n+1} = \left(\frac{1}{x}\frac{d}{dx}\right)^{n+1}\mathrm{J}_0$$

4.

Similarly we may easily verify that

$$\lambda x^p\mathrm{J}_{[0]}(x^{p^2}\lambda) = \frac{d}{dx}\left\{ x\mathrm{J}_{[1]}(x^p\lambda) \right\} \quad . \quad . \quad . \quad (19)$$

and in general that

$$\lambda x^{p^{[n]}}\mathrm{J}_{[n-1]}(x^{p^{n+1}}\lambda) = \frac{d}{dx}\left\{ x^{[n]}\mathrm{J}_{[n]}(x^{p^n}\lambda) \right\} \quad . \quad . \quad . \quad (20)$$

from which by repeating the operation

$$\begin{aligned} \lambda^{r+1}x^{p^{[n]}}\mathrm{J}_{[n-1]}(x^{p^{n+1}}\lambda) = & \frac{d}{dx}\left\{ x^{-\frac{1}{p}\frac{d}{d(x^{-p^2})}}\left\{ x^{-\frac{1}{p^2}\frac{d}{d(x^{-p^4})}}\left\{ \cdots \right.\right.\right. \\ & \left.\left.\left. \cdots\left\{ x^{-\frac{1}{p^{2r-1}}\frac{d}{d(x^{-p^{2r}})}}\left\{ x^{p^{-2r[n+r]}}\mathrm{J}_{[n+r]}(x^{p^{n-r}}\lambda) \right\}\right\}\cdots\right\}\right\} \quad (21) \end{aligned}$$

In the case $p=1$, this becomes

$$\lambda^{r+1}x^{n-1}\mathrm{J}_{n-1}(x\lambda) = \left(\frac{1}{x}\frac{d}{dx}\right)^{r+1}\left\{ x^{n+r}\mathrm{J}_{n+r}(x,\lambda) \right\}$$

5.

A very symmetrical differential equation is

$$x^{p^n}\frac{d}{d(x^{p^n})}\mathrm{J}_{[n]}(x,\lambda)-x^{p^{n-n}}\frac{d}{d(x^{p^{-n}})}\mathrm{J}_{[n]}(x,\lambda)=\frac{[2n]}{p^n}\left\{\mathrm{J}_{[n]}(x\lambda)-\mathrm{J}_{[n]}(x,\lambda p)\right\} \quad . \quad (22)$$

To prove this we must first show that

$$\mathrm{J}_{[n-1]}(x^p\lambda)-\mathrm{J}_{[n+1]}(x^p\lambda)=\frac{[2n]}{\lambda p^n x}\mathrm{J}_{[n]}(x,\lambda p) \quad . \quad . \quad . \quad (23)$$

The coefficient of $x^{p[n+2r-1]}$ in $\mathrm{J}_{[n-1]}(x^p\lambda)-\mathrm{J}_{[n+1]}(x^p\lambda)$ is

$$\frac{\lambda^{n+2r-1}}{[r]!\,[n+r-1]!\,(2)_r(2)_{n+r-1}}-\frac{\lambda^{n+2r-1}}{[r-1]!\,[n+r]!\,(2)_{r-1}(2)_{n+r}}$$

which is

$$\frac{\lambda^{n+2r}}{[r]!\,[n+r]!\,(2)_r(2)_{n+r}}\left\{\frac{1}{\lambda}[n+r](p^{n+r}+1)-\frac{1}{\lambda}[r](p^r+1)\right\}$$

$$=\frac{\lambda^{n+2r}\cdot p^{2r}[2n]}{[r]!\,[n+r]!\,(2)_r(2)_{n+r}\lambda}$$

The $r+1^{\text{th}}$ term of the resulting series is therefore

$$\frac{[2n]}{\lambda p^n}\,\frac{\lambda^{n+2r}p^{n+2r}x^{p[n+2r-1]}}{[r]!\,[n+r]!\,(2)_r(2)_{n+r}} \quad . \quad . \quad . \quad . \quad (24)$$

which since

$$x^{p[n+2r-1]}=\frac{1}{x}\cdot x^{[n+2r]}$$

shows us that the series is

$$\frac{[2n]}{\lambda p^n x}\mathrm{J}_{[n]}(x,\lambda p) \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and we have established the relation (23) between three successive functions, reducing when $p=1$ to

$$\mathrm{J}_{n-1}-\mathrm{J}_{n+1}=\frac{2n}{x}\mathrm{J}_n \quad . \quad . \quad . \quad . \quad (26)$$

Now it has been shown that

$$\lambda x^{p^{n-1}}\mathrm{J}_{[n-1]}(x^{p^{n+1}}\lambda)=\frac{d}{dx}\left\{x^{p^n}\mathrm{J}_{[n]}(x^{p^n}\lambda)\right\} \quad . \quad . \quad . \quad (27)$$

Changing the independent variable x to $x^{p^{-n}}$ we obtain

$$\lambda x^{p^{1-n}[n]}\mathrm{J}_{[n-1]}(x^p\lambda)=\frac{d}{d(x^{p^{-n}})}\left\{x^{p^{-n}[n]}\mathrm{J}_{[n]}(x\lambda)\right\} \quad . \quad . \quad . \quad (28)$$

which is

$$\mathrm{J}_{[n-1]}(x^p\lambda)=\frac{1}{\lambda}x^{-p^{1-n}[n]}\frac{d}{d(x^{p^{-n}})}\left\{x^{p^{-n}[n]}\mathrm{J}_{[n]}(x\lambda)\right\} \quad . \quad . \quad (29)$$

as can easily be verified independently.

We have already shown that

$$\mathrm{J}_{[n+1]}(x^p\lambda)=\frac{1}{\lambda}x^{p^{[n]}}\frac{d}{d(x^{p^n})}\left\{x^{p^n[-n]}\mathrm{J}_{[n]}(x\lambda)\right\} \quad . \quad . \quad . \quad (30)$$

Taking (30) from (29) we obtain by (23) that

$$\frac{[2n]}{\lambda p^n x}\mathrm{J}_{[n]}(x,\lambda p) = \frac{1}{\lambda}x^{-p^{1-n}[n]}\frac{d}{d(x^{p-n})}\left\{ x^{p-n[n]}\mathrm{J}_{[n]}(x\lambda)\right\} - \frac{1}{\lambda}x^{p[n]}\frac{d}{d(x^{p^n})}\left\{ x^{p^n[-n]}\mathrm{J}_{[n]}(x\lambda)\right\}$$

The expression on the right is equal to

$$\frac{1}{\lambda}\frac{[n]}{x}\mathrm{J}_{[n]}(x\lambda) + \frac{1}{\lambda}x^{p^{-n}-1}\frac{d}{d(x^{p-n})}\left\{ \mathrm{J}_{[n]}(x\lambda)\right\}$$

$$-\frac{1}{\lambda}\frac{[-n]}{x}\mathrm{J}_{[n]}(x\lambda) - \frac{1}{\lambda}x^{p^n-1}\frac{d}{d(x^{p^n})}\left\{ \mathrm{J}_{[n]}(x\lambda)\right\} \quad . \quad . \quad . \quad (31)$$

Now

$$\frac{1}{\lambda}\frac{[n]}{x}\mathrm{J}_{[n]}(x\lambda) - \frac{1}{\lambda}\frac{[-n]}{x}\mathrm{J}_{[n]}(x\lambda)$$

is equal to

$$\frac{[2n]}{\lambda p^n x}\mathrm{J}_{[n]}(x\lambda)$$

and therefore we have finally

$$\frac{[2n]}{p^n}\left\{ \mathrm{J}_{[n]}(x\lambda) - \mathrm{J}_{[n]}(x\,,\,\lambda p)\right\} = x^{p^n}\frac{d}{d(x^{p^n})}\mathrm{J}_{[n]}(x\lambda) - x^{p^{-n}}\frac{d}{d(x^{p^{-n}})}\mathrm{J}_{[n]}(x\lambda) \quad . \quad (32)$$

a very symmetrical equation.

From (23) it follows that

$$\mathrm{J}_{[n-1]}(x^p\lambda) = \frac{1}{\lambda p^n x}\left\{ [2n]\mathrm{J}_n(x\lambda p) + p^{-2}[2n+4]\mathrm{J}_{n+2}(x\lambda p) + \cdots\cdots\right\} \quad . \quad (33)$$

6.

The Function $\mathrm{P}_{[n]}(x\lambda)$.

If $\mathrm{P}_{[n]}(x)$ denote the series

$$\mathrm{A}\left\{ x^{[n]} - \frac{[n][n-1]}{[2][2n-1]}p^3x^{[n-2]} + \frac{[n][n-1][n-2][n-3]}{[2][4][2n-1][2n-3]}p^8x^{[n-4]} - \cdots\cdot\right\} \quad (34)$$

Then * $\mathrm{P}_{[n]}(x)$ satisfies the differential equation

$$px^{[2]}\frac{d^{(2)}y}{dx^{(2)}} - \frac{d^{(2)}y}{dx^{(2)}} + \left\{ 1-[n]-[-n-1]\right\}x\frac{dy}{dx} + [n][-n-1]y = \mathrm{P}^1_{[n-2]}(x) - \mathrm{P}^1_{[n-2]}(x^{p^2}) \quad (35)$$

Introducing a constant parameter λ we have more generally the series

$$\mathrm{A}\left\{ x^{[n]} - \frac{[n][n-1]}{[2][2n-1]}p^3\lambda^{-2}x^{[n-2]} + \frac{[n][n-1][n-2][n-3]}{[2][4][2n-1][2n-2]}p^8\lambda^{-4}x^{[n-4}} - \cdots\right\} \quad (36)$$

Satisfying

$$px^{[2]}\frac{d^{(2)}y}{dx^{(2)}} - \frac{1}{\lambda^2}\frac{d^{(2)}y}{dx^{(2)}} + \left\{ 1-[n]-[-n-1]\right\}x\frac{dy}{dx} + [n][-n-1]y = \mathrm{P}^1_{[n-2]}(x,\lambda) - \mathrm{P}^1_{[n-2]}(x^{p^2}\lambda) \quad (37)$$

Give to the arbitrary constant A the value

$$\frac{[2n]!\,\lambda^n}{(2)_n[n]!\,[n]!}$$

* *Proc. Edin. Math. Soc.*, vol. xxi.

Then throughout subsequent work the function $P_{[n]}(x,\lambda)$ will denote

$$\sum_{r=0}^{r=\infty}(-1)^r\frac{[2n-2r]!}{[n-r]![n-2r]![r]!\,(2)_{n-r}(2)_r}p^{r,r+2}\lambda^{n-2r}x^{[n-2r]} \quad . \quad (38)$$

$P^1_{[n-2]}(x\lambda)$ denotes the series

$$\sum_{r=0}^{r=\infty}(-1)^r\frac{[2n-2r]!}{[n-r]![n-2r-2]![r]!\,(2)_{n-r}(2)_r}\lambda^{n-2r-2}x^{[n-2r-2]} \quad . \quad . \quad (39)$$

We proceed to show that

$$\frac{d\cdot P_{[n+1]}(x\lambda)}{dx}-p\frac{dP_{[n-1]}(x\lambda)}{dx} = \frac{\lambda}{p^n}[2n+1]P_{[n]}(x',\lambda p) \quad . \quad . \quad (40)$$

In $P_{[n+1]}(x\lambda)+\kappa P_{[n-1]}(x\lambda)$ the terms involving $x^{[n-2r+1]}$ give us

$$p^{r,r+2}(-1)^r\frac{[2n-2r+2]!}{[n-r+1]![n-2r+1]![r]!\,(2)_r(2)_{n+1-r}}\lambda^{n+1-2r}x^{[n+1-2r]}$$

$$+\kappa p^{r-1,r+1}(-1)^{r-1}\frac{[2n-2r]!}{[n-r]![n-2r+1]![r-1]!\,(2)_{r-1}(2)_{n-2}}\lambda^{n+1-2r}x^{[n-2r+1]}$$

which gives us an expression for the coefficient of $x^{[n-2r+1]}$, viz.,

$$(-1)^r p^{r,r+2}\frac{[2n-2r]!}{[n-r]![n-2r]![r]!\,(2)_r(2)_{n-r}}\left\{\frac{[2n-2r+1]}{[n-2r+1]}-\kappa p^{-1-2r}\frac{[2r]}{[n-2r+1]}\right\}\lambda^{n-2r+1}$$

The part within the large bracket is

$$\frac{p^{2n-2r+1}-1-\kappa p^{-1-2r}(p^{2r}-1)}{(p-1)[n-2r+1]}$$

Putting $\kappa=-p$ this reduces to

$$\frac{p^{-2r}[2n+1]}{[n-2r+1]}$$

Therefore

$$P_{[n+1]}(x\lambda)-pP_{[n-1]}(x\lambda p) = \sum_0^\infty(-1)^r p^{r,r+2}\frac{[2n-2r]!\,p^{-2r}[2n+1]}{[n-r]![n-2r]![r]!\,(2)_r(2)_{n-r}[n-2r+1]}\cdot\lambda^{n-2r+1}x^{[n-2r+1]} \quad (41)$$

$$=\frac{\lambda[2n+1]}{p^n}\sum(-1)^r p^{r,r+2}\frac{[2n-2r]!}{[n-r]![n-2r+1]![r]!\,(2)_r(2)_{n-r}}\lambda^{n-2r+1}x^{[n-2r+1]}$$

from which we have at once

$$\frac{dP_{[n+1]}(x\lambda)}{dx}-p\frac{dP_{[n-1]}(x\lambda)}{dx} = \frac{\lambda}{p^n}[2n+1]P_{[n]}(x',\lambda p) \quad . \quad . \quad (42)$$

which leads to

$$p^n\frac{dP_{[n]}(x\lambda)}{dx}=\lambda p\left\{[2n-1]P_{n-1}(x',\lambda p)+p^2[2n-5]P_{n-3}(x',\lambda p)+p^6[2n-9]P_{n-5}(x',\lambda p)+\cdots\right\} \quad (43)$$

8.

The relation between three consecutive functions is

$$[n]P_{[n]}(x\lambda) = \lambda[2n-1]xP_{[n-1]}(x',\lambda)-p^{n+1}[n-1]P_{[n-2]}(x\lambda) \quad . \quad . \quad (44)$$

which corresponds to the ordinary relation

$$nP_n = (2n-1)xP_{n-1}-(n-1)P_{n-2}$$

Consider

$$\lambda[2n-1]xP_{[n-1]}(x',\lambda)-\kappa p[n-1]P_{[n-2]}(x,\lambda)$$

The terms involving $x^{[n-2r]}$ give us

$$\lambda(-1)^r[2n-1]p^{r.r+2}\frac{[2n-2r-2]!}{[n-r-1]![n-2r-1]![r]!(2)_{n-1-r}(2)_r}\lambda^{n-2r-1}x^{[n-2r]}$$

$$-\kappa p(-1)^{r-1}[n-1]p^{r-1.r+1}\frac{[2n-2r-2]!}{[n-r-1]![n-2r]![r-1]!(2)_{n-r-1}(2)_{r-1}}\lambda^{n-2r}x^{[n-2r]}$$

These may be written

$$(-1)^r p^{r.r+2}\frac{[2n-2r]!}{[n-r]![n-2r]![r]!(2)_{n-r}(2)_r}\left\{\frac{p^{3n-2r-1}-p^{n-2r}-p^{2n-1}+1-\kappa p^{-2r}(p^{n+2r-1}-p^{2r}-p^{n-1}+1)}{(p-1)^2[2n-2r-1]}\right\}$$

Putting $\kappa = p^n$ the large bracket reduces to

$$\frac{(p^n-1)(p^{2n-2r-1}-1)}{(p-1)^2[2n-2r-1]}$$

which is

$$[n]$$

so that the coefficient of $x^{[n-2r]}$ is

$$(-1)^r p^{r.r+2}[n]\frac{[2n-2r]!}{[n-r]![n-2r]![r]!(2)_{n-r}(2)_r}\lambda^{n-2r}$$

and the series is

$$[n]P_{[n]}(x\lambda)$$

establishing

$$[n]P_{[n]}(x\lambda) = \lambda[2n-1]xP_{[n-1]}(x^p\lambda) - p^{n+1}[n-1]P_{[n-2]}(x\lambda) \quad . \quad . \quad (45)$$

9.

Another property of the function is

$$\lambda^2 x^{[2]}\frac{dP_{[n]}(x^p\lambda)}{d(x^p)} - p\frac{dP_{[n]}(x^{\frac{1}{p}}\lambda)}{d(x^{\frac{1}{p}})} = \lambda^2[n]\left\{xP_{[n]}(x^p\lambda) - \frac{p}{\lambda}P_{[n-1]}(x\lambda)\right\} \quad . \quad (46)$$

By means of this we establish that

$$P_{[n]}(1\cdot p^{\frac{1}{2}}) = p^{\frac{1}{2}}P_{[n-1]}(1\cdot p^{\frac{1}{2}}) \quad . \quad . \quad . \quad . \quad (47)$$

and if n be integral

$$P_{[n]}(1\cdot p^{\frac{1}{2}}) = p^{\frac{n}{2}} \quad . \quad . \quad . \quad . \quad (48)$$

which corresponds to the theorem that the sum of the coefficients in LEGENDRE's series is equal to unity. The proof of (46) is as follows :

$$P_{[n]}(x^p\lambda) = \sum_0^\infty(-1)^r p^{r.r+2}\lambda^{n-2r}\frac{[n-2r]!}{[r]![n-r]![n-2r]!(2)_{n-r}(2)_r}x^{p[n-2r]} \quad . \quad (49)$$

therefore

$$\frac{d}{d(x^p)}\left\{P_{[n]}(x^p\lambda)\right\} = \sum_0^\infty(-1)^r p^{r.r+2}\lambda^{n-2r}\frac{[n-2r]!}{[r]![n-r]![n-2r]!(2)_{n-r}(2)_r}[n-2r]x^{p[n-2r-1]}$$

Similarly,

$$\frac{d}{d(x^{\frac{1}{p}})}\left\{P_{[n]}(x^{\frac{1}{p}}\lambda)\right\} = \sum_0^\infty(-1)^{r-1}p^{r-1.r+1}\lambda^{n-2r+2}\frac{[n-2r+2]!\cdot[n-2r+2]}{[r-1]![n-r+1]![n-2r+2]!(2)_{n-r+1}(2)_{r-1}}x^{[n-2r+1]}$$

The coefficient of $x^{[n-2r+1]}$ in the expression

$$\lambda^2 x^{[2]} \frac{d}{d(x^p)} \left\{ \mathrm{P}_{[n]}(x^p\lambda) \right\} - p \frac{d}{d(x^{\frac{1}{p}})} \left\{ \mathrm{P}_{[n]}(x^{\frac{1}{p}}\lambda) \right\}$$

is

$$(-1)^r p^{r.r+2} \lambda^{n-2r+2} \frac{[2n-2r]!}{[r]![n-r]![n-2r]!\,(2)_r(2)_{n-r}} \left\{ [n-2r] + p^{-2r} \frac{[2n-2r+1][2n-2r+2][2r][n-2r+2]}{[n-r+1][n-2r+1][n-2r+2](p^{n-r+1}+1)} \right\}$$

Now since

$$[n-r+1](p^{n-r+1}+1) = [2n-2r+2]$$

the expression within the large bracket reduces to

$$[n-2r] + p^{-2r} \frac{[2n-2r+1][2r]}{[n-2r+1]}$$

which is

$$\frac{p^{-2r}[n][n+1]}{[n-2r+1]}$$

and we may write

$$\lambda^2 x^{[2]} \frac{d\mathrm{P}_{[n]}(x^p\lambda)}{d(x^p)} - p \frac{d\mathrm{P}_{[n]}(x^{\frac{1}{p}}\lambda)}{d(x^{\frac{1}{p}})} = [n][n+1] \sum_{r=0}^{r=\infty} (-1)^r p^{r^2} \lambda^{n-2r+2} \frac{[2n-2r]!}{[r]![n-r]![n-2r+1]!\,(2)_r(2)_{n-r}} x^{[n-2r+1]} \quad (50)$$

Now consider

$$cx\mathrm{P}_{[n]}(x^p\lambda) - c_1\mathrm{P}_{[n-1]}(x\lambda)$$

The coefficient of $x^{[n-2r+1]}$ in this expression is

$$c(-1)^r p^{r.r+2} \lambda^{n-2r} \frac{[2n-2r]!}{[r]![n-r]![n-2r]!\,(2)_r(2)_{n-r}} - c_1(-1)^{r-1} p^{r-1.r+1} \lambda^{n-2r+1} \frac{[2n-2r]!}{[r-1]![n-r]![n-2r+1]!\,(2)_{r-1}(2)_{n-r}}$$

which may be written

$$(-1)^r p^{r^2} \lambda^{n-2r+2} \frac{[2n-2r]!}{[r]![n-r]![n-2r]!\,(2)_r(2)_{n-r}} \left\{ cp^{2r} \frac{1}{\lambda^2} + c_1 \frac{1}{p\lambda} \frac{[r](p^r+1)}{[n-2r+1]} \right\}$$

If now c be chosen as $\lambda^2[n]$

and c_1 ,, ,, $\lambda p[n]$

the large bracket reduces to

$$\frac{[n][n+1]}{[n-2r+1]}$$

and we have

$$\lambda^2[n]x\mathrm{P}_{[n]}(x^p\lambda) - p\lambda[n]\mathrm{P}_{[n-1]}(x\lambda) = \sum_{r=0}^{r=\infty} [n][n+1] p^{r^2} \lambda^{n-2r+2} \frac{[2n-2r]!}{[r]![n-r]![n-2r+1]!\,(2)_r(2)_{n-r}} x^{[n-2r+1]} \quad (51)$$

and this series has been shown in (50) to be

$$\lambda^2 x^{[2]} \frac{d\mathrm{P}_{[n]}(x^p\lambda)}{d(x^p)} - p \frac{d\mathrm{P}_{[n]}(x^{\frac{1}{p}}\lambda)}{d(x^{\frac{1}{p}})}$$

$$= \lambda^2[n] \left\{ x\mathrm{P}_{[n]}(x^p\lambda) - \frac{p}{\lambda} \mathrm{P}_{[n-1]}(x\lambda) \right\} \quad . \quad . \quad . \quad . \quad . \quad (52)$$

We see that if $\lambda = p^{\frac{1}{2}}$
and $x = 1$
the left side of the above expression becomes identically zero. Therefore when $\lambda = p^{\frac{1}{2}}$ and $x = 1$

$$\lambda^2[n]\left\{ xP_{[n]}(x^p\lambda) - \frac{p}{\lambda}P_{[n-1]}(x\lambda) \right\}$$

must also be zero, which is

$$P_{[n]}(1 \cdot p^{\frac{1}{2}}) = p^{\frac{1}{2}}P_{[n-1]}(1 \cdot p^{\frac{1}{2}}) \quad . \quad . \quad . \quad (53)$$

from which we have for integral values of n

$$\begin{aligned} P_{[1]}(1 \cdot p^{\frac{1}{2}}) &= p^{\frac{1}{2}}P_{[0]}(1 \cdot p^{\frac{1}{2}}) \\ P_{[2]}(1 \cdot p^{\frac{1}{2}}) &= p^{\frac{1}{2}}P_{[1]}(1 \cdot p^{\frac{1}{2}}) \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ P_{[n]}(1 \cdot p^{\frac{1}{2}}) &= p^{\frac{1}{2}}P_{[n-1]}(1 \cdot p^{\frac{1}{2}}) \end{aligned} \quad . \quad . \quad . \quad . \quad (54)$$

and so taking the product of the two sides of the equations

$$P_{[n]}(1 \cdot p^{\frac{1}{2}}) = p^{\frac{n}{2}}P_0(1 \cdot p^{\frac{1}{2}}) = p^{\frac{n}{2}} \quad . \quad . \quad . \quad (55)$$

which is

$$\frac{[2n]!\, p^{\frac{n}{2}}}{[n]!\,[n]!\,(2)_n}\left\{ 1 - p^2\frac{[n]\,[n-1]}{[2]\,[n-1]} + p^6\frac{[n]\,[n-1]\,[n-2]\,[n-3]}{[2]\,[4]\,[2n-1]\,[2n-3]} - \ldots . \right\} = p^{\frac{n}{2}} \quad (56)$$

that is

$$1 - p^2\frac{[n]\,[n-1]}{[2]\,[2n-1]} + \ldots . = \frac{[n]!\,[n]!\,(2)_n}{[2n]!} \quad . \quad . \quad . \quad (57)$$

the general term of the series being

$$p^{r.r+1}\frac{[n]\,[n-1]\,[n-2]\,\ldots .\,[n-2r+1]}{[2]\,[4]\,.\,.\,[2r]\,.\,[2n-1]\,.\,.\,.\,[2n-2r+1]}$$

If we put $p = 1$ this reduces to a well-known series

$$1 - \frac{n \cdot n-1}{2 \cdot 2n-1} + \frac{n \cdot n-1 \cdot n-2\, n-3}{2 \cdot 4 \cdot 2n-1 \cdot 2n-3} - \ldots = \frac{n!\,n!\,2^n}{2n!}$$

10.

The series

$$y = A\left\{ x^{[n-\nu]} - p^3\frac{[n-\nu]\,[n-\nu-1]}{[2]\,[2n-1]}\frac{p^{2\nu}}{\lambda^2}x^{[n-\nu-2]} + \ldots . \right\} \quad . \quad . \quad (58)$$

is a solution of a differential equation of the form

$$px^{[2]}\frac{d^{(2)}y}{dx^{(2)}} - \lambda^{-2}\frac{d^{(2)}y}{dx^{(2)}} + \left\{ 1 - [n-\nu] - [-n-\nu-1] \right\} x\frac{dy}{dx} + [n-\nu]\,[-n-\nu-1]y = f(x) - f(x^{p^2}) \quad (59)$$

for, assuming that y can be expressed as a convergent or finite series of the form

$$y = \sum Ax^{[m]}$$

performing the operations indicated on the left side of the differential equation we obtain

$$\{p[m_1][m_1-1]+\{1-[n-\nu]-[-n-\nu-1]\}[m_1]+[n-\nu][-n-\nu-1]\}A_1x^{[m_1]}-\lambda^{-2}[m_1][m_1-1]A_1x^{p^2[m_1-2]}$$

+similar expressions in $m_2, m_3, \ldots$ (60)

Since $$p[m_1][m_1-1]=[m_1]^2-[m_1]$$

the expression which is the coefficient of $A_1x^{[m_1]}$ in (60) reduces to

$$\{[m_1]-[n-\nu]\}\{[m_1]-[n-\nu-1]\}A_1$$

so that we have altogether the series

$$\begin{aligned}&\{[m_1]-[n-\nu]\}\{[m_1]-[-n-\nu-1]\}A_1x^{[m_1]}-\lambda^{-2}[m_1][m_1-1]A_1x^{p^2[m_1-2]}\\&+\{[m_2]-[n-\nu]\}\{[m_2]-[-n-\nu-1]\}A_2x^{[m_2]}-\lambda^{-2}[m_2][m_2-1]A_2x^{p^2[m_2-2]}\\&+\ldots\ldots\ldots\ldots\ldots\end{aligned} \quad (61)$$

Choose
$$\begin{aligned}m_2&=m_1-2\\m_3&=m_2-2\\&\ldots\ldots\\m_{r+1}&=m_r-2\end{aligned}$$

Also choose

$$\{[m_{r+1}]-[n-\nu]\}\{[m_{r+1}]-[-n-\nu-1]\}A_{r+1}=\lambda^{-2}[m_r][m_r-1]A_r$$

Let m_1 also be so chosen that the coefficient of $x^{[m_1]}$ may vanish, then

$$m_1=n-\nu \quad \text{or} \quad -n-\nu-1$$

For the value
$$\begin{aligned}m_1&=n-\nu\\m_{r+1}&=n-\nu-2r\end{aligned}$$

we have

$$A_{r+1}=-A_r\frac{p^{2\nu+2r+1}}{\lambda^2}\frac{[n-\nu-2r+2][n-\nu-2r+1]}{[2r][2n-2r+1]} \quad (62)$$

and for the value
$$\begin{aligned}m_1&=-n-\nu-1\\m_{r+1}&=-n-\nu-1-2r\end{aligned}$$

we have

$$A_{r+1}=A_r\frac{p^3}{\lambda^2}\frac{[n+\nu+2r-1][n+\nu+2r]}{[2r][2n+2r+1]} \quad (63)$$

From relation (62) we have the series

$$y=A\left\{x^{[n-n]}-p^3\frac{[n-\nu][n-\nu-1]}{[2][2n-1]}\left(\frac{p^n}{\lambda}\right)^2x^{[n-\nu-2]}+\ldots\ldots\ldots\right\} \quad (64)$$

a solution of the differential equation

$$px^{[2]}\frac{d^{(2)}y}{dx^{(2)}}-\lambda^{-2}\frac{d^{(2)}y}{dx^{(2)}}+\left\{1-[n-\nu]-[-n-\nu-1]\right\}x\frac{dy}{dx}+[n-\nu][-n-\nu-1]y=f(x)-f(x^{p^2}) \quad (65)$$

$f(x)$ denoting the function

$$[n-\nu][n-\nu-1]A\left\{x^{[n-\nu-2]}-p^3\frac{[n-\nu-2][n-\nu-3]}{[2][2n-1]}\frac{p^{2\nu}}{\lambda^2}x^{[n-\nu-4]}+\ldots\ldots\ldots\right\} \quad (66)$$

We also have from relation (63)

$$y = \mathrm{A}\left\{ x^{[-n-\nu-1]} + p^3 \frac{[n+\nu+1]\,[n+\nu+2]}{[2]\,[2n+3]} \lambda^{-2} x^{[-n-\nu-3]} + \ldots\ldots \right\} \quad . \quad (67)$$

a solution of the differential equation in which $f(x)$ denotes

$$\frac{[n+\nu+1]\,[n+\nu+2]}{p^{2n+2\nu+3}} \mathrm{A}\left\{ x^{[-n-\nu-3]} + p^3 \frac{[n+\nu+3]\,[n+\nu+4]}{[2]\,[2n+3]} \lambda^{-2} x^{[-n-\nu-5]} + \ldots\ldots \right\} \quad (68)$$

Since λ is quite arbitrary, replace it by λp^{ν} and we then have the series

$$y = \text{const.} \left\{ \lambda^{n-\nu} x^{[n-\nu]} - \frac{[n-\nu]\,[n-\nu-1]}{[2]\,[2n-1]} p^3 \lambda^{n-\nu-2} x^{[n-\nu-2]} + \ldots \right\} \quad . \quad (69)$$

a solution of

$$\lambda x^{[2]} \frac{d^{(2)}y}{dx^{(2)}} - \frac{1}{\lambda^2 p^{2\nu}} \mathrm{D}^{(2)} + \left\{ 1 - [n-\nu] - [-n-\nu-1] \right\} x \frac{dy}{dx} + [n-\nu]\,[-n-\nu-1] y = f(x) - f(x^{p^2}) \quad (70)$$

The series (69) is when ν is integral, and $< n$

$$\frac{d^{(\nu)} \mathrm{P}_{[n]}(x^{\frac{1}{p^{\nu}}} \lambda)}{d(x^{\frac{1}{p^{\nu}}})^{(\nu)}} \quad . \quad . \quad . \quad . \quad . \quad (71)$$

The series and differential equation are analogous to one given by Heine

$$(1-x^2) \frac{d^2y}{dx^2} - 2(\nu+1) x \frac{dy}{dx} + (n+\nu+1)(n-\nu) y = 0 \quad . \quad . \quad (72)$$

of which the primitive is

$$y = \mathrm{A} \frac{d^{\nu} \mathrm{P}_n}{dx^{\nu}} + \mathrm{B} \frac{d^{\nu} \mathrm{Q}_n}{dx^{\nu}} \qquad < n \quad . \quad . \quad (73)$$

The sum of the series of coefficients in Heine's series

$$1 - \frac{(n-\nu)(n-\nu-1)}{2 \cdot 2n-1} + \frac{(n-\nu)(n-\nu-1)(n-\nu-2)(n-\nu-3)}{2 \cdot 4 \cdot 2n-1 \cdot 2n-3} - \ldots\ldots$$

is shown in *Chrystal's Algebra*, Part II. page 185, or 209 (2nd edition), to be

$$2^{n-\nu} \frac{n!\,n+\nu!}{\nu!\,2n!} \quad . \quad . \quad . \quad . \quad . \quad (74)$$

The analogous theorem in the general series is

$$1 - p^{2\nu+2} \frac{[n-\nu]\,[n-\nu-1]}{[2]\,[2n-1]} + p^{4\nu+6} \frac{[n-\nu]\,[n-\nu-1]\,[n-\nu-2]\,[n-\nu-3]}{[2]\,[4]\,[2n-1]\,[2n-3]} - \ldots$$
$$= \frac{(2)_n}{(2)_{\nu}} \cdot \frac{[n]!\,[n+\nu]!}{[\nu]!\,[2n]!} \quad . \quad . \quad . \quad . \quad (75)$$

PART II.

1.

The series

$$y = A\left\{ x^{[n-\nu]} - p^3\frac{[n-\nu]\,[n-\nu-1]}{[2]\,[2n-1]}\,\frac{1}{\lambda^2}x^{[n-\nu-2]} + \ldots\ldots \right\} \quad . \quad . \quad (1)$$

has a general term

$$A(-1)^r p^{r.r+2}\frac{[n-\nu]\,[n-\nu-1]\ldots\ldots\ldots\ldots\ldots[n-\nu-2r+1]}{[2]\,[4]\ldots\ldots[2r]\cdot[2n-1]\,[2n-3]\ldots\ldots[2n-2r+1]}\,\frac{1}{\lambda^{2r}}x^{[n-\nu-2r]}$$

which may be written if $n-\nu$ be positive and integral

$$(-1)^r A\cdot p^{r.r+2}\frac{[n-\nu]!\,[n]!\,(2)_n[2n-2r]!}{[2n]!\,[n-r]!\,[n-\nu-2r]!\,[r]!\,(2)_r(2)_{n-r}}\,\frac{1}{\lambda^{2r}}x^{[n-\nu-2r]}$$

so that if we give to the arbitrary constant A the value

$$\lambda^{n-\nu}\frac{[2n]!}{[n]!\,[n-\nu]!\,(2)_n}$$

we have the series

$$y = \sum_{r=0}^{r=\infty}(-1)^r p^{r.r+2}\frac{[2n-2r]!}{[n-r]!\,[n-\nu-2r]!\,[r]!\,(2)_r(2)_{n-r}}\cdot\lambda^{n-\nu-2r}x^{[n-\nu-2r]} \quad . \quad (2)$$

which will be denoted by $P^{\nu}_{[n]}(x,\lambda)$

When n and ν are not integral

$$P^{\nu}_{[n]}(x.\lambda) = \frac{\Pi([2n])\lambda^{n-\nu}}{\Pi([n])\Pi([n-\nu])\cdot(2)_n}\left\{ x^{[n-\nu]} - p^3\frac{[n-\nu]\,[n-\nu-1]}{[2]\,[2n-1]}\cdot\frac{x^{[n-\nu-2]}}{\lambda^2} + \ldots \right\}$$

When $p=1$ this function reduces to the function denoted $\varpi(n\nu)$ by TODHUNTER (*Functions of Laplace, Lamé, and Bessel*, p. 80).

We see that

$$\frac{d^{(r)}P^{\nu}_{[n]}(x.\lambda)}{dx^{(r)}} = \lambda^r P^{\nu+r}_{[n]}(x^{p^r}.\lambda) \quad . \quad . \quad . \quad . \quad (3)$$

Also

$$P^{\nu}_{[n]}(x\lambda) = \lambda^{-\nu}\frac{d^{(\nu)}}{d\left(x^{\frac{1}{p^\nu}}\right)^{(\nu)}}\cdot\left\{ P_{[n]}\left(x^{\frac{1}{p^\nu}}.\lambda\right)\right\} \quad . \quad . \quad . \quad (4)$$

or

$$P^{\nu}_{[n]}\left(x^{\frac{1}{p^\nu}}.\lambda\right) = \lambda^{-\nu}\frac{d^{(\nu)}P_{[n]}(x.\lambda)}{dx^{(\nu)}} \quad . \quad . \quad . \quad . \quad (5)$$

For brevity $P^{\nu}_{[n]}$ will often be used to denote $P^{\nu}_{[n]}(x.\lambda)$.

The function $P^{\nu}_{[n]}$ satisfies the differential equation

$$px^{[2]}\frac{d^{(2)}P^{\nu}_{[n]}}{dx^{(2)}} - \frac{1}{\lambda^2 p^{2\nu}}\,\frac{d^{(2)}P^{\nu}_{[n]}}{dx^{(2)}} + \{1-[n-\nu]-[-n-\nu-1]\}x\frac{dP^{\nu}_{[n]}}{dx} + [n-\nu]\,[-n-\nu-1]P^{\nu}_{[n]}$$

$$= \frac{1}{p^{2\nu}}\left\{ P^{\nu+2}_{[n]}(x\lambda) - P^{\nu+2}_{[n]}(x^{p^2}\lambda)\right\}$$

$$= \frac{1}{\lambda^2 p^{2\nu}}\left\{ \frac{d^{(2)}P^{\nu}_{[n]}(x^{p^{-2}}\lambda)}{d(x^{p^{-2}})^{(2)}} - \frac{d^{(2)}P^{\nu}_{[n]}(x.\lambda)}{dx^{(2)}}\right\} \quad . \quad . \quad . \quad (6)$$

When $p=1$ the right side of the differential equation vanishes and the whole equation reduces to

$$(x^2-\lambda^{-2})\frac{d^2y}{dx^2}+2(\nu+1)\frac{dy}{dx}-(n-\nu)(n+\nu+1)y=0$$

2.

In this article we shall show that

$$1-p^2\frac{[n-\nu]\,[n-\nu-1]}{[2]\,[2n-1]}p^{2\nu}+p^6\frac{[n-\nu]\,[n-\nu-1]\,[n-\nu-2]\,[n-\nu-3]}{[2]\,[4]\,[2n-1]\,[2n-3]}p^{4\nu}-\ldots\ldots \quad (7)$$

of which the general term is

$$p^{r,r+1}\frac{[n-\nu]\,[n-\nu-1\ldots\ldots[n-\nu-2r+1}{[2]\,[4]\ldots[2r]\cdot[2n-1]\ldots\ldots[2n-2r+1]}p^{2r\nu}$$

may be expressed as a convergent infinite product

$$\frac{\Pi([n])\Pi([n+\nu])\cdot(2)_n}{\Pi([\nu])\Pi([2n])\cdot(2)_\nu} \quad . \quad . \quad . \quad . \quad (8)$$

reducing when n and ν are integers to

$$\frac{[n]!\,[n+\nu]!\,(2)_n}{[\nu]!\,[2n]!\,(2)_\nu} \quad . \quad . \quad . \quad . \quad (9)$$

as was anticipated in art. (10), Part I., by analogy with the sum of

$$1-\frac{n-\nu\cdot n-\nu-1}{2\cdot 2n-1}+\frac{n-\nu\cdot n-\nu-1\cdot n-\nu-2\cdot n-\nu-3}{2\cdot 4\cdot 2n-1\cdot 2n-3}$$
$$=\frac{n!\;n+\nu!\;2^{n-\nu}}{\nu!\;2n!}.$$

In the *Proceedings of the London Mathematical Society*, series 2, vol. i., "Series connected with the Enumeration of Partitions," the following theorem is obtained :—

$$[\mathrm{P}^{x+y}z]^m=[\mathrm{P}^xz]^m+\sum_{r=1}^{r=\infty}\mathrm{P}^{rx}\left\{{m\atop r}\right\}[\mathrm{P}^xz]^{m-r}(\mathrm{P}^y)_r z^r \quad . \quad . \quad . \quad (10)$$

$[\mathrm{P}^xz]^m$ denoting $\mathrm{L}_{\kappa=\infty}\dfrac{(\mathrm{P}^{x+\overline{m-1}l}z+1)(\mathrm{P}^{x+\overline{m-2}l}z+1)\ldots(\mathrm{P}^{x+\overline{m-\kappa}l}z+1)}{(\mathrm{P}^{x-l}z+1)(\mathrm{P}^{x-2l}z+1)\ldots(\mathrm{P}^{x-\kappa l}z+1)}$, $\mathrm{P}^l>1$

$(\mathrm{P}^y)_r$,, $(\mathrm{P}^y-1)(\mathrm{P}^{y+l}-1)\ldots(\mathrm{P}^{y+\overline{r-1}l}-1)$

$\left\{{m\atop r}\right\}$,, $\dfrac{\mathrm{P}^{ml}-1\cdot\mathrm{P}^{\overline{m-1}l}-1\ldots\ldots\mathrm{P}^{\overline{m-r+1}l}-1}{\mathrm{P}^l-1\cdot\mathrm{P}^{2l}-1\ldots\ldots\mathrm{P}^{rl}-1}$

P^i ,, $p_1^{l_1}\cdot p_2^{l_2}\cdot p_3^{l_3}p_4^{l_4}\ldots\ldots p_s^{l_s}$

$\mathrm{P}^{x+\overline{m-1}l}$,, $p_1^{x_1+\overline{m-1}l_1}p_2^{x_2+\overline{m-2}l_2}\ldots\ldots p_s^{x_s+\overline{m-1}}$

$l_1\ l\ \ldots\ldots l_s$

$p_1 p_2\ldots\ldots p_s$

$x_1 x_2\ldots\ldots x_s$

$y_1 y_2\ldots\ldots y_s$

being sets of s independent elements.

When m is integral and positive $[P^x z]^m$ reduces to the product of m factors

$$(P^x z+1)(P^{x+l} z+1) \ldots\ldots (P^{x+\overline{m-1}l} z+1)$$

Dividing both sides of (10) by $[P^x z]^m$ we have

$$\frac{[P^{x+y} z]^m}{[P^x z]^m} = 1 + \sum_{r=1}^{r=\infty} P^{rx} \frac{P^{ml}-1 \ldots\ldots P^{\overline{m-r+1}l}-1}{P^l-1 \ldots\ldots P^{rl}-1} \cdot \frac{P^y-1 \ldots\ldots P^{y+\overline{r-1}l}}{P^{x+\overline{m-1}l} z+1 \ldots\ldots P^{x+m-r+1l} z+1} z^r \qquad (11)$$

putting

$$\begin{aligned} l &= 2 \\ m &= \frac{n-\nu}{2} \\ z &= -1 \\ y &= -n+\nu+1 \\ x &= n+\nu+1 \end{aligned}$$

the general term of the series becomes

$$(-1)^r P^{r(n+\nu+1)} \frac{P^{n-\nu}-1 \ldots\ldots P^{n-\nu-2r+2}-1}{P^2-1 \ldots\ldots P^{2r}-1} \cdot \frac{P^{-n+\nu+1}-1 \ldots\ldots P^{-n+\nu+2r-1}-1}{1-P^{2n-1} \ldots\ldots 1-P^{2n-2r+1}}$$
$$(-1)^r P^{r^2+r} \frac{[n-\nu][n-\nu-1] \ldots\ldots [n-\nu-2r+1]}{[2][4] \ldots [2r] \cdot [2n-1] \ldots\ldots [2n-2r+1]} P^{2r\nu}$$

which is the $r+1^{\text{th}}$ term in, Const. $\times P^{\nu}_{[n]}(1 \,.\, p^{\frac{1}{2}-\nu})$.

The infinite product $\dfrac{[P^{x+y} z]^m}{[P^x z]^m}$, when we make the same substitutions as in the series, becomes

$$\underset{\kappa=\infty}{\mathrm{L}} \frac{(P^{n+\nu}-1)(P^{n+\nu-2}-1) \ldots (P^{n+\nu-2\kappa+2}-1) \cdot (P^{n+\nu-1}-1) \ldots (P^{n+\nu-2\kappa+1}-1)}{(P^{2\nu}-1)(P^{2\nu-2}-1) \ldots (P^{2\nu-2\kappa+2}-1) \cdot (P^{2n-1}-1) \ldots (P^{2n-2\kappa+1}-1)} \qquad (12)$$
$$P>1$$

which since

$$P^{2\nu}-1 \cdot P^{2\nu-2}-1 \ldots\ldots P^{2\nu-2\kappa+2}-1$$
$$= (P^{\nu}-1)(P^{\nu-1}-1) \ldots (P^{\nu-\kappa+1}-1) \cdot (P^{\nu}+1) \ldots (P^{\nu-\kappa+1}+1)$$

and

$$P^{2n-1}-1 \cdot P^{2n-2}-1 \ldots\ldots P^{2n-2\kappa+1}-1$$
$$= \frac{(P^{2n}-1)(P^{2n-1}-1) \ldots\ldots (P^{2n-2\kappa+1}-1)}{(P^{2n}-1)(P^{2n-2}-1) \ldots (P^{2n-2\kappa+2}-1)}$$

we are justified in writing

$$\frac{\Pi([n+\nu])\Pi([n]) \cdot (2)_n}{\Pi([2n])\Pi([\nu])(2)_{\nu}} \qquad (13)$$

by analogy with GAUSS's Π function.

Moreover, when n and ν are positive and integral, the infinite product reduces to

$$\frac{[n+\nu]!\,[n]!\,(2)_n}{[2n]!\,[\nu]!\,(2)_{\nu}}$$

We have now

$$\frac{\Pi([n+\nu])\Pi([n])(2)_n}{\Pi([2n])\Pi([\nu])(2)_{\nu}} = 1 - p^2 \frac{[n-\nu]\,[n-\nu-1]}{[2]\,[2n-1]} p^{2\nu} + \ldots\ldots \qquad (14)$$

and since

$$P^{\nu}_{[n]}(x\lambda) = \frac{\Pi([2n])\lambda^{n-\nu}}{\Pi([n])\Pi([n-\nu])(2)_n}\left\{ x^{[n-\nu]} - p^3\frac{[n-\nu][n-\nu-1]1}{[2][2n-1]}\frac{1}{\lambda^2}x^{[n-\nu-2]} + \ldots\ldots \right\}$$

putting

$$\lambda = p^{\frac{1}{2}-\nu}$$
$$x = 1$$

we have

$$P^{\nu}_{[n]}(1\cdot p^{\frac{1}{2}-\nu}) = \frac{\Pi([n+\nu]}{\Pi([\nu])\Pi([n-\nu])}p^{(\frac{1}{2}-\nu)(n-\nu)} \quad . \quad . \quad . \quad (15)$$

If $\nu = 0$ this gives

$$P_{[n]}(1\cdot p^{\frac{1}{2}}) = p^{\frac{n}{2}} \quad . \quad . \quad . \quad . \quad (16)$$

which was obtained in art. 9, Part I., by another method.

When $\nu = \frac{1}{2}$

$$P^{\frac{1}{2}}_{[n]}(1) = \frac{\Pi([n+\frac{1}{2}])}{\Pi([\frac{1}{2}])\Pi([n-\frac{1}{2}])} \quad . \quad . \quad . \quad . \quad (17)$$

which when $p = 1$ gives

$$P^{\frac{1}{2}}_{n} = \frac{1}{\sqrt{\pi}}\cdot\frac{\Pi(n+\frac{1}{2})}{\Pi(n-\frac{1}{2})} \quad . \quad . \quad . \quad . \quad (18)$$

3.

Consider

$$\frac{1}{[n]!}\frac{d^{(n+\nu)}}{dx^{(n+\nu)}}\left\{ \frac{x^{[2n]}\cdot\lambda^{2n}}{(2)_n} - p^3[n]\frac{x^{[2n-2]}\lambda^{2n-2}}{(2)_{n-1}(2)_1} + \ldots\ldots\ldots \right\} \quad . \quad . \quad (19)$$

the general term of the series within the bracket being

$$(-1)^r p^{r.r+2}\frac{[n]!}{[r]![n-r]!}\cdot\frac{x^{[2n-2r]}\lambda^{2n-2r}}{(2)_r(2)_{n-r}} \quad . \quad . \quad . \quad (20)$$

Then if we perform the operations indicated by $\frac{d^{(n+\nu)}}{dx^{(n+\nu)}}$ we obtain a series of which the general term is

$$+(-1)^r\frac{1}{[n]!}p^{r.r+2}\frac{[n]!}{[n-r]![r]!}\frac{[2n-2r]!}{[n-\nu-2r]!}\lambda^{2n-2r}\frac{x^{n+\nu[n-\nu-2r]}}{(2)_r(2)_{n-r}} \quad . \quad (21)$$

This reduces to

$$(-1)^r p^{r.r+2}\frac{[2n-2r]!}{[n-r]![n-\nu-2r]![r]!(2)_r(2)_{n-r}}\lambda^{2n-2r}x^{n+\nu[n-\nu-2r]}$$

viz., the general term of

$$\lambda^{n+\nu}P^{\nu}_{[n]}(xp^{n+\nu}\cdot\lambda)$$

$$\lambda^{n+\nu}P^{\nu}_{[n]}(xp^{n+\nu}\lambda) = \frac{1}{[n]!}D^{(n+\nu)}\left\{ \frac{x^{[2n]}\lambda^{2n}}{(2)_n} - p^3[n]\frac{x^{[2n-2]}\lambda^{2n-2}}{(2)_1(2)_{n-1}} + \ldots\ldots \right\} \quad . \quad (22)$$

reducing when $p = 1$ to

$$2^n\lambda^{\nu}P_n(x\lambda) = \frac{1}{n!}D^{n+\nu}\left\{ \lambda^2x^2 - 1 \right\}^n \quad . \quad . \quad . \quad (23)$$

The expression

$$\frac{x^{[2n]}\lambda^{2n}}{(2)_n} - p^3[n]\frac{x^{[2n-2]}\lambda^{2n-2}}{(2)_1(2)_{n-1}} + \ldots\ldots\ldots \quad . \quad . \quad (24)$$

cannot be expressed as a product of n factors analogous to $(\lambda^2x^2-1)^n$, but it possesses the property

$$\frac{\lambda^{2n}}{(2)_n}-p^3[n]\frac{\lambda^{2n-2}}{(2)_{n-1}(2)_1}+\ldots\ldots+(-1)^r p^{r.r+2}\frac{[n]!}{[r]![n-r]!}\frac{\lambda^{2n-2r}}{(2)_{n-r}(2)_r}-\ldots$$
$$\equiv\frac{(\lambda^2-p^3)(\lambda^2-p^5)\ldots\ldots(\lambda^2-p^{2n+1})}{(2)_n} \quad . \quad . \quad . \quad (25)$$

which is a particular case of

$$*\,[\mathrm{P},^x z]^m=1+\sum_{r=1}^{r=\infty}\mathrm{P}^{rz+\frac{r.r-1}{2}.l}\left\{\begin{matrix}m\\r\end{matrix}\right\}z^r \quad . \quad . \quad . \quad (26)$$
$$\mathrm{P}z<1$$

substituting

$$-\frac{1}{\lambda^2} \text{ for } z$$
$$3 \text{ ,, } x$$
$$2 \text{ ,, } l$$
$$n \text{ a positive integer for } m$$

$$\left[\mathrm{P}^3,-\frac{1}{\lambda^2}\right]^n=1+\sum_1^n(-1)^r p^{r.r+2}\frac{[2n][2n-2]\ldots[2n-2r+2]}{[2][4]\ldots[2r]}\cdot\frac{1}{\lambda^{2r}}$$

which may be differently expressed

$$\frac{(\lambda^2-p^3)(\lambda^2-p^5)\ldots\ldots(\lambda^2-p^{2n+1})}{(2)_n}=\frac{\lambda^{2n}}{(2)_n}-p^3[n]\frac{\lambda^{2n-2}}{(2)_{n-1}(2)_1}+\ldots\ldots\ldots \quad (27)$$

If we use the symbol $\mathrm{I}^{(n+\nu)}$ as the reverse of $\mathrm{D}^{(n+\nu)}$ we have from (22)

$$\mathop{\mathrm{I}}_{x=1}^{(n+\nu)}\mathrm{P}^{\nu}_{[n]}(x^{p^{n+\nu}}\cdot\lambda)=\frac{(\lambda^2-p^3)(\lambda^2-p^5)\ldots\ldots(\lambda^2-p^{2n+1})}{\lambda^{n+\nu}[n]!\,(2)_n} \quad . \quad . \quad (28)$$
$$=\frac{p^3-\lambda^2\cdot p^5-\lambda^2\ldots\ldots p^{2n+1}-\lambda^2}{p^2-1\cdot p^4-1\ldots\ldots p^{2n}-1}(-1)^n\frac{1}{\lambda^{n+\nu}}$$

of which particular cases are

$$\mathop{\mathrm{I}}_{x=1}^{(n+\nu)}\mathrm{P}^{\nu}_{[n]}(x^{p^{n+\nu}}\cdot p^{\frac{1}{2}})=p^{\frac{n-\nu}{2}} \text{ and } \mathop{\mathrm{I}}_{x=1}^{(n+\nu)}\mathrm{I}^{\nu}_{[n]}(x^{p^{n+\nu}}\cdot p)=p^{n-\nu}\frac{[2n]!}{[n]![n]!\,(2)_n(2)_n} \quad . \quad (29)$$

4.

It has been shown in art. 9, Part I., that if

$$y=\mathrm{A}\left\{x^{[-n-\nu-1]}+\frac{p^3}{\lambda^2}\frac{[n+\nu+1][n+\nu+2]}{[2][2n+3]}x^{[-n-\nu-3]}+\ldots\ldots\right\} \quad . \quad (30)$$

and $f(x)$ denote

$$p^{-2n-2\nu-3}[n+\nu+1][n+\nu+2]\mathrm{A}\left\{x^{[-n-\nu-3]}+\frac{p^3}{\lambda^2}\frac{[n+\nu+3][n+\nu+4]}{[2][2n+3]}x^{[-n-\nu-3]}+\ldots\ldots\right\}$$

then

$$px^{[2]}\frac{d^{(2)}y}{dx^{(2)}}-\frac{1}{\lambda^2}\frac{d^{(2)}y}{dx^{(2)}}+\left\{1-[n-\nu]-[-n-\nu-1]\right\}x\frac{dy}{dx}+[n-\nu][-n-\nu-1]y=f(x)-f(x^{p^2})$$

If we change λ to λp^{ν} this differential equation is identical in form with the differential equation satisfied by $\mathrm{P}^{[n]}_{\nu}(x,\lambda)$.

* *Proc. Lond. Math. Soc.*, "Series connected with the Enumeration of Partitions," series 2, vol. i.

The general term in the series $Q^{\nu}_{[n}(x.\lambda)$ is

$$\frac{[n+2r]!\,[n+r]!\,(2)_{n+r}}{[r]!\,[2n+2r+1]!\,(2)_r}p^{3r}\lambda^{-n-2r-1}x^{[-n-2r-1]}$$

so that

$$\frac{d^{(\nu)}Q_{[n]}(x\lambda)}{dx^{(\nu)}} = \sum\frac{[n+2r]!\,[n+r]!\,(2)_{n+r}[-n-2r-1]\ldots.[-n-2r-\nu]}{[r]!\,[2n+2r+1]!\,(2)_r}p^{3r}\lambda^{-n-2r-1}x^{p^{\nu}[-n-2r-\nu-1]} \quad (31)$$

Now the general term of the series (30) may be written

$$Ap^{3r}\frac{[2n+1]!\,[n+\nu+2r]!\,[n+r]!\,(2)_{n+r}}{[n+\nu]!\,[r]!\,[2n+2r+1]!\,(2)_r}x^{[-n-\nu-2r-1]}\lambda^{-2r} \quad . \quad . \quad (32)$$

so that if we give to the arbitrary constant A the value

$$\lambda^{-n-\nu-1}\frac{[n+\nu]!}{[2n+1]!}$$

and denote the series by $Q_{[n]}(x\lambda)$

then

$$Q^{\nu}_{[n]}(x\lambda) = \sum_0^{\infty}\frac{[n+\nu+2r]!\,[n+r]!\,(2)_{n+r}}{[2n+2r+1]!\,[r]!\,(2)_r}p^{3r}\lambda^{-n-\nu-2r-1}x^{[-n-\nu-2r-1]} \quad . \quad (33)$$

and

$$\frac{d^{(\nu)}Q_{[n]}}{dx^{(\nu)}} = \lambda^{\nu}p^{\frac{\nu.\nu+1}{2}}Q^{\nu}_{[n]}(x^{p^{\nu}}\cdot\lambda p^{\nu}) \quad . \quad . \quad . \quad (34)$$

If we had chosen $Q^{\nu}_{[n]}(x\lambda)$ to denote the series which satisfies

$$px^{[2]}\frac{d^{(2)}y}{dx^{(2)}} - \frac{1}{\lambda^2p^{2\nu}}\frac{d^{(2)}y}{dx^{(2)}} + \text{etc.}$$

instead of that which satisfies

$$px^{[2]}\frac{d^{(2)}y}{dx^{(2)}} - \frac{1}{\lambda^2}\frac{d^{(2)}y}{dx^{(2)}} + \text{etc.}$$

equation (34) would have taken the form

$$\frac{d^{(\nu)}Q_{[n]}(x\lambda)}{dx^{(\nu)}} = \lambda^{\nu}p^{\frac{\nu.\nu+1}{2}}Q^{\nu}_{[n]}(xp^{\nu}\lambda)$$

$Q^{\nu}_{[n]}(x,\lambda)$ satisfies

$$px^{[2]}\frac{d^{(2)}Q^{\nu}_{[n]}}{dx^{(2)}} - \frac{1}{\lambda^2}\frac{d^{(2)}Q^{\nu}_{[n]}}{dx^{(2)}} + \left\{1-[n-\nu]-[-n-\nu-1]\right\}x\frac{dQ^{\nu}_{[n]}}{dx} + [n-\nu][-n-\nu-1]Q^{\nu}_{[n]}$$
$$= \lambda^2p^3\left\{Q^{\nu+2}_{[n]}(x\cdot\lambda p^2) - Q^{\nu+2}_{[n]}(x^{p^2}\cdot\lambda p^2)\right\} \quad . \quad . \quad . \quad (35)$$

To find the sum of the coefficients of x in the series $Q^{\nu}_{[n]}$ we make the following substitutions,

$$m = -\tfrac{n+\nu+1}{2}$$
$$l = 2$$
$$y = n+\nu+2$$
$$z = -1$$
$$x = -n+\nu$$

in the series (11).

The general term

$$P^{rx}\frac{P^{ml}-1\ .\ .\ .\ .\ P^{\overline{m-r+1}l}-1}{P^{l}-1\ .\ .\ .\ .\ P^{rl}-1}\ \frac{P^{y}-1\ .\ .\ .\ .\ .\ .\ .\ P^{y+\overline{r-1}l}-1}{P^{x+\overline{m-1}l}z+1\ .\ .\ .\ P^{x+m-rl}z+1}z^{r}$$

becomes

$$P^{r(-n+\nu)}\frac{P^{-n-\nu-1}-1\ .\ .\ .\ .\ .\ P^{-n-\nu-2r+1}-1}{P^{2}-1\ .\ .\ .\ .\ P^{2r}-1}\cdot\frac{P^{n+\nu+2}-1\ .\ .\ .\ .\ .\ .\ P^{n+\nu+2r}-1}{1-P^{-2n-3}\ .\ .\ .\ .\ .\ 1-P^{-2n-2r-1}}(-1)^{r}$$

$$P^{2r}\frac{[n+\nu+1][n+\nu+2]\ .\ .\ .\ .\ .\ [n+\nu+2r]}{[2][4]\ .\ .\ .\ [2r]\cdot[2n+3]\ .\ .\ .\ .\ [2n+2r+1]}\quad .\quad .\quad (36)$$

and the infinite product

$$\frac{[P^{x+y}z]^{m}}{[P^{x}z]^{m}}\qquad \begin{matrix}\cdot\cdot & \cdot\cdot\\ \cdot\cdot & \cdot\cdot\end{matrix}$$

becomes

$$\mathop{\mathrm{L}}_{\kappa=\infty}\frac{[\nu-n-1][\nu-n-2]\ .\ .\ .\ .\ .\ .\ [\nu-n-2\kappa]}{[2\nu][2\nu-2][2\nu-4]\ .\ .\ .\ [2\nu-2\kappa+2]\cdot[-2n-3]\ .\ .\ [-2n-2\kappa-1]}\qquad (37)$$

so that we have this product

$$=1+p^{2}\frac{[n+\nu+1][n+\nu+2]}{[2][2n+3]}+p^{4}\frac{[n+\nu+1][n+\nu+2][n+\nu+3][n+\nu+4]}{[2][4][2n+3][2n+5]}$$

$$=\frac{\Pi([2n+1])}{\Pi([n+\nu])\Pi([n])\cdot(2)_{n}}p^{\frac{n+\nu+1}{2}}Q^{\nu}_{[n]}(1\cdot p^{\frac{1}{2}})\quad .\quad .\quad .\quad .\quad .\quad (38)$$

5.

Now take the differential equation

$$px^{[2]}\frac{d^{(2)}y}{dx^{(2)}}-\frac{1}{\lambda^{2}p^{2\nu}}\frac{d^{(2)}y}{dx^{(2)}}+\{1-[n-\nu]-[-n-\nu-1]\}x\frac{dy}{dx}+[n-\nu][-n-\nu-1]y$$
$$=f(x)-f(x^{p^{2}})\qquad (39)$$

and find a solution in the form of a series proceeding according to ascending powers of x.

Assume
$$y=A_{1}x^{[m_{1}]}+A_{2}x^{[m_{2}]}+\ .\ .\ .\ .\ .\ .\qquad .\quad .\quad .\quad (40)$$

Then performing the operations indicated on the left side of the differential equation (39) we obtain from a term $Ax^{[m]}$ the expression

$$p[m][m-1]Ax^{p^{2}[m]+[2]}-\frac{1}{\lambda^{2}p^{2\nu}}[m][m-1]Ax^{p^{2}[m-2]}+\{1-[n-\nu]-[-n-\nu-1]\}[m]A\cdot x^{p[m-1]+1}$$
$$+[n-\nu][n-\nu-1]Ax^{[m]}\qquad (41)$$

which is

$$\{p[m][m-1]+[m]-[m][n-\nu]-[m][-n-\nu-1]+[n-\nu][-n-\nu-1]\}Ax^{[m]}-\frac{1}{\lambda^{2}p^{2\nu}}[m][m-1]Ax^{p^{2}[m-2]}$$

which reduces further to

$$\{[m]-[n-\nu]\}\{[m]-[-n-\nu-1]\}Ax^{[m_{1}]}-\frac{1}{\lambda^{2}p^{2\nu}}[m][m_{1}-1]Ax^{p^{2}[m-2]}\quad .\quad (42)$$

So that from the whole series we obtain

$$\{[m_1]-[n-\nu]\}\{[m_1]-[-n-\nu-1]\}A_1x^{[m_1]}-\frac{1}{p^{2\nu}\lambda^2}[m_1][m_1-1]A_1x^{p^2[m_1-2]}$$

$$+\{[m_2]-[n-\nu]\}\{[m_2]-[-n-\nu-1]\}A_2x^{[m_2]}-\frac{1}{p^{2\nu}\lambda^2}[m_2][m_2-1]A_2x^{p^2[m_2-2]} \quad . \quad (43)$$

$$+ \quad . \ . \ . \qquad . \ . \ .$$
$$\qquad . \ . \ . \qquad . \ . \ .$$
$$\qquad . \ . \ . \qquad . \ . \ .$$

Choose

$$m_1 = m_2 - 2$$
$$m_2 = m_3 - 2$$
$$\cdots\cdots\cdots$$
$$\cdots\cdots\cdots$$
$$m_{r+1} = m_1 + 2r$$

Also choose $[m_1][m_1-1]=0$ so as to get rid of the term

$$\frac{1}{p^{2\nu}\lambda^2}[m_1][m_1-1]Ax^{p^2[m_1-2]}$$

Then, in order that the expression may be of the form $f(x)-f(x^{p^2})$, the coefficients $A_1\ A_2\ A_3\ \ldots$ etc. must be chosen so as to satisfy the relation

$$\frac{1}{p^{2\nu}\lambda^2}[m_{r+1}][m_{r+1}-1]A_{r+1} = \{[m_r]-[n-\nu]\}\{[m_r]-[-n-\nu-1]\}A_r$$

since $[m_1][m_1-1]=0$ either $m_1=0$ or $m_1=1$

for the value $m_1=0$, $m_{r+1}=2r$

and

$$A_{r+1} = -\frac{p^{2r-3}}{p^{n+\nu}}\lambda^2 p^{2\nu}\frac{[n-\nu-2r+2][n+\nu+2r-1]}{[2r][2r-1]}A_r$$

$$y = A\left\{1-\frac{\lambda^2}{p^{n-\nu+1}}\frac{[n-\nu][n+\nu+1]}{[2][1]}x^{[2]}+\ldots\ldots\ldots\right\} \quad . \quad . \quad (44)$$

$$f(x) = -\frac{[n-\nu][n+\nu+1]}{p^{2n+4}}A\lambda^2\left\{1-\frac{\lambda^2}{p^{n-\nu+1}}\frac{[n-\nu-2][n+\nu+3]}{[1][2]}+\ldots\ldots\right\} \quad (45)$$

the general term of the series y being

$$A\frac{p^{r.r-2}\lambda^{2r}}{p^{r(n-\nu)}}\frac{[n-\nu-2r+2]\ldots\ldots[n-\nu]\cdot[n+\nu+1]\ldots\ldots[n+\nu+2r-1]}{[2r]!}x^{[2r]} \quad (46)$$

When n is a positive integer and $n-\nu$ is even, the series (44) is C. $P^{\nu}_{[n]}(x\lambda)$ as is evident if we consider that there are $\frac{n-\nu}{2}$ terms in the series, and so by substituting $\frac{n-\nu}{2}-r$ for r we reverse the series. The general term becomes

$$Ap^{\left(\frac{n-\nu}{2}\right)^2-(n-\nu)+r^2+2r}\lambda^{n-\nu-2r}\frac{[2r+2][2r+4]\ldots[n-\nu]\cdot[n+\nu+1][n+\nu+3]\ldots[2n-2r-1]}{[1][2]\ldots\ldots[n-\nu-2r]\cdot}x^{[n-\nu-2r]} \quad (47)$$

$$p^{\frac{n^2-2n\nu+\nu^2}{4}-\frac{n-\nu}{2}}A\frac{[n-\nu][n-\nu-2]\ldots\ldots\ldots[2]}{[n-\nu-1][n-\nu-3]\ldots\ldots\ldots[1]}p^{r.r+2}\lambda^{n-\nu-2r}$$

$$\times\frac{[2n-2r]!}{[n-r]![n-\nu-2r]![r]!(2)_{n-r}(2)_r}x^{[n-\nu-2r]}$$

which is the general term of

$$Ap^{\frac{n-\nu \cdot n-\nu-2}{4}} \frac{[n-\nu][n-\nu-2] \ldots . [2]}{n-\nu-1][n-\nu-3] \ldots . [1]} \cdot P^{\nu}_{[n]}(x \cdot \lambda) \quad . \quad . \quad (48)$$

If we had used the value $m_1 = 1$

$$m_{r+1} = 2r+1$$

we should have obtained a series satisfying the differential equation (39).
The series being

$$y = A\left\{ x^{[1]} - \frac{\lambda^2}{p^{n-\nu}} \frac{[n-\nu-1][n+\nu+2]}{[3]!} x^{[3]} + \ldots \ldots \right\} . \quad . \quad (49)$$

the general term being

$$(-1)^r \frac{\lambda^{2r} p^{r^2-r}}{p^{r(n-\nu)}} A \frac{[n-\nu-2r+1] \ldots . [n-\nu-1] \cdot [n+\nu+2] \ldots . [n+\nu+2r]}{[2r+1]!} x^{[2r+1]} \quad (50)$$

If $n-\nu$ be an odd integer and the series is finite, the number of terms will be $\frac{n-\nu+1}{2}$

For r substitute $\frac{n-\nu-1}{2} - r$

The general term then becomes

$$\frac{A}{\lambda p^{\frac{n-\nu+3 \cdot n-\nu+1}{4}}} \frac{[n-\nu-1][n-\nu-3] \ldots . [2]}{[n+\nu][n+\nu-2] \ldots . [1} \lambda^{n-\nu-2r} \frac{[2n-2r]!}{[n-r]![n-\nu-2r]![r]!(2)_{n-r}(2)_r} x^{[n-\nu-2r]} \quad (51)$$

showing that the series is

$$\text{const.} \times P^{\nu}_{[n]}(x \cdot \lambda) \quad . \quad . \quad . \quad . \quad (52)$$

6.

In this and the following articles we shall give examples of the expansion of various functions in infinite series of the generalised Bessel's Functions. The three expansions to be considered are analogous to the following theorems in ordinary Bessel's Functions,

$$J_n(x) = \frac{x^n}{2^n \cdot n!} J_0(x) + \frac{x^{n+1}}{2^{n+1} \cdot n+1!} J_1(x) + \frac{x^{n+2}}{2^{n+2} \cdot n+2!} J_2(x) + \ldots \ldots \ldots . \quad (53)$$

$$S^m J_0(x) = 2^m \left\{ J_m(x) + \frac{m}{1} J_{m+2}(x) + \frac{m \cdot m+1}{2!} J_{m+4}(x) + \ldots . \right\} \quad . \quad . \quad (54)$$

$$\frac{x^n}{2^n \cdot n!} = J_n(x) + \frac{x}{2} J_{n+1}(x) + \frac{x^2}{2^2 \cdot 2!} J_{n+2}(x) + \ldots \ldots . \quad . \quad . \quad . \quad (55)$$

the symbol S^m denoting m successive integrations in which no arbitrary constants are introduced (TODHUNTER'S *Functions of Laplace, Bessel, and Legendre*, art's. 418–422).

When $[x]_m$ denotes $\frac{p^x-1}{p^l-1} \cdot \frac{p^{x-l}-1}{p^l-1} \ldots . \frac{p^{x-\overline{m-1}l}-1}{p^l-1}$

we have an identity

$$[x+y]_m=[x]_m+\sum_{s=1}^{s=m}p^{s(x-m-s)}\frac{p^{ml}-1\cdot p^{\overline{m-1}l}-1\ .\ .\ .\ .\ p^{\overline{m-s+1}l}-1}{p^l-1\cdot p^{2l}-1\ .\ .\ .\ p^{sl}-1}[x]_{m-s}[y]_s \quad . \quad (56)$$

Subject to a proper interpretation of $[x]_m$, this expansion holds for all values of m provided the series be convergent, the condition for which is $p^l>1$. If m be integral

$$[x]_{-m} \text{ denotes } \frac{1}{[x+ml][x+m-1l]\ .\ .\ .\ .\ .\ [x+l]} \quad . \quad . \quad (57)$$

The theorem in its generality is discussed in a paper on "Series connected with the Enumeration of Partitions," series 2, vol. i., *Proc. Lond. Math. Soc.* In the following work we require only the simple cases in which m is an integer positive or negative.

For y substituting $-2n$

x $2n+2r$

l 2

m r (a positive integer)

we have

$$[2n+2r-2n]_r=[2n+2r]_r+\sum_{s=1}^{s=r}p^{2s(n+s)}\frac{[2r][2r-2]\ .\ .\ .\ .\ [2r-2s+2]}{[2][4]\ .\ .\ .\ .\ [2s]}\times[2n+2r]_{r-s}[-2n]_s$$

Now $[2n+2r]_{r-s}$ is

$$[2n+2r][2n+2r-2]\ .\ .\ .\ [2n+2s+2] = [n+r][n+r-1]\ .\ .\ .\ [n+s+1]\cdot\frac{(2)_{n+r}}{(2)_{n+s}}$$

and $[-2n]_s = [-2n][-2n-2]\ .\ .\ .\ .\ [-2n-2s+2]$

$$= (-1)^s p^{-2ns-s^2+s}[2n][2n+2]\ .\ .\ [2n+2s-2]$$

$$= (-1)^s p^{-2ns-s^2+s}[n][n+1]\ .\ .\ [n+s-1]\cdot\frac{(2)_{n+s-1}}{(2)_{n-1}}$$

So that we have

$$(2)_r[r]! = [n+r][n+r-1]\ .\ .\ [n+1]\frac{(2)_{n+r}}{(2)_n}$$

$$+\sum_1^r(-1)^s p^{s^2+s}\frac{[r]!}{[r-s]![s]!}\frac{(2)_r}{(2)_{r-s}(2)_s}\cdot[n+r]\ .\ .\ .\ [n+s+1]\frac{(2)_{n+r}}{(2)_{n+s}}\times$$

$$[n][n+1]\ .\ .\ [n+s-1]\frac{(2)_{n+s-1}}{(2)_{n-1}}$$

Dividing throughout by $[n+r]!\,(2)_{n+r}[r]!\,(2)_r$ this reduces

$$\frac{1}{[n+r]!\,(2)_{n+r}}=\frac{1}{[n]!\,(2)_n[r]!(2)_r}+\sum_1^r(-1)^s p^{s^2+s}\frac{1}{[r-s]!\,[s]!\,(2)_{r-s}(2)_s[n+s]!\,(2)_{n+s}}\times$$

$$\frac{[n][n+1]\ .\ .\ .\ .\ [n+s-1]}{(2)_{n-1}}(2)_{n+s-1} \quad (58)$$

Dividing throughout again by $[r]!\,(2)_r$ this identity becomes

$$\frac{1}{[n+r]![r]!\,(2)_{n+r}(2)_r}=\frac{1}{[n]!\,(2)_n}\cdot\frac{1}{[r]!\,[r]!\,(2)_r(2)_r}+$$

$$\sum_{s=1}^{s=r}(-1)^s p^{s.s+1}\frac{1}{[n+s]!\,[s]!\,(2)_{n+s}(2)_s}\cdot\frac{[n][n+1]\ .\ .\ .\ [n+s-1]}{(2)_{n-1}}(2)_{n+s-1}$$

$$\times\frac{1}{[r-s]!\,[r]!\,(2)_{r-s}(2)_r} \quad . \quad . \quad . \quad . \quad (59)$$

From which we can see at once that

$$J_{[n]}(x\lambda) = \frac{\lambda^n x^{[n]}}{(2)_n[n]!}J_{[0]}(xp^n\cdot\lambda) - p^2\frac{\lambda^{n+1}x^{[n+1]}}{(2)_{n+1}[n+1]!}\cdot[n]\frac{(2)_n}{(2)_1(2)_{n-1}}J_{[1]}(xp^{n+1}\cdot\lambda)$$
$$+ \ldots\ldots\ldots\ldots \quad . \quad . \quad . \quad . \quad (60)$$

the general term being

$$(-1)^s p^{s.s+1}\ \frac{\lambda^{n+s}x^{[n+s]}}{(2)_{n+s}[n+s]!}\cdot\frac{[n][n+1]\ldots[n+s-1]}{[s]!}\cdot\frac{(2)_{n+s-1}}{(2)_s(2)_{n-1}}J_{[s]}(xp^{n+s}\lambda)$$

subject to the convergence of the series; for if out of each term of the series on the right side of (60) we pick out the part involving $\lambda^{n+2r}x^{[n+2r]}$, we get a set of $r+1$ terms

$$\lambda^{n+2r}x^{[n+2r]}\left\{\frac{1}{[n]!\,(2)_n}\cdot\frac{1}{[r]![r]!\,(2)_r(2)_r} - p^2\frac{1}{[n+1]!\,(2)_{n+1}}[n]\frac{(2)_n}{(2)_1(2)_{n+1}}\frac{1}{[r-1]![r]!\,(2)_{r-1}(2)_r}\right.$$
$$\left. + \ldots\ldots\right\} \quad . \quad . \quad . \quad . \quad (61)$$

which by (59) reduces to

$$\frac{\lambda^{n+2r}x^{[n+2]\cdot}}{[n+r]![r]!\,(2)_{n+r}(2)_r} \quad . \quad . \quad . \quad . \quad (62)$$

the general term of $J_{[n]}(x\lambda)$,—and theorem (60) is established.

A particular case of this is

$$J_{[2]}(x\lambda) = \frac{\lambda^2x^{[2]}}{[2][4]}J_{[0]}(x^{p^2}\lambda) - p^2\frac{\lambda^3x^{[3]}}{[2][4][6]}[2]\frac{(2)_2}{(2)_1(2)_1}J_1(x^{p^3}\lambda) + \ldots\ldots \quad . \quad (63)$$

analogous to

$$J_2(x) = \frac{x^2}{2\cdot4}\,J_0(x) + \frac{2x^3}{2\cdot4\cdot6}\,J_1(x) + \ldots\ldots\ldots \quad . \quad . \quad (64)$$

To investigate a theorem analogous to

$$S^m\cdot J_0(x) = 2^m\left\{J_m(x) + mJ_{m+2}(x) + \frac{m\cdot m+1}{2!}J_{m+4}(x) + \ldots\ldots\right\} \quad . \quad (65)$$

If, in the series on the right, we replace the Bessel's Functions by infinite series and collect the terms together according to powers of x, we find that the terms involving x^{m+2r} form an infinite series

$$2^m x^{m+2r}\left\{\frac{1}{m+r!\,r!\,2^{m+2r}} - \frac{m}{1!}\frac{1}{m+r+1!\,r-1!\,2^{m+2r}} + \frac{m\cdot m+1}{2!}\frac{1}{m+r+2!\,r-2!\,2^{m+2r}} - \ldots\ldots\ldots\right\} \quad (66)$$

which is

$$\frac{2^m x^{m+2r}}{2^{m+2r}r!\,r!}\left\{\frac{1}{m+r\cdot m+r-1\cdot r+1} - \frac{m}{1}\frac{r}{m+r+1\ldots\ldots\ldots r+1} + \ldots\ldots\right\}. \quad (67)$$

The series within the bracket is by an extension of the notation of VANDERMONDE'S theorem (*Proc. Lond. Math. Soc.*, vol. xxvi. p. 285),

$$(r)_{-m} - \frac{m}{1!}(r)_{-m-1}(r)_1 + \frac{m\cdot m+1}{2!}(r)_{-m-2}(r)_2 + \ldots\ldots \quad . \quad . \quad . \quad (68)$$

which is $(r+r)_{-m}$ subject to convergence conditions, viz., $2r+1>0$

$$(r+r)_{-m} = (2r)_{-m} = \frac{1}{m+2r\cdot m+2r-1\ldots\ldots 2r+1}$$

So the re-arrangement gives us an infinite series, of which the general term is

$$\frac{x^{m+2r}}{2^{2r}\cdot r!\,r!}\times\frac{1}{m+2r\cdot m+2r-1\ldots\ldots 2r+1}$$

which is the general term of $S^m J_0(x)$ . . . . . (69)
S^m indicating m successive integrations.

7.

The preceding analysis shows us how to construct the analogous series for the generalised Bessel's Functions. Consider the generalised form of VANDERMONDE's theorem

$$[x+y]_m=[x]_m+\sum_{s=1}^{s=\infty}p^{s(x-\overline{m-st})}\cdot\frac{p^{ml}-1p^{\overline{m-1}l}-1\ldots p^{\overline{m-s+1}l}-1}{p^l-1\cdot p^{2l}-1\ldots p^{sl}-1}[x]_{m-s}[y]_s \quad . \quad (70)$$

convergent for all values of m if $p^l>1$:
If m be a negative integer,

$$[x]_{-m}=\frac{1}{[x+ml][x+\overline{m-1}l]\ldots.[x+l]} \quad . \quad . \quad (71)$$

In the theorem (70) replace m by $-m$ a negative integer, l by 2, x by $2r$, and y by $2r$. Then

$$[2r+2r]_{-m}=[2r]_{-m}+\sum_{s=1}^{s=\infty}p^{s(2r+2m+2s)}\frac{p^{-2m}-1\ldots\ldots\ldots p^{-2m-2s+2}-1}{p^2-1\ldots\ldots p^{2s}-1}\cdot$$
$$\times\frac{[2r][2r-2]\ldots\ldots[2r-2s+2]}{[2r+2m+2s]\ldots.[2r+2]} \quad (72)$$

Now $[2r+2r]_{-m}$ is $[4r]_{-m}$

$$\frac{1}{[4r+2m][4r+2m-2]\ldots.[4r+2]}$$

and this may be written

$$\frac{(2)_{2r}}{(2)_{2r+m}[2r+m]\cdot[2r+m-1]\ldots[2r+1]}$$

Dividing (72) throughout by $[r]![r]!\,(2)_r\,(2)_r$ we obtain

$$\frac{1}{[r]![r]!\,(2)_r(2)_r}\times\frac{1}{[4r+2m]\ldots\ldots[4r+2]}=$$
$$\frac{1}{[r+m]![r]!\,(2)_{r+m}(2)_r}+\sum_{s=1}^{s=\infty}(-1)^s p^{s^2+s+2r}\frac{[m+s-1]!\,(2)_{m-s-1}}{[m-1]!\,(2)_{m-1}[s]![r-s]!\,(2)_s(2)_{r-s}[r+m+s]!\,(2)_{r+m+s}} \quad (73)$$

the analogue of

$$\frac{1}{2^{2r}\cdot r!\,r!}\frac{1}{m+2r\ldots.r+1}=2^m\left\{\frac{1}{m+r!\,r!\,2^{m+2r}}-\frac{m}{1!}\frac{1}{m+r+1!\,r-1!\,2^{m+2r}}+\ldots.\right\} . \quad (74)$$

used in art. 6 in the analysis of the series (65).

At this point we define the function $J^q_{[n]}(x\lambda)$ as the convergent infinite series

$$\sum_{r=0}^{r=\infty}\frac{\lambda^{n+2r}x^{\frac{q^{n+2r}-1}{q-1}}}{[n+r]![r]!\,(2)_r(2)_{n+r}} \quad . \quad . \quad . \quad . \quad (75)$$

Then

$$J^{p^2}_{[n]}(x\lambda) = \sum_{r=0}^{r=\infty} \frac{\lambda^{n+2r} x^{\overset{[2n+4r]}{[2]}}}{[n+r]!\,[r]!\,(2)_r(2)_{n+r}} \quad . \quad . \quad . \quad (76)$$

Consider the series

$$J^{p^2}_{[m]}(x\lambda) - p^{2-m}[m] \cdot \frac{(2)_m}{(2)_{m-1}(2)_1} J^{p^2}_{[m]}(x\cdot\lambda p) + \ldots\ldots\ldots \quad . \quad . \quad (77)$$

of which the general term is

$$(-1)^s p^{s^2+s-sm} \frac{[m]\,[m+1]\cdots[m+s-1]}{[s]!} \cdot \frac{(2)_{m+s-1}}{(2)_{m-1}(2)_s} \cdot J^{p^2}_{[m+2s]}(x\cdot\lambda p^s) \quad . \quad (78)$$

If we replace in (77) $J^{p^2}_{[m]}\ J^{p^2}_{[m+2]} \ldots\ldots$ by the infinite series which they represent and collect the terms according to powers of x, then the terms containing $x^{\overset{[2m+4r]}{[2]}}$ form the infinite series

$$\frac{\lambda^{m+2r} x^{\overset{[2m+4r]}{[2]}}}{[m+r]!\,[r]!\,(2)_{m+r}(2)_r} + \sum_{s=1}^{s=\infty} (-1)^s \frac{\lambda^{m+2r} x^{\overset{[2m+4r]}{[2]}} \cdot [m+s-1]!\,(2)_{m+s-1}}{[m-1]!\,(2)_{m-1}[s]!\,(2)_s[r-s]!\,(2)_{r-s}[m+r+s]!\,(2)_{m+r+s}} p^{s^2+s+2rs} \quad (79)$$

which reduces by (73) to

$$\frac{1}{[2m+4r]\,[2m+4r-2]\cdots[4r+2]} \cdot \frac{\lambda^{m+2r} x^{\overset{[2m+4r]}{[2]}}}{[r]!\,[r]!\,(2)_r(2)_r}$$

which is

$$\frac{\lambda^m}{[2]^m} \mathrm{I}^m J^{p^2}_{[0]}(x^{p^{2m}}\cdot\lambda) \quad . \quad . \quad . \quad . \quad (80)$$

I^m denoting the operation reversing

$$\frac{d}{d(x^{p^{2m-2}})}\left\{\frac{d}{d(x^{p^{2m-4}})}\left\{\ldots\left\{\frac{d}{d(x^{p^2})}\right\}\right\}\right\}$$

8.

It is well known that

$$\frac{x^n}{2^n \cdot n!} = J_n + \frac{x}{2} J_{n+1} + \frac{x^2}{2^2 \cdot 2!} J_{n+2} + \frac{x^3}{2^3 \cdot 3!} J_{n+3} \ldots\ldots \quad . \quad (81)$$

Consider the series

$$J_{[n]}(x\lambda) + \frac{\lambda x^{[1]}}{(2)_1} J_{[n+1]}(x^p\cdot\lambda) + p^2 \frac{\lambda^2 x^{[2]}}{(2)_2[2]!} J_{[n+2]}(x^{p^2}\cdot\lambda) + \ldots\ldots \quad . \quad (82)$$

of which the general term is

$$p^{r.r-1} \frac{\lambda^r x^{[r]}}{(2)_r[r]!} J_{[n+r]}(x^{p^r}\lambda) \cdot \quad . \quad . \quad . \quad . \quad (83)$$

$J_{[n]}(x\lambda)$ denoting

$$\sum (-1)^r \frac{\lambda^{n+2r} x^{[n+2r]}}{[n+r]!\,[r]!\,(2)_r(2)_{n+r}} \quad . \quad . \quad . \quad (84)$$

Replacing $J_{[n]}(x\lambda)\, J_{[n+1]}(x^p\lambda) \ldots$ in (82) by infinite series, and collecting the terms according to powers of x, we have for the terms involving $x^{[n+2r]}$ a group of $r+1$ terms, viz.,

$$(-1)^r \frac{\lambda^{n+2r} x^{[n+2r]}}{[n+r]!\,[r]!\,(2)_r(2)_{n+r}} + (-1)^{r-1} \frac{\lambda^{n+2r} x^{[n+2r]}}{[n+r]![r-1]!(2)_{r-1}(2)_{n+r}(2)_1}$$
$$+ \ldots + (-1)^{r-s} p^{s.s-1} \frac{\lambda^{n+2r} x^{[n+2r]}}{[n+r]![r-s]!(2)_{r-s}(2)_{n+r}(2)_s} \quad (85)$$

which is

$$(-1)^r \frac{\lambda^{n+2r} x^{[n+2r]}}{[n+r]!\,[r]!^s (2)_r (2)_{n+r}} \left\{ 1 - \frac{p^{2r}-1}{p^2-1} + p^2 \frac{p^{2r}-1 \cdot p^{2r-2}-1}{p^2-1 \cdot p^4-1} - \ldots . (-1)^s \frac{p^{2r}-1 \ldots . p^{2r-2s+2}-1}{p^2-1 \ldots . p^{2s}-1} + \ldots \right\} \quad (86)$$

which is identically zero.

The only term which does not vanish is the first term in $J_{[n]}(x\lambda)$ so that we have identically for all values of n

$$\frac{\lambda^n x^{[n]}}{[n]!\,(2)_n} = J_{[n]}(x\lambda) + \frac{\lambda x^{[1]}}{(2)_1 [1]!} J_{[n+1]}(x^p \lambda) + \ldots \ldots \quad . \quad . \quad (87)$$

If in 84 we had taken $J_{[n]}(x\lambda)$ as

$$\sum \frac{\lambda^{n+2r} x^{[n+2r]}}{[n+r]!\,[r]!\,(2)_r (2)_{n+r}}$$

the signs in the terms of (86) would not have been alternately + and −; (86) would not then be zero but

$$2(p^2+1) \ . \ . \ . \ (p^{2r-2}+1) \frac{\lambda^{n+r} x^{[n+2r]}}{[n+r]!\,[r]!\,(2)_r (2)_{n+r}}$$

Another expansion $x^{[n]}$ in terms of the P functions is

$$\frac{[2n]!\,\lambda^n x^{[n]}}{[n]!\,[n]!\,(2)_n} = P_{[n]} + p^3 \frac{[2n-3]}{[2]} P_{[n-2]} + p^6 \frac{[2n-1]\,[2n-7]}{[2]\,[4]} P_{[n-4]} + \ldots \quad . \quad (88)$$

analogous to

$$\frac{2n!\,x^n}{n!\,n!\,2^n} = P_n + \frac{2n-3}{2} P_{n-2} + \frac{2n-1 \cdot 2n-7}{2 \cdot 4} P_{n-4} + \ldots \ldots$$

9.

Various interesting theorems have been obtained with respect to Bessel's Functions when the variable is not x but $\sqrt{x}$. The analogous theorems for $J_{[n]}(x^{\frac{1}{p+1}} \cdot \lambda^{\frac{1}{2}})$ are given in the following work. It is well known that

$$\frac{d^m}{dx^m} \left\{ x^{-\frac{n}{2}} J_n(\sqrt{x}) \right\} = (-\tfrac{1}{2})^m x^{-\frac{n+m}{2}} J_{n+m}(\sqrt{x}) \quad . \quad . \quad (89)$$

$$\frac{d^m}{dx^m} \left\{ x^{\frac{n}{2}} J_n(\sqrt{x}) \right\} = (\tfrac{1}{2})^m x^{\frac{n-m}{2}} J_{n-m}(\sqrt{x}) \quad . \quad . \quad . \quad (90)$$

Let $D^{[m]}$ denote $\frac{d}{d(x^{p^{n+2m-2}})} \cdot \frac{d}{d(x^{p^{n+2m-4}})} \cdot \ldots \cdot \frac{d}{d(x^{p^{n+2}})} \cdot \frac{d}{d(x^{p^n})} \quad . \quad . \quad . \quad . \quad (91)$

Then

$$D^{[m]} \left\{ \lambda^{-\frac{n}{2}} x^{-\frac{[n]}{[2]}} J_{[n]}(x^{\frac{1}{[2]}} \lambda^{\frac{1}{2}}) \right\} = \frac{\lambda^{\frac{m-n}{2}}}{[2]^m} x^{-\frac{p^m [n+m]}{[2]}} J_{[n+m]}(x^{\frac{p^m}{[2]}} \lambda^{\frac{1}{2}}) \quad . \quad . \quad (92)$$

which reduces to (89) when $p = 1$

$$\lambda^{-\frac{n}{2}} x^{-\frac{[n]}{[2]}} J_{[n]}(x^{\frac{1}{[2]}} \lambda^{\frac{1}{2}}) =$$

$$\sum_0^{r=\infty} \frac{\lambda^{-\frac{n}{2}} x^{-\frac{p^n-1}{p^2-1}} \lambda^{\frac{n+2r}{2}} x^{\frac{p^{n+2r}-1}{p^2-1}}}{[r]!\,[n+r]!\,(2)_r (2)_{n+r}} \quad . \quad . \quad . \quad . \quad (93)$$

$$= \sum \frac{\lambda^r x^{p^n \frac{[2r]}{[2]}}}{[r]!\,[n+r]!\,(2)_r (2)_{n+r}} \quad . \quad . \quad . \quad . \quad (94)$$

Operating on this with $D^{[m]}$ we see that the operation reduces to zero all terms for which $r < m$, while the result of operating on the general term

$$\frac{\lambda^{m+r} (x^{p^n})^{\frac{[2m+2r]}{[2]}}}{[m+r]!\,[n+m+r]!\,(2)_{m+r}(2)_{n+m+r}} \quad . \quad . \quad . \quad (95)$$

is

$$\lambda^{m+r} \frac{\frac{[2m+2r]}{[2]} \cdot \frac{[2m+2r-2]}{[2]} \cdot \ldots \cdot \frac{[2m+2r-2m+2]}{[2]}}{[m+r]!\,[n+m+r]!\,(2)_r (2)_{n+m+r}} (x^{p^{n+2m}})^{\frac{[2m+2r-2m]}{[2]}} \quad . \quad (96)$$

which is easily reduced to

$$\frac{\lambda^{\frac{m-n}{2}} \left(x^{\frac{p^m}{p+1}}\right)^{-[n+m]}}{[2]^m} \frac{\lambda^{\frac{n+m+2r}{2}} \left(x^{\frac{p^m}{p+1}}\right)^{[n+m+2r]}}{[r]!\,[n+m+r]!\,(2)_r (2)_{n+m+r}} \quad . \quad . \quad . \quad (97)$$

viz., the general term of

$$\frac{\lambda^{\frac{m-n}{2}}}{[2]^m} x^{-p^{m\frac{[m+n]}{[2]}}} J_{[n+m]}\left(x^{\frac{p^n}{p+1}} \cdot \lambda^{\frac{1}{p}}\right) \quad . \quad . \quad . \quad . \quad (98)$$

In a similar way theorem (90) may be generalised.

We have shown that

$$[n] \cdot P_{[n]}(x\lambda) = \lambda[2n-1] x P_{[n-1]}(x^p,\lambda) - p^{n+1}[n-1] P_{[n-2]}(x\lambda) \quad . \quad . \quad (\alpha)$$

In the same way we can establish

$$p^{n+2}[n] Q_{[n]}(x\lambda) = \lambda[2n-1] x Q_{[n-1]}(x^p,\lambda) - [n-1] Q_{[n-2]}(x,\lambda) \quad . \quad . \quad (\beta)$$

analogous to the ordinary recurrence formulæ.

Multiplying (α) by $Q_{[n-1]}(x^p,\lambda)$ and (β) by $P_{[n-1]}(x^p,\lambda)$, then subtracting, we obtain

$$\begin{aligned}&[n]\{P_{[n]}(x,\lambda) Q_{[n-1]}(x^p,\lambda) - p^{n+2} Q_{[n]}(x,\lambda) P_{[n-1]}(x^p,\lambda)\} \\ &= [n-1]\{P_{[n-1]}(x^p,\lambda) Q_{[n-2]}(x\lambda) - p^{n+1} Q_{[n-1]}(x^p,\lambda) P_{[n-2]}(x,\lambda)\} \quad . \quad . \quad (\gamma)\end{aligned}$$

analogous to

$$n\{P_n Q_{n-1} - Q_n P_{n-1}\} = (n-1)\{P_{n-1} Q_{n-2} - Q_{n-1} P_{n-2}\}$$

So also multiplying (α) by $Q_{[n-2]}(x,\lambda)$ and (β) by $p^{n+1} P_{[n-2]}(x,\lambda)$, then subtracting, we obtain

$$\begin{aligned}&[n]\{P_{[n]}(x,\lambda) Q_{[n-2]}(x,\lambda) - p^{2n+3} Q_{[n]}(x\lambda) P_{[n-2]}(x\lambda)\} = \lambda[2n-1] x\{P_{[n-1]}(x^p\lambda) Q_{[n-2]}(x\cdot\lambda) - p^{n+1} Q_{[n-1]}(x^p,\lambda) P_{[n-2]}(x,\lambda)\} \\ &\frac{\lambda[2n-1][n]}{[n-1]} x\{P_{[n]}(x\lambda) Q_{[n-1]}(x^p,\lambda) - p^{n+2} Q_{[n]}(x,\lambda) P_{[n-1]}(x^p,\lambda)\} \quad . \quad . \quad (\delta)\end{aligned}$$

When we put $x = 1$ we obtain

$$[n]\{P_{[n]}(\lambda) Q_{[n-1]}(\lambda) - p^{n+2} Q_{[n]}(\lambda) P_{[n-1]}(\lambda)\} = [n-1]\{P_{[n-1]}(\lambda) Q_{[n-2]}(\lambda) - p^{n+1} Q_{[n-1]}(\lambda) P_{[n-2]}(\lambda)\} \quad (\epsilon)$$

and so by repetition, if n be integral,

$$= [1]\{P_{[1]}Q_{[0]} - p^3 Q_{[1]}P_{[0]}\}$$
$$= 1$$

the analogue of $P_{[n+1]}Q_n - Q_{n+1}P_n = \dfrac{1}{n+1}$

Similarly from (δ), putting $x = 1$ and repeating we obtain

$$P_{[n+1]}(\lambda)Q_{[n-1]}(\lambda) - p^{2n+5}Q_{[n+1]}(\lambda)P_{[n-1]}(\lambda) = \frac{\lambda[2n+1]}{[n+1]\cdot[n]} \quad . \quad . \quad (\eta)$$

From the recurrence formulæ we can obtain without difficulty

$$\sum_{r=0}^{r=n} \frac{[2r+1]}{p^{r.r+5/2}} P_{[r]}(\lambda)P_{[r]}(\lambda^1) = \frac{[n+1]}{p^{n+2}} \cdot \frac{P_{[n+1]}(\lambda)P_{[n]}(\lambda^1) - P_{[n+1]}(\lambda^1)P_{[n]}(\lambda)}{\lambda^1 - \lambda} \quad . \quad (\zeta)$$

A theorem analogous to NEUMANN's expansion of an arbitrary function in a series of Bessel Functions I have given in a paper, *L.M.S. Proceedings*; also the analogue of LOMMEL's theorem

$$\frac{2z}{\pi} = \{J_{\frac{1}{4}}(z)\}^2 + 3\{J_{\frac{3}{4}}(z)\}^2 + 5\{J_{\frac{5}{4}}(z)\}^2 + \ldots\ldots \textit{ad inf.}$$

II.—Certain Fundamental Power Series and their Differential Equations. By the Rev. F. H. Jackson, H.M.S. "Irresistible." *Communicated by* Dr W. Peddie.

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The series which will be discussed in this paper are of the general type

$$A_1x^{(m_1)} + A_2x^{(m_2)} + \ldots\ldots\ldots + A_rx^{(m_r)} + \ldots\ldots\ldots \quad . \quad . \quad (1)$$

Consider a sequence of elements $p_1, p_2, p_3, \ldots\ldots\ldots$ then (m_r) will denote the sum of the first m_r elements of the sequence. The simplest connection between the terms of the sequence is equality

$$p_1 = p_2 = p_3 = \ldots\ldots = p_\kappa = \ldots$$

The series expansions of the ordinary functions of analysis and the series which are solutions of ordinary linear differential equations belong to this simplest type. Another more general type of series is formed when the elements $p_1, p_2, p_3 \ldots$ are in geometrical progression as $a, ap, ap^2, \ldots\ldots$ In this case the index (n) denotes $a + ap + ap^2 + \ldots\ldots + ap^{n-1}$, which is $a\dfrac{p^n-1}{p-1}$, the limitation of n to positive integral values (where the sequence is geometrical) may be removed, as will be seen in the particular discussion of the functions $J_{[n]}(\lambda x)$, $P_{[n]}(\lambda x)$, $Q_{[n]}(\lambda x)$, $F([\alpha][\beta][\gamma]\lambda x)$. If p be made unity, the geometrical progression becomes a progression of equal elements and the properties of the general functions reduce to analogous properties of the simpler functions $F(\alpha\beta\gamma x)$; $P_n(x)$; $Q_n(x)$, etc. Euler's expansion

$$(1-x)(1-x^2)(1-x^3)\ ad\ inf. = \sum_{-\infty}^{+\infty}(-1)^n x^{\frac{1}{2}(3n^2+n)} \qquad (x < 1)$$

and Gauss's series

$$\frac{(1-x^2)(1-x^4)(1-x^6)\ \ldots\ ad\ inf.}{(1-x)(1-x^3)(1-x^5)\ \ldots\ ad\ inf.} = \sum_0^\infty x^{\frac{n.n+1}{2}} \qquad (x < 1)$$

are particular cases of the general series (1) for the sequences 1, 1, 3, 2, 5 . . . , $2n-1, n, \ldots$ and 1, 2, 3, 4, ... respectively.

The fundamental Hypergeometric Series is

$$F = 1 + \frac{\alpha_1.\alpha_2.\alpha_3 \ldots\ldots\ldots\ldots \alpha_s}{(1)\beta_1.\beta_2.\beta_3 \ldots\ldots\ldots\ldots \beta_t}x^{(1)} + \frac{\alpha_1(\alpha_1+(1)) \ldots\ldots\ldots\ldots \alpha_s(\alpha_s+(1))}{(1)(2)\beta_1(\beta_1+(2)-(1)) \ldots\ldots \beta_s(\beta_s+(2)-(1))}x^{(2)} + \ldots\ldots \quad (2)$$

In the case when there are only three elements $\alpha\,\beta\,\gamma$ in the Hypergeometric Series and the sequence of $p.s$ is $1, 3, 5, \ldots\ldots, 2n+1, \ldots$ the following series are interesting cases :

$$1+\frac{\alpha\cdot\beta}{1^2\cdot\gamma}x+\frac{\alpha\cdot\alpha+1\cdot\beta\cdot\beta+1}{1^2\cdot2^2\cdot\gamma\cdot\gamma+3}x^4+\ \ldots\ .+\frac{\alpha\cdot\alpha+1\ \ldots\ .\ \alpha+(n-1)^2\cdot\beta\cdot\beta+1\ \ldots\ .\ \beta+(n-1)^2}{1^2\cdot2^2\cdot3^2\ \ldots\ .\ n^2\cdot\gamma\cdot\gamma+3\ \ldots\ .\ \gamma-1+n^2}x^{n^2}+\ \ldots\ . \quad (3)$$

$$1+\frac{\alpha\beta}{1^2\cdot\gamma+1}x+\frac{\alpha\cdot\alpha+1^2\cdot\beta\cdot\beta+1^2}{1^2\cdot2^2\cdot\gamma+1^2\cdot\gamma+2^2}x^4+\ \ldots\ .\ .\ .\ . \quad (4)$$

$$1+\frac{\alpha\cdot\beta}{1^4}x+\frac{\alpha\cdot\alpha+1^2\cdot\beta\cdot\beta+1^2}{1^4\cdot2^4}x^4+\ \ldots\ .\ .\ .\ .\ . \quad (5)$$

It is well known that if $\Pi_s(\alpha+m)$ denote the product of s factors

$$(a_1+m)(a_2+m)(a_3+m)\ \ldots\ .\ (a_s+m) \qquad m \text{ integral}$$

an identity of the following kind can be established,

$$\Pi_s(\alpha+m)\equiv B_0+B_1m+B_2m(m-1)+\ \ldots\ .+B_sm(m-1)\ \ldots\ (m-s+1)$$

where $B_0\,B_1\ \ldots\ .\ B_s$ are constants, that is, are independent of m.

We proceed to establish a generalisation of this, on which all subsequent work will depend.

Let $\Pi_s(\alpha+(m))$ denote the product of s factors

$$(a_1+p_1+p_2+\ \ldots\ .+p_m)(a_2+p_1+p_2+\ \ldots\ +p_m)\ \ldots\ .\ (a_s+p_1+p_2+\ \ldots\ +p_m)$$

then $\Pi_s(\alpha+(m))$ is identically equal to

$$B_0+B_1\frac{(m)}{p_1}+B_2\frac{(m)((m)-(1))}{p_1p_2}+\ \ldots\ .\ .+B_s\frac{(m)((m)-(1))\ \ldots\ .\ ((m)-(s-1))}{p_1p_2p_3\ \ldots\ p_s} \quad (6)$$

which may be more conveniently written

$$B_0+B_1\frac{(m)}{p_1}+B_2\frac{(m)_2}{p_1p_2}+\ \ldots\ .\ .+B_s\frac{(m)_s}{p_s!} \quad (7)$$

The coefficients $B_0\,B_1$, etc. are independent of m and are given by

$$\frac{(n)!}{p_n!}B_n=\sum_{r=0}^{r=n}(-1)^r\frac{(n)!}{(n-r)!\,\{r\}!}\Pi(\alpha+(n-r)) \quad (8)$$

Before proceeding to obtain these coefficients it will be well to explain the notation clearly.

$(m-1)$ is not the same as $(m)-(1)$, for $(m-1)$ denotes $p_1+p_2+p_3+\ \ldots\ .+p_{m-1}$, while $(m)-(1)$ denotes $p_2+p_3+p_4+\ \ldots\ .+p_m$

$$(m)_s\equiv(p_1+p_2+\ \ldots\ .+p_m)(p_2+p_3+\ \ldots\ .+p_m)(p_3+\ \ldots\ .+p_m)\ \ldots\ .\ (p_s+\ \ldots\ .+p_m)$$

$$B_3\equiv\frac{p_3!}{(3)!}\Pi(\alpha+(3))-\frac{p_3!}{(2)!\,\{1\}!}\Pi(\alpha+(2))+\frac{p_3!}{(1)!\,\{2\}!}\Pi(\alpha+(1))-\frac{p_3!}{\{3\}!}\Pi(\alpha) \quad (9)$$

$$(3)!=(p_1+p_2+p_3)(p_2+p_3)p_3 \qquad \{3\}!=(p_3+p_2+p_1)(p_2+p_1)p_1$$
$$(2)!=(p_1+p_2)p_2 \qquad \{2\}!=(p_3+p_2)p_2$$
$$(1)!=p_1 \qquad \{1\}!=p_3 \quad (10)$$

In the expression for B_4 four elements p_1, p_2, p_3, p_4 appear and

$$(4)!=(p_1+p_2+p_3+p_4)(p_2+p_3+p_4)(p_3+p_4)p_4 \qquad \{4\}!=(p_4+p_3+p_2+p_1)(p_3+p_2+p_1)(p_2+p_1)p_1$$
$$(3)!=(p_1+p_2+p_3)(p_2+p_3)p_3 \qquad \{3\}!=(p_4+p_3+p_2)(p_3+p_2)(p_2)$$
$$(2)!=(p_1+p_2)p_2 \qquad \{2\}!=(p_4+p_3)p_3$$
$$(1)!=p_1 \qquad \{1\}!=p_4 \quad (11)$$

The symbol $\{\ \}!$ must not be dissociated from the expression B in which it occurs, because $\{r\}!$ as formed from n elements $p_1\, p_2\, \ldots\, p_n$ is not the same expression as $\{r\}!$ formed from m elements $p_1\, p_2\, \ldots\, p_m$.

In general, for n elements

$$\begin{aligned}
&\{1\}! = p_n \\
&\{2\}! = (p_n + p_{n-1})p_{n-1} \\
&\{3\}! = (p_n + p_{n-1} + p_{n-2})(p_{n-1} + p_{n-2})p_{n-2} \quad . \quad . \quad . \quad . \quad (12) \\
&\quad \ldots \qquad \ldots \qquad \ldots \\
&\quad \ldots \qquad \ldots \qquad \ldots \\
&\quad \ldots \qquad \ldots \qquad \ldots
\end{aligned}$$

At this point some properties of the coefficients

$$\frac{(n)!}{(n-r)!\,\{r\}!}$$

may be noted

$$\sum(-1)^r \frac{(n)!}{(n-r)!\,\{r\}!} \equiv 0 \quad . \quad . \quad . \quad . \quad (13)$$

which is the generalisation of

$$\sum(-1)^r \frac{n!}{r!\,n-r!} \equiv 0$$

As an example of this property take $(p_1, p_2, p_3, p_4) = (1, 7, 9, 5)$, then

$$\begin{aligned}
(4)! &= (1+7+9+5)(7+9+5)(9+5)5 = 22{\cdot}21{\cdot}14{\cdot}5 \\
(3)! &= (1+7+9)(7+9)9 = 17{\cdot}16{\cdot}9 \\
(2)! &= (1+7)7 = 8{\cdot}7 \\
(1)! &= 1 \\
\{1\}! &= 5 \\
\{2\}! &= (5+9){\cdot}9 = 14{\cdot}9 \\
\{3\}! &= (5+9+7)(9+7){\cdot}7 = 21{\cdot}16\,7 \\
\{4\}! &= (5+9+7+1)(9+7+1)(7+1){\cdot}1 = 22{\cdot}17{\cdot}8{\cdot}1
\end{aligned}$$

The expression

$$\frac{(4)!}{(4)!\,\{0\}!} - \frac{(4)!}{(3)!\,\{1\}!} + \frac{(4)!}{(2)!\,\{2\}!} - \frac{(4)!}{(1)!\,\{3\}!} + \frac{(4)!}{(0)!\,\{4\}!}$$

is

$$1 - \frac{1078}{408} + \frac{55}{12} - \frac{55}{4} + \frac{735}{68} \equiv 0$$

If $p_1\, p_2, \ldots$ be an arithmetical sequence $1,\ 1+a,\ 1+2a$, etc., then

$$\sum \frac{(n)!}{(n-r)!\,\{r\}!} = \frac{(2+(n-1)a)(2+na)\ \ldots\ (2+(2n-2)a)}{(1+a)(1+2a)\ \ldots\ (1+(n-1)a)} \quad . \quad . \quad (14)$$

a generalisation of

$$\sum \frac{n!}{r!\,n-r!} = 2^n$$

to which the identity reduces when the sequence $(p_1\, p_2\, p_3\, \ldots) = (1, 1, 1\, \ldots)$.

If the sequence is $(p_1\, p_2\, \ldots) = (1, 3, 5, \ldots)$

$$\sum \frac{(n)!}{(r)!\,\{n-r\}!} = 2^{2n-1} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The identity (14) is a particular case of $F(\alpha\,\beta\,\gamma\,\delta\,\epsilon\,1)$ and may be thrown into the form

$$(x)_n = (2x)_n + \sum \frac{n!}{r!\,n-r!}(2x-2r)_{n-r}(2x-r+1)_r \quad . \quad . \quad . \quad (16)$$

in which

$$(x)_n \text{ is } \frac{1}{x \cdot x-1 \cdot x-2 \ . \ . \ . \ x-n+1}$$

A more general form is

$$[x]_n = [2x]_n + \sum p^{r(x-n+1)} \frac{[n]!}{[r]!\,[n-r]!}[2x-2r]_{n-r}[2x-r+1]_r \quad . \quad . \quad . \quad . \quad (17)$$

$$[x]_n \equiv \frac{1}{[x]\,[x-1]\,.\,.\,.\,.\,.\,[x-n+1]} \qquad [x] \equiv \frac{p^x-1}{p-1}$$

These and other interesting theorems due to change of the sequence $(p_1\,p_2\,\ldots\,)$ must be left to another paper.

We now proceed to obtain the coefficients $B_0\,B_1$, etc.
Suppose that $\Pi_s(\alpha+(m))$ is capable of expansion in the form

$$\Pi_s(\alpha+(m)) \equiv B_0 + B_1\frac{(m)}{p_1} + B_2\frac{(m)_2}{p_2!} + \ . \ . \ . \ . \ . \ . + B_s\frac{(m)_s}{p_s!} \quad . \quad . \quad . \quad (18)$$

$(m)_s$ denoting $(p_1 + \ . \ . \ . \ + p_m)(p_2 + \ . \ . \ . \ + p_m) \ . \ . \ . \ . \ (p_s + \ . \ . \ . \ + p_m)$
$p_s!$ „ $p_1 \cdot p_2 \cdot p_3 \ . \ . \ . \ . \ p_s$

In (18) substitute $(m)=0$, then we have

$$B_0 = \Pi_s(\alpha)$$

Similarly, if we substitute $(m)-(1)=0$ we obtain

$$B_0 + B_1\frac{p_1}{p_1} = \Pi(\alpha+(1))$$

Continue the process of substitution by putting successively

$$(m)-(2)=0$$
$$(m)-(3)=0$$
. . . .
. . . .
. . . .

We obtain the following set of equations for determining the coefficients $B_0\,B_1$, etc.

$$\Pi(\alpha) = B_0$$

$$\Pi(\alpha+(1)) = B_0 + \frac{p_1}{p_1}B_1$$

$$\Pi(\alpha+(2)) = B_0 + \frac{p_1+p_2}{p_1}B_1 + \frac{p_1+p_2 \cdot p_2}{p_1 p_2}B_2 \quad . \quad . \quad . \quad . \quad (19)$$

. . . . . .
. . . . . .
. . . . . .

$$\Pi(\alpha+(n)) = B_0 + \frac{(n)}{p_1}B_1 + \frac{(n)_2}{p_2!}B_2 + \ . \ . \ . \ . + \frac{(n)_n}{p_n!}B_n$$

. . . . . .
. . . . . .
. . . . . .

From these equations we obtain

$$B_0 = \Pi(a)$$
$$B_1 = \Pi(a+(1)) - \Pi(a)$$
$$B_2 = \frac{p_1 p_2}{p_1 + p_2 \cdot p_2}\Pi(a+(2)) - \frac{p_1 p_2}{p_1 p_2}\Pi(a+(1)) + \frac{p_1 p_2}{p_1 \cdot p_2 + p_1}\Pi(a) \quad . \quad . \quad . \quad (20)$$
$$B_3 = p_3!\left\{\frac{\Pi(a+(3))}{(3)!\,\{0\}!} - \frac{\Pi(a+(2))}{(2)!\,\{1\}!} + \frac{\Pi(a+(1))}{(1)!\,\{2\}!} - \frac{\Pi(a)}{(0)!\,\{3\}!}\right\}$$

To establish the form of the coefficient B_n we assume that the law of formation holds for $B_0\, B_1\, B_2 \ldots B_{n-1}$, namely, that

$$B_{n-1} = p_{n-1}!\left\{\frac{\Pi(a+(n-1))}{(n-1)!\,\{0\}!} - \frac{\Pi(a+(n-2))}{(n-2)!\,\{1\}!} + \ldots\ldots + (-1)^{n-1}\frac{\Pi(a)}{(0)!\,\{n-1\}!}\right\} \quad . \quad (21)$$
$$B_{n-2} = p_{n-2}!\left\{\frac{\Pi(a+(n-2))}{(n-2)!\,\{0\}!} - \frac{\Pi(a+(n-3))}{(n-3)!\,\{1\}!} + \ldots\ldots + (-1)^{n-2}\frac{\Pi(a)}{(0)!\,\{n-2\}!}\right\}$$
$$\ldots \qquad \ldots \qquad \ldots$$
$$\ldots \qquad \ldots \qquad \ldots$$
$$\ldots \qquad \ldots \qquad \ldots$$

From (19)

$$\Pi(a+(n)) = B_0 + B_1\frac{(n)}{p_1} + \ldots\ldots + B_r\frac{(n)_r}{p_r!} + \ldots\ldots + B_n\frac{(n)_n}{p_n!}$$

Replacing $B_0\, B_1 \ldots B_{n-1}$ by the expressions (21) we obtain

$$B_n = \lambda\Pi(a+(n)) + \lambda_1\Pi(a+(n-1)) + \ldots\ldots + \lambda_s\Pi(a+(n-s)) + \ldots\ldots + \lambda_n\Pi(a) \quad . \quad (22)$$

where

$$\lambda = \frac{p_n!}{(n)!}$$
$$\lambda_1 = -\frac{p_n!}{(n)!}\frac{(n)_{n-1}}{(n-1)!\,\{0\}!}$$
$$\lambda_2 = \frac{p_n!}{(n)!}\left\{\frac{(n)_{n-1}}{(n-2)!\{1\}!} - \frac{(n)_{n-2}}{(n-2)!\{0\}!}\right\}$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\lambda_s = (-1)^{s+1}\frac{p_n!}{(n)!}\left\{\frac{(n)_{n-1}}{(n-s)!\,\{s-1\}!} - \frac{(n)_{n-2}}{(n-s)!\,\{s-2\}!} + \ldots\ldots + (-1)^s\frac{(n)_{n-s}}{(n-s)!\,\{0\}!}\right\} \quad . \quad (23)$$

$\{s-1\}!$ in the first term of λ_s is formed out of the set of elements $p_1\, p_2 \ldots p_{n-1}$ and is

$$(p_{n-1} + p_{n-2} + \ldots + p_{n-s+1})(p_{n-2} + \ldots + p_{n-s+1}) \ldots\ldots p_{n-s+1}$$

$\{s-2\}!$ in the second term of λ_s is formed out of the set of elements $p_1\, p_2 \ldots p_{n-2}$ and is

$$(p_{n-2} + \ldots + p_{n-s+1})(p_{n-3} + \ldots + p_{n-s+1}) \ldots\ldots p_{n-s+1}.$$

We see that λ_1 which is

$$\frac{p_n!\,(n)_{n-1}}{(n)!\,(n-1)!\,\{0\}!}$$

may be written

$$\frac{p_n!}{(n-1)!} \cdot \frac{(p_1 + \ldots + p_n)(p_2 + \ldots + p_n) \ldots (p_{n-1} + p_n)}{(p_1 + \ldots + p_n)(p_2 + \ldots + p_n) \ldots (p_{n-1} + p_n)p_n}$$
$$\equiv \frac{p_n!}{(n-1)!\,p_n} \equiv \frac{p_n!}{(n-1)!\,\{1\}!}$$

since p_n is $\{1\}!$ for the n elements $p_1\, p_2 \ldots p_n$.

Similarly λ_2 may be written

$$\frac{p_n!}{(n-2)!}\left\{\frac{(n)_{n-1}}{(n)!\,\{1\}!}-\frac{(n)_{n-2}}{(n)!\,\{0\}!}\right\}$$

which is

$$\frac{p_n!}{(n-2)!}\left\{\frac{1}{p_n\cdot p_{n-1}}-\frac{1}{p_{n-1}+p_n\cdot p_n}\right\}$$

$$\equiv \frac{p_n!}{(n-2)!\,p_{n-1}+p_n\cdot p_{n-1}} \equiv \frac{p_n!}{(n-2)!\,\{2\}!}$$

since $(p_{n-1}+p_n)p_n$ is $\{2\}!$ for the n elements $p_1 p_2 \ldots p_n$.

In the expression (23), if we reverse the order of the terms we have

$$\lambda_s = -\frac{p_n!}{(n-s)!}\left\{\frac{(n)_{n-s}}{(n)!\,\{0\}!}-\frac{(n)_{n-s+1}}{(n)!\,\{1\}!}+\ldots\ldots+(-1)^s\frac{(n)_{n-1}}{(n)!\,\{s-1\}!}\right. \quad . \quad (24)$$

and since

$$\frac{(n)_{n-1}}{(n)!} = p_n$$

in λ_s,

$$\frac{(n)_{n-2}}{(n)!} = p_n(p_n+p_{n-1})$$

$$\ldots\ldots$$

$$\ldots\ldots$$

$$\frac{(n)_{n-s}}{(n)!} = p_n(p_n+p_{n-1})\ldots\ldots(p_n+p_{n-1}+\ldots\ldots+p_{n-s+1})$$

also

$$\{0\}! = 1$$

$$\{1\}! = p_{n-s+1} \quad \text{(Because the term involving } \{1\}! \text{ was derived from } B_{n-s+1})$$

$$\{2\}! = (p_{n-s+2}+p_{n-s+1})p_{n-s+1}$$

$$\{3\}! = (p_{n-s+3}+p_{n-s+2}+p_{n-s+1})(p_{n-s+2}+p_{n-s+2})p_{n-s+1}$$

$$\{s-1\}! \quad (p_{n-1}+\ldots\ldots+p_{n-s+1})(p_{n-2}+\ldots+p_{n-s+1})\ldots\ldots\ldots p_{n-s+1}$$

We see that for the set of elements $p_n\, p_{n-1} \ldots\ldots p_{n-s+1}$

the expression

$$\frac{(n)_{n-s}}{(n)!\,\{0\}!}-\frac{(n)_{n-s+1}}{(n)!\,\{1\}!}+\ldots\ldots+(-1)^{s-1}\frac{(n)_{n-1}}{(n)!\,\{s-1\}!}$$

is

$$\frac{1}{(s)!}-\frac{1}{(s-1)!\,\{1\}!}+\ldots\ldots+(-1)^{s-1}\frac{1}{(1)!\,\{s-1\}} \quad . \quad . \quad . \quad (25)$$

but for any set of s elements in any order

$$\frac{1}{(s)!}-\frac{1}{(s-1)!\,\{1\}!}+\ldots\ldots\ldots+(-1)^s\frac{1}{\{s\}!} \equiv 0$$

so that

$$\frac{1}{(s)!}-\frac{1}{(s-1)!\,\{1\}!}+\ldots\ldots+(-1)^{s-1}\frac{1}{(1)!\,\{s-1\}!} = -(-1)^s\frac{1}{\{s\}!} \quad . \quad . \quad (26)$$

and we have

$$\lambda_s = (-1)^s\frac{p_n!}{(n-s)!\,\{s\}!}$$

$$B_n = \sum_{s=0}^{s=n}(-1)^s\frac{p_n!}{(n-s)!\,\{s\}!}\Pi(a+(n-s)) \quad . \quad . \quad . \quad (27)$$

which establishes the form of B_n in general.

The General Hypergeometric Series.

Let ϕ denote the differential operator

$$B_0 + B_1 x^{p_1} D^{(1)} + B_2 x^{p_1+p_2} D^{(2)} + \ldots\ldots + B_s x^{(s)} D^{(s)} \quad . \quad . \quad . \quad (28)$$

and ψ denote the operator

$$\frac{1}{x^{p_1}} \left\{ C_0 x^{(1)} D^{(1)} + C_1 x^{(2)} D^{(2)} + \ldots\ldots + C_t x^{(t)} D^{(t)} \right\} \quad . \quad . \quad . \quad (29)$$

B_n being $\sum_{r=0}^{r=n} (-1)^r \frac{p_r!}{(n-r)!\,\{r\}!} \Pi_s(\alpha + (n-r))$ . . . (30)

C_n ,, $\sum_{r=0}^{r=n} (-1)^r \frac{p_r!}{(n-r)!\,\{r\}!} \Pi_t(\beta + (n-r))$ . . . (31)

Then if y denote a series of the form $\Sigma A x^{(m)}$
that is

$$A_1 x^{(m_1)} + A_2 x^{(m_2)} + \ldots\ldots\ldots$$

and we operate on y with $\phi - \psi$ we obtain

$$\{\phi - \psi\} y = \sum A \cdot \Pi_s(\alpha + (m)) x^{(m)} - \sum \mathbf{A} \frac{(m)}{p_1} \Pi_t(\beta + (m) - (1)) x^{(m)-(1)} \quad . \quad . \quad . \quad (32)$$

The lowest index present in the series on the right is $(m_1) - (1)$, and the term in which it occurs is

$$A_1 \frac{(m_1)}{p_1} \Pi_t(\beta + (m_1) - (1)) x^{(m_1)-(1)}$$

Choose m_1 so as to make this term vanish, the possible values of m_1 are zero and $(1) - \beta$. Now, from the manner in which $(m_1)(m_2)\ldots$ are formed, we cannot have any such relation as $(m_r) = (m_{r+1}) - (1)$ unless the elements $p_1 p_2 p_2 \ldots$ are equal to one another, so we choose

$$\begin{aligned} (m_1) &= 0 \\ (m_2) &= (1) = p_1 \\ (m_3) &= (2) = p_1 + p_2 \\ &\ldots\ldots \\ &\ldots\ldots \\ &\ldots\ldots \end{aligned}$$

and

$$A_{r+1} \frac{(m_{r+1})}{p_1} \Pi_t(\beta + (m_{r+1}) - (1)) = A_r \Pi_s(\alpha + (m_r))$$

Then we have

$$F = 1 + p_1 \frac{\Pi_s(\alpha)}{(1)\Pi_t(\beta)} x^{(1)} + p_1^2 \frac{\Pi_s(\alpha)\Pi_s(\alpha + (1))}{(2)!\,\Pi_t(\beta)\Pi_t(\beta + (2) - (1))} x^{(2)} + \ldots\ldots \quad . \quad . \quad (33)$$

the general term being

$$p_1^n \frac{\Pi_s(\alpha)\Pi_s(\alpha + (1))\Pi_s(\alpha + (2)) \ldots\ldots\ldots \Pi_s(\alpha + (n-1))}{(n)!\,\Pi_t(\beta)\Pi_t(\beta + (2) - (1)) \ldots\ldots\ldots \Pi_t(\beta + (n) - (1))} x^{(n)} \quad . \quad . \quad (34)$$

and a differential equation

$$\{\phi-\psi\}\mathrm{F}=\Pi_s(a)\left\{\left(1+p_1\frac{(a_1+p_1)(a_2+p_1)\ .\ .\ .\ .\ .\ (a_s+p_1)}{(1)\beta_1\beta_2\beta_3\ .\ .\ .\ .\ .\ .\ .\ \beta_t}x^{p_1}+\ .\ .\ .\ \right)\right.$$
$$\left.-\left(1+p_1\frac{(a_1+p_1)(a_2+p_1)\ .\ .\ .\ .\ .\ .\ .\ (a_s+p_1)}{(1)\beta_1\beta_2\beta_3\ .\ .\ .\ .\ .\ .\ .\ \beta_t}x^{p_2}+\ .\ .\ .\ \right)\right\}\quad .\quad (35)$$

In the simple case

$$\mathrm{F}=1+p_1\frac{a\cdot\beta}{p\cdot\gamma}x^{p_1}+p_1{}^2\frac{a\cdot a+p_1\beta\beta+p_1}{p_1\cdot p_1+p_2\gamma\gamma+p_2}x^{p_1+p_2}+\ .\ .\ .\ .\quad .\quad .\quad (36)$$

the differential equation is

$$a\beta\cdot\mathrm{F}+p_1\{(a+\beta+p_1)x^{p_1}-\gamma\}\mathrm{D}\cdot^{(1)}\mathrm{F}+p_1p_2\{x^{p_1}-1\}x^{p_2}\mathrm{D}^{(2)}\mathrm{F}$$
$$=a\beta\left\{(1+p_1\frac{a+p_1\beta+p_1}{p_1\gamma}x^{p_1}+p_1{}^2\frac{a+p_1\cdot a+p_1+p_2\cdot\beta+p_1\cdot\beta+p_1+p_2}{p_1p_1+p_2\cdot\gamma\cdot\gamma+p_2}x^{p_1+p_2}+\ .\ .\ .\right.$$
$$\left.-(1+p_1\frac{a+\beta_1\beta+p_1}{p_1\gamma}x^{p_2}+p_1{}^2\frac{a+p_1a+p_1+p_2\cdot\beta+p_1\beta+p_1+p_2}{p_1p_1+p_2\gamma\cdot\gamma+p_2}x^{p_2+p_3}+\ .\ .\ .\ .)\right\}\quad .\quad (37)$$

Putting $x=1$ the expression on the right vanishes identically, therefore when $[\mathrm{D}^{(2)}\mathrm{F}]_{x=1}$ is finite or convergent we have

$$a\beta[\mathrm{F}]_{x=1}+p_1\{a+\beta-\gamma+p_1\}\left[\frac{d^{(1)}\mathrm{F}}{dx^{(1)}}\right]_{x=1}=0\quad .\quad .\quad .\quad (38)$$

which is

$$1+p_1\frac{a\beta}{p_1\gamma}+p_1{}^2\frac{a\cdot a+p_1\beta\cdot\beta+p_1}{p_1p_1+p_2\cdot\gamma\cdot\gamma+p_2}+\ .\ .\ .=\frac{p_1(\gamma-a-\beta-p_1)}{\gamma}\left\{1+p_1\frac{a+p_1\beta+p_1}{p_1\cdot\gamma+p_2}\right.$$
$$\left.+p_1{}^2\frac{a+p_1\cdot a+p_1+p_2\cdot\beta+p_1\cdot\beta+p_1+p_2}{p_1\cdot p_1+p_2\cdot\gamma+p_2\cdot\gamma+p_2+p_3}+\ .\ .\ .\right\}\quad .\quad (39)$$

subject to the convergence of the series.

When the progression of elements $p_1p_2\ .\ .\ .\ .\ .$ is 1 3 5 7 . . . . .

$$\mathrm{F}(a,\beta,\gamma+1,x)=1+\frac{a\beta}{1^2\cdot\gamma+1}x+\frac{a\cdot a+1\beta\beta+1}{1^2\cdot2^2\cdot\gamma+1\cdot\gamma+4}x^4+\ .\ .\ .\ .$$
$$\mathrm{F}(a,\beta,1,x)=1+\frac{a\beta}{1^4}x+\frac{a\cdot a+1\beta\beta+1}{1^4\cdot2^4}x^4+\frac{a\cdot a+1\cdot a+2^2\cdot\beta\beta+1\beta+2^2}{1^4\cdot2^4\cdot3^4}x^9+\ .\ .\ .\ .\ .\quad .\quad (40)$$

In connection with this series we can obtain from (39) the identity

$$1+\frac{n}{(1!)^4}x+\frac{n\cdot n-1}{(2!)^4}x\cdot x-1+\frac{n\cdot n-1^2\cdot n-2^2}{(3!)^4}x\cdot x-1x-2^2+\ .\ .\ .\ .\ .$$
$$=(x+n)\left\{1+\frac{n-1\cdot x-1}{1^2}\frac{}{2^2}+\frac{n-1^2\cdot n-2^2}{1^2\cdot2^2}\cdot\frac{x-1^2\cdot x-2^2}{2^2\cdot3^2}+\frac{n-1^2\cdot n-2^2n-3^2}{1^22^2\cdot3^2}\cdot\frac{x-1^2x-2^2x-3^2}{2^23^2\cdot4^2}+\ .\ .\ .\right\}\quad (41)$$

putting $n=-x=m^2$

$$1-\frac{m^4}{(1!)^4}+\frac{m^4\cdot m^4-1^4}{(2!)^4}-\ .\ .\ .\ .\ .\ .\equiv0$$

In the case when $p_1p_2\ .\ .\ .\ .\ .$ form a geometrical progression and $\mathrm{F}([a][\beta][\gamma]\lambda x)$ denotes the series

$$1+\frac{[a][\beta]}{[l][\gamma]}\lambda x^{[l]}+\frac{[a][a+l][\beta][\beta+l]}{[l][2l]\cdot[\gamma][\gamma+l]}\lambda^2x^{[2l]}+\ .\ .\ .\ .\ .\ .\quad .\quad .\quad (42)$$
$$[a]\equiv\frac{\mathrm{P}^a-1}{\mathrm{P}-1}$$

the following relations can be obtained :

$$\mathrm{F}([\alpha][\beta][\gamma]p^{\gamma-\alpha-\beta}) = \frac{[\gamma-l][\gamma-\alpha-\beta-l]}{[\gamma-\alpha-l][\gamma-\beta-l]}\mathrm{F}([\alpha][\beta][\gamma-l]p^{\gamma-\alpha-\beta-l})$$

$$\mathrm{F}([\alpha][\beta][\gamma]p^{\gamma-\alpha-\beta-l}) = \frac{[\gamma-l]\{[\gamma-\alpha-l]+p^{\gamma-l}[-\beta]\}}{[\gamma-\alpha-l][\gamma-\beta-l]}\mathrm{F}([\alpha][\beta][\gamma-l]p^{\gamma-\alpha-\beta-l}) \quad . \quad (43)$$

$$\mathrm{F}([\alpha][\beta][\gamma]p^{\gamma-\alpha-\beta}) = \frac{[\gamma-\alpha-\beta-l]}{[\gamma-\alpha-l]+p^{\gamma-l}[-\beta]}\cdot\mathrm{F}([\alpha][\beta][\gamma]p^{\gamma-\alpha-\beta-l})$$

$$\mathrm{F}([\alpha][\beta][\gamma]p^{\gamma-\alpha-\beta}) = \frac{1}{\mathrm{F}([-\alpha][\beta][\gamma-\alpha]p^{\gamma-\beta})}$$

and other similar relations found by interchanging α and β.

The expression of $\mathrm{F}([\alpha][\beta][\gamma]p^{\gamma-\alpha-\beta})$ and $\mathrm{F}([\alpha][\beta][\gamma]p^{\gamma-\alpha-\beta-l})$ in the form of infinite products analogous to $\dfrac{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}$ is effected at once by the above relations. The investigation of these I have given elsewhere, but note the results here as of interest in connection with the Fundamental Hypergeometric Equation discussed in this paper. Particular cases of these series are

$$\frac{\Gamma([x])}{\Gamma([x+\tfrac{1}{2}])} = \left\{\frac{p^x}{[x]} + \frac{[\tfrac{1}{2}]^2p^{2x}}{[1]\,[x]\,[x+1]} + \frac{[\tfrac{1}{2}]^2[\tfrac{3}{2}]^2p^{3x}}{[2]\,[x]\,[x+1]\,[x+2]} + \ldots\right\}^{\frac{1}{2}}$$

$$= \frac{[x\quad 1]^{\frac{1}{2}}}{\{[x-\tfrac{1}{2}]+p^x[-\tfrac{1}{2}]\}^{\frac{1}{2}}}\left\{\frac{p^x}{[x]} + \frac{[\tfrac{1}{2}]^2p^{2x-1}}{[1]\,[x]\,[x+1]} + \frac{[\tfrac{1}{2}]^2[\tfrac{3}{2}]^2p^{3x-2}}{[2]\,[x]\,[x+1]\,[x+2]} + \ldots\right\}^{\frac{1}{2}} \quad . \quad (44)$$

Bessel's Series :—

Consider now a progression

$$\ldots\ldots\ldots p_{-6}\ p_{-5}\ p_{-4}\ p_{-3}\ p_{-2}\ p_{-1}\ p_0\ p_1\ p_2\ p_3\ldots\ldots\ldots$$

denote

$$p_1+p_2+\ldots+p_n \quad \text{by} \quad (n)$$

and

$$p_0+p_{-1}+\ldots\ldots+p_{-n+1} \quad \text{by} \quad (-n)$$

Then the operator

$$\lambda p_1p_2x^{(2)}\mathrm{D}^{(2)} + \lambda\{(1)-(n)-(-n)\}p_1x^{(1)}\mathrm{D}^{(1)} + \lambda(n)(-n)\mathrm{D}^{(0)} \equiv \phi \quad . \quad . \quad (45)$$

operating on $\mathrm{A}_1x^{(m_1)}$ gives

$$\lambda[(m_1)((m_1)-(1)) + \{(1)-(n)-(-n)\}(m_1) + (n)(-n)]\mathrm{A}_1x^{(m_1)}$$

which is

$$\lambda[(m_1)-(n)]\,[(m_1)-(-n)]\mathrm{A}_1x^{(m_1)}$$

So that if we operate on a series $y = \sum \mathrm{A}x^{(m)}$

$$\phi\cdot y = \sum[(m)-(n)]\,[(m)-(-n)]\mathrm{A}x^{(m)}$$

choose

$$(m_1) = (n) \text{ and } (m_2) = (n+2)$$
$$(m_3) = (n+4)$$
$$\ldots\ldots$$
$$\ldots\ldots$$
$$\lambda\mathrm{A}_{r+1}[(m_{r+1})-(n)]\,[(m_{r+1})-(-n)] = \mathrm{A}^r$$

then

$$y = \mathrm{A}\left\{ x^{(n)} + \lambda \frac{x^{(n+2)}}{[(n+2)-(n)]\,[(n+2)-(-n)]} + \ldots\ldots \right\}$$

and

$$\phi{\cdot}y = \lambda\mathrm{A}\left\{ x^{(n+2)} + \lambda \frac{x^{(n+4)}}{[(n+2)-(n)]\,[(n+2)-(-n)]} + \ldots \right\} \quad . \quad . \quad . \quad (46)$$

when the elements are

$$\begin{aligned} p_1 = p_2 \;\; &= p_3 \ldots\ldots = \;\; 1 \\ p_0 = p_{-1} &= p_{-2} \qquad\quad = -1 \end{aligned}$$

The series is BESSEL'S Series and $\phi{\cdot}y = \lambda x^2 y$ is BESSEL'S Equation.

In the particular case of the progression of elements $p_1\, p_2 \ldots$ being

$$\begin{array}{l} \ldots\ldots\; p_{-3}\; p_{-2}\; p_{-1}\; p_0\; p_1\; p_2\; p_3 \ldots\ldots \\ \ldots\ldots\; -7\; -5\; -3\; -1\; 1\; 3\; 5 \ldots\ldots \qquad \text{and } \lambda = -1 \end{array}$$

$$y = \mathrm{A}\left\{ x^{n^2} - \frac{x^{(n+2)^2}}{(n+2)^4 - n^4} + \frac{x^{(n+4)^2}}{\{(n+2)^4 - n^4\}\{(n+4)^4 - n^4\}} - \ldots\ldots \right\} \quad . \quad (47)$$

III.—Magnetization and Resistance of Nickel Wire at High Temperatures.

By Professor C. G. Knott, D.Sc.

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In a recent paper published in these *Transactions* * I gave an account of certain experiments upon the change of electric resistance of nickel due to magnetization at different temperatures up to 100° C. In the closing sentence of that paper I pointed out the advisability of trying to push the temperature up to 400° C., the temperature at which nickel loses its pronounced magnetic properties. This has been accomplished in the experiments now to be described. These refer, meanwhile, to the effect of longitudinal magnetization on the resistance of the wire. The experiments on the effect of transverse magnetization are still incomplete, and are reserved for a future communication.

1. The Apparatus used.—Since the temperature was to be raised to about 400° C., it was necessary to use asbestos for insulation. Accordingly, two exactly similar anchor-ring coils with nickel-wire cores were constructed. These coils were about 18·3 cm. in diameter. Round the flat circular coil of nickel which formed the nucleus of the anchor-ring two independent layers of copper wire, carefully insulated throughout, were coiled, with the same number of turns in each. The ends of the two copper-wire coils could be joined in different ways, so that it was possible to have a strong current passing through both, and yet to have, at will, either strong magnetization within or none at all. The magnetizing force could thus be removed at a moment's notice by simply reversing the current through one of the coils of copper wire, while the heating effect of the current on the whole coil could be maintained unaltered. To preserve as constant a temperature as possible during any one set of observations was of the highest importance, for the change of resistance due to a very slight change of temperature was sufficient to mask completely the change due to magnetization. This change was measured by means of a Wheatstone Bridge arrangement. The galvanometer was made of a convenient sensitiveness for the purpose; and only when a very steady temperature was obtained during a set of observations was the galvanometer in a steady enough state of approximate balance to render measurements possible. It was for the purpose of further reducing the disturbances due to changing temperature that two anchor-ring coils were used, with equal lengths of nickel wire as the cores, and with the same magnetizing or merely heating current flowing through both double coils of copper wire.

* Vol. xl. pp. 535–545.

The nickel wires in these coils will be called L and M for ease of reference; and the same letters may, if occasion arise, be used each to designate the corresponding anchor-ring coil as a whole.

The wires L and M, with their stout nickel-bar terminals, formed two of the contiguous branches of the Wheatstone Bridge, the other branches λ and μ being also made of nickel wire, so as to minimise the possibility of thermoelectric effects in the circuit. The resistances in B. A. ohms of the various branches at 13° C. were as follows, L_0 and M_0 being the resistances of the parts of L and M included in the magnetizing coils.

Wire.	L	M	λ	μ	L_0	M_0
Resistance . .	3·1237	3·041	3·2413	3·1556	2·957	2·925

The lengths and number of turns of the four magnetizing copper coils were as follows:—

	L-coils		M-coils	
	Inner.	Outer.	Inner.	Outer.
Length of copper wire . .	899 cm.	1151 cm.	992 cm.	1237 cm.
Number of turns . .	172	172	171	171

In the experiments to be described it was the nickel wire L which was studied; hence, to reduce the magnetizing current in amperes in the coil L to fields in magnetic C.G.S. units we must multiply by 7·52.

The magnetizing current was measured on a Kelvin graded galvanometer, which was calibrated *in situ* by comparison with a Kelvin ampere balance.

The two anchor-ring coils L and M were enclosed in a porcelain vessel, through the side of which the various terminals were led. A quantity of asbestos wool was packed round the coils so as to reduce as far as possible the convection currents of hot air when the vessel was heated. The heating was effected by means of one, two, or more Bunsen burners, according to the temperature aimed at. After several hours' heating the temperature of both coils became fairly steady; and when the galvanometer indicated that a sufficiently steady temperature was reached, the necessary observations began to be made. The temperature of the wire L at any instant was indicated with great accuracy by the resistance of the wire, an independent experiment upon a wire cut

from the same piece giving data by means of which the measured resistance could be reduced to centigrade degrees. It is known * that the resistance of nickel changes rather curiously with temperature, of which more hereafter. In the present case the resistance temperature graph is, roughly speaking, a long sloping S form of curve, and can be represented for interpolation purposes very approximately by three straight lines. Thus, to reduce the measured resistance r of the wire (3 ohms at 15°) to temperatures the formulæ are

$$\begin{aligned} t &= 56{\cdot}7r - 163{\cdot}3 \text{ from } 15^\circ \text{ to } 200^\circ \\ t &= 33{\cdot}9r - 22{\cdot}7 \quad ,, \quad 200^\circ \;,, \; 350^\circ \\ t &= 95{\cdot}8r - 692{\cdot}5 \quad ,, \quad 350^\circ \;,, \; 400^\circ \end{aligned}$$

2. The Method of Experiment.—The resistance changes due to magnetization were measured by deflections on a delicate galvanometer after the bridge was approximately balanced, a steady current being supplied from a secondary cell in the battery branch. The galvanometer was calibrated in the following simple way. For any particular condition the current i through the galvanometer is given by the formula

$$Di = (L\mu - M\lambda)e \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where e is the electromotive force in the circuit, and D the well-known expression involving the resistances of the six conductors making up the Wheatstone Bridge. Its value is

$$D = BG(L+M+\lambda+\mu) + B(L+\lambda)(M+\mu) + G(L+M)(\lambda+\mu) + LM(\lambda+\mu) + \lambda\mu(L+M),$$

where B is the resistance in the battery branch and G that in the galvanometer branch. In the present experiments B was 6·2 ohms and G was 1·73 ohms.

The calibration of the galvanometer in each experiment was effected by making a known small change in λ; and the magnetic effect on resistance gave a slight change in L. We must therefore consider how the quantity D changes with slight changes of λ and L. Differentiation of D gives

$$\begin{aligned} \frac{dD}{d\lambda} &= BG + B(M+\mu) + (G+\mu)(L+M) + LM \\ \frac{dD}{dL} &= BG + B(M+\mu) + (G+M)(\lambda+\mu) + \lambda\mu. \end{aligned}$$

The particular values of these quantities will vary from experiment to experiment according to the temperature of the branches L and M. For temperatures higher than the temperature of the air, L was always at a higher temperature than M, since it lay lower down in the porcelain vessel. To obtain the approximate balance on the Wheatstone Bridge μ was appropriately varied so that the quantity $(L\mu - M\lambda)$ rarely differed more than 1/400th from the value of $L\mu$ or of $M\lambda$. When the difference was 1/200th the spot of light was driven off the galvanometer scale. In the following table the values

* See my paper on the Electrical Resistance of Nickel at High Temperatures, *Trans. R.S.E.*, 1886; also a paper by W. Kohlrausch, *Wiedemann's Annalen*, vol. xxxiii., 1888.

of the various resistances and of the quantities referred to above are given when the temperatures of L are about 15°, 180°, 300°.

Branches.	Resistances when the Temperatures of L are		
	15°	180°	300°
B	6 2	6·2	6·2
G	1·73	1·73	1·73
L	3·12	6	10
M	3·04	5	8
λ	3·24	3·24	3·24
μ	3 16	2·7	2·6
D	571·2	1009·9	2058·6
$dD/d\lambda$	88·8	126·4	234·4
dD/dL	89·9	96·4	141·7

In the calibration experiments the slight change in the resistance of λ was always effected by putting in a resistance of 30 ohms in multiple arc with a small part of λ whose resistance was 0·5125. That is to say, the change $d\lambda$ was a decrease of 0·008603 ohms, and $d\lambda/\lambda = -0{\cdot}002655$.

Returning now to equation (1), namely,

$$Di = (L\mu - M\lambda)e \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we get, in consequence of the small change $d\lambda$, the equation

$$Ddi + id\mathrm{D} = - Med\lambda \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Similarly, when by application of the magnetizing field L is changed to $L + \delta L$, we find

$$D\delta i + i\delta D = + \mu e\delta L \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Substituting for i its value as given in (1) we get for (2) and (3)

$$\left.\begin{aligned} Ddi &= - ed\lambda\left(M + \frac{L\mu - M\lambda}{D}\frac{d\mathrm{D}}{d\lambda}\right) \\ D\delta i &= + e\delta L\left(\mu - \frac{L\mu - M\lambda}{D}\frac{d\mathrm{D}}{dL}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

If, at the beginning of the experiments corresponding to equations (2) and (3), the bridge were accurately balanced, the current i would vanish and we should have

$$\frac{di}{\delta i} = - \frac{M}{\mu}\frac{d\lambda}{\delta L} = - \frac{L}{\lambda}\frac{d\lambda}{\delta L}$$

or

$$\frac{dL}{L} = - \frac{\delta i}{di}\frac{d\lambda}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This was the case in many instances, and in many others the value of i was so small as to make $(L\mu - M\lambda)$ less than the thousandth part of $L\mu$ or $M\lambda$. Under these

conditions, taking the largest value of the ratios $d\mathrm{D}/\mathrm{D}d\lambda$ and $d\mathrm{D}/\mathrm{D}d\mathrm{L}$, namely, 89/572, we find

$$\mathrm{D}di = -ed\lambda\left(\mathrm{M} + \mathrm{M}\lambda\frac{89}{572000}\right)$$
$$\mathrm{D}\delta i = +e\delta\mathrm{L}\left(\mu - \mu\mathrm{L}\frac{89}{572000}\right),$$

showing at once that the second terms in the brackets are negligible, and that equation (5) still holds.

Finally, consider the most extreme case of all, in which the applied field was so great as to produce a change $d\mathrm{L}$, which made the spot of light travel from end to end of the scale. It was necessary in this case to disturb the balance on the bridge, so that the initial current i produced a deflection 3·5 times that due to the imposed change $d\lambda$. That is to say, since the change $d\lambda$ meant an alteration of 0·002655 in the value of $\mathrm{L}\mu$ or $\mathrm{M}\lambda$ in the experiment corresponding to equation (2), the initial value of i in this extreme case of the experiment corresponding to (3) was such as to make

$$\mathrm{L}\mu - \mathrm{M}\lambda = 3{\cdot}5 \times {\cdot}002655\,\mathrm{L}\mu = {\cdot}0093\,\mathrm{L}\mu$$

and

$$\mathrm{M}\lambda = {\cdot}9907\,\mathrm{L}\mu.$$

Thus equations (4) become

$$\mathrm{D}di = -e\mathrm{M}d\lambda$$
$$\mathrm{D}\delta i = +e\mu\delta\mathrm{L}\left(1 - {\cdot}0093\frac{\mathrm{L}}{\mathrm{D}}\frac{d\mathrm{D}}{d\mathrm{L}}\right)$$

This extreme case occurred only at the ordinary temperature of 15° C. Putting in the corresponding values of L, D, and $d\mathrm{D}/d\mathrm{L}$, and taking the ratio, we find

$$\frac{di}{\delta i} = -\frac{\mathrm{M}}{\mu}\frac{d\lambda}{\delta\mathrm{L}}\frac{1}{1 - {\cdot}0093 \times 3{\cdot}12 \times 0{\cdot}158}$$
$$= -\frac{\mathrm{L}}{\lambda}\frac{d\lambda}{\delta\mathrm{L}}\frac{1}{1 - {\cdot}00458} \times {\cdot}9907 = -\frac{\mathrm{L}}{\lambda}\frac{d\lambda}{\delta\mathrm{L}} \times {\cdot}995.$$

Hence equation (5) is in error by 0·5 per cent.; and it will be noticed that this is due mainly to the factor by which we pass from the ratio M/μ to L/λ. The effect of the second terms in the brackets of equations (4) is in all cases negligible. Hence in every case we may write

$$\frac{di}{\delta i} = -\frac{\mathrm{M}}{\mu}\frac{d\lambda}{\delta\mathrm{L}},$$

and in the great majority of cases use the more convenient form (5).

3. Reduction of the Observations.—The results embodied in Table A were plotted on a large scale, the change of resistance $d\mathrm{L}$ being plotted in terms of the field. In any one series of experiments the temperature varied a little throughout; but it was easy to apply slight corrections by graphical interpolation so as to obtain a series of isothermal curves. From these curves the values of $d\mathrm{L}$ were read off for the fields 2, 4, 6, etc., up to 34, and were then divided by the appropriate value L_0 of the resistance of the nickel wire included in the coil. This was assumed to have the same

TABLE A.

Containing the reduced results immediately deducible from the individual observations of the experiments described above, and arranged in order of date, H *being the magnetic force in the heart of the anchor-ring coil,* t *the temperature of the nickel coil at the instant, as determined by the measured resistance of the wire* L, *and* dL *being the change of resistance due to the magnetization.*

H	*t*	*d*L	H	*t*	*d*L	H	*t*	*d*L
	(Feb. 13)			(Feb. 16)			(Feb. 18)	
33·6	14	0·0298	32·2	179·1	0·0262	32·6	127·5	0·0291
26·2	15·3	222	25·8	183·1	207	25·8	130	236
21·8	16·3	176	21·2	185	183	21·2	128·1	198
17·8	17·5	142	12·5	180·8	110	16·1	124	152
13·5	18·7	96	8·9	179	79	10·2	121·5	88
9·4	19·2	48	6·8	176·9	51	7·8	117	54
7·1	18·9	25	5	176·3	25	5·1	113·2	20
5·2	16·9	11	3·2	176	6	3·2	111·1	5
3·5	16·3	24						
	(Feb. 19)			(Feb. 21)			(Feb. 23)	
30·8	241·5	0·0173	30·4	317	0·0044	34	16·7	0·0296
24·8	242·7	150	24·4	322	20	26·6	19·7	238
20·7	242·1	133	20·2	322	25	21·8	20·4	194
15·3	241	107	14·9	319·7	21	16·6	18·7	126
12·3	240·5	88	11·9	322	19	13·0	18·1	96
10·1	239·5	76	10	323	14	10·8	17	68
7·1	237·7	54	7·4	323·5	11	8	16·3	31
4·9	236·1	27	4·8	323·5	5	5·8	15·3	14
3·6	234·8	16				4·2	14·6	6
	(Mar. 12)			(Apr. 1)			(May 6)	
31·8	279	0·0117	31·8	328	0·0043	31·8	299·3	0·0097
21·4	278·3	96	21·6	328	28	31·6	303·4	71
12·9	276·3	69	10·6	326·4	6	15·7	302·7	49
8·9	274·6	51		(May 10)		10·3	301	38
5·9	272·7	34	30·4	342·3	46	7·1	300	3
3·6	271	23	22·0	344·3	15	4·5	299	2
2·3	269·1	6	4·8	342·2	7	2·7	299	08
	(May 14)			(May 15)			(May 18)	
32	17·9	0·0331	33·3	67·7	0·0301	34·1	16·3	0·0280
17·8	16·6	130	26·2	68·6	241	26·8	18·7	225
14·1	15·4	109	21·5	68·8	195	22	18	186
13	15·3	93	16·5	67·2	152	16·3	14·7	127
12·8	16·3	91	13	66	111	12·8	15·4	90
10·6	15·8	67	10·7	64·9	77	10·8	15	66
10	15·5	54	8·4	63·5	53	8·1	14·7	33
8	15·1	34	6·1	61·0	26	4·7	14·4	7
6·5	14 1	19	4·1	62·3	11			
4·4	13·8	5	3·4	60·3	6			
			10·7	60	81			
			21·6	63·5	206			

ratio to the whole resistance L at all temperatures. It is possible, however, that with the strongest currents, whose heating effect is quite apparent in the slight rise and then fall of temperature during one set of readings as shown in Table A, the part of the nickel wire included within the magnetizing coil might be slightly higher in temperature than the short parts outside the coil which joined the nickel wire to the stout nickel terminals. This would make the ratio L_0/L a little greater in the highest fields, so that the quantity dL/L_0 would be a little smaller. It is clear, however, that any small error due to this cause will not materially affect the broad conclusions to be drawn from the experiments.

In Table B are given the final reductions, each column corresponding to a particular temperature, and each horizontal row to a particular field. The numbers entered are the increments of resistance per 100,000 ohms.

TABLE B.

Showing resistance changes per 100,000 *ohms of nickel wire in various longitudinal fields and at various temperatures.*

Magnetic Field.	Resistance Changes at Temperatures								
	15°	65°	125°	180°	240°	280°	300°	328°	342°
34	1040	816	621	475	253	141	86	45	5
32	970	776	594	450	244	138	83	44	...
30	901	729	565	426	235	133	81	43	...
28	825	679	529	401	226	128	79	41	4
26	749	629	498	375	216	123	76	38	...
24	682	579	463	350	205	119	73	35	...
22	613	529	425	324	193	113	69	32	2
20	543	479	392	298	179	106	66	29	...
18	474	429	356	270	166	100	61	24	...
16	411	379	317	242	151	93	57	18	1·6
14	341	318	275	215	135	84	52	12	...
12	268	255	231	186	112	75	48	6	...
10	192	187	181	158	103	64	42	4	1
8	119	124	123	121	85	52	37	3	...
6	53	63	65	70	56	38	30	2	...
4	17	23	21	23	27	27	19	1	...
2	3	5	5	5	5·5	5	4	...	...

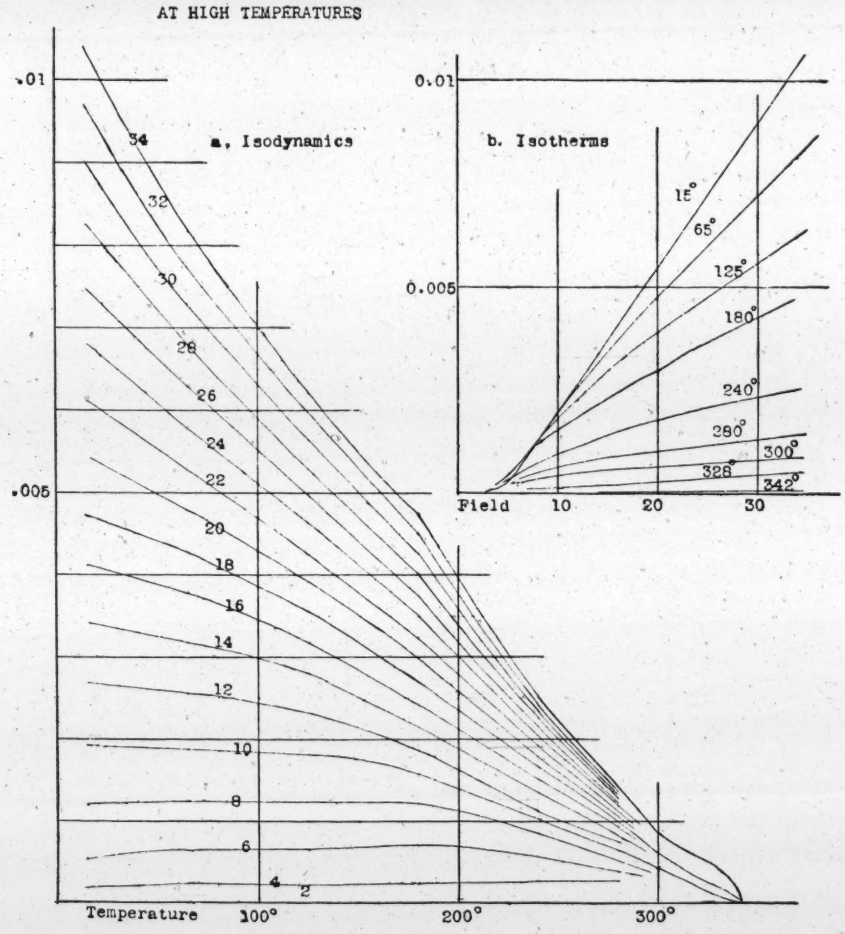
The numbers in the last column under temperature 342° C. were just measurable; anything under 3 is, in fact, barely outside the errors of observation.

4. DISCUSSION OF THE RESULTS.—These numbers give two sets of graphs—namely, the isothermals showing the relation between magnetizing force and resistance change at the various temperatures, and the isodynamics showing the relation between the

resistance change and the temperatures in the various fields. These sets of curves, marked b, a, are given in the accompanying Plate.

The first obvious result is the diminution of the resistance change in the higher fields as the temperature rises. Thus the effect in various fields at temperature 15° is from 200 to 300 times the effect at temperature 342°. So rapid is the final drop above 300° C. that we may safely regard the effect as practically non-existent at temperature 350° C. It is just at this temperature that nickel loses its strong magnetic properties,

RESISTANCE CHANGES ACCOMPANYING MAGNETIZATION OF NICKEL AT HIGH TEMPERATURES



the permeability being practically unity. Thus we learn that the change of resistance of nickel wire due to the application of a longitudinal magnetic field is mainly a function of the magnetization or induction in the material, and not of the magnetizing force.

In fields below 5, there is first increase of the resistance change as the temperature rises. In fact, all the isothermals from 65° to 300° begin above the isothermal of 15°, and then cross it as the field increases. This is particularly well marked in the case of the isothermals 65°, 125°, and 180°. This phenomenon may be connected with the fact that, up to a certain limit, the induction curve for nickel rises more abruptly and reaches its 'wendepunkt' in lower fields the higher the temperature. In other

words, the *first* effect of rise of temperature is to increase the permeability in lower fields, probably because of the greater ease with which the molecular groupings assume new configurations. But anything which tends to increase permeability must tend to increase the effect on resistance. As the magnetization approaches its saturation value, rise of temperature diminishes the permeability, and rapidly so as the critical temperature of 350° is approached; and very similar is the effect of rise of temperature on the change of resistance due to a given field.

The isodynamic curves indicate the existence of a further peculiarity which declares itself at or near the temperature of 180° by a kind of cusp-like peak in the graphs of the higher fields. This peculiarity is also well brought out by calculating the differences between the resistance changes corresponding to the successive temperatures in the preceding table, and dividing these by the change of temperature. These average differences per degree will correspond to the mean of the extreme temperatures; and their values for five of the fields are given in the following subsidiary Table C.

TABLE C.

Showing differences per degree calculated from Table B.

Magnetic Field.	Differences per Degree at Temperatures							
	40°	95°	152°·5	210°	260°	290°	314°	335°
34	4·5	3·3	2·7	3·7	2·8	2·7	1·5	2·9
28	2·9	2·5	2·3	2·9	2·5	2·5	1·4	2·6
20	1·3	1·5	1·7	2·0	1·8	2·0	1·3	1·4
10	0·01	0·01	0·4	0·9	1·0	1·1	1·4	0·2
6	−0·02	−0·003	−0·9	0·7	0·45	0·4	1·0	...

From these few examples we see that there is at or near the temperature 200° a peculiarity which shows itself by an increase in the rate per unit rise of temperature at which the resistance change due to a given field is diminishing.

5. COMPARISON WITH RESULTS FORMERLY OBTAINED.—When we compare the results here given with those obtained in the earlier experiments a considerable discrepancy declares itself. From the results given in the final table in the earlier paper (*Trans. R.S.E.*, vol. xl. p. 543), we readily find by interpolation the resistance changes at the three temperatures 12·7, 57·5, and 93·5, corresponding to the fields 30, 22, and 14. Then, from the table given above (p. 45) we can interpolate values corresponding to the same fields and temperatures. These are compared in the following short table (D), the earlier and later results being distinguished by the Roman numerals I. and II.

TABLE D.

Field.	12° to 15°		57°·5		93°·5	
	I.	II.	I.	II.	I.	II.
30	349	901	376	754	405	647
22	215	613	233	543	260	477
14	109	341	121	321	140	297

When this discrepancy was noticed, the first idea was that some fundamental mistake had been made in reducing one of the series of experiments. The mistake might have been made either in calculating the field in C.G.S. units or in calculating the changes of resistance.* It is important to make sure that no such mistake has been made. I shall therefore reproduce exactly the observed numbers in two distinct experiments, one made on 19th June 1902 with the first apparatus, the other made on 13th February 1903 with the second apparatus. Anchor-ring coil I. had 846 turns, with a mean radius of 3·6 cm.; and anchor-ring II. had 344 turns, with a mean radius of 9·1 cm. Hence the reducing factors which enable us to calculate field from current are in the ratio of about 6·2 to 1. Now, in the two experiments mentioned above, the measured deflections of the currents on the Kelvin graded galvanometer were, respectively, 9·9 with the magnet at mark 44·35, and 22 with the magnet at mark 16; that is, the currents were in the ratio of $44{\cdot}35 \times 22$ to $9{\cdot}9 \times 16$; that is, about 6·3 to 1. Hence in these two experiments the fields were nearly the same, somewhere in the neighbourhood of 26. The following are the readings as jotted down in the experimental note-book, with the deflections and differences immediately deduced from them.

Experiment I.—Field = 26·5.

(1) Calibration of galvanometer.

Shunt in λ.	Reading on Scale when Current is			Deflection.	Difference of Deflections.
	Direct.	Reversed.	Direct.		
inf.	134	173	134	+39	...
20	166	141·2	165·7	−24·7	64·8
inf.	133·3	174	133·1	+40·8	...
20	165	142	165	−23	64·3
inf.	132·7	174·5	132·6	+41·8	...
				mean =	64·55

* The constancy of the resistances of the various coils in use and the steadiness of the results obtained in all cases quite preclude the possibility of any error in the estimation of the constants arising from faulty insulation.

(2) Measurement of resistance change due to applied field.

Direction of Field.	Deflection when Field is			Difference.
	Off.	On.	Off.	
+	113	169·8	114·4	56·1
−	117	173	119·3	55·4
+	121	176	120·9	55
−	122	177·9	122·5	55·5
			mean =	55·5

Experiment II.—Field = 26·2.

(1) Calibration of galvanometer.

Shunt in λ.	Reading on Scale when Current is			Deflection.	Difference of Deflections.
	Direct.	Reversed.	Direct.		
inf.	134	162·7	134	− 28·7	...
30	169	127·8	168·5	+ 40·95	69·96
inf.	133·5	163	133·7	− 29·4	...
30	169	127·7	169	∓ 41·3	70·53
inf.	133·9	162·9	133·8	− 29·05	...
				mean =	70·25

(2) Measurement of resistance change.

Magnetic Field.	Reading on Scale when Current is			Deflection.	Difference of Deflections.
	Direct.	Reversed.	Direct.		
off	150·5	146·5	149	+ 3·75	...
on −	239·8	57·5	234·7	+ 179·75	185·5
off	141·5	155·9	140	− 15·25	...
on +	231·5	64·5	229	+ 165·75	186·2
off	136·8	160	136	− 23·6	...
				mean =	185·9

In the calibration experiments conducted as described on p. 41, the column headed 'shunt' indicates that in the second and fourth lines the resistance λ was slightly altered in value by joining in multiple arc a fairly large resistance and a small part of λ. In Experiment I., 20 ohms resistance was joined up with a resistance of 0·3085; in Experiment II., resistance 30 was joined up with resistance 0·5125. The

corresponding changes in λ* were 0·0909/20·3015 and 0·2627/30·5125. Thus we find by equation (4):—

in I.

$$dL/L = \frac{\cdot 0909}{20\cdot 3015 \times 3\cdot 089} \cdot \frac{55\cdot 5 \times 2}{64\cdot 45} = \cdot 0025\,;$$

in II.

$$dL/L = \frac{\cdot 2627}{30\cdot 5125 \times 3\cdot 2413} \cdot \frac{185\cdot 9}{70\cdot 25} = \cdot 00703\,.$$

Notice that in I. (1), II. (1) and (2), the deflections are really double the true values, whereas in I. (2) the deflection is given at once. For in the last case the approximate balance is altered by the magnetizing force being put on. In the other cases the deflections are due to the reversal of the current supplied by the single cell in the battery branch of the Wheatstone Bridge.

The two cases here given in detail prove that there can be no doubt as to a difference of effect under apparently similar magnetic conditions. The nickel wires used in the experiments were cut originally from the same piece of wire. The only difference between the two forms of apparatus lay in the manner of winding. In the first small anchor-ring the nickel core was a small compact closely-wound coil of twenty windings of silk-covered wire; in the second large anchor-ring the nickel core was a loosely-wound coil of some 10 or 11 turns, with asbestos paper interwoven. It is possible that in the compactly-wound coil the inner turns were screened from the full magnetic action of the applied field by the outer windings. This view receives some corroboration from the manner in which the discrepancy established by the figures given above diminishes as the temperature rises. Taking the ratios of the corresponding changes in II. and I. we get the following results:—

Field.	Ratio of Resistance Changes (II : I) at		
	12°–15°	57°·5	93°·5
30	2·58	2·01	1·60
22	2·85	2·33	1·83
14	3·13	2·65	2·12

Thus the measured effect in the earlier experiment deviates more from the same effect in the later experiment the lower the field and the lower the temperature. But this is just what would be expected if the discrepancy were due to magnetic screening, which is well known to become less evident in higher fields. There are no experiments, so far as I am aware, as to the effect of temperature on the screening effect; but we have every reason to expect that it will diminish as the temperature rises. The

* In the earlier paper λ was called n; its value was less than the value of λ in the present paper.

question here raised would probably repay further experimental investigation; and it obviously suggests a new method for studying magnetic screening.

6. Comparison with other peculiarities of nickel at 200° C.—That some kind of peculiarity should occur at about this temperature was not unexpected. It was indeed with the expectation of getting some such effect that these experiments were originally planned more than a dozen years. It was my good fortune as an undergraduate to assist the late Professor Tait in the thermoelectric investigations which occupied his attention during the years 1872–4.* Probably the most remarkable results established by these investigations were those in connection with iron and nickel. The thermoelectric lines for all metals save iron and nickel are approximately straight through great ranges of temperature. Their inclinations in the properly constructed thermoelectric diagram give the Thomson Effects in the corresponding metals. In every case of a pure metal except those mentioned, the Thomson Effect retains the same sign throughout. In the case of iron and nickel, however, it changes sign—at a dull red heat in the case of iron, and at about 180°–200° in the case of nickel. But the nature of the phenomenon is the same in both. The Thomson Effect, which is negative at ordinary temperatures, becomes positive at higher temperatures; and finally, when the temperature is raised still higher, negative again. The second change of sign occurs in each case at the temperature at which the metal ceases to be strongly paramagnetic. In the case of iron, another phenomenon is known to occur at the temperature of dull red, namely, the sudden expansion during cooling discovered by Gore, and the accompanying reglow discovered by Barrett. No similar effect has been observed in the case of nickel, possibly not because it does not exist, but because the temperature is too low to admit of a visible 'reglow.' In any case these phenomena point to a curious molecular change occurring both in iron and in nickel at a temperature well below that at which the magnetic permeability becomes unity.

Mention has already been made as to the rather curious manner in which the resistance of nickel changes with temperature. In my paper on the electric resistance of nickel at high temperatures, referred to above, this peculiarity is established, and the difficulty in working at high enough temperatures prevented me establishing the existence of the same peculiarity in iron, although there was indication of its existence. This, however, was done shortly afterwards by W. Kohlrausch. The peculiarity in the case of nickel is shown by the interpolation formulæ given above, p. 41. The rate of increase of resistance with temperature undergoes a sudden increase at a temperature of about 180°–200° C., and then diminishes as abruptly again at about 400° C. Once again, then, we have another set of phenomena indicating a peculiar molecular change in nickel at 200° as well as at 400°.

In the present investigation the relation that is being studied involves the measure-

* See Tait, *Trans. Roy. Soc. Edin.*, vol. xxvii. pp. 125–140; also *Scientific Papers*, vol. i. pp. 218–233.

ment of small changes, which are indeed changes of the second order, namely, the change per unit rise of temperature of the change due to a given applied field. It would be utterly impossible, in the present state of knowledge regarding molecular groups, to make any prediction as to how the molecular change indicated by the thermoelectric and resistance peculiarities should show itself in the present case. A glance, however, at the isodynamics, with their cusp-like point in the higher fields, and a tendency to a maximum in the lower fields, seems to indicate some peculiarity at this critical temperature of 200°. What seems to be indicated is, that about this temperature the change of resistance with magnetization begins to fall off more quickly as the temperature is raised.

IV.—The Glacial Deposits of Northern Pembrokeshire.

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I. Introduction.

The area embraced in this paper consists of that part of Pembrokeshire which lies to the north and north-east of St Bride's Bay. Bounded on the west by St George's Channel and on the north by Cardigan Bay, it extends to the north-east as far as the mouth of the river Teifi, near Cardigan.

That part of the country which lies in the immediate neighbourhood of St David's has, through the laborious researches of the late Dr Hicks and others, become well-known to geologists, and may now be regarded as classic ground. The solid geology of this promontory has given rise to much discussion, and has, perhaps, attracted more attention than that of any other part of the Principality. The reason for this great interest is to be sought in the facts that the rocks of this area are of a very great antiquity, and that the sedimentary series contain the remains of some of the earliest organic forms yet found in the earth's crust, whilst the igneous rocks are also displayed in great abundance and variety, and present us, in the words of Sir Archibald Geikie, with "the oldest well-preserved record of volcanic action in Britain."

The geology of the district lying immediately to the north-east of the St David's promontory has not been the subject of so much attention, but the investigations carried on by De la Beche and the other officers of the Geological Survey before the middle of last century have recently to some extent been revised by Mr Cowper Reed, and his results are published in a paper entitled "The Geology of the Country around Fishguard," which appeared in the *Quart. Journ. Geol. Soc.* (vol. li., 1895, p. 149).

But while so much has been written concerning the ancient rocks of this country, very little attention has been paid to the more recent geological deposits. Owing to

the facts that geographically the region lies further south, and that it does not present such a great elevation of the land above sea-level, it is not to be expected that Pembrokeshire will show such marked traces of the former presence of glacial conditions as are to be met with in North Wales. Nevertheless, it has been known for a long time that this region is to a large extent covered by more or less loose and unconsolidated material, which is usually spoken of as Drift. And references are found scattered in the geological literature of the district relating to travelled boulders and other possible glacial phenomena there seen. But hitherto no attempt has been made to give a connected description of the glacial deposits with a view to the unfolding of the sequence of events which occurred during and after the Glacial epoch, and of correlating the results obtained by an examination of this area with those derived from a study of glacial deposits in North Wales and other regions. The need for further investigation will be evident to anyone who compares the map (plate i.) in Professor James Geikie's work on *The Great Ice-Age*, illustrating the British Isles during the Epoch of Maximum Glaciation, with the late Mr Carvill Lewis' "Sketch Map of England and Wales showing the Edge of Land Ice," which is reproduced in Professor Bonney's *Ice-Work*. In the former the southern boundary of the great ice-sheet is made to pass beyond Wales and run along the Bristol Channel; and the northern ice which overwhelmed Anglesea is marked as crossing the western end of the Lleyn promontory of Carnarvonshire, and, joining the Irish Sea, it fills up St George's Channel and crosses the extreme tip of Pembrokeshire at St David's Head. In the latter the land-ice is shown as not extending over the whole of South Wales to the Bristol Channel, but with its southern edge extending no further south than is indicated by a line drawn eastwards from the St David's promontory, and the glaciation of Northern Pembrokeshire is attributed solely to local ice—the northern ice apparently extending no further south off the Welsh coast than the Lleyn promontory.

The results obtained during the investigations carried on by the present writer will at any rate serve to settle the dispute with regard to the southward extension of the Northern or Irish Sea Glacier.

II. Previous Literature.

References to the surface deposits and surface features of Pembrokeshire are meagre and scanty in the extreme.

Sir R. Murchison, in *The Silurian System* (p. 520), makes the following remarks: "The detritus which appears on the surface of most parts of Pembrokeshire is of a simple character and, as in other parts of South Wales, is of *local origin*. It consists of fragments of greenstone, porphyry, carboniferous grits, etc., all of which can be traced to the various mountains forming the crest of the country. In some parts this detritus is exceedingly coarse. . . . In other tracts, as north of Haverfordwest, we meet with finely comminuted gravel; but this is rare."

SYMONDS, in his book entitled *Records of the Rocks* (p. 53), refers to the fact that the country near St David's must in former years have been largely covered by boulders. These have now to a large extent been cleared away from the surface and used for building rough walls which serve the purposes of fences and hedges. He adds that in his opinion "these boulders are all local, and have travelled over a slope of ice and snow which once reached from the Trap Hills of Precelly down to the sea." And in another place (p. 181) he speaks of the Precelly hills in North Pembrokeshire as being "studded with ice-carried boulders, which were used for cromlechs and tumuli by a prehistoric race of men."

Messrs HOWARD and SMALL, in their "Geological Notes on Skomer Island," which appeared in the *Trans. Cardiff Naturalists' Soc.* (vol. xxviii., 1896), assert that distinct evidences of the action of ice are seen on the mainland opposite Skomer. And on Skomer Island itself blocks are found which have travelled from the St David's district and some possibly from North Wales. Flints were also seen scattered about.

Professor M'KENNY HUGHES, in his paper "On the Drifts of the Vale of Clwyd" (*Quart. Journ. Geol. Soc.*, vol. xliii., 1887), remarks that "the low-lying plateau at St David's is covered by a gravel containing flints." But he found no traces of shells there.

In a paper by Professor BONNEY "On the So-called Diorite of Little Knott (Cumberland), with further Remarks on the Occurrence of Picrite in Wales" (*Quart. Journ. Geol. Soc.*, vol. xli., 1885), some observations are found communicated by Dr HICKS relating to the glaciation of the St David's region. A boulder of picrite was found on the promontory just to the east of Porth-lisky, "resting immediately on Dimetian rock, surrounded by an uncultivated area overgrown by gorse and heather." The striæ along the coast are said to run usually from north-west to south-east. He adds: "But it is clear that very many of the boulders scattered over it must have come from the high land in the north-east of Pembrokeshire, the Precelly range. There is ample evidence of local till, and in places (at considerable elevations) of marine sand with transported boulders, fragments of flint being common among them." Dr HICKS was of opinion that "this points to the derivation of some of the materials, including possibly certain boulders, from a north-west source."

The most important communication which has appeared on this subject is a very short report read by Dr HICKS at the Cardiff meeting of the British Association in 1891, "On the Evidences of Glacial Action in Pembrokeshire, and the Direction of Ice-Flow." This report is reproduced in the *Geol. Mag.* for that year. He there refers to the presence of ice-scratched rocks and of northern erratics in the district. The direction of the glacial striæ and the probable presence of erratics from North Wales and from Ireland "would tend to the conclusion that glaciers from these areas coalesced in St George's Channel, and that the ice which overspread Pembrokeshire was derived from both these sources, as well, probably, as from a flow extending down the Channel from more northern areas." By far the majority of the boulders are said to be of local origin, but he notes a large boulder of granite and another of picrite found on Porth-

lisky farm. "The picrite boulder has been shown by Professor BONNEY to resemble masses of that rock exposed in Carnarvonshire and Anglesea, and the granite boulder, which before it was broken must have been over 7 feet in length and 3 to 4 feet in thickness, is identical with a porphyritic granite exposed in Anglesea, but not found anywhere in Pembrokeshire." He found clear evidences showing that St Bride's Bay was overspread by a great thickness of drift from the hills immediately to the north. "The intervening preglacial valleys were also filled by this drift, and the plains and rising grounds up to heights of between 300 and 400 feet still retain evidences of its former presence, and many perched blocks." Chalk flints were found at heights of over 300 feet, and have probably come from Ireland. He refers also to the crushing and bending of the strata at places, and to some well-marked examples of "crag and tail," but he does not locate these phenomena.

The late Professor PRESTWICH, in his paper on "The Raised Beaches and 'Head' or Rubble-Drift of the South of England" (*Quart. Journ. Geol. Soc.*, vol. xlviii., 1892), refers to the possible occurrence of this rubble-drift on the coast of Pembrokeshire. He thought that he had detected traces of a raised beach and 'head' near Porth Clais, and again at Whitesand Bay.

Professor BONNEY, in his *Ice-Work* (p. 161), states that "In Pembrokeshire and the adjoining districts erratics are often abundant, as may be seen near St David's. At present no systematic attempt has been made to trace them up to their sources, but they have probably come from the higher ground inland, that is to say, roughly, from the north-east." And again (p. 165) he refers to the possibility that the northern ice travelled down the bed of the Irish Sea, and perhaps ultimately overflowed St David's Head.

Dr WRIGHT, in his book on *Man and the Glacial Period*, remarks that "At St David's peninsula, Pembrokeshire, striæ occur coming in from the north-west, and, taken with the discovery of boulders of northern rocks, may point to a southward extension of a great glacier produced by confluent sheets that choked the Irish Sea" (p. 143).

Mr COWPER REED, in the paper already referred to, mentions the fact that "drift or boulder-clay causes a difficulty in tracing the boundaries or determining the characters of the underlying beds" in the Fishguard district.

III. PHYSICAL FEATURES AND GEOLOGY OF THE DISTRICT.

The county of Pembrokeshire lies in the extreme south-west corner of the Principality, and that part of it which is under consideration in this paper extends further to the westward than any other part of England and Wales, with the exception of the extremity of Cornwall. The promontory of St David's is washed on three sides by the sea which has eaten into the land so as to give rise to a variety of recesses and bays. It is the presence of hard igneous rocks that has enabled it to resist the ceaseless action

of the waves, which, owing to the direction of the prevalent winds, often beat upon the coast with great fury. The softer Lower Palæozoic slates and Carboniferous shales to the south have succumbed to the encroaching sea, and there given rise to the broad and wide bay of St Bride's. The islands lying to the west consist partly of hard igneous rocks, and no doubt were once joined to the mainland. The coast scenery is magnificent, and throughout this region the rocky cliffs rise steeply out of the sea, and sandy beaches are only found here and there, such as at Whitesand Bay, Abereiddy Bay, Abermawr, Goodwick Bay, and Newport Bay. The cliffs in places are nearly perpendicular, and everywhere exhibit excellent sections of the rocks. The outline of the coast is very jagged, reefs and stacks of rock sticking out here and there, whilst, on the other hand, the sea has penetrated in so as to form caves and coves, and small narrow *ria*-like channels, such as that seen at Porth Clais at the mouth of the Alan river, and that at the mouth of the Solva river. A silted-up estuary occurs at Abermawr to the west of Strumble Head, and a larger one on the east side at Goodwick Bay; they now form swampy ground. The only other estuaries of importance lie on the north-east side of Strumble Head, at the mouths of the rivers Gwaen, Nevern, and Teifi. That of Gwaen is also somewhat *ria*-like in character.

On the north coast Strumble Head is a prominent feature and stands out boldly to sea, and a little further to the north-east a small but well-marked headland occurs at Dinas.

The one-inch Geological Survey maps of this part of Wales were prepared before the end of the first half of last century, and no revision has yet been made. Since that time a more complete knowledge of the fossil contents of the sedimentary series has been obtained, and improved methods for the study of igneous rocks, especially with regard to their microscopic structure, have been introduced. The need for a fresh survey is generally recognised, but much new light has been thrown on the geology of St David's promontory during the last thirty years through the researches carried on by Dr Hicks, Sir A. Geikie, and others. And Mr Cowper Reed has examined and described within recent years the geology of the area around Fishguard. But the region is a very complicated one, and much of the geology still remains obscure. The rocks of this part of Pembrokeshire are almost entirely of Lower Palæozoic age, and a remarkable variety of both the igneous and sedimentary kinds is there displayed. In the St David's region a very full development of Cambrian rocks is exhibited, and these are underlain by a series of volcanic rocks—both series often showing signs of metamorphic changes. The base of the Cambrian was taken by Dr Hicks to be marked by a conglomerate in which are enclosed pebbles of the underlying rocks. The volcanic tuffs and breccias which underlie the conglomerate were taken to be pre-Cambrian. Underneath these again comes a granitoid mass, which he regarded as still older. Later, the district was visited by Sir A. Geikie, who, after an examination of the ground, arrived at the conclusion that the granite is an intrusive mass, and that there is no break between the Lower Cambrian rocks and the volcanic series underlying them.

The granite covers an area lying immediately to the south of St David's, and there is another wedge-shaped mass a little to the south-west, reaching the coast on the eastern side of Porth-lisky. The St David's mass graduates into a spherulitic quartz-porphyry and felsite at its northern end. The granite is surrounded on all sides but the south by rocks of the volcanic series which are marked as Andesites on the index-map. These form a ridge running E.N.E. and W.S.W., stretching from Llanhowell, past the city of St David's, to reach the coast at the southern end of Ramsay Sound. Two detached masses are marked further east, about Llanreithan.

The volcanic group consists largely of bedded tuffs; but lavas also occur, and give rise to prominent crags to the west of St David's.

Dykes and sheets of diabase traverse the other formations.

These igneous rocks are flanked on the west and south-west by the Cambrian conglomerate, and this is followed by green, purple, and gray flaggy sandstones, with intercalated red shales. Towards the base, fragments of *Olenellus* were found by Dr Hicks. In a south-east direction these are followed by the gray and black flagstones and shales of the Menevian series, and these again by gray and bluish flagstones and slates of the Lingula Flag series. Still further eastwards, a small tract of Tremadoc beds is found near Tremainbir.

Beds of the Menevian and Lingula Flag series also occur at the south of Whitesand Bay. The Lingula flags run as a continuous band from the bay inland as far as Crug-las, and on the north a band of Tremadoc flagstones and earthy slates runs parallel to a point south of Abereiddy Bay.

The north coast from St David's Head to Abereiddy is made up of slates and shales and flagstones of Arenig age. Masses of gabbro occur at St David's Head, and a little east of this, diabase masses, giving rise to rugged eminences, are seen.

North-east of Abereiddy the Llandeilo flags succeed the Arenig series, and consist of black slates and flags, sometimes calcareous, and some felspathic tuffs.

Numerous bands of "felspathic trap" are seen to occur in the tract bordering Abereiddy Bay.

Eastwards from the St David's promontory, right into mid-Wales, the sedimentary rocks are marked in one colour on the Survey maps, and are referred to as "Lower Silurian (including Upper Silurian not yet separated)." They consist of shales, slates, and gritty sandstones, with some flagstones. Mr Cowper Reed found that beds of Llandeilo and Bala age form Strumble Head. Immediately around Fishguard we meet with Llandeilo and Arenig beds, and further east with Upper Llandeilo and Bala beds again. Dinas Island is composed of sandstones, slates, and conglomerates of Llandovery age.

The country around Fishguard is rich in igneous rocks. Felsites, tuffs, and agglomerates contemporaneous with the Llandeilo and Bala beds occur, and intruded into these are sills and masses of "greenstone." The latter include basalts, dolerites, diabases, and gabbros.

Contemporaneous volcanic rocks and intrusive sheets of diabase occur in the district between Fishguard and Newport, and further south in the Precelly mountains strips of felspathic rock are indicated on the Survey maps.

East of Newport the igneous rocks die out.

Blown sands are heaped up at places on the coast, and are generally full of land shells. To the east of Whitesand Bay they rise to heights of 150 feet and are prolonged inland for nearly a mile, giving rise to the tract known as "The Burrows" or "Towyn." They are also seen at Abermawr, and cover an extensive area at Newport, and again at the mouth of the Teifi, where they form dunes known as the "Towyn Warrens."

Towards the western end the country, as seen from a height, presents the aspect of a flattened tableland or plateau, having for the greater part of its extent no great elevation above sea-level, but here and there having rocky knobs and masses jutting out, especially towards the western and north-western extremities of the promontory. The view, as seen from one of the hills on Ramsay Island, suggested to Sir ANDREW RAMSAY that in this part of Wales we have the remnant of an old plain of marine denudation, which is continued into Cardiganshire and further inland. The rugged masses of Carn Llidi and Pen Berry, which rise so boldly just east of St David's Head, and have relatively the appearance of considerable mountains, only attain heights of 595 feet and 576 feet respectively. Further east the rough eminences which stand out so prominently on Strumble Head are mostly under 600 feet in height—Garn Gelli alone exceeding that limit and attaining 625 feet. Garn Fawr, famous for the remains of ancient fortifications there found, is just short of 600 feet high.

Between Fishguard and Newport is a ridge of high land, reaching elevations of over 1000 feet at Mynydd Melyn, Mynydd Caregog, and Carn Ingle. To the south of these rise the Precelly mountains, which have a somewhat smooth outline and attain heights up to 1500 feet. From Newport to Cardigan the country is hilly, but not mountainous—the highest point being Pen Creigiau, which is 642 feet above sea-level.

The highest ground is formed of diabase and other intrusive igneous masses; the volcanic rocks occupy ground above the average level, while the low-lying ground consists of the more easily denuded slaty and shaly beds.

Passing from west to east, the land generally becomes more hilly, and the higher grounds from Newport to Cardigan are composed of hard sandstones or arenaceous slates.

The main streams of the district occupy pre-glacial valleys, and have cut their way through the drift which once filled them. The estuaries of the rivers Solva, Alan, and the Gwaen are *ria*-like in appearance, and it is probable that an arm of the sea once extended for some way up the lower course of each; for the estuaries are trench-like, with steep rocky walls on either side for a considerable distance inland, and it is hardly conceivable that these have been cut out altogether by the action of the streams.

On the northern coast there are two peninsulas of a very peculiar character, for they are separated from the mainland by trench-like valleys, which, though now never

occupied by the sea, look as if they had been so in comparatively recent times. They are both spoken of locally as islands: one lies between Abereiddy and Porth Gain, and is known as Barry Island, and the other is Dinas Island, west of Fishguard. The valley between Dinas and the mainland is particularly striking, being only a few feet above sea-level, whilst that at Barry Island is not much less noteworthy, though its bottom attains a somewhat higher level. These peculiar valleys, together with the indications shown of the former presence of the sea up the inlets at the mouths of many of the streams, and the occurrence of swampy estuaries such as those seen at Abermawr and Goodwick, seem to point to a slight rise of the coast within recent times, causing a retreat of the sea.

But at the present day the sea seems to be gaining once more on the land. At several places along the coast peat is seen at low tide, and in most of the bigger bays evidences of buried forests are sometimes seen. Giraldus Cambrensis, who wrote in the twelfth century, says with regard to St Bride's Bay at Newgall:—"When Henry II. was in Ireland an unusually violent storm on that sandy coast blowing back the sand discovered the appearance of the land concealed for so many ages; stumps of trees standing in the sea, with the marks of the hatchet as if done but the day before, a very black earth and wood like ebony, so that it appeared not so much like a sea-coast as a grove."* And George Owen, in *The Description of Pembrokeshire*, written in 1603, gives an account of a very similar occurrence. He says: "About xij or xiij yeeres past were seene on the sandes at *Newgall*, by reason as it seemeth the violence of the sea or some extreeme freshe in the winter, washed awaye the sandes (w^{ch} dayelye is and was overflowen with the tyde), soe lowe that there appeared in the sandes infinitte nomber of buttes and trees in the places where they had been growinge, and nowe euerye tyde overflowen: there appeared the verye strookes of the hatched at the fallinge of these tymber, the sandes being washed in the winter, the buttes remained to be seene all the sommer followinge, but the next yeere the same was covered againe with the sandes: by this it appeareth that the sea in that place hath intruded upon the lande."† Again about sixteen years ago a big storm washed away the sand and exposed roots of great trees in Whitesand Bay. Huge logs of oak trees were carried away by the neighbouring farmers, some of which are still stored, and were shown to the writer. Twigs and branches of hazel were found in abundance, although no hazel grows now near St David's. The writer is also informed that horns of deer were picked up.

Similar evidence of a buried forest has been discovered in Goodwick Bay, and all along the coast up Cardigan Bay. All this reminds one of the old Welsh tradition regarding a great inundation of a land called Cantref Gwaelod, situated in the region now covered by Cardigan Bay, which is usually attributed to the fifth century. There is also a very old tradition that St Bride's Bay was formed by an inrush of the sea.

* *Rolls*, ed. vi. 100.

† Page 247 in the Cymmrodorion reissue.

Referring to some of the shallow valleys and peculiar bays seen in Pembrokeshire, Sir ANDREW RAMSAY, in his essay "On the Denudation of South Wales and the adjacent Counties of England," contributed to volume i. of the *Memoirs of the Geological Survey*, says (p. 329): "In numerous instances valleys opening to the sea end in bays, and it will be found that not infrequently the headlands on either side of these bays were composed of comparatively unyielding materials. Depress the land, and these valleys become arms of the sea; raise it, and the bays become a continuation of the valleys. If Pembrokeshire were elevated but 60 or 70 feet, Milford Haven would become a shallow valley of this nature, with occasional pools or small lakes in its hollows, through which would wind the water now flowing into the haven." And similarly, if the county was depressed but 60 or 70 feet, the valleys at the lower part of their course would form sea-inlets like Milford Haven, and Barry and Dinas would be separated as islands from the mainland. It is practically certain that movements of elevation and of depression have taken place within comparatively recent times, but it is a difficult problem to ascertain the extent and duration of these movements.

The land of the St David's promontory was long ago described by GIRALDUS CAMBRENSIS as a "stony, barren, unimprovable territory, undecked with woods, undivided by rivers, unadorned with meadows, exposed only to winds and storms." Since his time the land has yielded somewhat to the continuous treatment of generations of farmers, and a great part of it is now under cultivation. But much rough uncultivated moorland still remains, which in places is overgrown with gorse, and sometimes shows a boggy nature. Such are the commons seen round about St David's, and also further east. Parts of them are occupied by shallow sheets of water, as at Trefeithan common and Dowrog common. As a rule these commons are of a clayey nature, and in places some peat formation is seen. Peat becomes more evident in the country lying to the south of Fishguard and south-west of the Precelly hills, and it is dug even right up on the Precelly hills themselves. The country from Newport to Cardigan is well cultivated for most part.

Throughout the area under consideration in this paper much of the soil is of a distinctly sandy nature. Hence the land is very dry and needs much rain. It is an old saying in this part of Pembrokeshire, that "in summer rain every day is too much, and every second day too little."

The greater part of the land is covered by a blanket of superficial material, which may all be included under the name of Drift. This is somewhat variable in character, but near the surface a sandy element seems to predominate. However, as traced laterally, the sand often passes abruptly into clay or clayey loam, and *vice versa*. This drift near its upper part is usually stuck full of boulders and rock-fragments of all kinds, and of all shapes and sizes. Good sections are seen along the coast in some of the bays, but it is very rarely that one meets with a good exposure inland. Smoothed, polished, and ice-scratched boulders can be picked out of the drift in plenty, and occur throughout the district. It is more difficult to meet with examples of striated and

smooth rock-surfaces, and this is partly owing to the fact that so much of the country is covered by the superficial accumulations; and the igneous rocks which project above the surface are so weathered and worn from exposure for such a great length of time, that any marks of glaciation which they may once have shown must have become almost entirely obliterated.

Reference has already been made to Dr HICKS' observations on the striæ seen on the rock-surfaces at places along the coast near St David's. The best example noted by the present writer is at Whitesand Bay, just at the northern corner. Here the rock, as it appears from under the cliff of drift, presents a distinctly hummocky appearance, and is smooth, polished, and well striated—the striæ having a course from north-west to south-east. It is interesting to note that out of the drift above, striated stones can be readily picked: these are generally sub-angular, with blunted angles and somewhat smooth and rounded edges. In the small valley coming down near Porth Melgan, and separating the rocks of St David's Head from Carn Llidi, a pavement of sedimentary rocks has been exposed by the removal of some turf, and this pavement shows distinct marks of glaciation. Another very good example of a glaciated rock-surface is seen quite at the other end of the district at Gwbert, on the coast to the north of Cardigan. Here, emerging from beneath the drift again, a smooth and striated rock is seen—the direction of the striæ being from a little west of north to a little east of south, showing that the ice must have come on to the land from the region of Cardigan Bay.

The igneous rocks of St David's Head and those lying further east, especially at Pen Berry, appear to be somewhat *moutonné* on their northern aspect, but no unmistakable glacial striæ were seen, and this is no doubt due to the fact that the rock-faces are so much weathered.

On the greater heights there is a general absence of perched blocks and big erratics. It is quite possible that perched blocks may have been common in former times, but they have in all probability been removed by man, for the region is full of traces of defences prepared by primitive man, and these usually take the form of great collections of boulders and stones gathered together and heaped up in the form of dykes. Splendid examples of such ancient entrenchments are seen near St David's Head, and again on Strumble Head. The few blocks seen on the high ridges are almost invariably of the same nature as the underlying and surrounding rocks.

South-west of St David's, in the Treginnis tract, some huge boulders are seen. A big one lies on the hill above the cliffs at Penmaen-melyn, but it consists of a somewhat coarse andesitic rock, which is found *in situ* at no great distance away. The boulders of granite and of picrite found near Porth-lisky farm by Dr HICKS have been already mentioned. The whole country was once strewn with boulders; and although many still remain scattered over the land, most have been cleared away and used for building dykes, etc. An examination of the stones in the dykes shows that they are almost entirely of rocks found in the locality, as might be expected. Boulders of the St David's Head gabbro are found carried in a south-eastern direction, and are plentiful in

the neighbourhood of Caerfai Bay and Caer Bwdy Bay. In the country between St David's and Strumble Head boulders are also common. Blocks of "greenstone" are found around Mathry : the rock is a diabase, probably occurring at no great distance away. On Strumble Head the boulders are very plentiful, and the writer found a boulder of picrite not far from the extreme head, near the coastguard station, and another further south, near Tre-Seissyllt, together with a remarkably fresh olivine-gabbro, of a kind which is not found in the district. These will be referred to more fully in another section of the paper. East and south of Fishguard, boulders of a dark "greenstone" and of a volcanic rock which weathers white are abundant. They are found on the hill-slopes, on roadsides, and in the fields, but many have been cleared away as before. The Precelly hills are free of boulders as compared with the hills lying further north.

It may be mentioned that vestiges of antiquity abound in this part of Pembrokeshire. Hut circles, ancient entrenchments, cromlechs, British camps, etc. occur at places all the way from St David's Head to Cardigan, being particularly evident at St David's Head and Strumble Head. Some of the ancient British towns or settlements, traditions of which are to be found in the old Welsh romances, called The Mabinogion, are situated within this area, and one has been identified as occurring on the Garn Fawr, Strumble Head.

IV. Description of the Deposits.

The glacial deposits of the district vary a great deal as traced laterally from place to place. Owing to the want of good exposures inland, it is generally impossible to mark out the limits of the different kinds of superficial detritus. One has to depend for most part on a study of the sand-pits and clay- or marl-pits which are dug in places all over the district. But it is not often that these go down to any great depth ; and when occasionally a deep pit has been dug out for the purpose of obtaining clay or sand for the land or for building, it is invariably filled up again with surface-rubbish, so as not to be a danger to animals. Where a pit appeared to be of peculiar interest, the writer employed a man for digging, and in this way obtained some valuable sections. At the eastern limit near Cardigan there are brickworks, and here it is that the best sections are to be seen. The writer hopes in the near future to continue his investigations eastwards in the neighbourhood of Cardigan. The deposits which occur in the district are the following :—

3. Upper Boulder-Clay and Rubbly-Drift.
2. Sands and Gravels.
1. Lower Boulder-Clay.

1. *The Lower Boulder-Clay.*— This is a typical boulder-clay which is met with in patches throughout the district, but is best and most fully developed towards the east. It has received no attention within recent times, but a very quaint and, on the whole, a very accurate description of it is found in the works of a writer who lived in

the time of Shakespeare. In an article on Sir RODERICK MURCHISON's Silurian System in the *Edinburgh Review*, 1841 (vol. lxxiii. p. 3), it is stated that "one of the oldest inquirers connected with the geology of this ancient region is GEORGE OWEN of Henllys, in Pembrokeshire, who has been called the patriarch of English geologists." This worthy Welshman left behind him a manuscript work on the topography of his native country—a book of great value and interest. It was published in the Cambrian Register, 1793, and has recently been reproduced, under the editorship of Mr HENRY OWEN, in the Cymmrodorion Record Series. The book has been already referred to, and is entitled *The Description of Pembrokeshire.*

His observations on the boulder-clay are so good that they are well worth quoting. Writing of "the naturall helpe and amendementes the soil it selfe yealdeth, for betteringe and mendinge the lande," he refers to what he calls "Claye Marle." "This kind of Marle is digged out of the Earthe, where it is found in great quantitie, and thought to be in rounde great heapes and lompes of Erthe as bigg as round hills, and is of nature fatt, toughe, and Clamye. . . . The opinion of the Countrie people where this Marle is founde is that it is the fattness of the Earthe gathered at *Noes* flood, when the Erthe was Covered withe the said flood a whole yeare, and the surginge and tossinge of the said flood, the fattness of the Earth being clamye and slymie of nature did gather together, and by rowlinge vpon the Earthe became round in forme, and when the flood departed from the face of the earthe, the same was left drie in sondrie partes, which is nowe this Marle that is found, and how the Common people Cam to this opinion I knowe not, but it is verye like to be true, for wheresoever the same is founde, it is loppie (loose) and covered with sande, gravell, and round peblestones, such as you shall finde at the sea side verie plaine, appearing that the stones hath ben worne by the sea or some swift river."

"Also in the harte of the Marle is founde diverse sortes of shells, of fishe, as Cogle shells, Muskell shells, and such like, some altogether rotten & some yet unrotted, as also you shall therein finde peaces of tymber that ben hewen with edge tools & fire brandes, the one ende burned and diverse other thinges which hath ben before tyme vsed, & this XX^{tie} foote and more deepe in the Earth in places that never haue been digged before, and over the which great oakes are now growinge; and this seaven or eight myles from the sea, so that it is verie probable that the same came into these places at the tyme of the great and generall flood. . . . ."

"This marle is of couler with vs most commonlie blwe and in some place redd." "It is verie hard to digg by reason of the toughness, much like to waxe: and the pickax or mattock beinge stroken into it, is hardlie drawne out againe, so fast is it holden: it is alsoe verie heavie as ledd." "This Marle is founde in Kemes and both Emlyns from *Dynas* vpp to *Penboyr* in Carmerthen sheere, beinge about twentie myles in lengthe and about fowre myles in bredeth in most places to the sea syde, and out of this compasse I cannot heare that the same ys founde: I thinke more for want of Industrie than otherwise" (pp. 71, 73). He ends up his remarks on the Clay Marle

thus :—"And, who so list to learne more of this Marle: let him pervse a pamphlett which I have written thereof, wherein I have declared the nature of the marle, how to know yt and finde yt, and the order at Lardge of digginge and layeinge yt on the lande; of the severall sortes thereof for what yt is good, and for what yll. And so for brevyties sake I Cesse to writte any More thereof." It is a great pity that this pamphlet has never been published, for it would be of great interest to geologists, as perhaps the earliest attempt to give a full description of the boulder-clay. It seems that the treatise was finished in 1577, and consists of twelve chapters. A footnote to the preface of Mr Henry Owen's edition of his *Pembrokeshire* (p. xxiv) states that a copy of the pamphlet lies in the *Vairdre Book* at Bronwydd, "written out of a copy in his own hand, by me, John Owen of Berllan, 1684."

The Kemes mentioned in the above extract is that part of Pembrokeshire lying between Dinas and Cardigan.

The present writer can bear testimony to the general accuracy of the description given by George Owen. It is at the brickworks, Cardigan, that the best exposure is seen. There in the pit a vertical section of this boulder-clay 20 feet deep is seen. It is dark-bluish in colour, but after drying becomes more of a light bluish-gray. It is a remarkably tough and tenacious clay, and can only be dug with great difficulty, for no crevices or fissures are seen and no trace of bedding. The whole mass is strikingly homogeneous and uniform in character, and has evidently been subjected to great pressure. The bottom is not reached in the section, and so the depth attained by it at this place is not known. For most part it is very free from stones, but a little further east in the same pit these are rather more commonly met with. Some beautifully glaciated sub-angular and blunted boulders were seen, with the striæ running principally in the direction of their longer axes. Many of these boulders are of Carboniferous Limestone, and these interfere very much with the manufacture of the bricks, and are, as far as possible, picked out by the workmen. Boulders of conglomerate, grit, shaly and slaty rocks were also noted, and many of igneous rocks, which are foreign to the district. These will be dealt with again below, in another section.

One of the most characteristic features of this Lower Boulder-Clay is the presence of marine shells scattered irregularly through it. They seem to occur chiefly in its upper part, and are invariably much broken and worn, and therefore very difficult to identify. The fragments are also extremely friable. Occasionally small waterworn pebbles of quartz, etc. are seen in the clay; but most of the stones included are ice-worn rather than waterworn in the Cardigan pit. Another striking feature is the presence of fragments of woody matter in the clay, sometimes at a depth of 15 to 18 feet.

Above the boulder-clay in the brickyard occurs 2 or 3 feet of sand and gravel and a yellowish stony clay, and towards its north end this stony clay increases in thickness to at least 7 feet, passing in places into yellow sand.

At Cardigan the height of the boulder-clay above sea-level is under 50 feet. Between Cardigan and Dinas this blue clay is seen in patches underlying small tracts of moorland, and it attains a height of nearly 600 feet a little south of Pen-Creigau, where, at a short distance below the road to Cardigan, it may be seen, though the exposures are very poor.

Just south-west of Dinas, near the roadside, clay-pits occur on Rhos-Isaf showing a depth of 6 feet. The same stiff, compact, bluish boulder-clay is here seen, full of comminuted shell-fragments. Boulders are fairly common, mostly ice-worn and scratched, but some water-worn. One example of a slaty rock showed not only fine striæ but a wide groove smoothed out by ice action. The clay gets darker as traced downwards, but the bottom is not seen. Workmen stated that it reaches a depth of at least 15 feet, and occasionally a thin seam or stratum—no more than half an inch in diameter—of fine gravel is said to occur. But no trace of bedding occurs in the clay. It is capped for 2 feet by a yellowish clay with boulders. The pits are 240 feet above sea-level. Small exposures are seen in some of the fields on Dyffryn farm, about a mile south of Goodwick. Owing to drying and weathering, it is of a light bluish-gray colour, and is here full of fragments of the local lavas and tuffs. Most of this farm is underlain by this clay. A little further south, in a field belonging to Drim farm, is a small pit of a similar character. No shell fragments were to be seen in these exposures.

Similar tough bluish boulder-clay is seen in clay-pits on Tregroes moor, at a height of over 250 feet. In fact, in all the moors lying to the south of Fishguard, with the exception of those found on mountain-slopes, this boulder-clay can be found, but it is unfortunate that there are no deep pits or good sections to be seen. But the engineers of the Great Western Railway are making borings in this neighbourhood for a tunnel, and they very kindly supplied the writer with all the information in their possession which might be of interest. A boring has been made at Trebrython farm, to the south-west of Tregroes moor, and about 150 yards from the railway. In the boring the following succession was obtained :—

5 feet of earthy clay.
5 feet of yellowish clay, with rock fragments.
10 feet of a somewhat tough greyish-blue clay.
Slate rock.

The greyish-blue clay is the Lower Boulder-Clay, and it is seen that it only attains a thickness of 10 feet here. The yellowish clay may be partly the bluish clay weathered, but most probably consists for most part of the equivalent of the Upper Boulder-Clay. The boring is made at a height of nearly 300 feet above sea-level. About three-quarters of a mile down the railway towards Goodwick is a railway cutting passing through part of the superficial deposits, and a boring has been made here also. A full section of this cutting and boring is given on another page, but it may be mentioned here that typical tough blue boulder-clay is there shown which attains a depth of

18 feet, and is followed above by Sand and Upper Boulder-Clay. The Lower Boulder-Clay is partly exposed in the cutting, and is full of fragments of marine shells.

There is also a small exposure in the moor just south of Letterston.

That the boulder-clay occurs far inland is proved by an examination of a pit at Llyn, near Llangolman, not far from Maenclochog. When seen the pit was only 5 feet deep, but it is opened up by the farmers periodically to a depth of from 15 to 20 feet, and ladders have to be used to get in and out. The clay is bluish in colour and very tough, and is usually spoken of by the natives as "indiarubber clay." It occurs in boggy land at the bottom of a small valley, at a level of over 400 feet. On both sides the land rises to a higher level, and is covered by a mantle of sands and gravels.

West of Goodwick the blue clay comes to the surface just below St Nicholas Church, at Clyn Bach moor. The depth was not ascertained. Fragments of broken shells occur here, though somewhat rarely. Some boulders of igneous rocks, foreign to the district, were found, and among them an unmistakable boulder of the Ailsa Craig riebeckite micro-granite.

The bluish clay is seen at places as far as the western end of the promontory—such as in the moors around Trefain, and in the shallow pits at Henllys, below Llanrian, where very fine examples of chalk flints were seen included. In the pits of Trefeithan moor, west of St David's, this boulder-clay is exposed to a depth of 6 feet and is of the usual character. No shells were seen, but some vegetable matter occurs in the clay. The boulders were few, and those noted were all of local rocks. Somewhat similar but shallower pits may be seen on Dowrog moor, Tretio moor, and Caer-farchell moor.

It is thus evident that this Lower Boulder-Clay occurs throughout the district, but it appears to thicken as we pass from west to east, and to be best developed to the east of Strumble Head. Fragments of marine shells were found in the clay at Cardigan, at Dinas, at the railway cutting near Tregroes, and at St Nicholas, and for the determination of these and other shells mentioned in this paper the writer is indebted to Mr Henry Woods, M.A., St John's College, Cambridge. Owing to their fragmentary condition it has been difficult to identify with certainty the shells found in the clay. But the following species appear to be represented:—

Cardigan Clay-pit—

Pectunculus glycimeris, L.
Astarte sulcata, Da Costa.
Mytilus, sp.

Dinas Clay-[illegible]—

Astarte (*Nicania*) *compressa*, Mont.
(?) *Cyprina islandica*, L.

Railway Cutting (between Tregroes moor and Manor-owen)—

Pectunculus glycimeris, L.
Cardium islandicum? Chem.
Vulsella modiola? L.
Astarte sulcata, Da Costa.
Astarte (*Nicania*) *compressa*, Mont.
Venus (*Ventricola*) *casina*, L.
Cyprina islandica, L.

Some of the boulder-clay from the boring near Tregroes was washed and examined under the microscope. A good deal of very fine sandy material—mostly quartz—was observed, and a few foraminifera could be seen.

2. *The Sands and Gravels.*—Above the Lower Boulder-Clay comes a series of aqueous deposits, consisting of sands and gravels, which are sometimes stratified and sometimes show hardly any traces of stratification. These deposits vary very much in thickness, and are apt to die out suddenly when traced laterally. They usually occupy a higher level than that attained by the Lower Boulder-Clay, and are often seen banked on the lower slopes of the hills. In places where sections are seen passing through the different deposits, no gradual passage can be traced from the lower stiff blue clay to the sands and gravels above—and this suggests that the sands and gravels lie upon an eroded surface of the clay. The sands are as a rule yellowish and yellowish-brown in colour, and have all the appearance of being marine: they are very variable in texture, and show all gradations from very fine sand to coarse gritty sand and gravel. And the gravels are often coarse and pebbly, resembling the shingle collected on beaches. At some places the sands are charged with worn and broken fragments of marine shells: these usually occur more abundantly in the fine gravel or coarse gritty sand than in the fine sand, although minute flakes can often be detected in the latter. They seem to be collected together in the stratified beds at certain spots, and to be absent in somewhat similar beds exposed only a short distance away. For instance, shells are plentiful in the Manor-owen sand-pit, whilst not a trace of shells can be seen in the Cnuc sandy pit, which lies only about a quarter of a mile further south: and 50 yards or so beyond the Cnuc sandy pit shell-fragments are again found in sand exposed in the railway cutting. In the pits where shell-fragments are found it may often be noticed that small pieces are cemented to the surface of a rounded stone. This is doubtless due to a deposit of carbonate of lime derived from the decay of some of the shells.

Like the Lower Boulder-Clay, the deposits of sand and gravel become better developed when traced from west to east, and the best sections and pits can be seen in that part of the district lying to the east of Strumble Head. Though the St David's promontory is largely covered by a loamy sand, no sections were seen showing deposits of the well-marked marine-like sands and gravels found further east. At Ty-llwyd there are small pits reaching a depth of 5 to 6 feet, where the loamy sand is well

shown. The bottom of the deposit is not seen, but as traced downwards the material becomes very sandy. It is ferruginous, and has a reddish colour.

The sands and gravels become more evident in the neighbourhood of Mathry. A little to the east lies a rugged mass of igneous rock known as Y Graig, and on the south side of this there is a newly dug pit, exposing 8 feet of pebbly gravel and gritty sand, with streaks of fine yellow sand. The deposit gets more sandy as traced downwards, but the bottom is not seen. A tendency to a rough bedding is shown in the section. At Pont Duan, north of the roadside, a very similar gravel-pit is seen. Pen Cnuc, at Castle Morris, marks the site of a mound of fine yellow sand, most of which has now been carried away.

Gravel and sand is exposed, but only to a depth of 4 feet, in a field 200 yards north of Bridge-end, and again at Heathfield, south of the house. Further north, Tre-gwynt lies on sand and gravel, and much sand is seen between Tre-gwynt and Trellys.

East of St Nicholas Church the sands and gravels cover most of the land, as may be seen in pits in many of the fields. The writer employed a man to dig here in order to ascertain if possible the depth attained by the sand and gravel. At 8 feet the bottom was not reached, but the gravel became wet, and it is probable that the blue clay lies a few feet lower down, for it crops out in the moor below St Nicholas Church. The pebbles in the gravel-pit were all well rounded, and chalk-flints were seen. Gravels and sands seem to cover much of the ground on the Strumble Head promontory, but at places a yellowish earthy clay replaces them at the surface, though they may here occur with the clay. In a boring made for a well at Llandruidion farm sand was brought to the surface, in which comminuted shells were seen. The boring reached a depth of over 20 feet, and rock was not reached.

In the farmyard at Tre-howell, near the northern extremity of Strumble Head, in sinking for a well, no sand was passed through—all was earthy clay ; but in a field 250 yards further north a sand-pit occurs, where 2 feet of fine yellowish sand are seen, covered by 2 or 3 feet of a loamy and somewhat stony clay. Mounds of gravel and sand occur on Caergowil, on the heights above Goodwick, and sections 7 feet deep are exposed. They are of the usual character.

South and east of Goodwick and Fishguard deposits of sand and of gravel are frequently met with. They do not occur in the form of kames or eskers, but are found lying in the slopes of the minor hills, and sometimes spreading to the top. It would be almost impossible to map them, as their occurrence is so irregular and patchy ; they are apt to die out laterally in a sudden way, passing into clay or rubbly-drift. At many places they are overlain by an Upper Boulder-Clay. Perhaps the most interesting of all the sections is one seen in the Manorowen sand-pit, which lies in a small wood on the roadside, immediately south of the farm buildings, at a height of nearly 200 feet above sea-level, and two miles distant from Goodwick Bay. When visited a section of only 5 feet was seen, but means were taken to deepen it down to 12 feet. A somewhat diagrammatic view of this section is shown in fig. 1.

The materials were well stratified, and occur in the following order from above downwards :—

A. Soil and rubbly-drift . . . . . 1–3 feet.

B. Very fine light-brown sand, with the beds somewhat contorted, passing down into thin beds of darker sand, followed by coarse sand with some pebbles. Near the bottom is a thin band of sandy clay, 3 inches . . . . . 4½ feet.

C. Pebbly sand, variable in thickness . . . . . 1 to 1½ feet.

D. Coarse grey sand or fine gravel, showing bedding. Some layers are more distinctly pebbly, and here and there fine sand occurs. The fine gravel is full of fragments of marine shells . . . . . 6 feet.

Bottom not reached.

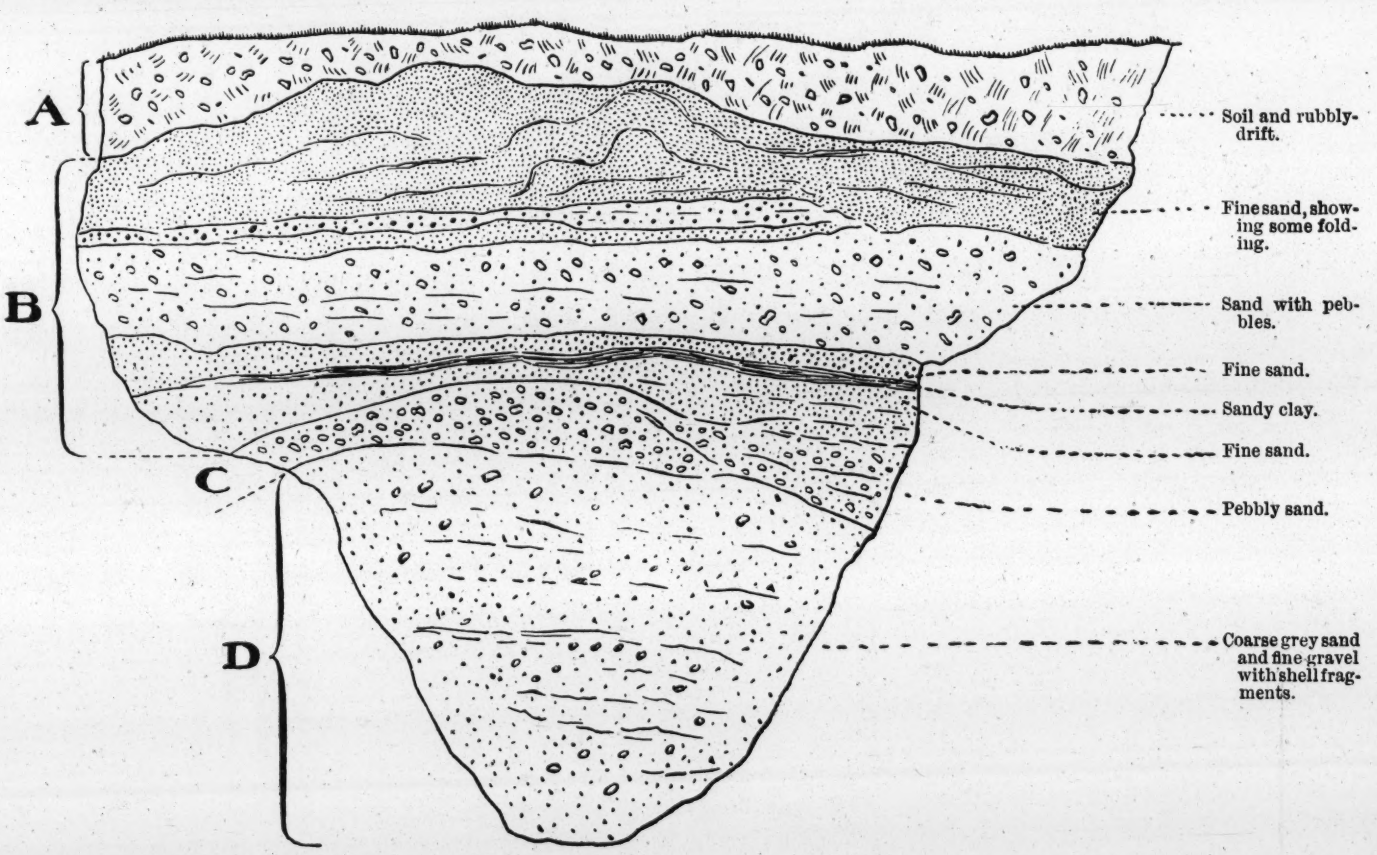

FIG. 1.—Diagrammatic Section of the Manorowen Sand-pit.

Many of the shells have been identified, and are discussed below. Chalk-flints are common. On the opposite side of the road, below the churchyard, and at a lower level, there is a small exposure—3 to 4 feet deep—which consists entirely of a coarse gravel ; but above, towards Manorowen Hill, the gravels and sand are replaced at the surface by clay.

About a quarter of a mile further south, sand is seen again at Cnuc Sandy. There is a big pit just in front of the cottage, 8 feet deep. What is seen here is for most part very fine yellow sand. Gritty and gravelly streaks and layers occur here and there, dying out as traced horizontally. No traces of shells were found here. The pit was at

one time 15 feet deep, and the bottom of the sand was not then reached. 1 to 3 feet of stony-drift cap the sandy beds at the surface.

At a distance of 50 yards further south the railway cutting has passed through 10 feet of similar very loose sand, in which fragments of marine shells occur plentifully.

The town of Fishguard is, in part at any rate, built on sandy deposits, and a good exposure is seen in a quarry on the roadside going down Fishguard Hill to Goodwick Bay. It consists of yellowish sand and fine gritty gravel of the usual kind, which near the top becomes more of a loamy, stony drift. North of the town gravel-pits are common in the fields, and at Pwll Landdu on the coast, east of Castle Point, the cliff is largely made up of gravel and a ferruginous sand, capped by a yellowish boulder-clay full of stones.

A little north of the valley of the Gwaen, at Tre-llan, near Llanllawer, fine yellow sand occurs on the lower slope of the hill of Ceunant.

It would be useless to mention every spot where the sands and gravels are to be seen. They occur in patches all the way to Cardigan, being especially well seen in Llwyn-y-Gwaer Park.

The highest level attained in this part of Pembrokeshire by the sands and gravels is at Pen Creigiau Cemmaes, just off the road leading from Nevern to Cardigan, and four miles distant from the latter place. Sand occurs at the top of the hill, at an elevation of 640 feet. Most of the hill-top is evidently of sandy material, and in a pit a section 8 feet deep is seen, showing very fine yellowish sand passing downwards into darker and more gritty material. There is only a faint trace of bedding. The bottom is not seen. This spot is nearly three miles distant from the coast. Chalk-flints and well-rounded pebbles of quartz are found. A few yards down the hill on the northern side are other small exposures, about 4 feet deep, showing more pebbly sand with rounded boulders; and on the southern side, immediately below the main road to Cardigan, is a gravel-pit, in which are seen rounded and sub-angular stones, some a foot in length. Chalk-flints and pebbles of white quartz were common, and a boulder of Millstone Grit and of a reddish granitoid rock foreign to the district were picked up. Also two pebbles of a muscovite granite. These will be referred to again below. A rough kind of stratification could be seen—layers of small gravelly pebbles separating beds of coarse shingle. The pit is 8 to 10 feet deep.

At Pant-gwyn, half a mile north of Pen Creigiau, sand is seen in a pit, and it is darker and more gritty than that on the hill-top.

Deposits of material resembling marine sands are met with even north of Cardigan, as at Banc-y-warren, but this is outside the area embraced in this paper.

Similar deposits are found far inland, even south of the Precelly hills. A few yards south of Rose-bush there is a sand-pit on the western side of the railway. A diagrammatic section of it is shown in fig. 2. The lower part is hidden by a talus slope. Above this comes 4 to 5 feet of fine yellow sand, very ferruginous in places. The sand becomes a little clayey or loamy in the eastern half of the section, and is

traversed by patches and imperfect layers of a blackish hard pan-like material, which is probably organic in nature. Above this comes 3 to 4 feet of rubbly material, full of fragments of slaty and other rocks of local origin.

Fine yellow sand is seen also near Llangolman, a little east of Maenclochog, on the slopes of the ground rising from the moor, and at Charing Cross there is a pit showing roughly-bedded gravel and sand to a depth of 8 feet. A smaller pit of a similar nature lies at Cefn Ithyn, just north of Maenclochog.

Further south, in the neighbourhood of Trefgarn, sand of quite a different character occurs. There is an exposure of 12 to 15 feet in a big gravel-pit on the roadside opposite the Chapel-of-ease, near Nant-y-coe mill. The fine gravel and sand here is dark grey in colour, and consists largely of minute flattened flakes. The sand is not

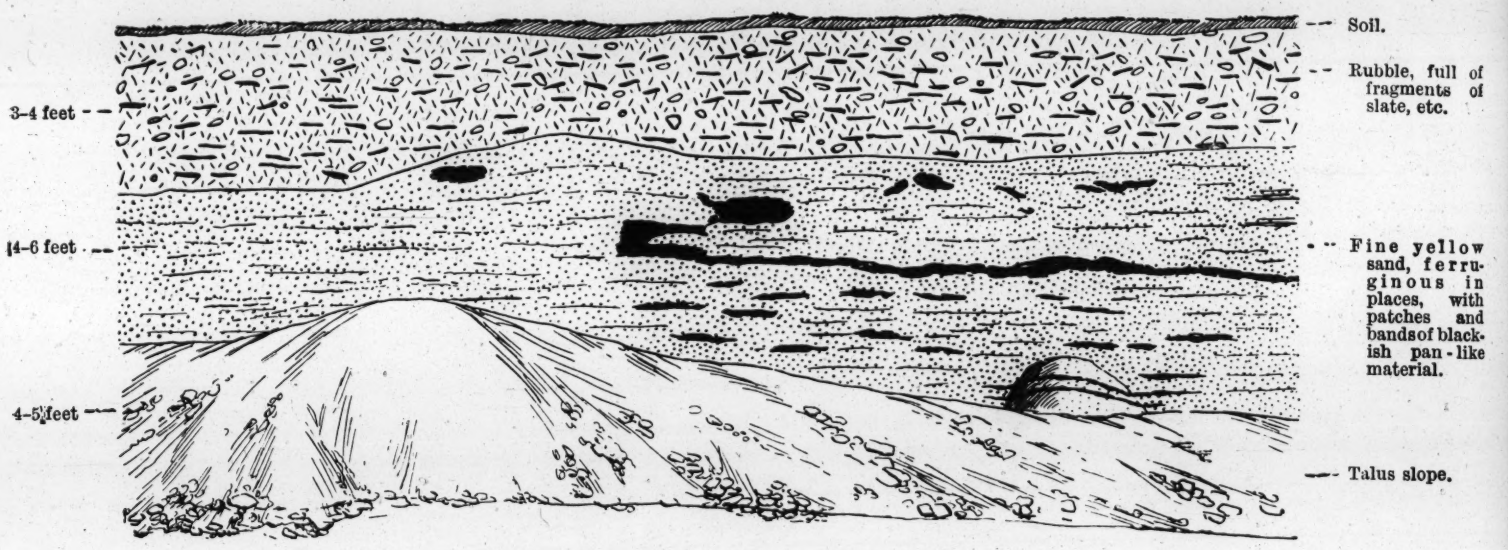

Fig. 2.—Diagrammatic Section of the Rose-bush Sand-pit.

yellow nor brown like that found further north, which the natives speak of as "Demerara-sugar" sand. In Trefgarn Hall park there is a pit where the material seen is somewhat similar, but much coarser. The stones are rounded and sub-angular. The deposits here do not remind one so much of the sand and gravel found on sea-shores.

*List of Mollusca found in the Manorowen Sand and Gravel Pit.**

LAMELLIBRANCHIA.

Nuculana pernula, Mull.	rare.
Pectunculus glycimeris, L.	very common.
Barbatia lactea, L.	very rare.
Mytilus edulis, L.	rare.
Vulsella modiola? L.	very rare.
Astarte sulcata, Da Costa	very common.
Astarte (*Nicania*) *compressa*, Mont.	moderately rare.
Astarte (*Tridonta*) *arctica*, Gray (= *borealis*)	moderately rare.

* The specimens which I collected from this place, together with some obtained subsequently by Mr V. M. Turnbull, are now in the Sedgwick Museum, Cambridge.

Cyprina islandica, L.	common.
Tellina (*Macoma*) *balthica*, L.	moderately rare.
Mactra (*Spisula*) *solida*, L.	rare.
Venus (*Ventricola*) *casina*, L.	moderately rare.
Tapes (*Amygdala*) *decussatus*? L.	very rare.
Cardium islandicum? Chem.	very rare.
Mya truncata, L.	rare.

Scaphopoda.

Dentalium entalis? L.	rare.

Gasteropoda.

Puncturella noachina, L.	very rare.
Natica clausa? Brod. and Sow.	rare.
Turritella communis, Lam.	common.
Buccinum undatum, L.	common.
Tritonofusus gracilis, Da Costa	very rare.
Ocinebra erinacea, L.	very rare.
Trophon (*Boreotrophon*) *clathratus*, L.	common.
„ „ *scalariformis*, Gould	moderately rare.
Nassa (*Hima*) *incrassata*, Ström.	very rare.
Bela turricula, Mont.	rare.
Bela rufa, Mont. (elongate form)	rare.

Fragments of other shells were also found, but they were too broken for identification. Some of the shell fragments found were very thick, especially pieces of *Cyprina islandica*. Many are rolled, and the majority very broken. Entire single valves of *Astarte compressa* were found; and of the univalve shells, *Ocinebra erinacea* and *Trophon clathratus* occurred in nearly perfect condition.

The fauna appears to contain a mixture of species belonging to different climates: *Astarte borealis*, *Trophon clathratus*, and *Trophon scalariformis* are Arctic and Scandinavian species, not now found living in British seas.

Astarte compressa, *Cyprina islandica*, *Buccinum undatum*, and *Puncturella noachina* belong to a northern type of British species which inhabit Arctic and Scandinavian seas in common with our own.

The shells are in very much the same condition as those which have been obtained at Moel Tryfan and at Gloppa, and most of the forms found at Manorowen occur in the other two places also. But *Nuculana*, which occurs rarely at Manorowen, is common at both of the other places.

Pectunculus glycimeris is abundant at Manorowen, but very rare at Moel Tryfan and Gloppa. *Venus casina*, though frequent at Manorowen, is also rare at the other places.

Samples of the sand from several places were examined microscopically, and they all showed a very close resemblance to marine sands. Most of the grains were of

quartz, and the smaller ones were angular, while the bigger ones tended to be more rounded.

3. *Upper Boulder-Clay and Rubbly-Drift.*—The sands and gravels are in many places covered by a yellowish-brown boulder-clay, quite different in character from the bluish boulder-clay which underlies them. This Upper Boulder-Clay is sometimes fairly tough, and is generally much more stony than Lower Boulder-Clay. It varies very much in thickness and character. Inland it often only occurs as a thin covering a few feet deep, but on the coast, where the best exposures are seen, much greater depths are attained. Sometimes it is a tumultuous unstratified till, with boulders of all shapes and sizes scattered pell-mell throughout the matrix. At other places—and it may be at no great distance away—it has more of the character of a rubble-drift, and as seen in section, has the appearance of an agglomeration of coarse and more or less angular débris, showing a rude kind of bedding. It is evident that most of it consists of material which has been re-arranged to some extent, and afterwards modified by sub-aerial agencies. It is impossible to separate the more typical unstratified boulder-clay from the rough semi-stratified clayey and sandy rubble-drift. The included boulders are derived in the main from the rocks of the district, but many far-travelled stones are also found, and these will be discussed in the next section. Ice-scratched stones are fairly common. These are usually sub-angular, with blunted angles and rounded edges. Rounded waterworn stones are also common, especially in the re-sorted rubbly-drift. No traces of marine shells are seen in the Upper Boulder-Clay and Rubbly-Drift. On the coast it is found capping the rocky cliffs at places, and in the bays fine sections, sometimes over 20 feet deep, are exposed. The foreshore is often covered with big boulders derived from the neighbouring cliffs. As the sea is now gradually gaining on the land, the cliffs of drift on the coast are being continually undermined, and the included stones and boulders are washed out and strewn over the shores. The beaches on the bays are rich in boulders and stones of rocks foreign to the district. These have undoubtedly been derived from the cliffs of drift, which are constantly undergoing a process of degradation owing to the action of the waves and of sub-aerial agencies.

By far the best exposures of these upper deposits are shown on the coast-line of the St David's promontory. Figs. 4 and 5 (Plate) represent sections seen in Whitesand Bay, and give a very good general idea of the appearance of these Upper Drift deposits. Fig. 4 shows a section of the cliff near the north end of the bay. Here the cliff is about 20 feet high and consists entirely of drift. At this spot the drift is a typical till or boulder-clay, showing no bedding, but full of stones and boulders, big and small, which are scattered confusedly through it. Most of the boulders are angular and sub-angular, and some are well glaciated. A few rounded pebbles and stones also occur. The boulders are mostly of local rocks, though some erratics are seen. Loose sandy soil occurs at the top. At the base the rock does not appear, but the shore is covered with stones derived from the cliff. A few yards further north slaty rock is seen

emerging from underneath the boulder-clay, which becomes thinner in this direction. At places the rock shows a hummocky surface, marked with glacial striæ, which run from north-west to south-east. At the base of the small promontory called Trwyn Hwrddyn, on the north side, a rubble of very coarse fragments lies between the solid rock and the drift. Here the drift shows a rough sort of stratification, and has much sandy and pebbly material intermingled with boulder-clay.

Fig. 5 shows a section in Whitesand Bay seen further south. This is also about 20 feet in height, but here it consists more of a rubbly-drift. The matrix is earthy and sandy, and is choke-full of small flakes of slaty and other rocks, which have a rude kind of arrangement, especially towards the lower part. Boulders of larger size occur here and there, and consist for most part of grit and conglomerate and slaty rocks, with some blocks of the local gabbros and diabases. Pebbles of white and yellowish-white quartz are common. The top is covered by loose yellow sand, probably wind-blown, and just underneath this are traces of a pebbly bed. Though rock does not appear at the bottom just at this spot, it crops out on both sides a short distance away.

Drift of a similar kind is seen in sections, and capping the cliffs at other places on the western coast of the promontory.

Boulder-clay is seen at Porth-lisky stuck full of stones, many of which are smoothed, polished, and striated; and a boulder of the St David's Head gabbro, measuring roughly $3 \times 2 \times 2$ feet, lies on the beach below. A stony boulder-clay or a more rubbly drift is seen at various places on the south coast, the best exposures being above Caerfai Bay and at Caerbwdi Bay. At Caerbwdi the cliff is over 20 feet in height, but the base is hidden by talus: the matrix is here rather sandy, and streaks and pockets of rather fine sand are seen here and there. The included stones are often pebbly, but some are sub-angular and ice-marked. They are made up almost entirely of rocks found in the neighbourhood. Near the top flaky fragments are very plentiful, and these are derived from the local purple flagstones and slates. A big boulder of the coarse gabbro from St David's Head lies at the base of the cliff.

No good section is seen at Porth-y-Rhaw, but the drift caps the hills and cliffs to the south-east.

Very similar sections are seen on the north coast at Abereïddy and above Traeth Llyfn. At the latter place a rubble of big boulders is seen; most of these are of local igneous rock, very iron-stained and decomposed. Towards the top the section is freer of big boulders, and is full of little flakes of sedimentary and cleaved rocks.

One of the finest sections on the coast is seen at Aber-mawr, west of Strumble Head. At the northern end the rock is seen capped by 10 to 15 feet of stony-drift. As traced southwards the drift thickens to about 40 feet, then tails off rapidly. Where thickest the lower part shows some tendency to a rough kind of bedding, and is full of small flakes and little stones, more or less pebbly. This passes above into a rubbly clay, full of boulders of all sizes, most of which are angular and sub-angular, and derived from rocks of the locality. The cliffs on Strumble Head in many places are covered by a

mantle of stony till, but good sections are rare. The sections of drift exposed on the coast between Strumble Head and Cardigan are not so good as those seen west of Strumble Head, but where seen they are of the usual character.

Inland, good sections are not often met with. Much of the country in the neighbourhood of St David's is covered by drift, which consists of a sandy matrix full of boulders. But often the matrix is clayey, as may be seen in sections on the roadside near Castell, south-west of St David's, and again in a pit just off the road near Pont Clegyr, two miles east of St David's. In fact a large part of the country south-west of Strumble Head is covered by material which has been to a large extent re-arranged, and which cannot be defined accurately either as boulder-clay or as sand and gravel, though the tendency is for the sand and gravel to become more marked at a depth of a few feet from the surface. Much rubbly-drift, becoming more sandy as traced downwards, is spread out on Strumble Head, especially on the moorland above Goodwick. The sands and gravels occurring south and east of Fishguard Bay are, as already mentioned, usually overlain by a few feet of rubbly-drift or more typical stony boulder-clay. In the railway cutting between Tregroes moor and Manorowen 7 feet of stiff yellowish-brown boulder-clay is seen covering the shelly sand. This clay is spread out over much of the high land skirting the railway here on the west side. A little further south the sands and gravel die out, and the Upper Boulder-Clay seems to lie directly on the Lower Boulder-Clay, and this occurs possibly in the boring at Tre-bython already referred to, where yellowish clay is succeeded by tough bluish clay.

It is very rarely that one has an opportunity of finding all the deposits succeeding one another in the same section, and of ascertaining the depth of each. But the engineer of the Great Western Railway at Goodwick supplied the author with particulars of the boring made in the railway cutting between Tregroes Moor and Manorowen, just about Cnuc Sandy. A complete section of the railway cutting, together with the results obtained by boring, are given on the next page. From above downwards, the deposits passed through were—

(4)	Stiff yellowish-brown clay with fragments of slate-rock	7 feet
(3)	Fine yellow sand with shell-fragments	10 „
(2)	Stiff dark-blue boulder-clay with shell-fragments	18 „
(1)	Gravel	5 „
	Rock.	

This section proves very clearly the presence of an Upper and a Lower Boulder-Clay. These are separated here, as in many other places, by a deposit of sand. Most of the Lower Boulder-Clay is below the surface of the railway cutting. And a glance at the section shows that before the cutting was made, the surface of the ground was covered to a depth of 7 feet by the Upper Boulder-Clay. So it is quite possible that the Lower Boulder-Clay spreads over a much wider area than is evident at the surface, and that much of it is hidden by more superficial deposits. It is interesting to note that underneath the Lower Boulder-Clay there lies 5 feet of gravel. On comparison this was

found to be very similar to underlying grit rock when this is broken up. So it may represent material ground out of the solid rock by the movement of land-ice.

Further inland a good example of the Upper Drift is seen in a cutting on the roadside between Puncheston village and the railway station. It is a stony and rather rubbly clay. Close at hand, on Puncheston common, the Lower Boulder-Clay crops to the surface.

Further east, up to Cardigan, the Upper Boulder-Clay and Rubbly-Drift are seen in many places overlying sands and gravels or lying immediately on the blue clay.

The line of division between this Upper Boulder-Clay and the sands and gravels is not so marked as that between the sands and gravels and the Lower Boulder-Clay.

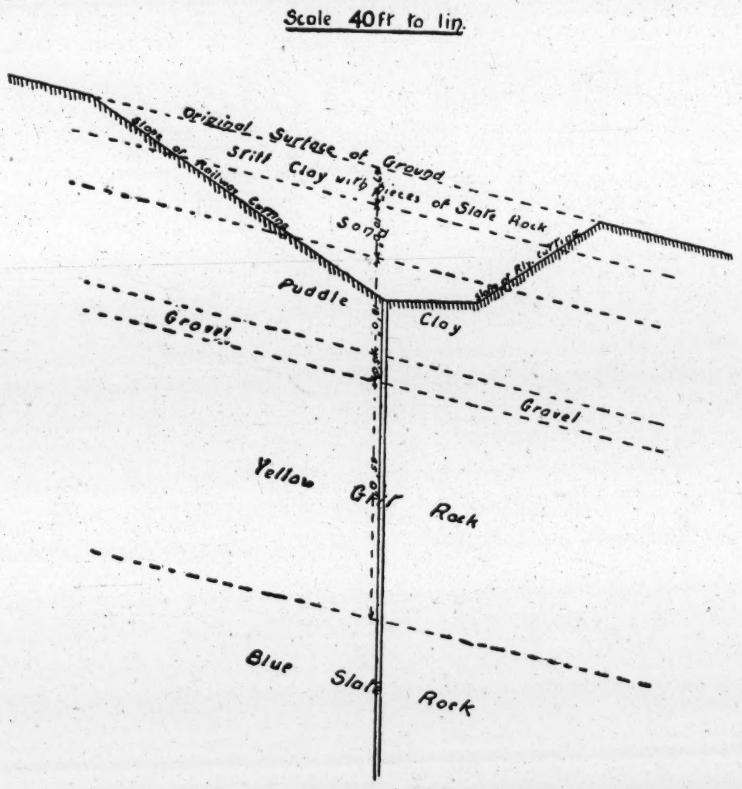

FIG. 3.—Section at Boring No. 1 on Goodwick to Letterston (existing) Railway, between Tregroes Moor and Manorowen.

V. THE BOULDERS AND ERRATICS.

The transport of boulders is of great importance as indicating the general direction of ice-movement. Throughout northern Pembrokeshire boulders may be seen scattered over the surface, and are especially common on waste or uncultivated land. Many of those found on the slopes of the hills have no doubt rolled from the parent rock above, but many, though not foreign to the district, have yet been carried for some distance by ice.

Where the land is cultivated an immense number of boulders must have been removed, but even in the fields it is common to meet with big blocks of igneous rocks, which, as a rule, have been left in the position where they were originally stranded in order that cattle may have something to rub against, or have been left on account

of their immense size, though these latter are becoming gradually destroyed by blasting operations. At many places, especially on the St David's plateau, huge standing stones, cromlechs, or other ancient remains are seen, and it is more reasonable to believe that the immense blocks used for these purposes were found as boulders near at hand than that they were quarried from the parent mass, which often lies at a considerable distance away. And this is rendered the more probable as blocks of similar size are not infrequently seen dotted over the surface. An examination of the stone dykes will show how plentifully boulders of all sizes must at one time have been studded over the ground, and what a variety of rocks is represented. Towards the western end of the area the boulders are largely found to have been dispersed from the igneous rocks on the north coast of the promontory. And everywhere the great majority are derived from parent masses found in the district. Boulders of diabase may often be seen resting on volcanic or sedimentary rocks, and *vice versa*, proving that there has been some transport. It has already been mentioned that blocks of the St David's Head gabbro are found lying to the south-east at Caerfai Bay, and Caerbwdi Bay, and on the cliffs above. This implies that there was a movement of ice from a north-westerly direction, and it agrees with the evidence shown by the glacial striæ which are seen on the coast. Erratics are met with often in the drift and on the shores, but the number which has been noted on the surface is not great. A further study of the stone dykes would doubtless bring more to light

1. *Erratics seen on the surface of the ground.*—The detection of those about to be mentioned is mostly due to the fact that they had been broken up by the farmers through blasting or other agency, so as to expose fresh surfaces. Hicks states (*Geol. Mag.*, 1891, p. 501) that he observed many northern erratics in the St David's district. The granite boulder which he discovered on Porth-lisky farm "before it was broken must have been over 7 feet in length and 3 to 4 feet in thickness, and is identical with a porphyritic granite exposed in Anglesea." He found another of picrite which he thus describes: "The boulder is somewhat rounded; its longer axis, which lies nearly south-east and north-west, measures about a yard. A transverse section is slightly triangular, the shorter sides measuring respectively about 16 inches and 22 inches. It lies on the promontory forming the east side of Porth-lisky harbour, resting immediately on Dimetian rock, surrounded by an uncultivated area overgrown by gorse and heather" (*Quart. Journ. Geol. Soc.*, vol. xli. p. 519). It was submitted to Professor Bonney for examination, and he states that it is wonderfully like the boulders found at Pen-y-Carnisiog, Anglesea, which had been previously described by him (*Quart. Journ. Geol. Soc.*, vol. xxxvii., 1881, p. 137). In a later paper (*Quart. Journ. Geol. Soc.*, vol. xli., 1885, p. 518), Professor Bonney remarks that "the lithological evidence rather favours the derivation of the Anglesea boulders from dykes in that island." A hornblende picrite of a somewhat similar character occurs also *in situ* at Penarfynydd, on the south-west coast of Carnarvonshire. So it is probable that the boulder found near St David's has been carried by the agency of ice from Carnarvonshire or Anglesea.

The writer discovered a big boulder of picrite on Strumble Head, on a piece of uncultivated ground a little north of Tre-sinwen farm, by the side of the pathway leading to the coastguard station at the Head. The boulder had been broken, and now lies in several pieces—the biggest piece measuring roughly $3\frac{1}{2} \times 3 \times 2$ feet—and in its original condition it must have been much bigger. Another boulder of picrite, very similar in appearance and size, and also broken, was discovered on a field lying a little to the north-west of Tre-seissyllt, between the farm and the coast north of Aberbach. A microscopic section of this rock revealed the presence of the following minerals :—brown hornblende, strongly dichroic ; augite, nearly colourless ; olivine, not very abundant, and always very much serpentinised and of a greenish colour ; magnetite ; a chloritic mineral, which is evidently an alteration product, and a little plagioclase felspar. The chloritic mineral is greenish in colour and markedly dichroic. The general appearance of the rock as seen in section was very different to that of the St David's picrite : the augite does not show such perfect forms in the former as in the latter, and there is rather more olivine in the St David's rock. The Strumble Head picrite boulder is rather more like some of the Penarfynydd specimens as seen in section, but there is not so much olivine in the former, and the pœcilitic structure which often characterises the latter is not seen in the former. But it is highly probable that the Strumble Head picrites have also been borne from Lleyn or Anglesea. The two specimens found on Strumble Head lie about three miles apart, in a line whose direction is north-north-east to south-south-west. Near the Tre-seissyllt boulder of picrite lay a boulder of olivine-gabbro, also broken to pieces by blasting. The newly-exposed faces were remarkably striking, and the crystals are very fresh. The rock is quite unlike the gabbros found in Pembrokeshire. A microscopic section showed beautifully fresh olivine crystals, and the rock is undoubtedly of Tertiary age, and has probably come either from the Western Isles of Scotland or from the north-east of Ireland.

2. *Erratics in the Drift.*—As might be expected, the majority of the boulders found in the drift deposits are of local origin. They occur abundantly in the Upper Boulder-Clay and Rubbly-Drift, and in the sands and gravels, and to a less extent in the Lower Boulder-Clay. The grits, shales, and slaty rocks of Pembrokeshire are very similar in appearance to rocks of a like nature from North Wales, and the same is true of some of the igneous rocks, especially the diabases and some of the lavas. It is thus quite possible that among the boulders found imbedded in the drift many North Wales rocks may be represented, though there is no means of distinguishing them readily from the boulders of local origin. This was suggested to the writer by the discovery of boulders of a diabase rock in the boulder-clay exposed at Cardigan. To the naked eye this diabase seemed very like that found to the south-west in Pembrokeshire. But no such rock is known to occur anywhere nearer Cardigan than Newport—nine or ten miles to the south-west—and so these boulders must have come from the north. This is made all the more probable by the discovery of boulders of

what are undoubtedly northern rocks associated with these boulders of diabase at Cardigan.

Of the erratics the most important discovery in the drift was that of a small boulder of the Ailsa Craig riebeckite rock, or paisanite. This was found in the bluish boulder-clay, near the surface, at Clun-Bach moor, St Nicholas, near the south-western end of Strumble Head. The specimen was sliced, and as seen under the microscope it is identical with specimens obtained from Ailsa Craig.

Boulders of hornblende-porphyrites from the south-west of Scotland occur in the Lower Boulder-Clay as far east as Cardigan, and are found even oftener in the Upper stony Boulder-Clay. Several varieties are seen, all of which can be matched in Wigtownshire and Kirkcudbrightshire. But the erratics which are most commonly met with in the drift are reddish granophyres, quartz-porphyries and micro-granites. Many of these are dyke-rocks, and it is very difficult to trace them to their source. Some have certainly come from Ireland, and some most probably from the south-west and west of Scotland. Lake District rocks and North Wales rocks are not so well represented.

In the clay-pits of Rhos Isaf, near Dinas, excavated in the Lower Boulder-Clay, a reddish granophyre was found, which, under the microscope, resembles very much some of the granophyres of Mull. And in the Pen Creigiau gravel-pit boulders of granophyre occur, which have come from the Carlingford district, Ireland. One was sliced, and the microscopic characters seen were identical with that of the granophyre of Barnavaine, Carlingford. These reddish granophyres are found in the drift throughout the area, but seem to be rather scarcer in the extreme west. As the writer had not much opportunity of comparing them with Irish rocks, he sent them to Prof. Watts, of Birmingham University, who very kindly examined them. Many of the granophyres, he thinks, can be matched in the Carlingford mountains; others bear more resemblance to the Tertiary granophyres of the Inner Hebrides. The reddish quartz-porphyries appeared to him to be like the varieties of the quartz-porphyry of Cushendale in Antrim; and among the boulders of micro-granite he noted two which are likely to have come from the mass of micro-granite at Cushendun in Antrim. He is of opinion also that the Old Red conglomerate of Cushendun might, when broken up, present examples of many of the types of boulders found in the drift of Pembrokeshire. A pebble of muscovite-granite, probably the Foxdale granite of the Isle of Man, was obtained from the gravel-pit at Pen Creigiau, over 600 feet above sea-level. Examples of Millstone Grit were obtained here also, and in the Cardigan clay-pit boulders of Carboniferous Limestone are common, often with the fossils well preserved. On the beach at Gwbert, not far from Cardigan, boulders and pebbles of Carboniferous Limestone are common. These must have come from Ireland or from carboniferous rocks which are hidden at the bottom of the Irish Sea. It is hardly likely that they have come from the small exposures bordering the Menai Straits in North Wales.

Chalk-flints occur everywhere—in both boulder-clays, in the sands and gravels, and

on the beaches. These also must be derived from the north-east of Ireland and from rocks hidden under the Irish Sea.

From the cliff of Upper Boulder-Clay at Porth-lisky a boulder of olivine-dolerite was obtained, microscopic sections of which show very fresh olivine. It is certainly a Tertiary rock, and has probably come either from the north-east of Ireland or from the western isles of Scotland.

3. *Erratics found on the shores along the coast.*—These are especially abundant at those places where cliffs of boulder-clay or drift are seen. Many of them are found lying just at the foot of the cliffs, having only recently fallen from them; and others which were picked up as pebbles on the beach have doubtless, for most part, been derived from the drift also.

Ailsa Craig, Riebeckite Rock or Paisanite . . .	found on	Abermawr beach (frequent), Aberfelin beach, Porth-y-Rhaw beach.
(It is interesting to note that a boulder of this was also found in the Lower Boulder-Clay at St Nicholas, not far from Abermawr.)		
Granites from the Dalbeattie area, several varieties .	found at	Pwll Gwaelod beach (frequent), Aberbach beach (near Dinas).
Granites of Galloway type	„	Whitesand Bay, Aberbach (near Dinas), Pwll Gwaelod, Gwbert (near Cardigan).
A fine specimen of a Mica-hornblende-Granite, identical with that of Auchencairn, Kirkcudbrightshire,	„	Abermawr.
Mull of Galloway Granite	„	Pwll Gwaelod.
Another variety from same area	„	Gwbert (near Cardigan).
A Gneissose Granite from Criffel	„	Pwll Gwaelod.
Granite or Quartz-Diorite from head of Loch Doon, South of Scotland,	„	Pwll Gwaelod.
Biotite Granite, Loch Dee, South of Scotland . .	„	Abermawr, Gwbert (near Cardigan).
A Diorite identical with that of a dyke near Gutchen Isle, Colvend shore, south of Dalbeatie,	„	Abermawr, Whitesand Bay.
Other Diorites from the Galloway area	„	Abermawr (frequent), Aberbach (near Dinas), Abereiddy.
Hornblende-porphyrite identical with one found south of Castle Douglas, Kirkcudbrightshire,	„	Pwll Lan-ddu.
Other Hornblende-porphyrites of the Galloway country .	„	Pwll Gwaelod, Aberbach (near Dinas), Abermawr (frequent), Whitesand Bay.
Hornblende-biotite-porphyrite, Wigtownshire . . .	„	Pwll Gwaelod.
Silurian grits, South-West of Scotland	„	Abermawr.
Muscovite-granite, Foxdale, Isle of Man . . .	„	Gwbert (near Cardigan).
Andesites, Rhyolites, and altered Tuffs of the Borrowdale series,	„	Abermawr, Aberbach (near Dinas).
Reddish Quartz-porphyry, probably from Cushendale, Antrim,	„	Porth-y-Rhaw, Abermawr, Whitesand Bay.
Reddish granophyres and micro-granites, mostly North-East Ireland, but some possibly from West of Scotland,	„	Pwll Lan-ddu, Gwbert (near Cardigan), Aberbach (near Dinas).
A gneissose Grit—locality unknown	„	Porth Sele
Carboniferous Limestone	„	Gwbert (near Cardigan).
Gannister	„	Abermawr.
A Muscovite-granite, with microcline and some biotite—locality unknown.	„	Abermawr.

The most striking fact in connection with the erratics is that so many of them can be traced to the south-west of Scotland. The Ailsa Craig paisanite has been obtained in the boulder-clay, and is frequently met with on some of the beaches, especially at Abermawr. The granites, diorites, and porphyrites of the Galloway country are also well represented, boulders being found which represent the three principal massifs, namely, (1) Dalbeattie and Criffel, (2) Cairns Muir of Fleet and New Galloway, and (3) Loch Doon and Loch Dee, and in addition some from smaller exposures, such as that of the Mull of Galloway.

The other region from which the boulders have travelled is the north-east of Ireland, and its rocks are represented in Pembrokeshire by reddish granophyres, quartz-porphyries, and micro-granites.

A few boulders are found also which have almost certainly come ultimately from the Western Isles of Scotland.

It is a noticeable feature that the Lake District rocks are but poorly represented, and the same is apparently true of those of North Wales.

Many of the boulders and pebbles, such as those of Carboniferous Limestone and the chalk-flints, may have been torn up from the bed of the Irish Sea.

VI. General Conclusions.

The facts adduced in this paper prove conclusively that northern Pembrokeshire has been the theatre of glacial action to an extent greater than had previously been supposed. Glacial deposits cover the ground in that part much in the same way as they do further north, and present very similar characteristics. Here also we meet with a tripartite division of the deposits, namely, a Lower Boulder-Clay, Intermediate Sands and Gravels, and an Upper Boulder-Clay and Rubble-Drift, reminding us of the tripartite division found at so many places further north on both the east and west sides of England, and in North Wales. But in the present state of our knowledge it is very difficult to correlate the deposits found in one area with those found in another area; and it is not safe to assume that the Sands and Gravels always represent any definite horizon in the glacial series.

Of the deposits which have been described it is the Lower Boulder-Clay which has the widest extension; it covers much of the lower grounds inland, and is often hidden under the other accumulations. It follows the slope of the ground, and a little below Pen Creigiau Cemmaes it attains an elevation of nearly 600 feet above sea-level. The series of sands and gravels is a very variable one. Often they taper or die away suddenly into a stony or loamy drift, and at places are absent altogether. They attain their greatest elevation near the east end of the area at Pen Creigiau, where the sands reach a level of over 640 feet, and are followed immediately below by coarse shingly gravel. The Upper Boulder-Clay, where the sands and gravels are absent, is sometimes seen to rest immediately upon and coalesce with the Lower Boulder-Clay, so that

it becomes somewhat difficult to separate them. But the distribution of the Upper Drift is not so wide; it is only met with here and there, and the true Upper Boulder-Clay is often replaced laterally by Rubbly-Drift. The more sporadic occurrence of the Upper Boulder-Clay is probably due in part to the fact that it has suffered more from denudation.

The Lower Boulder-Clay is undoubtedly the product of an ice-sheet, and it has all the characteristics of a true ground-moraine. It is remarkably tough and homogeneous, and shows no traces of stratification, and it has all the appearance of having undergone great compression. The included stones are often intensely glaciated, and are sub-angular rather than rounded in form. The fact that fragments of marine shells occur in the clay proves that the ice which gave rise to it must have travelled over a sea-bottom. On the other hand, the bits of woody matter sometimes seen embedded in the tough clay to a depth of 18 or 20 feet suggest that vegetation grew on the land bordering the sea, before the advent of the ice or during an interglacial period, and that some fragments of this found their way, by means of streams or otherwise, to the sea-bottom, where they lay in the path of the ice. Or they may have been derived from the remains of a submerged forest. The included erratic stones help us to follow the direction from which the ice came, and the occurrence of boulders from the south-west of Scotland and from the north-east of Ireland in the Lower Boulder-Clay and Drift as far east as Cardigan, and the discovery of fragments of marine shells in the Lower Boulder-Clay exposed at the brickworks near that town, make it clear that the whole of northern Pembrokeshire was buried underneath an ice-sheet coming from the north. The view held by Carvill Lewis, that the Irish Sea glacier (as he termed it) extended no further south than the extremity of Lleyn in Carnarvonshire, is shown to be inaccurate. And though Professor James Geikie makes the *mer de glace* which overwhelmed Anglesea flow down St George's Channel, to a limit reaching beyond the south-west of Wales, he only indicates it as crossing the extreme west of Pembrokeshire at St David's Head. But the facts just mentioned show that this *mer de glace* must have passed over a great deal more of Pembrokeshire than St David's Head. It invaded northern Pembrokeshire along its whole extent, and even encroached on Cardiganshire to the east, and its trail is evident in the tough dark-blue homogeneous boulder-clay, with its northern erratics and the broken shells derived from the sea-bottom over which the ice travelled. How much further south this typical boulder-clay or ground-moraine extends is a point which must be left to future investigation.

This *mer de glace* was of course the southward extension of that ice-sheet which filled the northern basin of the Irish Sea, and which has been described by Professor James Geikie and other workers in Glacial Geology. The latest results published are those of the investigations of Mr Lamplugh in the Isle of Man, and these have appeared in his Survey Memoir on the Geology of that Island. His observations on the Irish Sea Glacier are of great interest and importance, and throw light even on

what occurred to the south of his area. Speaking of the conditions which obtained in the northern part of the Irish Sea at the beginning of the Glacial period he says, "Along the shores an ice-foot probably formed in the winter and broke away in the summer into floes, which distributed their burden of rock-fragments broadcast over the sea-floor. This seems to be the explanation of the universally wide dispersal of the fragments from Ailsa Craig, which have been recognised in the drift almost all round the northern part of the Irish Sea basin, in Ireland and Wales, as well as in the Isle of Man. The sea-girt precipices of splintering rock in Ailsa would not fail to cast off a load upon an ice-foot below; and thus these fragments became strewn over the sea-floor almost as widely as the shells, and were subsequently carried by the ice-sheet into nearly every district to the southward where the shells were carried" (p. 370). This helps to explain also in a satisfactory way the occurrence of fragments from Ailsa Craig and of boulders from the north-east of Ireland, from the south-west of Scotland, and even possibly from the Inner Hebrides, in the drift, and on the beaches of northern Pembrokeshire. For the ice-sheet as it advanced would pick up any such fragments which had been previously strewn over the sea-bottom by ice-floes, and would carry them southwards on to the land as it carried the shell-fragments.

Lamplugh estimates that in the neighbourhood of the Isle of Man the ice-sheet, at its maximum, must have attained an elevation of not less than 2000 to 3000 feet above the present level of the sea, and the general direction of the ice-movement was from north-north-west to south-south-east. He points out that "the West-British Ice-sheet probably attained its ultimate dimensions mainly from the accretion of snowfall upon its surface, and in only a minor degree from the inflow of tributary glaciers." He calls attention to certain results which help us to understand the southward extension of the ice-sheet far enough to overwhelm Pembrokeshire. "As the British ice-sheets must always have received their increment principally from the moist Atlantic winds, it seems probable that, without any change of climate, the centre of greatest accumulation, and consequently of maximum glaciation, would tend to shift steadily westward and south-westward as the icy plateau rose higher in the path of the moisture-laden winds and compelled their earlier precipitation. This effect would, moreover, be accentuated by the obliteration of the open water in the sea-basins to the eastward. The West-British sheet might from this cause go on increasing, while its East-British and Pennine equivalents were already diminishing from lack of sufficient snowfall. . . . For the above reason, the shrinkage of the ice-sheet covering the Isle of Man is likely to have commenced while the Welsh and Ivernian sheets were still increasing." Although the ice in the southern part of the Irish Sea basin did not probably attain such a great thickness as the ice in the northern area did, nevertheless all the evidence goes to show that even as far south as Pembrokeshire it must have reached a considerable elevation. The presence of drift material at Pen Creigiau, at an altitude of over 600 feet above sea-level, indicates that the ice-sheet here was in all probability not much less than 1000 feet in thick-

ness, even if we allow that the land at that time stood at a somewhat lower level than it does at the present day. Mr LAMPLUGH gives a sketch-map of the Irish Sea (as far south as the Lleyn promontory), showing glacial striæ and probable direction of the ice-movement. The ice which streamed over the Isle of Man from the north is shown as usual to have travelled south and to have overwhelmed Anglesea, being here diverted so as to move more to the south-west on account of the opposition of the ice coming down to meet it from the mountains of Snowdonia. On the western side of the Irish Sea basin the striæ indicate that the ice moved from the land on the eastern seaboard of Ireland, and took a course from north-north-west to south-south-east, and coalesced with that which passed over Anglesea and Lleyn. Its course southwards from this limit is not shown. But in Professor GEIKIE's map the Irish ice is made to bend back to the south as a result of its meeting with that part of the ice-sheet which flowed over Anglesea, and the northern ice is shown as passing down to the west of Cardigan Bay, on account of the presence of the ice flowing west from Merionethshire and Central Wales. But the investigations carried out on the glaciation of Pembrokeshire make it clear that the Irish ice was not bent back so sharply, but, on the contrary, it continued in its original course from north-north-west to south-south-east, whilst the ice from the north was forced to invade Cardigan Bay, and must therefore have shouldered in the Welsh ice again upon the mainland. This is proved by the direction of the striæ seen along the coast, as well as by the presence of boulders of igneous rock from Ireland and the south of Scotland in the drift as far east as Cardigan. Again the presence of chalk-flint throughout the area is evidence in the same direction, for these must have come from the north-east of Ireland or from the bed of the Irish Sea; and it is possibly from this bed that the boulders of Carboniferous Limestone which are seen so abundantly at Cardigan have come. In this connection it is interesting to recall the presence of fragments of Millstone Grit in the gravels at Pen Creigiau. Our knowledge of the glaciation of Ireland is as yet very imperfect, and it is difficult to estimate what volume of ice passed seawards from its eastern border. At the present day the rainfall over Ireland is very excessive, and so it seems probable that the snowfall was likewise excessive during glacial times. This would give rise to a proportionately large ice-sheet moving outwards in all directions, and so it is quite possible that the amount of ice which found its way into the Irish Sea basin was considerably greater than has been generally supposed. And in the southern part of the basin it would to some extent oppose the passage of the western ice which overflowed Anglesea and the end of Lleyn in a south-westerly direction, and cause it to turn a little more to the south so as to travel over Cardigan Bay. The confluent sheet, forming by the junction of the Northern ice, the Irish ice, and to some extent the Welsh ice, would invade northern Pembrokeshire in the direction which is shown by the striæ along the coast near St David's, namely, from north-west to south-east, or perhaps from north-north-west to south-south-east. This would explain the transport of boulders from the St David's Head gabbro south-

eastwards to the neighbourhood of Caer-bwdi Bay. In this connection it may be mentioned that Mr J. HARRIS, in a report on erratics in South Wales, which appeared in the British Association Reports, 1898, refers to some boulders found at Pencoed, near Bridge-end, Glamorganshire. Microscopic sections of some of these were prepared and sent to petrologists for examination. The result of this is given as follows:—"One was identified with the gabbro of St David's Head; a felsite bore some resemblance to the pre-Cambrian rocks of Pembrokeshire; two or three acid rocks, brecciated felsites, and tuffs are very like those of the Lleyn promontory." From these data it is concluded that the transport of boulders was from the west or north-west. If one of the boulders found near Bridge-end is accurately identified as belonging to the St David's Head gabbro, it is a most remarkable fact. It is hardly safe to draw any conclusion until it has some further corroboration.

The more or less loose materials covering the bottom of the sea, which existed before the advance of the ice, would become incorporated into the lower layers of the ice-sheet. And as the ice was very thick, and moved onward slowly, it would exert a great pressure over its bed, with the result that much of the rocky floor would be torn away, and much of the material ground up and pulverised to form the typical ground moraine. The shell-banks which occurred on the sea-bottom would be destroyed, and the marine detritus would be carried forward under the ice or in the ice. This accounts for the presence of shell-fragments at places in the Lower Boulder-Clay. And the most natural explanation of the shelly sands and gravels is that they represent the material of a sea-bottom, carried onwards and upwards to their present position by an ice-sheet, and re-arranged by fluvio-glacial action. That is to say, they are *remaniés* derived from the bottom-moraine of an ice-sheet which had travelled over a sea-floor. Similar sands and gravels have been found at many other places on the west side of the island, and they have given rise to much discussion—notably those found at Moel Tryfan in Carnarvonshire. The Pembrokeshire series differ from those of Moel Tryfan in that they are found overlying the well-marked stiff Lower Boulder-Clay. The most remarkable feature in connection with the Moel Tryfan beds is the great elevation at which they are found—1350 feet above sea-level. In Pembrokeshire the greatest height at which they have been met with is 642 feet at Pen Creigiau, four miles south-west of Cardigan. The mode of origin of such sands and gravels has been one of the most vexed questions in Glacial Geology. Some writers, such as MACKINTOSH, T. M. READE, and others, have argued that the sands and gravels represent marine deposits laid down in place during a great submergence. It is admitted, even by the opponents of that theory, that a partial submergence took place during Glacial times, but to what extent is not known, and there is no evidence to show that it meant a sinking of the land in Carnarvonshire to as much as 1350 feet below its present level. And the partial subsidence which is allowed is generally thought to have diminished towards the south. In Pembrokeshire no evidence can be seen along the coast which would lead us to

believe that there had been a subsidence of 640 feet below the present level, so as to account for the marine deposition of the beds found at Pen Creigiau. Further, if the advocates of a submergence point to the presence of marine shells at places in the sands and gravels as a proof, on the ground that the shells must have been altogether destroyed if carried beneath the ice in the morainic débris, one asks then how they can account for the presence of fragments of shells—some large enough to be identified—in the tough blue boulder-clay underlying the sands and gravels? No one who looks upon that clay, as exposed, for instance, at the Cardigan brickworks, can doubt for a moment that it is the product of an ice-sheet. Not a trace of stratification can be seen in it, nor is there any character which suggests even the possibility of its being the result of marine deposition. And yet marine shells are seen imbedded in the clay. This, as has been already pointed out, is due to the fact that the ice-sheet, of which this clay is the bottom-moraine, travelled over a pre-existing sea-bottom. The fact that everywhere the shells are very broken and much rolled is hardly compatible with the view that they are now found in or near the positions in which the molluscs themselves lived. And it is worthy of note that the Lamellibranch shells obtained in these sands and gravels are never found with the two valves in apposition, as one might expect to find if they lie in ordinary sea deposits. Again it has often been pointed out that it is a significant fact that deposits of this kind only occur in glaciated areas, and that wherever broken shells are found, with them there also we find far-travelled erratics present. And this is to a marked extent the case in Pembrokeshire.

Mr J. F. Blake (*Geol. Mag.*, vol. x., 1893, p. 267) concluded that the shelly sand at Moel Tryfan had been pushed up in front of the advancing glacier, and that, as a result of this glacier meeting that which came out of the Bettws Garmon valley, the sand got pushed into a protected corner and was left there. But in northern Pembrokeshire the sands and gravels are found scattered in patches over a wide area, and are frequently well bedded. Here they are the products of the washing and re-sorting of infra- and intra-glacial detritus. This may have gone on partly under the ice, but it would no doubt take place to a great extent at the time of the melting of the ice-sheet, when large streams would issue from the margins of the glacier and re-arrange much of the superficial deposits left on the surface of the land.

The Upper Boulder-Clay is so sporadic in its occurrence that it is difficult to draw any definite conclusions with regard to it. It may possibly represent a second advance of the ice-sheet after an interval of less severe glacial conditions. It is far more stony than the Lower Boulder-Clay, and in places passes into Rubbly-Drift. This Rubbly-Drift is very similar to that found by Lamplugh in the Isle of Man, and is probably "the *remanié* deposit of the ice-sheet modified by sub-aerial agencies." At the time of the final disappearance of the ice, torrential waters must have overflown parts of the surface, and the rubble is probably to be attributed in part to the action of these waters. Morainic material would become mixed up with rock débris, formed by ordinary weathering processes, and the whole mass would be re-arranged, and in places sifted by the waters.

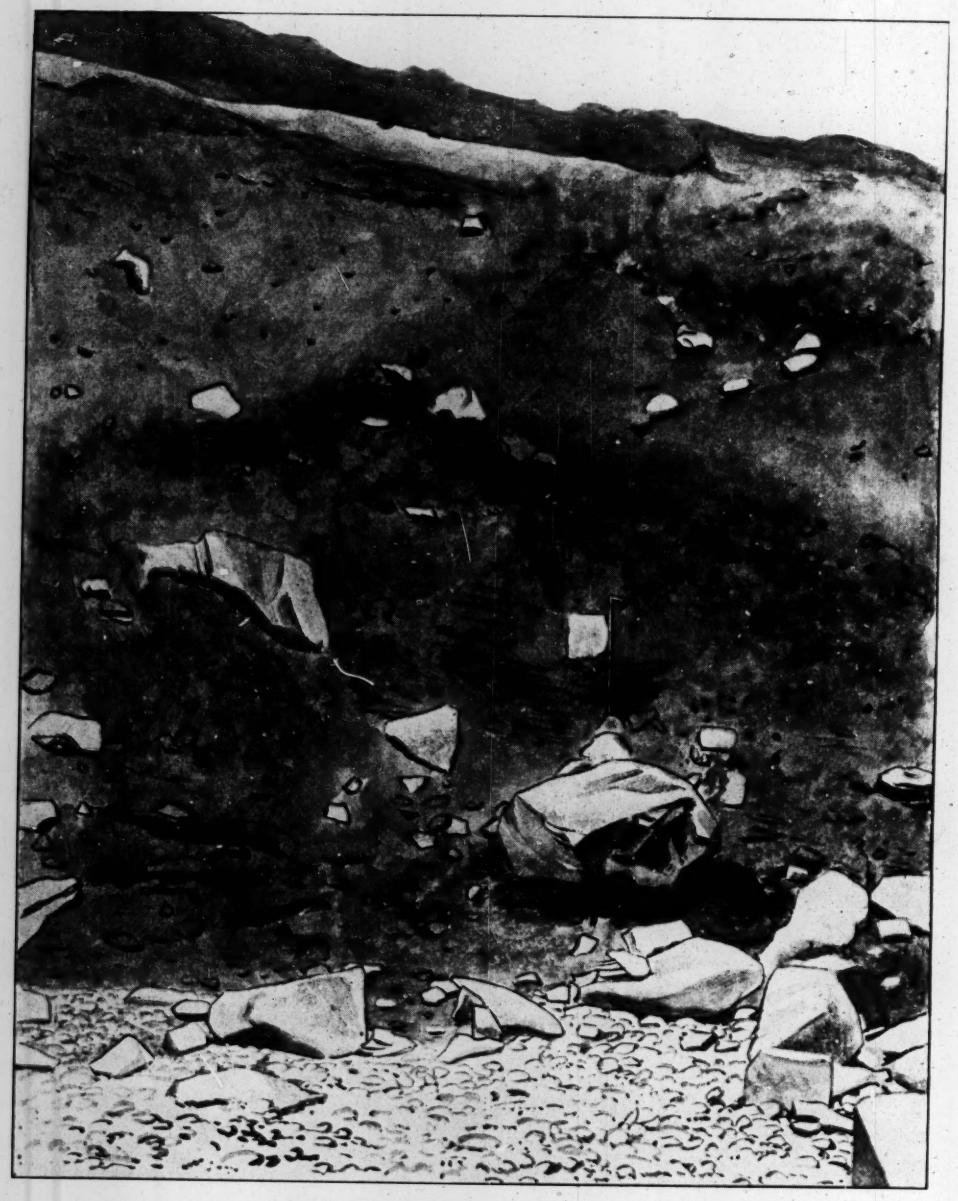

Fig. 4.—Section of Drift seen at northern end of Whitesand Bay.

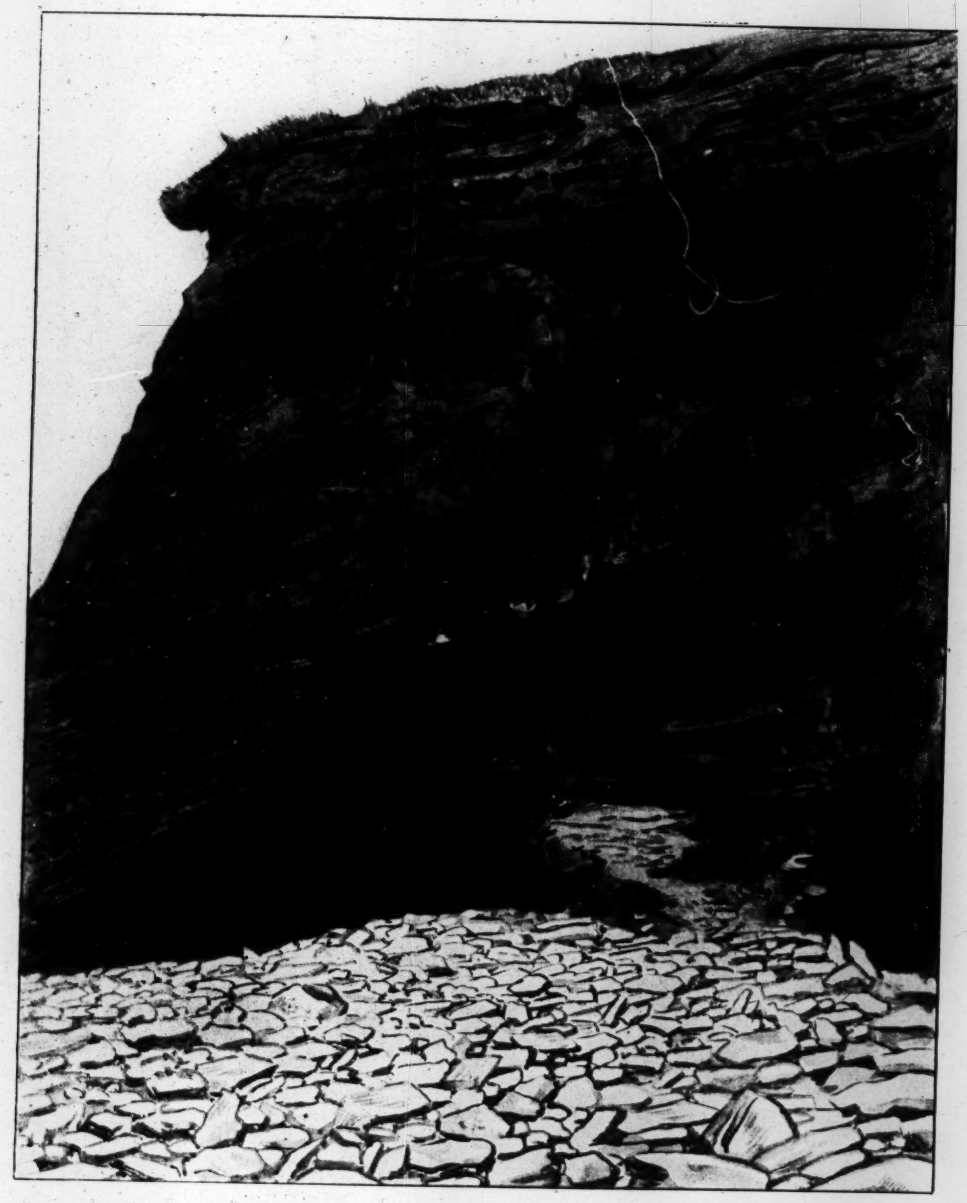

Fig. 5.—Section of Drift seen towards the southern end of Whitesand Bay.

BRITISH MUSEUM
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V.—Spectroscopic Observations of the Rotation of the Sun.

By Dr J. Halm, Assistant Astronomer at the Royal Observatory, and Lecturer in Astronomy at the University, Edinburgh. *Communicated by* The Astronomer Royal for Scotland.

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The causes of the peculiarities of the solar rotation exhibited by the superficial layers of the sun's body must still be considered unknown, notwithstanding recent interesting attempts at explanation. This is no doubt partly due to the difficulties of the hydrodynamical problem placed before the mathematician in an investigation of such complexity as the movements of particles in a rotating fluid subject to energetic convection. But it must also be conceded that the observational data available for a basis of mathematical, or even speculative, research are still so scanty, that for this reason alone we may perhaps not feel surprised at the failure so far of theoretical attempts.

The first empirical demonstration of the peculiar law which apparently governs the rotation of our luminary was given by Carrington, whose important investigations were taken up and extended by Spoerer. Their observations brought to light the main character of the peculiarity of solar rotation, viz., the decrease of the angular velocity from the equator towards the poles. An objection has, however, been raised against the value of their results with regard to the general rotation of the sun's photosphere, on the ground that they were obtained from the observed movements of solar spots. It was justly urged that spots have "proper motions" which preclude their adoption as points of reference. Besides, we must remember that these spots were visible only within an equatorial zone of about $\pm 50^\circ$ or 40° latitude, and that therefore the polar regions remained inaccessible by this method. This difficulty and limitation was overcome by the ingenious application of the spectroscope to the problem, which we owe to Professor Dunér. His conclusions are based on the displacements shown by the Fraunhofer lines at the solar limb, where the gases producing these absorptions are carried by the rotation either towards us or from us. His results are, it is true, not directly comparable to those derived from the movements of the spots, because both refer most probably to different levels, and therefore perhaps to different conditions of motion; but the great advantage of the spectroscopic method seems to me to lie in the fact that we always measure at the same level, wherever this level may be—a point on which we are by no means certain in the case of the spots. Besides, we are independent of the uncontrollable vicissitudes of proper motions, and we are able to extend the investigations from the equator to the immediate vicinity of the poles. The results obtained by Professor Dunér may be summarised by the statement that the retardation of the angular velocity discovered by Carrington and Spoerer was found to be also shared by the photospheric layer emitting the Fraunhofer lines, and that the amount of this

"lag" appeared to increase continually towards the poles. So far, then, as the general character of this peculiarity goes, the question appears to be empirically settled. But there remains still another, and, as I think, not less important question. Are we allowed to suppose that the surface rotation of the sun remains unaffected by the periodic changes of solar activity? Judging the question from a purely logical point of view, we are almost bound to answer it in the negative. It seems to me difficult to imagine that such violent disturbances of the normal conditions of convection as we perceive in solar eruptions and spots, and the consequent displacements of matter in the solar globe, should have no influence on the distribution of the rotational velocities at the surface. A careful study of the behaviour of the solar rotation during a cycle of activity may probably teach us far more about the causes and the seat of these solar

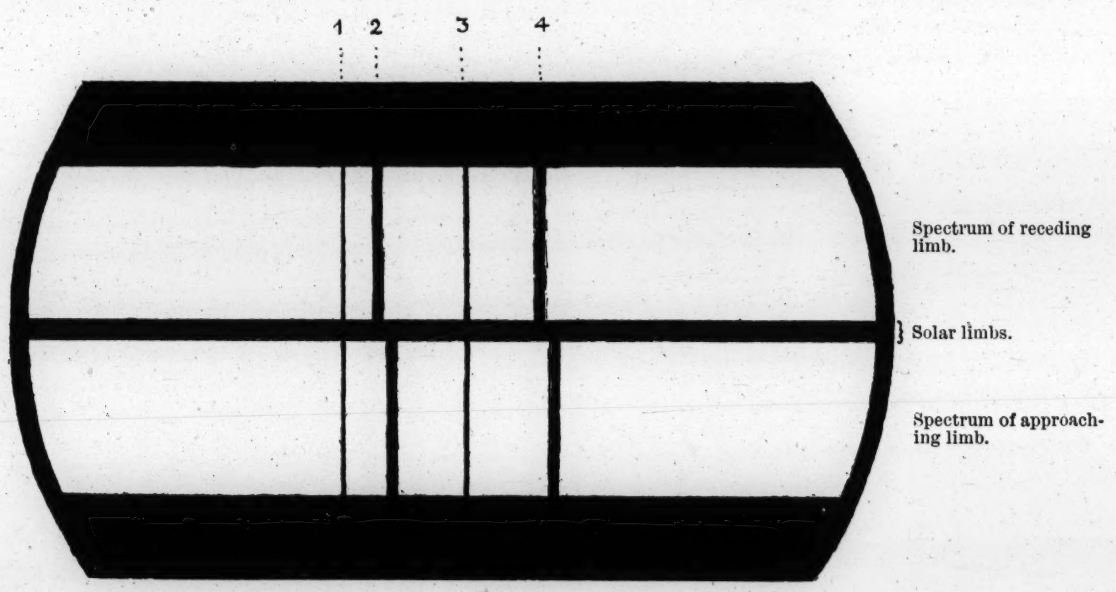


FIG. 1.

Group of lines as seen in the viewing telescope. (1 and 3 are telluric, 2 and 4 solar lines.)

disturbances than the whole array of statistical facts regarding the periodic displays of dynamical phenomena at the surface which are now in our possession. Professor DUNÉR's observations, covering a period of three years, during which next to no change took place in the activity of the sun, cannot give an answer to this question. I therefore thought it a promising venture, the success of which seemed to me in some way guaranteed by the great accuracy and consistency of DUNÉR's results, to extend these observations over a time of more pronounced changes of solar activity. The following contains a description of the results so far obtained, and of the instrument employed in my investigations.

The observations were begun in August 1901, and so far carried on to the end of 1903. Although this interval of time is not greater than that covered by DUNÉR's work, there is this essential difference, that the year 1903 was characterised by an abrupt and violent increase of solar disturbances after a pronounced and persistent calm during 1901–2. This fact, as we shall see, has an important bearing on the results, which

seem to me so remarkable and unexpected, that I trust this early publication may induce other observatories to take part in the work, at this opportune time of the beginning of what seems to be an energetic sunspot cycle.

As I intended to make the following observations directly comparable to those of Professor DUNÉR, I have used the same group of lines. The wave-lengths of these are given on page 55 of his treatise, "Recherches sur la rotation du Soleil."* The measurements therefore refer to the same photospheric level. Fig. 1 represents this group of lines as seen in the viewing telescope. The displacements of the solar lines shown in the figure correspond approximately to the shift at the solar equator.

With regard to the instrument, some deviations from DUNÉR'S arrangement have been suggested by the apparatus at my disposal. Since they increase the stability of the instrumental plant, these alterations may be considered as essential improvements. The most important of them was attained by the use of a siderostat. DUNÉR'S spectroscope was mounted on the great refractor of the Lund Observatory. The extraordinary dimensions of the apparatus gave rise to flexures which were bound to lessen the accuracy of the observations. The influence of such flexures is, of course, avoided if the whole spectral apparatus, including the front telescope, is mounted on fixed and insulated tables, and if the solar light is thrown upon the object-glass of the telescope by means of a siderostat. The considerable advantage of such an arrangement is indeed shown by the fact that the probable error of a single observation appears to be only half the probable error of one of Professor DUNÉR'S measurements. This increase of accuracy has to be ascribed chiefly to the greater stability of the instrument, and perhaps also to the easy and comfortable position of the observer during the observations.

A second essential departure from the design of the apparatus used by Professor DUNÉR is to be found in the arrangement by which the focal images of the two opposite limbs of the sun are thrown upon the slit of the spectroscope. DUNÉR employed a system of right-angled prisms arranged according to a device previously suggested by LANGLEY. By successive inner reflections from the hypotenuse surfaces of these prisms the light of the solar limbs can be thrown upon neighbouring points near the centre of the slit, which lies in the optical axis of the telescope. In a much simpler way, however, this same purpose can be attained by using as a front telescope a heliometer of sufficient optical power. By separating the halves of the object-glass, we can at once bring opposite points of the solar limb into contact, and these images may be thrown upon the centre of the slit without any further auxiliary apparatus. Besides, by altering the position angle of the heliometer, all the opposite points of the solar disc can be successively brought into contact. Thus we are enabled to determine the rotational velocity for any desired heliographic latitude simply by turning the heliometer into a position which corresponds to that latitude.

It seemed advantageous to throw the two solar images at each observation into such a position that the line joining their centres coincided with the slit. This could be

* *Nova Acta Regiae Societatis Scientiarum Upsaliensis*, 3rd series, vol. xiv. *fasc.* ii., 1891.

effected by bringing a large-sized, right-angled prism into the cone of light between the object-glass of the heliometer and the focus. By turning the prism, the solar images projected upon the slit-plate could be brought into the desired position, which is represented in the accompanying fig. 2. I may say that in order to attain sufficient accuracy of this adjustment, two lines parallel to the slit and at equal distances from it had been engraved upon the slit-plate. The solar images were brought into such a position that the two lines cut off equal segments. This arrangement was quite sufficient to guarantee the heliographic latitudes of the points measured within a fraction of a degree. Such small errors in the adjustment have, however, no appreciable effect on the observations, because they displace the measured point on the one limb exactly as much towards the equator as they displace the point measured on the other limb towards the pole. The total displacement therefore still agrees practically with that we should have obtained if no such error had been present. This consideration, however, does not apply to points exactly on the equator.

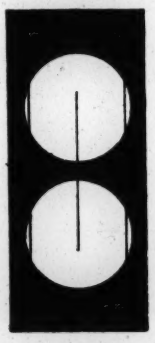


Fig. 2.

The heliometer employed in these observations has been kindly lent to this Observatory by the Hon. Lord M'Laren, Judge of the High Court of Session of Scotland. It is the instrument used by Sir David Gill in the Mauritius Expedition, 1874, and a full description of it may be found in vol. ii. of the Dunecht Observatory publications. Its aperture is 4·2 inches, and the focal length about 64 inches. The optical quality of the glass is exceedingly good. For the purpose of the observations the eye-end part of the heliometer had to be removed, so that the focus came to fall *outside* the main tube. The instrument is mounted upon a cast-iron table, adjustable in two horizontal directions at right angles, with the optical axis parallel to the line of the meridian, the object-glass towards the north. As a collimator, a telescope is used of 4 inches aperture and 50·5 inches focal length. The eye-end tube, which carries the slit, is adjustable by means of a focussing screw. The cylinder of rays, emanating from the object-glass of the collimator, is thrown upon a Rowland plane grating of speculum metal, 5 inches long, $3\frac{1}{2}$ inches broad, with 14,438 lines to the inch. The surface of the grating is perpendicular to the axis of the collimator. I have used the spectrum of the third order, which, upon the one side of this grating, is remarkably bright and well-defined. The disturbing effect of the overlapping violet spectrum of the higher order was found to be sufficiently elim-

inated by a plane glass of purple colour fixed in front of the eye-piece. The grating is mounted upon a stand, turning on a vertical axis in such a way that the axis of rotation coincides exactly with the plane of the grating. This stand carries a horizontal circle, divided from 5 to 5 minutes, which can be read by two diametrically opposite microscopes. By this arrangement a differential determination of wave-lengths is made possible.

The viewing telescope, which is horizontally mounted on another isolated stand, has an object-glass of 4·1 inches aperture and 60 inches focal length. A Cooke wire micrometer is used for measuring the distances between the lines. After various preliminary trials with crossed and parallel wires, the pointing on the spectral lines by means of sufficiently close parallel wires seemed to me to yield the best results. The observation consists in bringing, by turning the micrometer screw, the centre of the space between two close wires exactly upon the line to be measured, in the same way as the meridian circle observer sets the division of the circle between the wires of the microscope. Care has to be taken, however, that the setting is made exactly on the solar limb. I have found that this may be done without the least difficulty. The errors of the micrometer screw have been repeatedly determined, with the result that the progressive error may be considered as negligible, but that there are indications of a small periodic error. In order to eliminate the effect of the latter directly from the observations, I have invariably observed the lines with *two* pairs of wires which were at a distance of exactly $1\frac{1}{2}$ turns of the screw. As is well known, the periodic error is practically eliminated from the arithmetical mean of the observations made with two sets of wires thus arranged. The value of a revolution of the screw, expressed in wave-lengths, has been determined by measurements of the distances of neighbouring spectral lines, by a method which is fully described in Professor DUNÉR'S paper. Such measurements will be continued in the future, but there can be no doubt already that the provisional value here adopted must be sufficiently correct, and that therefore the computed values of the velocities may not have to be altered in the final discussion. The value of the screw is subject to small variations, chiefly owing to temperature fluctuations. These have, however, been taken into account in the reductions in the following way :—Each set of observations yields a very accurate determination of the normal distances between the four lines of the group measured. Since the wave-lengths of these lines are known, the value of a revolution of the screw expressed in wave-lengths may be found by dividing the measured distances of each pair of lines into the difference of their wave-lengths. Thus values of a turn of the screw are found for different points within the group, and from these the value to be applied in the reduction can be easily derived.

The computation of the position angles of the various points measured along the solar limb requires the knowledge of the rotation of the field of the siderostat at the time of the observation. Obviously, what is required is the diurnal rotation of the hour circle of the sun, and therefore also of the pole of the heavens, round the centre of the field. The data required for this computation have been supplied by CORNU in his

paper on the "Law of Diurnal Rotation of the Optical Field of the Siderostat and Heliostat" (*Astrophysical Journal*, vol. xi., 1900, pp. 148–162). It appears from his calculations that in our case, where a siderostat oriented in the meridian has been used, the angle **Y** at the centre of the field between the reflected image of the meridian and that of the hour circle of the sun is expressed by the formula,

$$\tan \tfrac{1}{2}\mathbf{Y} = \mathrm{K} \tan \tfrac{1}{2}t,$$

where t is the hour angle, reckoned positive towards the west, and

$$\mathrm{K} = \frac{\sin \frac{1}{2}(\phi - p)}{\sin \frac{1}{2}(\phi + p)},$$

ϕ being the latitude of the place and p being the polar distance of the sun. These equations determine the amount and the direction of the rotation. We are thereby enabled to fix the position of the north point of the solar disc as it appears in the field of the siderostat for any time of the day. The values **Y** can be tabulated with the arguments t and declination of the sun. The position circle of the heliometer is so oriented that the diameter 0°–180° falls in the plane of the meridian. The angular distance of the point under observation from the north point of the solar disc, that is, the position angle P of the point, is then found by the expression

$$\mathrm{P} = 90° - \Pi + \mathbf{Y},$$

where Π is the reading of the heliometer position circle. The value of P being found, the heliographic latitude of the point observed can be directly computed from the formulæ given by Professor Dunér in his treatise, where he also exhibits extensive tables, greatly facilitating these computations. I need not, therefore, enter more fully upon this part of the reductions.

I shall now briefly describe the way in which observations were performed with this instrument. The sunlight reflected from the siderostat mounted on the main platform of the Observatory is thrown into a meridional direction upon a window of 6 inches aperture in the north wall of the great optical room, and falls upon the object-glass of the heliometer placed immediately behind this window. The halves of the object-glass are screwed apart until the two solar images are nearly in contact, but still separated by a narrow space, which in the viewing telescope appears as a dark horizontal band between the spectra of the two limbs. The right-angled prism mentioned above is then turned and the heliometer adjusted so that the solar images fall upon the slit in the position indicated in fig. 2. After reading the position circle of the heliometer, the observer commences the measurements by pointing the first pair of the micrometer wires on the four lines of the upper of the two spectra seen in the viewing telescope. He begins, say, with the left-hand line of the group, and proceeds towards the right. He then measures with the same pair the lines of the lower of the two spectra in the direction from right to left. The observations are afterwards repeated with the second pair of wires; this time, however, they are begun on the lower spectrum from left to right, and finally the upper spectrum is measured from right to left. This arrangement, though it may

seem perhaps somewhat pedantic, has certainly the advantage of referring the mean of the observations on each limb to one and the same moment of time. Besides, approaching a line alternately in opposite directions must decrease the amount of the personal error of the observer in his judgment of the bisection. After the completion of these measurements, the halves of the heliometer object-glass are reversed and the time noted. The observations are then repeated in the same order as before. A complete set therefore consists of 32 pointings, 16 on each limb. When the set is finished, the heliometer is turned 10° in position angle, and the solar images are again brought into the position of fig. 2. The observation of a full set, including the necessary adjustments of the instrument, requires on the average from 10 to 15 minutes.

The observations of Professor DUNÉR refer to six equidistant points on the solar limb between the equator and 75° latitude. The selection of these fixed heliographic latitudes was perhaps necessitated by the arrangement of his instrument, which required a previous computation of the difference of declination between the north limb of the sun and the point to be observed. Since, however, a previous knowledge of the heliographic latitude is not required in our case, I have preferred to proceed from 10 to 10 degrees on the position circle of the heliometer, without considering at all the heliographic position of the points thrown upon the slit. The readings of the position circle, together with the time of the observation, are sufficient to evaluate afterwards the true heliographic latitude. In this way, a uniform and continuous distribution of observations over the whole quadrant, from the equator to the pole, may be secured, and we are, I think, in a better position to ascertain the character of the velocity-curve by this method than by the observations of six single points of the quadrant.

After this general description of the instrument and the method of observation, I shall now turn to the results. The measurements were commenced on 13th August 1901 and extended to 6th November 1903. During this time 564 determinations of the rotational velocity were made. The values obtained, expressed in kilometres per second, were divided into two groups; the first group comprising the measurements made during 1901–2, the second those of the year 1903. The reason for this division is that the period 1901–2 was characterised by a low and protracted minimum of solar activity, while in the early part of 1903 the commencement of a new solar cycle was vigorously manifested by the appearance of large spots. It was therefore to be expected that, if indeed solar activity has an influence on the rotation of the sun, such a division into groups as I have made would show this influence more clearly than any other. In each group the individual values were arranged according to their heliographic latitudes, and from the materials thus collected normal values were formed by adequate combinations of single values into arithmetic means. The figures thus obtained are exhibited in the following table.

TABLE I.

1901–2.			1903.		
Heliographic Latitude.	Linear Velocity per sec.	No. of Obs.	Heliographic Latitude.	Linear Velocity per sec.	No. of Obs.
	km.			km.	
1·6°	1·908	33	3·7°	1·898	16
5·2	1·894	30	9·1	1·883	15
8·3	1·871	30	15·5	1·831	16
12·8	1·802	30	21·1	1·753	15
18·0	1·720	30	27·7	1·631	16
23·8	1·594	30	34·0	1·512	16
30·6	1·488	30	40·3	1·365	15
38·7	1·265	30	47·1	1·201	16
47·3	1·061	30	54·7	1·014	16
55·8	0·840	30	62·3	0·797	15
65·1	0·560	30	69·2	0·564	16
75·5	0·307	30	76·0	0·408	15
			83·0	0·187	14

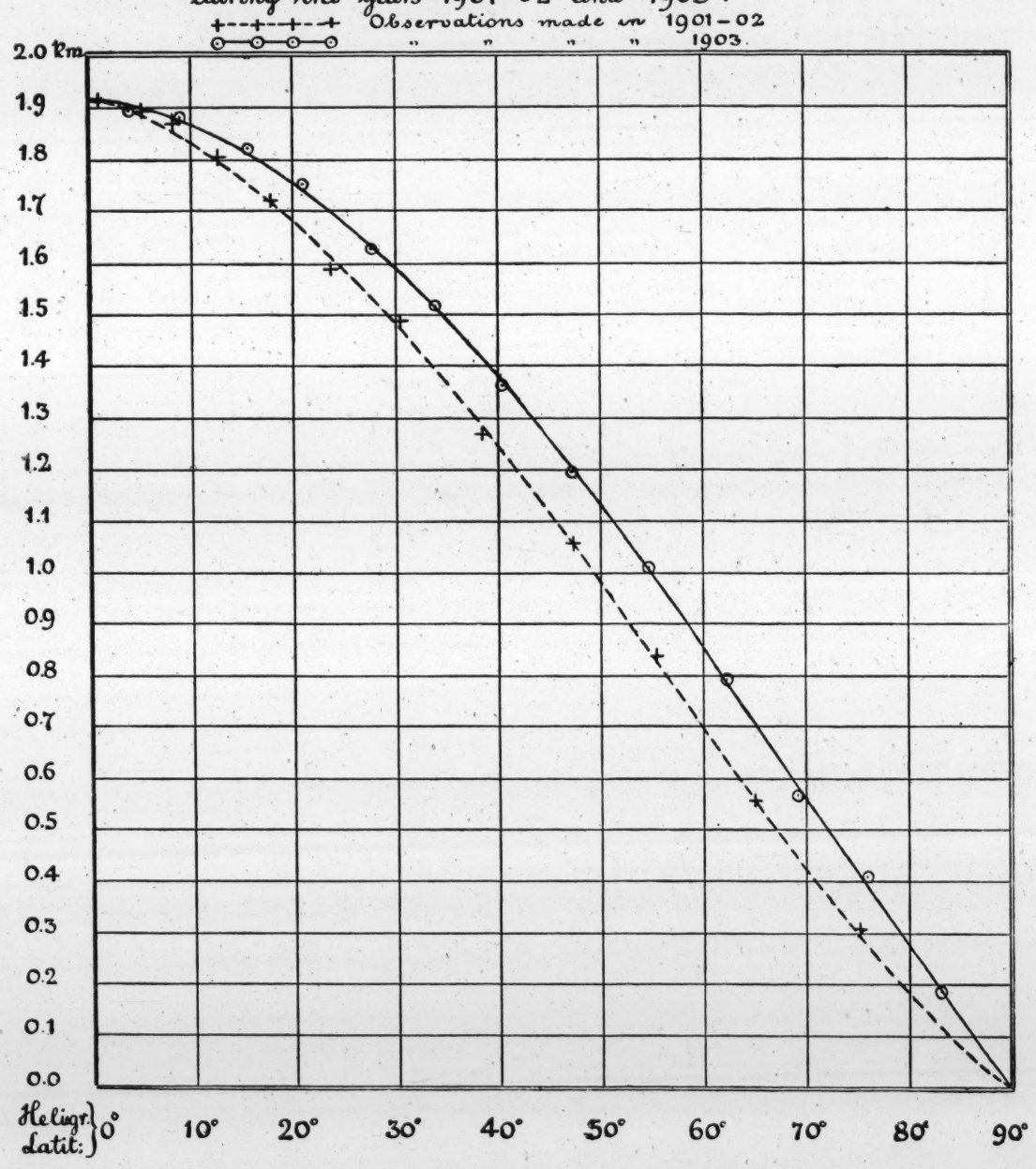
The values of the linear velocities were now plotted upon squared paper as ordinates, with the heliographic latitudes as abscissæ, and in each group a curve was drawn representing, as closely as possible, the observed ordinates. In this graphical form the results are represented in the accompanying fig. 3. It is seen at first glance that the two groups differ materially from each other. They agree fairly well only at the equator, but in higher latitudes the observations of 1903 show considerably larger values than those of the two preceding years. There is not a single exception to this rule. The evidence of these curves leaves therefore no doubt as to the reality of this systematic difference, which is also apparent from a consideration of the probable error. An investigation of the observations of 1903 shows the probable error of a single observation to be $\pm 0{\cdot}070$ km., a value which appears to be very nearly constant for all heliographic latitudes. From this we find the probable error of a mean value for the first group $\pm 0{\cdot}013$ km., and for the second group $\pm 0{\cdot}018$ km., while the observed systematic differences between the groups attain the considerable value 0·16 km. in middle latitudes.

Professor DUNÉR has also computed the probable error of his observations. I find from his figures that of a single observation to be $\pm 0{\cdot}138$ km., also nearly constant for all latitudes. Considering that his single observations comprise twelve to twenty-four pointings of each line, as compared with only eight in my observations, we may conclude that the latter show at least double the accuracy of those of Professor DUNÉR. As already pointed out, this favourable result is in my opinion only due to the greater stability and the more comfortable management of the instrumental plant.

The influence of a possible error in the position angles shown by the heliometer has

also been closely investigated. The careful mounting and orientation of the instruments, it is true, did not warrant the assumption that such an error was to be seriously apprehended. Besides, the arrangement of the observations was such that an error of this kind, even though it existed, should practically disappear from the mean values. This

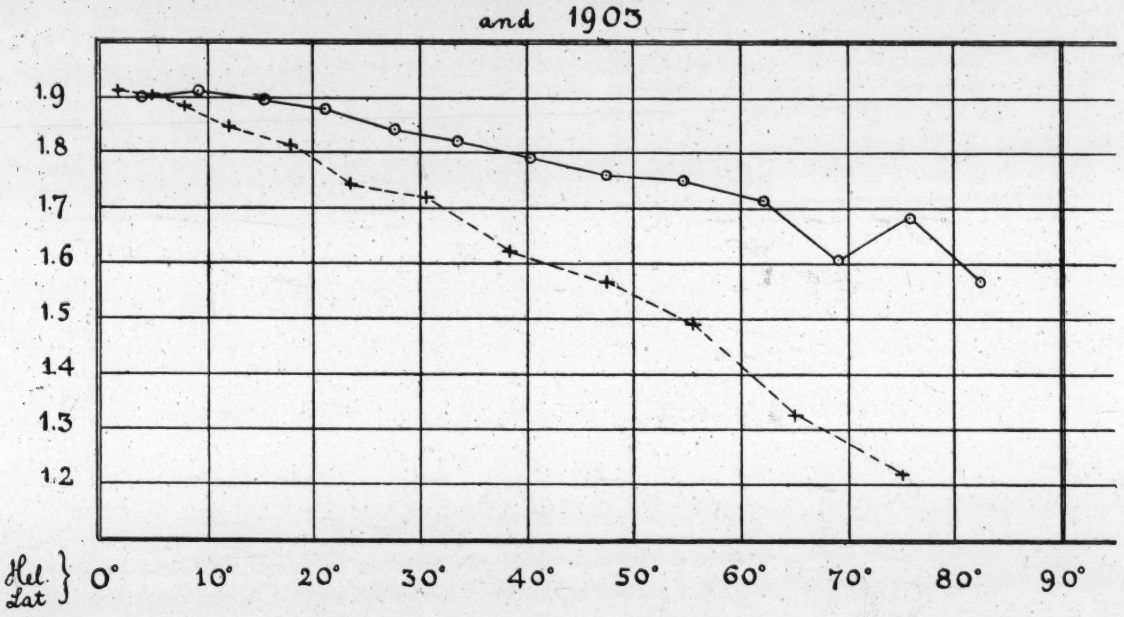
Fig. 3. Curves showing the linear rotational velocities observed during the years 1901–02 and 1903.



may be seen from the following consideration. Let the error of the position angle be Δp, so that a point on the solar limb whose computed distance from the equator, measured in the direction of the position angle, is Π, is actually at a heliographic latitude $\Pi + \Delta p$. If, now, we pass to the opposite side of the pole to a point whose computed distance from the equator, again measured in the direction of the position angle, is $180° - \Pi$, such a point would actually be in the heliographic latitude $\Pi - \Delta p$. Hence

our first point would be nearer the pole and the second nearer the equator than is shown by computation, and the arithmetic mean of both must therefore agree with the actual latitude. If, then, we arrange, as has indeed been done in these observations, the measurements so that equal numbers of points are observed on opposite sides of the poles, the mean values of the heliographic latitudes are practically uninfluenced by a systematic error of the position angle. The question, however, whether such an error actually exists, is readily answered by the observations. We have only to separate the

Fig. 4 showing the values $v \cdot \sec \beta$ in the two periods 1901-2 and 1903



observations into two groups in such a manner that the heliographic latitudes *increase* with the position angle in one group, and *decrease* in the other. The error would thus show itself by the fact that the same values of the velocity would not correspond to exactly the same latitudes in the two groups; or, to put it in other words, that corresponding latitudes should show slightly different values of the velocities. I have made such investigation for the whole time from 1901 to 1903, with the following result:—

TABLE II.

Heliographic Latitude.	Group I. km.	Group II. km.
82·5°	0·21	0·22
75·0	0·34	0·41
65·0	0·65	0·64
55·0	0·92	0·91
45·0	1·18	1·17
35·0	1·43	1·42
25·0	1·64	1·63
15·0	1·77	1·79
5·0	1·89	1·89

With the exception of latitude 75°, the differences between the two groups are far within the limits of their probable errors. This shows, as was anticipated, that practically no error of the position angle was present.

We have now to deal with the question, How are these remarkable differences between the two groups to be interpreted? Is their cause indeed to be sought for in the sun itself? We are, of course, not yet in a position to attempt an answer to this question, but there is a probability of the solar origin of this phenomenon, for which I should like to give some reasons. First, we have to take into consideration that the observations have been made throughout by the same observer, with the same entirely unaltered instrument, and at the same place,—that means, under the same atmospheric conditions. While it must be granted that deficiencies in the instrument and in the method of observation, or the personal perception of the line-displacements, which may be different for different observers, but most of all atmospheric conditions, which shall be discussed later—that all these circumstances may vitiate the results, there seems, however, to be no reason for the assumption that these influences should have altered so decidedly from one year to the next.

Secondly, that a real change of the rotation must have taken place from the one group to the other seems to be indicated by a significant peculiarity of the *angular* velocities which have been derived from the linear velocities v contained in Table I. By multiplying the values of v by sec. β, β being the heliographic latitude, we obtain numbers which obviously must be proportional to the angular velocities. This computation having been made, the values were plotted down in fig. 4, again for both groups separately. At first glance we recognise the retardation of the angular velocity from the equator towards the poles, but it will be remarked that the amount of this retardation during the second period is considerably smaller than during the first. The most significant fact, however, appears to be this: if in both groups the values of the curve for corresponding latitudes are subtracted from the equatorial velocity, these differences can be made to agree perfectly if the values for 1901–2 are multiplied by the factor 0·4. This result seemed to me so remarkable that I decided to test its correctness by a special investigation, including also the results of Professor DUNÉR's observations.

First of all I endeavoured to find an empirical formula which should represent the angular velocities in every group in a satisfactory manner. After various attempts at representing these velocities in the usual way by the sine and cosine functions of the heliographic latitude, the method had to be abandoned, as my observations could be represented only by extremely complicated expressions of such a form. Accidentally, however, I arrived at a formula very different from those hitherto used, which satisfies the observations in all three groups with a high degree of accuracy, and which has the further advantage of being extremely simple. This formula can be expressed in the following way—

$$\xi = a - bc^{\beta},$$

where ξ is the angular velocity in latitude β, and a, b and c are constants. When

computing the constants of this entirely empirical formula, I found that c had the same value for all the three groups, but that a and b showed considerable differences. The value of c was found to be 1·01447 if β is expressed in degrees, and 2·2784 if expressed in units of the sun's radius. The constants a and b in each of the three groups are given in the following table, which also contains the comparison between the observed values of v and those computed by means of the preceding formula.

TABLE III.

GROUP	a	b
1887–9 (Dunér)	2·349	0·354
1901–2 (Halm)	2·292	0·370
1903 do.	2·066	0·148

GROUP 1887–9.

β	v (comp.)	v (observed)	obs. – comp.
0·4°	1·99 km.	1·98 km.	– 0·01
15·0	1·85	1·85	0·00
30·0	1·56	1·58	+ 0·02
45·0	1·19	1·19	0·00
60·0	0·76	0·74	– 0·02
74·8	0·34	0·34	0·00

GROUP 1901–2.

β	v (comp.)	v (observed)	obs. – comp.
1·6°	1·908	1·908	0·000
5·2	1·881	1·894	+ 0·013
8·3	1·851	1·871	+ 0·020
12·8	1·797	1·802	+ 0·005
18·0	1·720	1·720	0·000
23·8	1·617	1·594	– 0·023
30·6	1·474	1·488	+ 0·014
38·7	1·281	1·265	– 0·016
47·3	1·055	1·061	+ 0·006
55·8	0·822	0·840	+ 0·018
65·1	0·566	0·560	– 0·006
75·5	0·299	0·307	+ 0·008

GROUP 1903.

β	v (comp.)	v (observed)	obs. – comp.
3·7°	1·906	1·898	– 0·008
9·1	1·874	1·883	+ 0·009
15·5	1·813	1·831	+ 0·018
21·1	1·741	1·753	+ 0·012
27·7	1·633	1·631	– 0·002
34·0	1·513	1·512	– 0·001
40·3	1·374	1·365	– 0·009
47·1	1·209	1·207	– 0·008
54·7	1·008	1·014	+ 0·006
62·3	0·792	0·797	+ 0·005
69·2	0·592	0·564	– 0·028
76·0	0·393	0·408	+ 0·015
83·0	0·192	0·187	– 0·005

The probable value of a difference "obs. – comp." is in all the groups approximately the same, viz., ±0·010 km., whereas the probable errors of an observed single value of v are respectively ±0·013, 0·013, 0·018. The formula therefore represents the observations in all the groups with sufficient accuracy. The constancy of c is a somewhat significant feature. If it should be confirmed by future observations, it would almost certainly point to the conclusion that the formula is not purely empirical, but has some physical meaning. A decision on this question would be highly important, for if the rotation of the surface layers be actually governed by a law which has its analytical expression in the formula here adopted, we may conclude that *surface* phenomena are the cause of these peculiarities. I shall not dwell, however, on this at present, but shall now draw attention to the result of these computations with regard to the constants a and b of our equation. Here we find a satisfactory agreement between the values for 1887–9 and 1901–2. These are also the two groups which have exactly the same position in the sun-spot cycle, both embracing a time of minimum activity. On the other hand, as already pointed out, the year 1903 was characterised by vigorous displays of spots and solar eruptions. Simultaneously with this activity we notice an enormous change in the values of the constants of our equation. The "retardation" of the higher latitudes appears now to be reduced to less than half its former amount. This is a novel and, I am sure, unexpected result. So far, it is true, it can only be said to represent a coincidence in time, but I trust the discussion has made it sufficiently clear that the discrepancies observed, since they can in no way be ascribed to observational errors, must, in all probability, be interpreted as a new and very peculiar feature of the still mysterious mechanism of the sun. Care has been taken throughout not only to investigate all possible errors which may arise from deficiencies of the instrument, and to eliminate such errors from the reduced values of the observations, but the measurements were also arranged in such a way that the possible remainders of such errors which could not be elicited in the reductions should have a minimum effect on the results.

The possible bearing of an investigation of this kind on important questions of solar physics makes it very desirable that the observations should not only be continued at this Observatory, but that other observatories should also take part in them. Next to the question of possible alterations in the amount of solar heat, an answer to which may now be expected from the ingenious researches of Professor LANGLEY and his staff, the problem of solar rotation should command the greatest attention from the part of solar physicists, for no other seems to me so well adapted to give us information on the mechanism of the solar forces. That the periodic play of these forces should in some manner be conducive to changes of the distribution of the moments of rotation is a logical conclusion on which I trust astronomers are unanimous. It is well to remark in this connection that the idea has already received theoretical consideration in an interesting paper by Mr EMDEN.

The construction of the apparatus required for these observations is well within the

reach of our large astrophysical observatories, and there are several beautiful heliometers which might be resuscitated for this purpose from their present unprofitable state of repose. Nor is the present the only investigation which can be taken up with an instrument of the kind here described. The construction of the apparatus, combined with its high dispersive power, makes it possible to separate by a mere glance the telluric lines from the solar ones, and at the same time to determine by a single observation the wave-length of a line with an accuracy of about $\pm 0{\cdot}05$ tenth-metres. The instrument appears therefore to be particularly adapted for an investigation of the telluric spectrum in accordance with the method first suggested by CORNU.

As regards the accuracy of the determination of the rotational velocities, I am confident that, under more favourable atmospheric conditions, the annual output of observations can be considerably increased, and thereby the probable error of the annual means correspondingly lessened. It does not seem to me impossible that in this way changes of only one to two hundredths of a kilometre could be traced with certainty.

I must not conclude this paper, however, without drawing attention to an inconvenience encountered in this spectroscopic determination of the rotational velocity. With a hazy atmosphere, a not inconsiderable quantity of scattered daylight, reflected from the particles of the aqueous vapour contained in the air, is thrown upon the slit of the spectroscope. As a certain percentage of this light will have emanated from the interior of the sun's disc, the solar lines of this "day" spectrum appear less displaced than the lines of the two solar limbs. By a superposition of the two spectra, the intensity of the lines will therefore be lessened on the one side and increased on the other, the effect always being to bring the lines nearer to their normal position. The velocities obtained under such conditions must therefore be too small. I have often had occasion to observe this phenomenon when, after a measurement made under favourable atmospheric conditions, the sky was suddenly overspread with a veil of haze, a meteorological feature not at all infrequent in the climate of Edinburgh. When the observation was then repeated, the measured displacement was invariably found less than that obtained before. Observations under misty conditions of the sky were therefore carefully avoided. A reliable scale for the transparency of the atmosphere was supplied by the optical appearance of the dark band separating the spectra of the two limbs in the viewing telescope. Whenever this space became so bright as to show traces of the absorption lines of the solar spectrum, the observation was broken off. All the measurements used in this discussion were made at moments when the band mentioned was uniformly dark, so that the spectrum of the diffused daylight cannot have seriously influenced the measured displacements.

Table IV.

Single Values of Linear Rotational Velocities of the Sun observed during the Years 1901–1903.

Lat.	Vel.
1901.	
Aug. 13.	
24·2	1·41
38·1	1·23
50·4	0·84
64·1	0·55
77·0	0·22
79·2	0·12
64·6	0·71
48·5	0·97
32·9	1·17
17·0	1·49
Aug. 15.	
23·0	1·52
Aug. 16.	
8·5	2·03
5·6	1·77
19·8	1·76
33·3	1·45
47·6	1·17
60·7	0·67
74·0	0·34
Aug. 19.	
11·5	1·70
2·5	1·85
16·7	1·60
30·8	1·23
44·5	1·20
58·3	0·57
71·8	0·16
77·9	0·33
65·0	0·51
51·0	1·08
37·1	1·28
23·6	1·43
9·4	1·48
4·5	1·75
Aug. 20.	
1·0	2·03
2·2	1·80
11·4	1·59
25·2	1·39
39·0	1·08
52·8	0·84
66·5	0·32
4·5	1·93
5·5	1·80
21·1	1·52
Aug. 21.	
2·3	1·79
3·4	1·69
5·5	1·75
14·9	1·65
23·7	1·58
2·3	1·75
31·4	1·31
40·4	1·13
49·3	0·83
58·4	0·66
67·2	0·35
75·9	0·32
5·5	1·77
10·5	1·83
Aug. 22.	
0·8	2·07
8·2	1·87
17·0	1·84
25·5	1·83
34·3	1·46
43·2	1·19
52·2	1·03
61·1	0·75
69·7	0·42
5·4	2·06
6·6	2·11
Aug. 24.	
3·7	2·06
5·0	2·11
3·4.	1·96
11·1	1·82
20·5	1·67
29·1	1·54
34·9	1·20
44·2	1·08
Aug. 25.	
78·6	0·11
68·8	0·52
58·1	0·89
47·4	1·19
36·7	1·41
25·8	1·64
14·6	1·87
3·8	1·98
7·2	2·05
7·9	2·04
26·7	1·84
17·8	1·86
Aug. 27.	
6·6	2·00
8·1	1·85
0·5	2·05
16·9	2·09
Aug. 28.	
10·0	1·89
18·6	1·92
27·4	1·83
36·3	1·59
43·5	1·33
52·4	1·07
51·0	0·72
1·1	1·90
Aug. 30.	
19·3	1·70
2·5	1·88
8·3	1·89
Aug. 31.	
1·9	2·17
1·1	1·95
Sept. 1.	
6·5	1·90
19·3	1·97
10·3	2·03
1·8	2·13
7·0	2·09
15·9	1·98
Sept. 2.	
50·6	1·03
59·2	0·76
68·0	0·32
2·2	1·94
8·1	1·88
1·0	2·17
9·5	1·92
18·3	1·83
27·0	1·59
Sept. 4.	
28·8	1·53
17·4	1·71
5·7	2·03
5·9	2·21
Sept. 5.	
0·8	2 02
Sept. 7.	
7·7	1·85
41·1	1·34
48·9	1·19
Sept. 10.	
62·9	0·77
52·2	1·04
41·4	1·20
30·8	1·59
19·7	1·87
8·7	1·92
Sept. 11.	
15·9	1·99
24·4	1·77
33·3	1·67
42·4	1·37
51·3	1·00
60·0	0·87
68·6	0·56
Oct. 26.	
6 9	1·98
1 8	2·11
10·3	1·84
18·9	1·49
Nov. 5.	
39·6	1·30
48·2	0·99
56·3	0·86
64·6	0·63
73·2	0·39
1·2	2·00
9·0	1·73
16·8	1·67
1902.	
Feb. 12.	
13·0	1·72
5·3	1·89
2·7	2·01
10·5	1·73
18·9	1·68
Feb. 13.	
7·9	2·14
Feb. 15.	
3·3	1·85
14·6	1·79
26·1	1·46
37·2	1·07
48·9	0·98
60·7	0·77
71·4	0·42
Feb. 17.	
67·8	0·31
55·0	0·63
Feb. 18.	
35·7	1·39
27·1	1·35
18·9	1·46
10·3	1·71
1·8	1·67
6·8	1·74
14·8	1·75
23 0	1·67
31·5	1 52
39·8	1·14
Mar. 7.	
46·0	0·96
54·5	0·78
63·0	0·51
71·4	0·35
Mar. 15.	
76·2	0·23
65·9	0·55
54·7	0·88
43·3	1·25
32·3	1·43
13·9	1·90
2·9	2·05
Mar. 16.	
21·0	1·64
9·2	1·90
1·8	1·89
26·7	1·47
35·5	1·22
43·9	1·04
51·5	0·84
Mar. 20.	
9·5	1·77
6·8	1·84
18·1	1·76
Mar. 21.	
17·6	1·66
Mar. 22.	
6·5	1·85
0·2	1·77
7·6	1·90
16·0	1·83
24·3	1·66
32·1	1·54
40·7	1·32
49·2	0·99
Mar. 24.	
62·0	0·64
Mar. 25.	
70·6	0·44
59·8	0·75
48·8	1·10
37·8	1·28
26·4	1·61
15·5	1·90
4·2	1·96
7·3	1·85
Mar. 29.	
21·1	1·58
Apr. 1.	
6·7	1·89
Apr. 2.	
0·4	1·77
8·4	1·89
16·5	1·75
25·6	1·50
31·4	1·48
40·1	1·14
Apr. 7.	
50·1	0·98
Apr. 10.	
52·6	1·04
61·5	0·86
70·6	0·67
78·5	0·16
Apr. 16.	
79·5	0·24
71·1	0·46
75·9	0·13
65·5	0·61
54·9	0·85
44·3	1·14
33·6	1·46
21·8	1·55
10·8	1·63
0·4	1·83
Apr. 18.	
5·1	1·85
16·0	1·80
Apr. 25.	
18·0	1·61
27·3	1·73
Apr. 28.	
27·9	1·72
17·1	1·66
6·4	1·84
5·0	1·74
16·1	1·47
27·5	1·51
Apr. 30.	
18·1	1·78
May 29.	
0·6	1·92
14·9	1·87
25·5	1·41
36·5	1·28
47·0	1·07
57·6	0·82
68·7	0·64
June 27.	
52·7	0·69
63·0	0·43
73·8	0·22
44·2	1·02
34·7	1·30
25·2	1·53
15·5	1·76
June 30.	
10·4	1·79
1·0	1·88
8·8	1·75
18·0	1·59
27·3	1·48
36·8	1·13
46·2	0·93
55·8	0·63
July 5.	
62·3	0·70
51·8	0·98
41·5	1·23
31·1	1·41
20·6	1·66
8·0	1·73
0·3	1·85
10·8	2·00
21·5	1·76
32·3	1·39

TABLE IV.—*continued.*

Lat.	Vel.
1902.	
July 5 *contd.*	
44·2	1·02
54·4	0·83
64·9	0·58
65·3	0·56
75·5	0·29
Aug. 21.	
49·8	1·00
Aug. 23.	
59·8	0·67
50·2	0·92
41·5	1·16
33·0	1·23
23·8	1·49
14·7	1·73
Aug. 25.	
5·9	1·71
2·5	1·54
10·2	1·75
17·7	1·66
Aug. 26.	
35·6	1·41
24·9	1·65
11·2	1·65
0·7	1·73
22·4	1·69
3·6	1·78
37·1	1·44
Aug. 27.	
6·4	1·72
Aug. 28.	
8·2	1·92
17·2	1·85
25·7	1·66
34·8	1·44
43·6	1·31
52·8	0·96
61·5	0·69
70·5	0·40
78·7	0·15
78·4	0·37
Aug. 29.	
74·4	0·41
Sept. 8.	
79·1	0·39
74·2	0·13
64·0	0·63
53·5	0·89
42·9	1·14
30·4	1·40
19·7	1·65
8·9	1·80
Sept. 9.	
18·6	1·72
8·0	1·86
Sept. 13.	
2·9	1·81
11·0	1·86
Sept. 17.	
12·2	1·90
1·4	2·01
9·4	1·85
Sept. 23.	
9·9	1·80
1·0	1·95
6·9	1·89
15·1	1·77
21·1	1·71
29·8	1·52
Oct. 4.	
36·7	1·50
45·5	1·17
54·5	1·08
62·4	0·71
70·8	0·33
Oct. 19.	
2·1	1·70
4·5	1·89
13·3	1·76
21·9	1·64
30·7	1·50
39·4	1·17
48·3	1·02
Oct. 30.	
62·1	0·57
70·4	0·29
78·3	0·13
79·0	0·48
Nov. 1.	
80·0	0·41
72·0	0·53
Nov. 4.	
59·7	0·84
51·0	1·05
42·1	1·32
33·4	1·54
25·7	1·64
21·2	1·61
Nov. 10.	
77·7	0·29
1903.	
Feb. 25.	
24·0	1·65
38·8	1·30
Feb. 27.	
35·4	1·52
17·2	1·87
9·4	1·94
Mar. 15.	
14·5	1·83
2·3	2·02
8·9	1·99
20·5	2·00
31·8	1·68
52·8	1·19
63·1	0·87
Apr. 17.	
7·2	1·85
18·3	1·78
29·3	1·65
Apr. 19.	
16·3	1·79
28·3	1·60
Apr. 24.	
9·9	2·13
20·8	2·01
31·6	1·70
41·9	1·34
53·0	1·03
64·0	0·74
74·3	0·42
83·8	0·16
81·7	0·24
71·5	0·53
59·7	0·81
48·6	1·03
May 23.	
3·9	1·84
7·1	1·86
17·8	1·80
28·5	1·65
38·7	1·29
49·0	1·17
59·7	0·77
70·9	0·53
81·3	0·26
74·8	0·48
64·3	0·75
53·7	1·04
43·1	1·35
32·5	1·46
May 26.	
46·6	1·25
36·3	1·41
25·9	1·69
15·0	1·77
3·9	1·91
6·8	1·83
17·3	1·93
27·9	1·74
38·4	1·54
48·9	1·20
59·6	0·99
69·9	0·63
May 27.	
64·8	0·89
75·3	0·56
86·3	0·19
83·1	0·14
72·9	0·33
61·5	0·79
51·0	1·02
40·5	1·35
28·8	1·61
18·4	1·68
7·7	1·84
2·8	1·91
13·1	1·93
May 28.	
4·4	1·93
15·1	1·86
25·9	1·66
36·1	1·46
47·3	1·26
57·6	0·95
68·1	0·53
May 29.	
72·8	0·49
83 3	0·24
85·9	−0·14
75·8	0·33
65·4	0·68
54·8	0·82
44·2	1·13
34·0	1·56
23·5	1·73
13·2	1·78
2·9	1·84
7·7	1·73
June 3.	
5·5	1 82
4·1	1·84
13·5	1·67
June 12.	
20·6	1·95
30·1	1·63
39·6	1·31
June 20.	
33·3	1·44
42·9	1·26
52·5	1·07
62·1	0·77
71·5	0·33
81·2	0·09
79·2	0·55
69·9	0·52
Aug. 14.	
41·4	1·46
31·9	1·41
23·0	1·69
13·5	1·88
4·4	1·98
5·1	1·90
14·1	1·96
23·4	1·81
32·3	1·67
Aug. 17.	
40·0	1·68
48·1	1·34
57·4	1·10
66·1	0·75
74·7	0·58
81·2	0·31
75·5	0·14
Aug. 18.	
69·2	0·39
60·5	0·65
51 2	0·92
42·3	1·13
32·9	1·47
23·5	1·56
24·5	1·68
Aug. 27.	
1·9	1·81
Sept. 14.	
70·9	0·56
78·8	0·09
Sept. 15.	
16·7	1·59
30·0	1·36
38·4	1·28
44·4	1·28
53·7	1·00
Sept. 18.	
73·5	0·73
64·0	0·92
54·9	1·13
46·0	1·17
37·6	1·57
28·4	1·73
19·3	1·72
10·0	1·93
0·2	1·96
Sept. 19.	
10·9	1·92
19·4	1·66
28·6	1·58
37·2	1·33
46·3	1·20
52·5	1·20
60·1	0·87
68·8	0·42
77·3	0·23
Sept. 27.	
66·9	0·64
Oct. 11.	
67·5	0·76
75·0	0·33
81·8	0·28
80·5	0·20
Oct. 13.	
66·0	0·78
78·7	0·32
83·6	0·16
79·8	0·46
70·0	0·55
59·1	0·71
47·2	1·20
35·9	1·45
25·1	1·67
Oct. 21.	
18·1	1·90
6·4	1·94
5·5	1·83
17·4	1·84
Oct. 23.	
11·2	1·94
3·2	1·85
Oct. 27.	
11·9	1·79
20·7	1·72
29·3	1·61
37·6	1·42
42·5	1·16
50·6	0·97
58·5	0·77
Oct. 31.	
67·2	0·68
56·3	0·94
44·3	1·28
33·4	1·49
21·3	1·51
9·8	1·82
Nov. 3.	
6·8	1·82
15·6	1·89
23·9	1·66
32·4	1·57
40·4	1·38
48·8	1·11
57·2	1·04
Nov. 4.	
3·1	1·99
12·0	1·85
20·5	1·82
27·1	1·60
Nov. 6.	
41·0	1·57
49·9	1·28
58·1	1·01
65·4	0·69
73·8	0·57
81·6	0·33
86·0	0·16

VI.—Theorems relating to a Generalisation of the Bessel-Function.

By the **Rev. F. H. Jackson**, H.M.S. "Irresistible." *Communicated by* Dr W. PEDDIE.

(MS. received February 17, 1904. Read March 21, 1904. Issued separately May 27, 1904.)

1.

In this paper, theorems which are extensions of the following, are discussed :—

$$J_0(x+y) = J_0(x)J_0(y) - 2J_1(x)J_1(y) + 2J_2(x)J_2(y) - \ldots\ldots\ldots \text{ ad inf.} \qquad (\alpha)$$

$$1 = \{J_0(x)\}^2 + 2\{J_1(x)\}^2 + 2\{J_2(x)\}^2 + \ldots\ldots\ldots\ldots \text{ ad inf.} \qquad (\beta)$$

$$J_n(x) = (-1)^m \left\{ J_{2m+n}(x) - \frac{2m(m+n)}{x} J_{2m+n-1}(x) + \frac{2^2 m(m-1)(m+n)(m+n-1)}{2!\,x^2} J_{2m+n-2}(x) - \ldots\ldots \right\} \qquad (\gamma)$$

$$J_{-n}(x) = (-1)^n J_n(x)$$

$$J_{n+\frac{1}{2}}(x) = (-1)^n \frac{(2x)^{n+\frac{1}{2}}}{\Pi^{\frac{1}{2}}} \frac{d^n}{d(x^2)^n} \left\{ \frac{\sin x}{x} \right\} \qquad (\delta)$$

We define $J_{[n]}(\lambda, x)$ as

$$\sum_{r=0}^{r=\infty} \frac{1}{[r]!\,[n+r]!\,(2)_r (2)_{n+r}} \lambda^{n+2r} x^{[n+2r]}$$

In this expression

$$[n+r]! \text{ is } \Gamma_p([n+r+1]) \quad \text{or} \quad \Pi_p([n+r])$$

$$[n] \text{ is } \frac{p^n - 1}{p-1}$$

$$(2)_{n+r} = [2]^{n+r} \frac{\Gamma_{p^2}([n+r+1])}{\Gamma_p([n+r+1])}$$

The function $\Pi_p([n])$ is defined in the previous paper (*Trans. Roy. Soc. Edin.*, vol. xli. part i.). If λ be changed to $i\lambda$ in $J_{[n]}$, the function will then be more strictly analogous to J_n.

In Weierstrassian form

$$\frac{1}{\Gamma_p([x])} = [x] e^{Px} \prod_{n=1}^{n=\infty} \left\{ \left(1 + p^{-x}\frac{[x]}{[n]}\right) e^{-\frac{x}{[n]}} \right\}$$

$$P = 1 + \frac{1}{[2]} + \frac{1}{[3]} + \ldots\ldots - \log\frac{p}{p-1}$$

$$\{2n\}! = (2)_n \Gamma_p([n+1])$$

being in the case of n positive and integral

$$\{2n\}! = [2][4][6] \ldots [2n]$$

analogous to

$$2 \cdot 4 \cdot 6 \ldots\ldots 2n = 2^n . n!$$

This notation enables us to write shortly

$$J_{[n]} = \sum \frac{(-1)^r \lambda^{n+2r}}{\{2n+2r\}!\,\{2r\}!} x^{[n+2r]}$$

2.

If we invert the base element p, we see that $[r]!$ is transformed into $p^{-r(r-1)/2}[r]!$ and that $(2)_r$ becomes transformed into $p^{-r(r-1)/2}(2)_r$. These transformations hold whether r be integral or not. Inverting the base p in the series $J_{[n]}(\lambda)$, we obtain

$$p^{n^2}\sum_{r=0}^{r=\infty}\frac{(-1)^r p^{2r(n+r)}}{[r]!\,[n+r]!\,(2)_r(2)_{n+r}}\left(\frac{\lambda}{p}\right)^{n+2r} \quad . \quad . \quad . \quad . \quad (1)$$

which we denote

$$p^{n^2}\mathfrak{J}_{[n]}\left(\frac{\lambda}{p}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Lommel has shown that

$$J_n(\lambda)J_m(\lambda) = \sum_{r=0}^{r=\infty}(-1)^r\frac{\binom{m+n+2r}{r}}{\Gamma(m+r+1)\Gamma(n+r+1)}\left(\frac{\lambda}{2}\right)^{m+n+2r} \quad . \quad . \quad . \quad (3)$$

The function $\mathfrak{J}_{[n]}$ was formed while seeking to extend the above theorem. The extension was found to be

$$J_{[n]}(\lambda)\mathfrak{J}_{[m]}(\lambda) = J_{[m]}(\lambda)\mathfrak{J}_{[n]}(\lambda) = \sum_{r=0}^{r=\infty}\frac{(-1)^r\Gamma_{p^2}([m+n+2r+1])}{\Gamma_{p^2}([m+n+r+1])\Gamma_{p^2}([m+r+1])\Gamma_{p^2}([n+r+1])\Gamma_{p^2}([r+1])}\left(\frac{\lambda}{[2]}\right)^{m+n+2r} \quad (4)$$

The relation between J and $\mathfrak{J}$ was surmised from the following simple but similar theorem.

$$E_p(\lambda) = 1 + \frac{\lambda}{[1]!} + \frac{\lambda^2}{[2]!} + \ldots\ldots$$

$$E_{\frac{1}{p}}(\lambda) = 1 + \frac{\lambda}{[1]!} + p\frac{\lambda^2}{[2]!} + \ldots\ldots$$

$$E_p(\lambda)E_{\frac{1}{p}}(\lambda) = 1 + \frac{2\lambda}{[1]!} + \frac{2(1+p)\lambda^2}{[2]!} + \ldots\ldots$$

which suggested that $\mathfrak{J}_{[n]}$ might be derived from $J_{[n]}$ by inverting the base p. As I have given the proof of (4) as an example of the use of generalised Gamma-functions in a paper communicated to the Royal Society, London, it will be sufficient to say here, that the theorem may be proved by showing, that the coefficients of the powers of λ on both sides of the equation are identical, being cases of the extension of Vandermonde's theorem (*Proc. Lond. Math. Soc.*, series 2, vol. i. p. 63).

In the notation of Art. (1) we may write the theorem

$$J_{[m]}(\lambda)\mathfrak{J}_{[n]}(\lambda) = \sum_{r=0}^{r=\infty}\frac{(-1)^r\{2m+2n+4r\}!}{\{2m+2n+2r\}!\{2m+2r\}!\{2n+2r\}!\{2r\}!}\lambda^{m+n+2r} \quad . \quad . \quad (5)$$

3.

Consider the series

$$J_{[0]}(\lambda)\mathfrak{J}_{[0]}(\lambda) + \frac{[4]}{[2]}J_{[1]}(\lambda)\mathfrak{J}_{[1]}(\lambda) + \ldots\ldots + p^{s.s-1}\frac{[4s]}{[2s]}J_{[s]}(\lambda)\mathfrak{J}_{[s]}(\lambda) + \ldots\ldots\ldots \quad (6)$$

By means of (5) we write this—

$$\left\{1-\frac{[4][2]}{[2][2][2][2]}\lambda^2+\ldots\ldots\ldots\ldots(-1)^r\frac{\{4r\}!}{\{2r\}!\{2r\}!\{2r\}!\{2r\}!}\lambda^{2r}+\ldots\ldots\ldots\right\}$$

$$+\frac{[4]}{[2]}\left\{\frac{[4][2]}{[4][2][2][2]}\lambda^2-\ldots\ldots\ldots\ldots+\frac{\{4r\}!}{\{2r+2\}!\{2r\}!\{2r\}!\{2r-2\}}\lambda^{2r}+\ldots\ldots\ldots\ldots\right\} \quad (7)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$p^{r(r-1)}\frac{[4r]}{[2r]}\left\{\frac{\{4r\}!}{\{4r\}!\{2r\}!\{2r\}!\{0\}!}\lambda^{2r}+\ldots\ldots\ldots\ldots\ldots\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

We see by inspection that the coefficient of λ^2 vanishes: the coefficient of λ^{2r} is the expression

$$(1-)^r\frac{\{4r\}!}{\{2r\}!\{2r\}!\{2r\}!\{2r\}!}\left\{1-\frac{[4]}{[2]}\frac{[2r]}{[2r+2]}+p^2\frac{[8]}{[4]}\frac{[2r][2r-2]}{[2r+2][2r+4]}-\ldots\ldots\right.$$

$$\left.+(-1)^rp^{r(r-1)}\frac{[4r]}{[2r]}\frac{[2r][2r-2]\ldots\ldots\ldots[4][2]}{[2r+2][2r+4]\ldots\ldots\ldots[4r]}\right\}\ldots\ldots\quad(8)$$

The series within the large brackets is easily summed term by term. The sum of the first two terms is $-p^2\frac{[2r-2]}{[2r+2]}$, which is a factor of the third term. The sum of $-p^2\frac{[2r-2]}{[2r+2]}$ and the third term is

$$p^6\frac{[2r-4][2r-2]}{[2r+2][2r+4]}$$

Continuing in this way, we obtain that the sum of the first r terms is

$$(-1)^{r-1}p^{r(r-1)}\frac{[2r-2][2r-4][2r-6]\ldots\ldots\ldots\ldots[4][2]}{[2r+2][2r+4][2r+6]\ldots\ldots\ldots\ldots[4r-2]}$$

which is equal to the last term, but is of opposite sign. The coefficient of λ^{2r} is zero, and only the constant unity is left. Therefore

$$1=J_{[0]}(\lambda)\mathfrak{J}_{[0]}(\lambda)+\frac{[4]}{[2]}J_{[1]}(\lambda)\mathfrak{J}_{[1]}(\lambda)+\ldots\ldots+p^{s,s-1}\frac{[4s]}{[2s]}J_{[s]}(\lambda)\mathfrak{J}_{[s]}(\lambda)+\ldots\ldots\quad(9)$$

which, if $p=1$, reduces to

$$1=\{J_0\}^2+2\{J_1\}^2+2\{J_2\}^2+\ldots\ldots\ldots\quad(10)$$

4.

Consider now the series

$$J_{[0]}(\lambda)\mathfrak{J}_{[0]}(\lambda)-\frac{[4]}{[2]}J_{[1]}(\lambda)\mathfrak{J}_{[0]}(\lambda)+\ldots\ldots\ldots+p^{r,r-1}\frac{[4r]}{[2r]}J_{[r]}(\lambda)\mathfrak{J}_{[r]}(\lambda)-\ldots\ldots\quad(11)$$

Referring to expression (8), we see that the coefficient of λ^{2r} is

$$(-1)^r\frac{\{4r\}!}{\{2r\}!\{2r\}!\{2r\}!\{2r\}!}\left\{1+\frac{[4]}{[2]}\frac{[2r]}{[2r+2]}+p^2\frac{[8]}{[4]}\frac{[2r][2r-2]}{[2r+2][2r+4]}+\ldots\ldots\ldots\ldots\right\}\quad(12)$$

The series within the large brackets, although simple in form, offers considerable difficulty in summation. The sum is

$$2(p^2+1)(p^4+1)\ldots.(p^{2r-2}+1)\cdot(p^2+1)(p^4+1)\ldots(p^{2r}+1)\frac{\{2r\}!\{2r\}!}{\{4r\}!}. \quad . \quad (13)$$

The sum in general for all values of r is

$$2\frac{[4]^{2r}}{[2]^{2r}}\cdot\frac{[2r]}{[4r]}\cdot\frac{\Gamma_{p^4}([r+1])\Gamma_{p^4}([r+1])}{\Gamma_{p^2}([2r+2])} \quad . \quad . \quad . \quad . \quad (14)$$

The reason that the simple series $(p=1)$

$$1+\frac{4}{2}\cdot\frac{2r}{2r+2}+\frac{8}{4}\cdot\frac{2r(2r-2)}{(2r+2)(2r+4)}+\ldots.$$

is easily summed as

$$2^{2r}\frac{r!\,r!}{2r!}$$

while the general series offers difficulty, is that the functions Γ_p are present, both to the base p^4 and the base p^2 in the series (12) and in expression (14).

HEINE has shown in his "*Kugelfunctionen*" that

$$\phi[a,b,c,p,x]=1+\frac{(1-a)(1-b)}{(1-p)(1-c)}x+\frac{(1-a)(1-ap)(1-b)(1-bp)}{(1-p)(1-p^2)(1-c)(1-cp)}x^2+\ldots\ldots$$

$$=\prod_{n=0}^{n=\infty}\frac{(1-bxp^n)(1-\frac{c}{b}p^n)}{(1-xp^n)(1-cp^n)}\ \phi\left[b,\frac{abx}{c},bx,p,\frac{c}{b}\right] \quad . \quad . \quad . \quad . \quad (15)$$

Consider now the series

$$1+\frac{[2]}{[1]}\frac{[r]}{[r+1]}+p\frac{[4]}{[2]}\frac{[r][r-1]}{[r+1][r+2]}+\ldots\ldots\ldots+p^{s(s-1)/2}\frac{[2s]}{[s]}\frac{[r][r-1]\ldots[r-s+1]}{[r+1][r+2]\ldots[r+s]}+\ldots. \quad (16)$$

Since

$$\frac{[2]}{[1]}=\frac{p^2-1}{p-1}=p+1$$

$$\frac{[4]}{[2]}=\frac{p^4-1}{p^2-1}=p^2+1$$

. . . .

. . . .

we write series (16) as the sum of two HEINE's series

$$S_1+p\frac{p^r-1}{p^{r+1}-1}S_2=\left\{1+\frac{p^r-1}{p^{r+1}-1}+p\frac{(p^r-1)(p^{r-1}-1)}{(p^{r+1}-1)(p^{r+2}-1)}+\ldots.\right\}$$

$$+p\frac{p^r-1}{p^{r+1}-1}\left\{1+\frac{p^{r-1}-1}{p^{r+2}-1}p^2+\ldots.+p^{s(s+3)/2}\frac{(p^{r-1}-1)\ldots(p^{r-s}-1)}{(p^{r+2}-1)\ldots(p^{r+s+1}-1)}+\ldots\right\} \quad . \quad (17)$$

These series we transform by means of (15).

First, for the series S_1 we put

$$a=p$$
$$b=p^{-r}$$
$$c=p^{r+1}$$
$$x=-p^r$$

and obtain, after obvious reductions,

$$S_1 = 1 + \frac{p^r - 1}{p^{r+1} - 1} + p\frac{(p^r - 1)(p^{r-1} - 1)}{(p^{r+1} - 1)(p^{r+2} - 1)} + \ldots\ldots$$

$$= \frac{2(1+p)(1+p^2)\ldots\ldots(1+p^{r-1})}{(1-p^{r+1})(1-p^{r+2})\ldots\ldots(1-p^{2r})}\left\{1 - \frac{p^{2r}-1}{p-1}\cdot\frac{p}{2} + \frac{(p^{2r}-1)(p^{2r-2}-1)}{(p^2-1)(p^2-1)}\frac{p^4}{2}\right.$$

$$\left. - \frac{(p^{2r}-1)(p^{2r-2}-1)(p^{2r-4}-1)}{(p^2-1)(p^3-1)(p^4-1)}\cdot\frac{p^9}{2} + \ldots\ldots\right\} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In the same way, if we put

$$a = p$$
$$b = p^{1-r}$$
$$c = p^{r+2}$$
$$x = -p^{r+1}$$

we obtain, after reductions

$$S_2 = 1 + \frac{p^{r-1} - 1}{p^{r+2} - 1}p^2 + \ldots\ldots$$

$$= \frac{(1+p^2)(1+p^3)\ldots\ldots(1+p^r)}{(1-p^{r+2})(1-p^{r+3})\ldots\ldots(1-p^{2r})}\left\{1 - \frac{p^{2r-2}-1}{(p-1)(p^2+1)}p^3 + \frac{(p^{2r-2}-1)(p^{2r-4}-1)}{(p-1)(p^2-1)(p^2+1)(p^3+1)}p^8 - \ldots\ldots\right\} \quad (19)$$

So that $S_1 + p\frac{p^r - 1}{p^{r+1} - 1}S_2$ may be written

$$\frac{(1+p)(1+p^2)\ldots\ldots(1+p^{r-1})}{(1-p^{r+1})(1-p^{r+2})\ldots\ldots(1-p^{2r})}\left\{2 - \frac{p^{2r}-1}{p-1}p + \frac{(p^{2r}-1)(p^{2r-2}-1)}{(p^2-1)(p^2-1)}p^4 - \ldots\ldots\ldots\right.$$

$$\left. - \frac{p^{2r}-1}{p+1}p + \frac{(p^{2r}-1)(p^{2r-2}-1)}{(p^2-1)(p^2+1)}p^4 - \ldots\ldots\ldots\right\} \quad . \quad (20)$$

Adding the terms with like numerators together, we obtain

$$\frac{(1+p)(1+p^2)\ldots\ldots(1+p^{r-1})}{(1-p^{r+1})(1-p^{r+2})\ldots\ldots(1-p^{2r})}\left\{2 - 2p^2\frac{p^{2r}-1}{p^2-1} + 2p^6\frac{(p^{2r}-1)(p^{2r-2}-1)}{(p^2-1)(p^4-1)} - \ldots\ldots\right\} \quad . \quad (21)$$

The series within the large brackets is the simplest type of series, and its sum is well known to be

$$2(1-p^2)(1-p^4)\ldots\ldots(1-p^{2r}) \quad . \quad . \quad . \quad . \quad (22)$$

We have therefore

$$1 + \frac{[2]}{[1]}\frac{[r]}{[r+1]} + p\frac{[4]}{[2]}\frac{[r][r-1]}{[r+1][r+2]} + \ldots\ldots$$

$$= \frac{2(1+p)(1+p^2)\ldots(1+p^{r-1})\cdot(1-p^2)(1-p^4)\ldots(1-p^{2r})}{(1-p^{r+1})(1-p^{r+2})\ldots\ldots(1-p^{2r})} \quad . \quad . \quad (23)$$

Changing the base p to p^2 we obtain the series whose sum was sought

$$1 + \frac{[4]}{[2]}\frac{[2r]}{[2r+2]} + p^2\frac{[8]}{[4]}\frac{[2r][2r-2]}{[2r+2][2r+4]} + \ldots\ldots$$

$$= \frac{2(1+p^2)(1+p^4)\ldots(1+p^{2r-2})\cdot(1-p^4)(1-p^8)\ldots(1-p^{4r})}{(1-p^{2r+2})(1-p^{2r+4})\ldots\ldots\ldots(1-p^{4r})} \quad . \quad . \quad (24)$$

The coefficient of λ^{2r} is obtained by multiplying this sum by

$$\frac{\{4r\}!}{\{2r\}!\,\{2r\}!\,\{2r\}!\,\{2r\}!}$$

which gives us

$$(-1)^r \frac{2(1+p^2)(1+p^4)(1+p^6)\ldots(1+p^{2r-2})\cdot(1+p^2)(1+p^4)\ldots(1+p^{2r})}{\{2r\}!\,\{2r\}!} \quad . \quad . \quad (25)$$

as the coefficient of λ^{2r}. If r be not integral, the infinite products in HEINE's transformation do not reduce to finite products but to expressions in terms of the Γ_p functions, ultimately giving the sum of the series in the form (14).

Having obtained the coefficient of λ^{2r}, we have established, subject to convergence, the theorem

$$J_{[0]}(\lambda)\mathfrak{J}_{[0]}(\lambda) - \frac{[4]}{[2]}J_{[1]}(\lambda)\mathfrak{J}_{[1]}(\lambda) + p^2\frac{[8]}{[4]}J_{[2]}(\lambda)\mathfrak{J}_{[2]}(\lambda) + \ldots\ldots$$

$$= 1 - \frac{2(1+p^2)\lambda^2}{\{2\}!\,\{2\}!} + \ldots\ldots (-1)^r \frac{2(1+p^2)\ldots(1+p^{2r-2})\cdot(1+p^2)\ldots(1+p^{2r})\lambda^{2r}}{\{2r\}!\,\{2r\}!} - \ldots\ldots \quad (26)$$

which is the extension of

$$J_0(2\lambda) = \{J_0(\lambda)\}^2 - 2\{J_1(\lambda)\}^2 + 2\{J_2(\lambda)\}^2 - \ldots\ldots\ldots\ldots \quad . \quad . \quad (27)$$

This is a particular case of the addition theorem for J_0.

$$J_0(\lambda+\lambda_1) = J_0(\lambda)J_0(\lambda_1) - 2J_1(\lambda)J_1(\lambda_1) + \ldots\ldots$$

5.

Defining * $\mathrm{Sin}_p(\lambda)$ and $\mathrm{Cos}_p(\lambda)$ as

$$\mathrm{Sin}_p(\lambda) = \lambda - \frac{\lambda^3}{[3]!} + \frac{\lambda^5}{[5]!} - \ldots\ldots$$

$$\mathrm{Cos}_p(\lambda) = 1 - \frac{\lambda^2}{[2]!} + \frac{\lambda^4}{[4]!} - \ldots\ldots$$

we obtain

$$\mathrm{Sin}_p(\lambda)\,\mathrm{Cos}_{\frac{1}{p}}(\lambda_1) + \mathrm{Cos}_p(\lambda)\,\mathrm{Sin}_{\frac{1}{p}}(\lambda_1) = (\lambda+\lambda_1) - \frac{(\lambda+\lambda_1)(\lambda+\lambda_1 p^2)(\lambda+\lambda_1 p^4)}{[3]!} + \ldots\ldots$$

$$\mathrm{Sin}_p(\lambda)\,\mathrm{Cos}_{\frac{1}{p}}(\lambda) + \mathrm{Cos}_p(\lambda)\,\mathrm{Sin}_{\frac{1}{p}}(\lambda) = 2\lambda - \frac{2(1+p^2)(1+p^4)\lambda^3}{[3]!} + \ldots\ldots$$

This suggests that the extension of the addition theorem of $J_0(\lambda+\lambda_1)$ will be on similar lines.

Consider now the series

$$J_{[0]}(\lambda)\mathfrak{J}_{[0]}(\lambda_1) - \frac{[4]}{[2]}J_{[1]}(\lambda)\mathfrak{J}_{[1]}(\lambda_1) + \ldots\ldots\ldots (-1)^r p^{r(r-1)}\frac{[4r]}{[2r]}J_{[1]}(\lambda)\mathfrak{J}_{[1]}(\lambda_1) + \ldots\ldots \quad . \quad (28)$$

and the product

$$J_{[n]}(\lambda)\mathfrak{J}_{[n]}(\lambda_1)$$

$$= \left\{ \frac{\lambda_n}{\{2n\}!} - \frac{\lambda^{n+2}}{\{2n+2\}!\{2\}!} + \ldots + \frac{\lambda^{n+2r}}{\{2n+2r\}!\{2r\}!} - \ldots \right\} \left\{ \frac{\lambda_1{}^n}{\{2n\}!} - \ldots\ldots + \frac{\lambda_1{}^{n+2r}p^{2r(n+r)}}{\{2n+2r\}!\{2r\}!} - \ldots \right\} \quad (29)$$

From series (28) we are to form a new series, of which the successive terms will be homogeneous in $\lambda\lambda_1$ and of degrees 0, 2, 4, 6, ... $2r$, ... respectively.
The first term of (28) gives rise to the constant, unity.
The terms of the second degree arising from $J_{[0]}\mathfrak{J}_{[0]}$ are

$$-\frac{\lambda^2}{\{2\}!\{2\}!} \text{ and } -\frac{\lambda_1{}^2p^2}{\{2\}!\{2\}!}$$

The term of the second degree arising from $J_{[1]}\mathfrak{J}_{[1]}$ is

$$-\frac{[4]}{[2]}\frac{\lambda\lambda_1}{\{2\}!\{2\}!}$$

There are no other terms of the second degree; the sum of these terms is

$$\frac{1}{\{2\}!\{2\}!}\left\{\lambda^2 + \frac{[4]}{[2]}\lambda\lambda_1 + p^2\lambda_1{}^2\right\}$$
$$= \frac{1}{\{2\}!\{2\}!}\left\{\lambda^2 + (p^2+1)\lambda\lambda_1 + p^2\lambda_1{}^2\right\}$$
$$= \frac{(\lambda+\lambda_1p^2)(\lambda+\lambda_1)}{\{2\}!\{2\}!}$$

Terms of the fourth degree arise only from the first, second, and third terms of (28), being respectively

$$\frac{\lambda^4}{\{4\}!\{4\}!} + \frac{\lambda^2\lambda_1{}^2p^2}{\{2\}!\{2\}!\{2\}!\{2\}!} + \frac{\lambda_1{}^4p^8}{\{4\}!\{4\}!}$$
$$\frac{[4]}{[2]}\left\{\frac{\lambda^3\lambda_1}{\{4\}!\{2\}!\{2\}!} + \frac{\lambda\lambda_1{}^3p^4}{\{4\}!\{2\}!\{2\}!}\right\} \quad . \quad . \quad . \quad . \quad (30)$$
$$p^2\frac{[8]}{[4]}\frac{\lambda^2\lambda_1{}^2}{\{4\}!\{4\}!}$$

Remembering that

$$\{4\}! = [4][2] \quad \text{and} \quad \{2\}! = [2]$$

we write the sum of (30)

$$\frac{1}{[4][2][4][2]}\left\{\lambda^4 + \frac{[4][4]}{[2][2]}\lambda^3\lambda_1 + \frac{[4][4]}{[2][2]}\lambda^2\lambda_1{}^2p^2 + \frac{[8]}{[4]}\lambda^2\lambda_1{}^2p^2 + \frac{[4][4]}{[2][2]}\lambda\lambda_1{}^3 + p^8\lambda_1{}^4\right\}$$

Replacing

$$\frac{[4]}{[2]} \text{ by } (p^2+1) \quad \text{and} \quad \frac{[8]}{[4]} \text{ by } p^4+1$$

the expression within the large brackets reduces to

$$(\lambda+\lambda_1)(\lambda+p^2\lambda_1)(\lambda+p^2\lambda_1)(\lambda+p^4\lambda_1)$$

The term of the sixth degree in λ, λ_1 I have verified as

$$-\frac{1}{[6][4][2][6][4][2]}\left\{(\lambda+\lambda_1)(\lambda+\lambda_1p^2)(\lambda+\lambda_1p^4)\cdot(\lambda+\lambda_1p^2)(\lambda+\lambda_1p^4)(\lambda+\lambda_1p^6\right\} \quad . \quad (31)$$

The term of degree $2r$ is the following expression—

$$\left\{ \frac{\lambda^{2r}}{\{2r\}!\{2r\}!} + \frac{\lambda^{2r-2}\lambda_1{}^2p^2}{\{2r-2\}!\{2r-2\}!\{2\}!\{2\}!} + \ldots\ldots\ldots + \frac{\lambda_1{}^{2r}p^{2r^2}}{\{2r\}!\{2r\}!} \right\}$$

$$+\frac{[4]}{[2]}\left\{ \frac{\lambda^{2r-1}\lambda_1}{\{2r\}!\{2r-2\}!\{2\}!} + \frac{\lambda^{2r-3}\lambda_1{}^3p^4}{\{2r-2\}!\{2r-4\}!\{4\}!\{2\}!} + \ldots\ldots\ldots + \frac{\lambda_1{}^{2r-1}\lambda p^{2r(r-1)}}{\{2\}!\{2r-2\}!\{2r\}!} \right\} \quad (32)$$

$$+p^2\frac{[8]}{[4]}\left\{ \frac{\lambda^{2r-2}\lambda_1{}^2}{\{2r\}!\{2r-4\}!\{4\}!} + \frac{\lambda^{2r-4}\lambda_1{}^4p^6}{\{2r-2\}!\{2r-6\}!\{6\}!\{2\}!} + \ldots\ldots\ldots + \frac{\lambda_1{}^{2r-2}\lambda^2p^{2r(r-2)}}{\{4\}!\{2r-4\}!\{2r\}!} \right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+p^{r(r-1)}\frac{[4r]}{[2r]}\left\{ \frac{\lambda^r\lambda_1{}^r}{\{2r\}!\{2r\}!} \right\}$$

We have shown in Art. (4) that in case $\lambda = \lambda_1$ this expression is

$$\frac{(\lambda+\lambda)(\lambda+\lambda p^2)(\lambda+\lambda p^4)\ldots(\lambda+\lambda p^{2r-2})\cdot(\lambda+\lambda p^2)(\lambda+\lambda p^4)\ldots\ldots(\lambda+\lambda p^{2r})}{\{2r\}!\{2r\}!} \quad (33)$$

It has been directly verified that for particular values of r (1, 2, 3) the forms, in case λ be not equal to λ_1, are

$$-\frac{(\lambda+\lambda_1)(\lambda+\lambda_1p^2)}{\{2\}!\{2\}!}$$

$$\frac{(\lambda+\lambda_1)(\lambda+\lambda_1p^2)(\lambda+\lambda_1p^2)(\lambda+\lambda_1p^4)}{\{4\}!\{4\}!}$$

$$-\frac{(\lambda+\lambda_1)(\lambda+\lambda_1p^2)(\lambda+\lambda_1p^4)\cdot(\lambda+\lambda_1p^2)(\lambda+\lambda_1p^4)(\lambda+\lambda_1p^6)}{\{6\}!\{6\}!}$$

respectively. This indirectly establishes the form of the coefficient of degree $2r$ in λ and λ_1. A direct proof of the algebraic identity would, however, be preferable. Writing now

$$\sum_{n=0}^{n=\infty}(-1)^np^{n(n-1)}\frac{[4n]}{[2n]}J_{[n]}(\lambda)J_{[n]}(\lambda_1) = 1+\sum_{n=1}^{n=\infty}(-1)^n\frac{(\lambda+\lambda_1)(\lambda+\lambda_1p^2)\ldots(\lambda+\lambda_1p^{2n-2})\cdot(\lambda+\lambda_1p^2)\ldots\ldots(\lambda+\lambda_1p^{2n})}{\{2n\}!\{2n\}!} \quad (34)$$

If $p=1$, we obtain the addition theorem of J_0

$$J_0(\lambda+\lambda_1) = J_0(\lambda)J_0(\lambda_1) - 2J_1(\lambda)J_1(\lambda_1) + \ldots\ldots\ldots \quad (35)$$

6.

The analogue of LOMMEL's theorem

$$J_n(\lambda) = (-1)^m\left\{ J_{2m+n}(\lambda) - \frac{2m(m+n)}{\lambda}J_{2m+n-1}(\lambda) + \ldots\ldots\ldots \right\}$$

I have shown by two distinct methods * that

$$\lim_{\kappa=\infty}\frac{(p^{\gamma-\alpha}-1)(p^{\gamma-\alpha+1}-1)\ldots(p^{\gamma-\beta+\kappa-1}-1)\cdot(p^{\gamma-\beta}-1)(p^{\gamma-\beta+1}-1)\ldots\ldots(p^{\gamma-\beta+\kappa-1}-1)}{(p^{\gamma-\alpha-\beta}-1)(p^{\gamma-\alpha-\beta+1}-1)\ldots(p^{\gamma-\alpha-\beta+\kappa-1}-1)\cdot(p^{\gamma}-1)(p^{\gamma+1}-1)\ldots(p^{\gamma+\kappa-1}-1)}$$

$$= 1+p\frac{[\alpha][\beta]}{[1][\gamma]}+p^2\frac{[\alpha][\alpha+1][\beta][\beta+1]}{[1][2][\gamma][\gamma+1]}+\ldots\ldots \quad (36)$$

* *Proc. Lond. Math. Soc.*, series 2, vol. i. pp. 71, 72, 1903, and *Amer. Jour. Math.*, vol. xxvi., 1904.

In terms of the function Γ_p of this paper, this theorem is

$$\frac{1}{p^{\alpha\beta}} \frac{\Gamma_p([\gamma-\alpha-\beta])\Gamma_p([\gamma])}{\Gamma_p([\gamma-\alpha])\Gamma_p([\gamma-\beta])} = 1 + p\frac{[\alpha][\beta]}{[1][\gamma]} + \ldots . \quad (37)$$

Change the base p to p^2 and put

$$\alpha = -m$$
$$\beta = -m-n$$
$$\gamma = r-m+1$$

we obtain

$$\frac{\Gamma_{p^2}([r+m+n+1])\Gamma_{p^2}([r-m+1])}{\Gamma_{p^2}([r+1])\Gamma_{p^2}([r+n+1])} = 1 + \frac{[2m][2m+2n]}{[2][2r-2m+2]}p^{2-4m-2n} + \ldots .$$
$$+\frac{[2m]\ldots[2m-2s+2]\cdot[2m+2n]\ldots[2m+2n-2s+2]}{[2][4]\ldots[2s]\cdot[2r-2m+2]\ldots[2r-2m+2s]}p^{2s^2-2s(2m+n)} + \ldots . \quad (38)$$

Now consider

$$J_{[2m+n]}(\lambda) - \frac{p^2}{p^{4m+2n}}\frac{[2m][2m+2n]}{[2]\lambda}J_{[2m+n-2]}(\lambda) + \frac{p^8}{p^{8m+4n}}\frac{[2m][2m-2][2m+2n][2m+2n-2]}{[2][4]}J_{[2m+n-4]}(\lambda) + \ldots . \quad (39)$$

The coefficient of λ^{n+2r} is the infinite series

$$\frac{(-1)^{r-m}}{\{2n+2m+2r\}!\{2r-2m\}!}\left[1 + \frac{p^2}{p^{4m+2n}}\frac{[2m][2m+2n]}{[2][2r-2m+2]} + \ldots . \right] \quad (40)$$

which by (38) reduces to

$$\frac{(-1)^{r-m}}{\{2n+2m+2r\}!\{2r-2m\}!} \times \frac{\Gamma_{p^2}([m+n+r+1])\Gamma_{p^2}([r-m+1])}{\Gamma_{p^2}([r+1])\Gamma_{p^2}([r+n+1])} \quad (41)$$

Now remembering

$$\{2s\}! = [2]^s\Gamma_{p^2}([s+1]) = (2)_s\Gamma_p([s+1])$$

the expression (40) reduces to

$$(-1)^{r-m}\frac{p^{2m(m+n)}}{\Gamma_p([r+1])\Gamma_p([n+r+1])(2)_r(2)_{n+r}} \quad (42)$$

which is $(-1)^{-m}p^{2m(m+n)}$ × coefficient of λ^{n+2r} in the series $J_{[n]}$.
This establishes

$$(-1)^m p^{2m(m+n)}J_{[n]}(\lambda) = J_{[2m+n]}(\lambda) - \frac{p^2}{p^{4m+2n}}\frac{[2m][2m+2n]}{[2]\lambda}J_{[2m+n-1]}(\lambda) + \ldots . \text{ ad inf.} \quad (43)$$

an extension of

$$J_n = (-1)^m\left\{J_{2m+n} - \frac{2m(m+n)}{\lambda}J_{2m+n-1} + \ldots\ldots\ldots\right\} \quad (44)$$

Lommel defined J_n for negative integral values of n, so as to make this theorem always hold: for example, suppose n a negative integer, and put it equal to $-m$, then we have by this theorem

$$J_{[-n]} = (-1)^n J_{[n]} \quad (45)$$

extending

$$J_{-n} = (-1)^n J_n$$

also

$$\mathfrak{J}_{[-n]} = (-1)^n \mathfrak{J}_{[n]}$$

as may be shown by inverting the base p in expression (43).

7.

$$J_{n+\frac{1}{2}}(x) = (-1)^n \frac{(2x)^{n+\frac{1}{2}}}{\Pi^{\frac{1}{2}}} \frac{d^n}{d(x^2)^n} \left\{ \frac{\sin x}{x} \right\}$$

If we now define

$$\mathrm{Sin}_p(\lambda,x) \text{ as } \lambda x - \frac{\lambda^3 x^{[3]}}{[3]!} + \frac{\lambda^5 x^{[5]}}{[5]!} - \ldots\ldots \quad . \quad . \quad . \quad (46)$$

then

$$\frac{\mathrm{Sin}_p(\lambda,x^{\frac{1}{p}})}{\lambda x^{\frac{1}{p}}} = 1 - \frac{\lambda^2 x^{[2]}}{[3]!} + \ldots\ldots + (-1)^{n+r} \frac{\lambda^{2n+2r} x^{[2n+2r]}}{[2n+2r+1]!} + \ldots\ldots$$

Operating on this with $\Delta^{[n]}$ which is

$$\left\{ \frac{d}{d(x^{p^{2n}-2[2]})} \ldots\ldots\ldots \left\{ \frac{d}{d(x^{p^2[2]})} \cdot \left\{ \frac{d}{d(x^{[2]})} \right\} \right\} \ldots \right\} . \quad . \quad . \quad (47)$$

the first n terms of the series are destroyed, while the term involving $x^{[2n+2r]}$ is reduced to

$$(-1)^{n+r} \lambda^{2n+2r} \frac{1}{[2]^n} \frac{[2n+2r][2n+2r-2] \ldots\ldots [2r+2]}{[2n+2r+1]!} x^{p^{2n}[2r]} \quad . \quad . \quad (48)$$

Taking

$$J_{[n+\frac{1}{2}]}(\lambda,x) = \sum (-1)^r \frac{\lambda^{n+\frac{1}{2}+2r}}{[n+r+\frac{1}{2}]! \, [r]! \, (2)_r (2)_{n+r+\frac{1}{2}}} x^{[n+\frac{1}{2}+2r]}$$

$$[n+r+\tfrac{1}{2}]! = [n+r+\tfrac{1}{2}][n+r-\tfrac{1}{2}] \ldots\ldots [\tfrac{3}{2}]\Gamma_p([\tfrac{3}{2}]$$

$$(2)_{n+r+\frac{1}{2}} = (p^{n+r+\frac{1}{2}}+1)(p^{n+r-\frac{1}{2}}+1) \ldots\ldots (p^{\frac{3}{2}}+1) \cdot (2)_{\frac{1}{2}}$$

and

$$(2)_{\frac{1}{2}}\Gamma_p([1+\tfrac{1}{2}]) = [2]^{\frac{1}{2}}\Gamma_{p^2}([1+\tfrac{1}{2}])$$

therefore

$$[n+r+\tfrac{1}{2}]! \, (2)_{n+r+\frac{1}{2}} = [2n+2r+1] \ldots\ldots [3] \times [2]^{\frac{1}{2}}\Pi_{p^2}([\tfrac{1}{2}]) \quad . \quad . \quad (49)$$

So we obtain

$$J_{[n+\frac{1}{2}]}(\lambda,x) = \frac{\lambda^{n+\frac{1}{2}} x^{[n+\frac{1}{2}]}}{[2]^{\frac{1}{2}}\Gamma_{p^2}([\frac{3}{2}])} \sum (-1)^r \frac{\lambda^{2r} x^{p^{n+\frac{1}{2}}[2r]}}{[2n+2r+1] \ldots [5][3] \cdot [2][4] \ldots [2r]} \quad . \quad (50)$$

and by a change of the variable x

$$\lambda^{2n} J_{[n+\frac{1}{2}]}(\lambda,x^{p^{n+\frac{1}{2}}}) = (-1)^n \frac{\{[2]\lambda\}^{n+\frac{1}{2}} x^{p^{n+\frac{1}{2}}[n+\frac{1}{2}]}}{[2]\Gamma_{p^2}([\frac{3}{2}])} \Delta^{[n]} \left\{ \frac{\mathrm{Sin}_p(\lambda,x)}{\lambda x} \right\} . \quad . \quad (51)$$

$\Delta^{[n]}$ denoting the operator

$$\left\{ \frac{d}{d(x^{p^{2n}-1[2]})} \ldots\ldots \left\{ \frac{d}{d(x^{p^3[2]})} \cdot \left\{ \frac{d}{d(x^{p[2]})} \right\} \right\} \ldots \right\} \quad . \quad . \quad (52)$$

For further properties of Sin_p, Cos_p, and their connection with symbolical solutions of certain differential equations, reference may be made to a paper on "Basic Sines and Cosines" (*Proc. Edin. Math. Soc.*, 1904).

CONTINUATION OF PAPER—

"THEOREMS RELATING TO A GENERALISATION OF THE BESSEL-FUNCTION."

(MS. received April 19, 1904.)

8.

The theorem

$$J_{[0]}(a)\mathfrak{J}_{[0]}(b) - \frac{[4]}{[2]}J_{[1]}(a)\mathfrak{J}_{[1]}(b) + \ldots\ldots + (-1)^s p^{s(s-1)}\frac{[4s]}{[2s]}J_{[s]}(a)\mathfrak{J}_{[s]}(b) - \ldots\ldots$$

$$= 1 - \frac{(a+b)(a+bp^2)}{[2]^2} + \frac{(a+b)(a+bp^2)(a+bp^2)(a+bp^4)}{[2]^2[4]^2} - \ldots\ldots \quad . \quad . \quad (\alpha)$$

discussed in the first part of this paper may be obtained very naturally from the properties of a certain function analogous to the exponential function. Elsewhere,* by means of the function E_p I have obtained

$$J_{[0]}(a)\mathfrak{J}_{[0]}\left(\frac{b}{p}\right) - 2pJ_{[1]}(a)\mathfrak{J}_{[1]}\left(\frac{b}{p}\right) + \ldots\ldots + (-1)^s 2p^{s^2}J_{[s]}(a)\mathfrak{J}_{[s]}\left(\frac{b}{p}\right) - \ldots\ldots$$

$$= 1 - \frac{(a+b)^2}{[2]^2} + \frac{(a+b)^2(a+p^2b)^2}{[2]^2[4]^2} - \ldots\ldots\ldots \quad . \quad . \quad . \quad (\beta)$$

We naturally expect to find some general form to which both (α) and (β) will belong, as particular cases. The following is the general theorem which will be obtained from the function E_p, just as the addition theorem for Bessel coefficients is obtained by means of the exponential function. Exp. $\left(\frac{x}{2}\left(t - \frac{1}{t}\right)\right)$

$$J_0^\nu(a, b) = J_{[0]}(a)\mathfrak{J}_{[0]}(bp^{\nu-1}) - p^{1-\nu}\frac{[4\nu]}{[2\nu]}J_{[1]}(a)\mathfrak{J}_{[1]}(bp^{\nu-1}) + p^{2(2-\nu)}\frac{[8\nu]}{[4\nu]}J_{[2]}(a)\mathfrak{J}_{[2]}(bp^{\nu-1}) - \ldots\ldots \quad (\gamma)$$

$$J_0^\nu(a, b) = 1 - \frac{(a+b)(a+bp^{2\nu})}{[2]^2} + \frac{(a+b)(a+bp^2)(a+bp^{2\nu})(a+bp^{2\nu+2})}{[2]^2[4]^2} - \ldots\ldots$$

In case $\nu = 0$ we have the quasi-addition theorem (β). If, however, $\nu = 1$ we have the quasi-addition theorem (α). The corresponding theorems for the function $J_n^\nu(a, b)$ will be briefly noticed.

$$J_n^\nu(a, b) = \frac{(a+b)(a+bp^2)\ldots\ldots(a+bp^{2n-2})}{[2][4]\ldots[2n]}\left\{1 - \frac{(a+bp^{2n})(a+bp^{2\nu})}{[2n+2][2]}\right.$$

$$\left. + \frac{(a+bp^{2n})(a+bp^{2n+2})(a+p^{2\nu}b)(a+p^{2\nu+2}b)}{[2n+2][2n+4][2][4]} - \ldots\right\} \quad . \quad . \quad . \quad (\delta)$$

The expression for $J_n^\nu(a, b)$ will be given also in the case when n is not a positive integer.

* *Proc. Lond. Math. Soc.*, shortly to be published.

9.

In this article certain results will be obtained which will be required in subsequent work. We define the function $\mathrm{E}_p(a)$ as

$$\mathrm{E}_p(a)^* = 1 + \frac{a}{[1]!} + \frac{a^2}{[2]!} + \ldots\ldots$$

If we invert the base p

$$\mathrm{E}_{\frac{1}{p}}(a) = 1 + \frac{a}{[1]!} + p\frac{a^2}{[2]!} + \ldots\ldots + p^{s(s-1)/2}\frac{a^s}{[s]!} + \ldots\ldots$$

without difficulty we have

$$\mathrm{E}_p(a)\mathrm{E}_{\frac{1}{p}}(b) = 1 + \frac{(a+b)}{[1]!} + \frac{(a+b)(a+bp)}{[2]!} + \ldots\ldots\ldots \quad (\epsilon)$$

Changing p to p^2

$$\mathrm{E}_{p^2}(a)\mathrm{E}_{p^{-2}}(b) = 1 + \frac{(a+b)}{p^2-1} + \frac{(a+b)(a+bp^2)}{p^2-1}\cdot\frac{}{p^2-1} + \ldots\ldots$$

$$= 1 + \frac{[2](a+b)}{[2]} + \frac{[2]^2(a+b)(a+bp^2)}{[2][4]} + \ldots\ldots$$

$$\mathrm{E}_{p^2}\left(\frac{a}{[2]}\right)\mathrm{E}_{p^{-2}}\left(\frac{b}{[2]}\right) = 1 + \frac{(a+b)}{[2]} + \frac{(a+b)(a+bp^2)}{[2][4]} + \ldots\ldots\ldots \quad (\eta)$$

In part (1) we have established

$$\mathrm{J}_{[-n]}(a) = (-1)^n\mathrm{J}_{[n]}(a) \quad (\theta)$$

Inverting the base p we obtain also from this

$$\mathfrak{J}_{[-n]}(a) = (-1)^n\mathfrak{J}_{[n]}(a) \quad (\kappa)$$

10.

A consideration of the product of the two absolutely convergent series

$$\mathrm{E}_{p^2}\left(\frac{at}{[2]}\right) = 1 + \frac{at}{[2]} + \frac{a^2t^2}{[2][4]} + \ldots\ldots + \frac{a^nt^n}{\{2n\}!} + \ldots\ldots\ldots$$

$$\mathrm{E}_{p^2}\left(-\frac{at^{-1}}{[2]}\right) = 1 - \frac{at^{-1}}{[2]} + \frac{a^2t^{-2}}{[2][4]} - \ldots\ldots(-1)^n\frac{a^nt^{-n}}{\{2n\}!} + \ldots\ldots$$

shows us that

$$\mathrm{E}_{p^2}\left(\frac{at}{[2]}\right)\mathrm{E}_{p^2}\left(-\frac{at^{-1}}{[2]}\right) = \sum_0^{\infty}\mathrm{J}_{[n]}(a)t^n + \sum_1^{\infty}(-1)^n\mathrm{J}_{[n]}(a)t^{-n}$$

$$= \sum_{-\infty}^{+\infty}\mathrm{J}_{[n]}(a)t^n \quad (\lambda)$$

In precisely the same manner, if we consider the product of

$$\mathrm{E}_{\frac{1}{p^2}}\left(\frac{bt}{[2]}\right) = 1 + \frac{bt}{[2]} + p^2\frac{b^2t^2}{[2][4]} + \ldots\ldots + p^{n(n-1)}\frac{b^nt^n}{\{2n\}!} + \ldots\ldots$$

$$\mathrm{E}_{\frac{1}{p^2}}\left(-\frac{p^{2\nu}bt^{-1}}{[2]}\right) = 1 - \frac{p^{2\nu}bt^{-1}}{[2]} + p^2\frac{p^{4\nu}b^2t^{-2}}{[2][4]} - \ldots\ldots(-1)^sp^{n(n-1)}\frac{p^{2n\nu}b^nt^n}{\{2n\}!} + \ldots\ldots$$

* *Proc. Edin. Math. Soc.*, vol. xxii.

we obtain

$$\mathrm{E}_{\frac{1}{p^2}}\left(\frac{bt}{[2]}\right)\mathrm{E}_{\frac{1}{p^2}}\left(-\frac{p^{2\nu}bt^{-1}}{[2]}\right) = \sum_{n=0}^{n=\infty} p^{n(n-\nu)}\mathfrak{J}_{[n]}(bp^{\nu-1})t^n + \sum_{n=1}^{n=\infty}(-1)^n p^{n(n+\nu)}\mathfrak{J}_{[n]}(bp^{\nu-1})t^{-n}$$

$$= \sum_{-\infty}^{+\infty} p^{n(n-\nu)}\mathfrak{J}_{[n]}(bp^{\nu-1})t^n \qquad \ldots\ldots\ldots \qquad (\mu)$$

We have now, on taking the product of (λ) and (μ),

$$\sum_{-\infty}^{+\infty}\mathrm{J}_{[n]}(a)t^n \times \sum_{-\infty}^{+\infty} p^{n(n-\nu)}\mathfrak{J}_{[n]}(bp^{\nu-1})t^n = \mathrm{E}_{p^2}\left(\frac{at}{[2]}\right)\mathrm{E}_{p^2}\left(-\frac{at^{-1}}{[2]}\right)\mathrm{E}_{\frac{1}{p^2}}\left(\frac{bt}{[2]}\right)\mathrm{E}_{\frac{1}{p^2}}\left(-\frac{p^{2\nu}bt^{-1}}{[2]}\right)$$

The product of the four basic-exponential functions on the right of this expression is the product of two convergent series

$$\left\{1+\frac{(a+b)t}{[2]}+\frac{(a+b)(a+p^2b)t^2}{[2][4]}+\ldots\ldots\right\}\times\left\{1-\frac{(a+p^{2\nu}b)t^{-1}}{[2]}+\frac{(a+p^{2\nu}b)(a+p^{2\nu+2}b)}{[2][4]}\,t^{-1}\right\} \qquad (\mathrm{B})$$

This result follows from result (η) of article (9).

If now we equate coefficients of the various powers of t in (B) with the corresponding coefficients in

$$\sum_{-\infty}^{+\infty}\mathrm{J}_{[n]}(a)t^n \times \sum_{-\infty}^{+\infty} p^{n(n-\nu)}\mathfrak{J}_{[n]}(bp^{\nu-1})t^n$$

remembering that

$$\mathrm{J}_{[n]} = (-1)^n\mathrm{J}_{[-n]}$$
$$\mathfrak{J}_{[n]} = (-1)^n\mathfrak{J}_{[-n]}$$

we obtain from the terms which are independent of t

$$\mathrm{J}_{[0]}(a)\mathfrak{J}_{[0]}(bp^{\nu-1}) - (p^{1-\nu}+p^{1-\nu})\mathrm{J}_{[1]}(a)\mathfrak{J}_{[1]}(bp^{\nu-1}) + (p^{2(2-\nu)}+p^{2(2+\nu)})\mathrm{J}_{[2]}(a)\mathfrak{J}_{[2]}(bp^{\nu-1}) - \ldots\ldots \qquad (\rho)$$

$$= 1 - \frac{(a+b)(a+p^{2\nu}b)}{[2]^2} + \ldots\ldots\ldots$$

which by an obvious reduction becomes

$$\mathrm{J}_{[0]}(a)\mathfrak{J}_{[0]}(bp^{\nu-1}) - p^{1-\nu}\frac{[4\nu]}{[2\nu]}\mathrm{J}_{[1]}(a)\mathfrak{J}_{[1]}(bp^{\nu-1}) + \ldots\ldots + (-1)^r p^{r(r-\nu)}\frac{[4r\nu]}{[2r\nu]}\mathrm{J}_{[r]}(a)\mathfrak{J}_{[r]}(bp^{\nu-1}) + \ldots\ldots \qquad (\sigma)$$

$$= \mathrm{J}_0^{\nu}(a, b)$$

Equating the coefficients of t^n we obtain

$$\sum_{m=-\infty}^{m=+\infty}\mathrm{J}_{[m]}(a)\mathfrak{J}_{[n-m]}(bp^{\nu-1})p^{(n-m)(n-m-\nu)} = \mathrm{J}_n^{\nu}(a, b) \qquad \ldots\ldots \qquad (\tau)$$

the expression for J_n^{ν} being that given in article (8) expression (δ).

11.

When n is not a positive integer the expression

$$(a+b)(a+p^2b)\ldots\ldots(a+p^{2n-2}b) \text{ in } \mathrm{J}_n^{\nu}(a, b)$$

must be replaced by

$$\mathrm{L}_{\kappa=\infty}\frac{(a+b)(a+p^2b)(a+p^4b)\ldots\ldots(a+p^{2\kappa-2}b)}{(a+p^{2n}b)(a+p^{2n+2}b)\ldots\ldots(a+p^{2n+2\kappa-2}b)}a^n$$

$p<1$

If, however, $p > 1$,

$$\mathrm{L}_{\kappa=\infty} \frac{(a+bp^{2n-2})(a+bp^{2n-4}) \ldots (a+bp^{2n-2\kappa})}{(a+p^{-2}b)(a+p^{-4}b) \ldots (a+p^{-2\kappa}b)} a^n \qquad p > 1$$

is the effective representative of the product $(a+b)(a+p^2b) \ldots$ to n factors. This corresponds to the change of $n!$ in the Bessel coefficients into $\Gamma(n+1)$ in the case of Bessel-Functions. The series expansions of the products given above may be found in *Proc. L.M.S.*, series 2, vol. i. pp. 63–88. The theorem analogous to NEUMANN'S theorem

$$\mathrm{J}_0(a^2+b^2+2ab\cos\theta) = \mathrm{J}_0(a)\mathrm{J}_0(b) + 2\sum(-1)^s \mathrm{J}_s(a)\mathrm{J}_s(b)\cos s\theta \qquad (\xi)$$

I have investigated in a paper (*Proc. L.M.S.*). The function E_p being used in a manner similar to the use of the exponential (pp. 25, 26, 27, Gray and Matthew's *Treatise on Bessel-Functions*), gives us a rather complicated extension of (ξ).

VII.—On Some Points in the Early Development of Motor Nerve Trunks and Myotomes in *Lepidosiren paradoxa* (Fitz.).

By J. Graham Kerr, Regius Professor of Zoology in the University of Glasgow. (With Six Plates.)

(Read January 18 ; MS. received February 9, 1904. Issued separately July 1, 1904.)

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Introductory.

My main purpose in the following short paper is to publish figures illustrating some of the more important facts of the early development of myotomes and motor nerves in *Lepidosiren*. The bearing of some of the observations of nerve development upon current theories renders it particularly desirable that they should be illustrated by untouched photographs of the sections. A few photographs illustrating the more important stages * in the development of the motor nerve trunks are given on Plate I. For the preparation of the photographs here published, as well as several others, I am indebted to the skill of my friend Dr T. H. Bryce, and of Mr Fingland, our University photographer.

Development of the Motor Trunks of the Spinal Nerves.

In describing the observed phenomena it will be convenient to begin with a late stage in development and work backwards to the earlier stages, and so pass from the better known and more familiar to the less known and less familiar.

Stage 34.—Fig. 1 illustrates a considerable length of motor nerve from stage 34.† Here the nerve (*n.t.*) consists of a cylindrical mass of nerve fibres about 13μ in diameter. On the surface of this the nuclei of the protoplasmic sheath are conspicuous. The greater part of the sheath itself is so thin as to be invisible even under the 3 mm. immersion objective, except in the neighbourhood of each nucleus, where it swells out to form a thick mass containing the nucleus.

Stage 31.—At this stage the nerve rudiment on superficial examination presents the appearance simply of a chain of nuclei placed end to end in a strand of protoplasm.

* By stage *n* I mean the stage represented by fig. *n* in my paper on the external features during development, *Phil. Trans. Roy. Soc. B.*, vol. cxcii. p. 299.

† *Cf.* Raffaele's fig. in *Anat. Anz.*, 1900, p. 340 (Per la genesi dei nervi da catene cellulari). *Cf.* also Kölliker's remarks on this, *op. cit.*, p. 511.

Appearances, in fact, support the *Kettentheorie*—suggesting a chain of "cells" placed end to end. More careful examination shows the presence within the protoplasmic strand of a cord, faintly fibrillated longitudinally, and differentiated from the simple protoplasm by its affinity for eosin. The nerve runs downwards along the inner face of the myotome, and it is difficult to make out with certainty its connections with the cells of the myotome.

Stage 29.—At this stage (figs. 2 and 7) the conducting part of the nerve (*n.t.*) is in a similar condition to that described for stage 30—a distinctly fibrillated band—usually about 2·5 to 3μ in thickness close to its root—which stains deeply with eosin. It slopes outwards and downwards from the ventrolateral angle of the spinal cord to the inner surface of the myotome, along which it proceeds in a ventral direction. The protoplasmic sheath (*p.s.*), however, is now far more conspicuous than in later stages. It is a great, irregularly thickened mass of granular protoplasm, sharply marked off from the true nerve by its being stained more greyish in colour in hæmatoxylin and eosin preparations, in sharp contrast to the deep red of the nerve trunk.

Scattered through the protoplasm of the sheath are large nuclei rich in chromatin, yolk granules, and here and there vocuoles.

Stage 27.—At this stage (figs. 3 and 8) the most conspicuous difference from stage 29 is that the nerve trunk (*n.t.*) is now naked for the greater part of its length. At its outer end it spreads out into a number of strands arranged in a conical fashion. In the case of the strands near the axis of the cone—*i.e.* in the case of the strands which pursue a direct course towards the inner surface of the myotome—it may be clearly seen under a high power of the microscope that each strand passes perfectly continuously and by insensible gradations into the granular protoplasm, which forms a tail-like process of a myoepithelial cell of the myotome. In the case of many of the motor trunks at this stage there is to be seen a mass of mesenchymatous protoplasm (*p.s.*) richly laden with yolk, and containing numerous nuclei, concentrated in the neighbourhood of the nerve towards its outer end. This is the rudiment of the mesenchymatous sheath which in stage 29 we saw had spread out over the surface of the nerve. The nerve trunk itself is about the same thickness as in stage 29, though I find considerable variation in this respect.

Stage 25 (figs. 4 and 9).—A little behind the middle of the body at this stage the myotome is seen to be just commencing its recession from the spinal cord, mesenchyme (*me.*) cells* richly laden with yolk having begun to migrate in between the two structures. The nerve rudiment (*n.t.*) in the section figured is about 7μ thick. It is distinctly fibrillated, and at its lower end expands into a cone as in stage 27—the base of the cone, however, here being less expanded. Traces of longitudinal fibres are already visible on the ventrolateral surface of the spinal cord.

* Here as elsewhere I use the word "cell" merely as a substitute for the more cumbrous expression "nucleated mass of protoplasm" without in the least implying that it is separate from its neighbours. As a matter of fact the "cells" of the mesenchyme are merely the enlarged and nucleated nodes of an irregular continuous protoplasmic spongework such as SEDGWICK describes in Selachians.

Stage 24.—Transverse sections about the middle of the body at this stage (figs. 5 and 6) show the myotome still in contact with the wall of the neural canal, mesenchyme either not having intruded itself between the two at all, or only having done so to a very slight extent as yet except in the anterior part of the body. In embryos which have been excised while alive in normal salt solution, spread out in one plane like that figured to illustrate this stage in my account of the external features of development,* and fixed in this position, the myotomes are pulled slightly away from the neural tube. It is seen in transverse sections of such embryos that the rudiments of the motor nerve trunks already exist as soft thin bridges (*n.t.*) metamerically arranged and connecting spinal cord rudiment and myotome as shown in figs. 5 and 6.

The nerve rudiments at this early stage are formed of granular protoplasm either without yolk or containing only very minute granules, without obvious fibrillar structure when stained by the ordinary methods. The rudiment is quite naked, the richly yolked mesenchymatous sheath of later stages being conspicuous by its absence. Nor are there any nuclei in contact with the nerve rudiment. There can happily be no doubt about this in *Lepidosiren*, where the nerve rudiment in this early condition is of relatively small size compared with the dimensions of a single nucleus!

The nerve rudiment—composed as it is of soft protoplasm—is at first extremely fragile, gradually becoming tougher as development proceeds. Consequently in a straightened out embryo of this period we find the more posterior nerve rudiments—*i.e.* those in an earlier stage of development—show a greater and greater tendency to be torn away from the myotome in prepared sections. A nerve rudiment which has become torn away from the myotome is shown in fig. 10, which brings out a point more difficult to observe in the uninjured condition, that the protoplasmic mass forming the nerve rudiment spreads out over the inner face of the myotome. How far this expansion extends, whether—as is probable—it really covers the whole face of the myotome, is a point almost impossible to decide definitely by actual observation. Similarly I am deterred by the unreliability of observations made on a spinal cord so laden with yolk in its early stages from saying anything as to the connections of nerve rudiments with neuroblasts or other cellular elements in the substance of the spinal cord.

At this stage the spinal cord is without any obvious mantle of fibres.

The motor nerve trunk has thus been traced back to a period in which it is represented by a bridge of soft granular protoplasm connecting spinal cord and myotome at a stage when these structures are in close apposition. As the myotome becomes pushed outwards by the development of mesenchyme, it remains connected with the spinal cord by the ever-lengthening strand of nerve. As the nerve develops it soon loses its simple granular protoplasmic character and assumes a fibrillated appearance. Richly yolked masses of mesenchymatous protoplasm become aggregated round the nerve, which till now has been quite naked. At first this protoplasm forms an irregular mass towards the outer end of the nerve trunk, but it soon spreads along it and forms a definite sheath.

* *Phil. Trans. B.*, vol. cxcii., pl. 10, fig. 24.

This is at first very thick and conspicuous, but it gradually thins out, its yolk is consumed, and eventually the only parts remaining conspicuously visible are the nuclei dotted along the surface of the nerve trunk.

Into the subsequent history of the motor nerve—which is of minor morphological interest—I do not propose at this time to enter in detail.

The protoplasmic sheath grows into the nerve trunk, dividing it up into separate bundles of fibrils. The nerve trunk, as has been shown, spreads out at an early stage in conical fashion over the inner face of the myotome. As the myotome grows in surface this cone becomes broken up into distinct strands which become more and more divergent.

As the adult condition is reached the part of the nerve trunk proximal to the point of divergence—*i.e.* to the apex of the cone—increases relatively little in length. The distal portion, on the other hand, with its individual branches, increases enormously in length and the branches become more and more changed in direction as the muscles to which they are attached become pushed about by differential growth.

Development of the Myomeres.

The general features in the development of a typical myomere as seen in transverse sections are shown in text-figures A to H,* and in detail in Pl. III. and IV. figs. 11–14.

The protovertebra, at first (text-fig. A) a solid diverticulum of the enteric rudiment, develops a myocœlic cavity through the breaking down of its central cells about stage 21 (text-fig. B).

By stage 24 (text-fig. C) the myotome is beginning to show signs of a flattening from without inwards. The myocele is obliterated, and the cells of its mesial wall have become flattened in form. At their outer ends they interdigitate with the inner ends of the cells of the outer wall, so that the line separating the two walls in a transverse section is a zigzag one.

In stage 27 (text-fig. D, Pl. III. fig. 11) the inner wall cells have become more regular in shape; forming rectangular parallelepipeds flattened in an obliquely dorso-ventral direction, so that their larger faces slope inwards and downwards. Contractile fibres (fig. 11, *c.f.*) have now appeared in the body of these cells, running longitudinally and forming most frequently a layer close to each of the dorsal and ventral surfaces, the two layers becoming frequently continuous with one another externally, and sometimes internally. Very often, however, the arrangement of fibres at their first appearance does not show this regular arrangement. The contractile fibres appear to be rounded in section, and are easily distinguished by their highly refringent character, and by their peculiarly deep stain with Heidenhain's hæmatoxylin. The cells of the myotome are at this stage still laden with yolk, and this naturally is a difficulty in the way of observation.

The muscle fibrillæ are striped almost from the beginning. One can often see in

* These have been printed as separate Plates—V. and VI.

Heidenhain hæmatoxylin preparations that the young fibril appears to be made up of discrete particles arranged end to end.* But between these the protoplasmic matrix shows a more or less distinct fibrillar structure, the fibrils causing it having a longitudinal course and being continuous. These continuous fibrils appear to be an earlier stage of the contractile fibrils. The outer wall of the myotome in transverse section has the appearance of being composed of a single layer of large cubical epithelial cells. In horizontal sections the cell boundaries are less easy to make out, but when visible they show that the cells form a truly cubical epithelium. During mitosis these cells become more or less spherical.

Stage 31.—At this stage (text-fig. E, and figs. 12 and 12A) the outer portion of the muscle cell has increased in size, so that the outer limit of the contractile fibres is relatively much further removed from the outer end of the cell than it was before. In this outer end of the cell the cytoplasm assumes a clear transparent appearance, and in the preserved specimens large clear vacuoles are seen which possibly in the fresh condition contained glycogen. The inner part of the cell is now almost filled with contractile fibres, the protoplasm being reduced to the matrix between them. In this matrix yolk granules are still abundant, and it is noteworthy that the muscle cells are now becoming multinucleate, the original nucleus having divided repeatedly. The division is mitotic. As the divisions only take place at relatively long intervals, a little patience is required in hunting for the mitotic figures. The resting nuclei lie free in the protoplasm of the myoblast. It is interesting, however, that during the period of mitotic activity the nucleus becomes surrounded by a sharply delimited more or less spherical mass of protoplasm, simulating the appearance of a cell within the myoblast. So striking is this appearance (Pl. IV. fig. 14) that it suggested at first sight that more or fewer of the nuclei of the myoblast were really the nuclei of cells which had wandered into its substance from the mesenchyme without, just as such cells wander in later between the muscle fibres. On the whole, however, the balance of evidence is in favour of the cell-like structure round the nucleus being merely a temporary phenomenon due in some way to the influence of the mitotic activity of the nucleus on the surrounding cytoplasm—a phenomenon of the same nature as the rounding off into a spherical shape during mitosis of cells which in the resting condition are of more irregular outline.

The appearance of the myotome of this stage, as shown in horizontal sections, is indicated in Pl. IV. fig. 12A.

At about this period a striking change comes over the outer wall of the myotome. Numerous mitotic figures are observed in it. Its cells subdivide rapidly, so that the outer wall becomes several cells thick. The innermost of the cells so arising become squeezed in between the rounded ends of the primary muscle cells. At this stage it is often difficult to draw a line of demarcation between the outermost cells and those of the cutis which is now beginning to appear between myotome and skin. I am not, however, prepared to assert definitely that they actually give rise to cutis.

* *Cf.* GODLEWSKI, *Arch. Mikr. Anat.*, Bd. lx., 1902.

The outer wall cells now become converted into elongated irregularly cylindrical cells which stretch continuously from one muscle septum to the next. In the protoplasm of these there begin to appear about stage 31+ (Pl. IV. figs. 13 and 13A, and text-fig. F) contractile fibrils of a similar nature to those which have been long present in the inner wall cells. About stage 31 mesenchyme cells wander in between the myotomes. These give rise to the substance of the septum. Some also wander in between the muscle cells of the outer wall.

By stage 34 the layer of muscle cylinders arising from the outer wall much exceeds the inner wall in thickness. It is distinguished from the latter at the first glance, its muscle cells being slender cylinders instead of flattened parallelepipeds.

The general appearance in transverse section of a myotome of this stage will be gathered from text-fig. G. It will be seen that the lateral branch of the vagus nerve with its surrounding mesenchyme has formed an immovable obstacle, so that the myotome as it increases in thickness, and as it is pushed outwards by the increase of mesenchyme on its inner side, becomes gradually divided into two portions, a dorsal and a ventral, the two remaining for a time connected by a thin isthmus but being eventually completely separated. Along the mesial face of the myotome are seen the muscle cells of the inner wall, now reduced greatly in size in proportion to the whole thickness of the myotome.

The whole thickness of the myotome outside this consists of the derivatives of the outer wall. In its extreme outer portion it consists of cylindrical cells still in the condition described for stage 31, in which contractile fibres are just beginning to appear. From this in a mesial direction the muscle cells are seen to become of greater and greater diameter, and their contained contractile substance increases, especially in size, showing that the muscle cells are older as they are farther removed from the outer surface.

This, together with the fact that mitotic figures are numerous in the external layer, shows that this latter is the region in which growth in thickness of the myotome takes place.

The myoepithelial cells of the inner wall remain distinct up to about stage 35, though constantly becoming more and more insignificant as compared with the great mass of the myotome composed of muscle cylinders derived from the outer wall. About the stage mentioned, however, the myoepithelial cells begin to break down, portions becoming segmented off from their outer ends. These resemble the muscle cells of the outer layer in character, being long cylinders. This process goes on, and soon the once conspicuous myoepithelial cells have become entirely resolved into these cylindrical elements, and the myotome is composed of apparently similar elements throughout its thickness (text-fig. H).

General Remarks.

1. *Motor Nerve Trunks.*

As is well known, there are three main views regarding the development of motor nerves in the vertebrata, which may be shortly stated as follows :—

(1) Each nerve fibre develops as an independent outgrowth from a ganglion cell which gradually grows out towards, and finally and secondarily becomes united to, its special muscle. The sheath of protoplasm surrounding the nerve is an accessory structure of independent origin developed from mesenchyme. This view is associated especially with the name of His, and is the view favoured by the majority of embryologists.

(2) The nerve trunk is multicellular in origin, consisting at first of a chain of cells, in the substance of which the nerve fibres are developed later, as fine fibres passing continuously from one cell body to another. The elements forming the original chain are most frequently looked on as ectodermal elements which have wandered out from the spinal cord rudiments. The protoplasmic sheath is derived from parts of the original cell chains which retain their protoplasmic character (Balfour, Goette, Beard, Dohrn, v. Wijhe, and others).

(3) The nerve trunk is not a secondarily formed bridge between spinal cord and motor end organ. It has existed from the first, and in subsequent development it merely undergoes elaboration from its at first simple protoplasmic beginning (v. Baer, Hensen, Sedgwick, Fürbringer and others).

It is clear that the facts of development in *Lepidosiren*, at least in the motor nerves of that animal, give strong support to the last-mentioned view as regards the nerve trunk itself, and to the second view as regards the protoplasmic sheath. It has been shown that by the examination of earlier and earlier stages the motor trunk can be traced back, without, I think, any possibility of error, to a simple protoplasmic bridge which already connects the substance of the medullary tube with that of the myotome at a stage when they are still in contact.

As regards the origin of the protoplasmic sheath the evidence of *Lepidosiren* is equally emphatic. In its early stages the motor trunk is perfectly naked. About stage 27 masses of mesenchymatous protoplasm laden with yolk become applied to the nerve trunk, at first over only a small portion of its length, and these masses of protoplasm gradually spread over the whole trunk, remaining, however, for some time clearly distinguishable from the nerve trunk by their difference in staining reaction. As development goes on the yolk becomes used up, the protoplasm with its nuclei extends into the substance of the nerve trunk—doubtless to keep up the proper proportion between the bulk of the nerve trunk and its nutritive surface in contact with the sheath protoplasm. The protoplasm itself becomes less and less conspicuous, and eventually is only to be detected in the immediate vicinity of the nuclei.

Lepidosiren offers no evidence, so far, as to the ultimate origin of the nerve fibrils. They appear gradually in an at first simple protoplasmic matrix. They seem to form, as it were, a kind of picture in morphological symbols of already existent physiological activities. It would seem that the continued passage backwards and forwards between centre and end organ of a stream of nerve impulses gradually finds expression in the marking out of the original simple-looking granular protoplasm into definite fibrillar impulse tracks,* the undifferentiated protoplasm remaining as the perifibrillar substance. Such a view of the nature of the fibrils is supported by BETHE's remarkable observation† that in the chick embryo about the end of the sixth day a nerve trunk may lose its fibrillar character in the immediate neighbourhood of a mitotic figure, to reassume it on the completion of mitosis.

It is pretty clear that the great function of the sheath is to serve as a nutritive organ for the nerve trunk. We see how its protoplasm is at first laden with yolk which gradually becomes used up as the nerve trunk develops within it.

That the main function of nuclei, apart from reproduction, is to control cytoplasmic metabolism is well recognised. The nuclei of the sheath are able to exercise this control over the active metabolism of the developing nerve trunk which is without nuclei of its own. Connected with this relation of the sheath nuclei to the metabolism of the nerve trunk is no doubt the active multiplication of these nuclei observed in early stages of nerve regeneration.‡ In such regeneration it may well be that the protoplasmic matrix of the nerve simply repeats the process of its original development, increasing in size and then developing nerve fibrils within itself. If these fibrils represent merely the differentiated paths of nerve impulses passing through the substance of the protoplasm, it would of course happen naturally that the regenerated fibres would be formed in continuity with those of the undegenerated stumps. On this view the process which takes place when the peripheral part of a cut nerve degenerates and then regenerates is somewhat as follows:—The fibrils, no longer subject to the stimulus of passing nerve impulses, revert to their protoplasmic condition. The protoplasmic sheath becomes highly active.§ It increases in thickness: its nuclei divide actively. Its protoplasm digests the remains of the medullary sheath.‖ It thus comes to contain stored-up food material as in its original embryonic heavily yolked condition. The protoplasmic matrix representing the degenerated axis cylinder lies imbedded within the sheath.¶ Controlled and supplied with nourishment by the activities of the surrounding sheath the protoplasm behaves just as it does in ontogenetic development: (1) it *grows*—probably slowly—and

* Were this the case, it might well be that the formation of fibrils might tend as a rule to spread from the end of the nerve from which came the most active and frequent nerve impulses.

† BETHE, *Allgemeine Anatomie und Physiologie des Nervensystems*, p. 244.

‡ BÜNGNER, *Ziegler's Beiträge z. Path. Anat.*, Bd. x., 1891.

§ WIETING, *op. cit.*, Bd. xxiii., 1898.

‖ This view of nerve regeneration, which my ontogenetic work inclines me towards, appears to agree most closely with that enunciated by NEUMANN (*Arch. Path. Anat. u. Phys.*, Bd. clviii. p. 466).

¶ This protoplasmic strand within the protoplasmic sheath could only be demonstrated with extreme difficulty.

so gaps are bridged over; and (2) as soon as it has become continuous, nerve impulses beginning to play backwards and forwards in its substance cause again a differentiation into fibrils. As part of the impulse tracks persist as the stumps of the fibrils, the regenerate parts of the fibrils will naturally develop in exact continuity with these.

I have no intention of entering into a general discussion either as to the data in connection with the development of nerves in other Vertebrates, or as to the general conclusions which have been based on them. My purpose now is merely to emphasize the observations of the phenomena as they occur in *Lepidosiren*. I may, however, be allowed to point out that such observations as those of BETHE upon the Chick,* although apparently supporting the cell chain view, are in no way irreconcilable with the observations here chronicled. The soft protoplasmic bridges which form the first distinguishable rudiment of the motor nerve are difficult to observe in *Lepidosiren*, whose histological features are upon a relatively large and coarse scale; and how much more delicate and difficult to observe are the corresponding structures likely to be in the chick! It may well be that further research will demonstrate the existence of a delicate protoplasmic bridge about which are clustered the "nerve cells" (in APATHY'S sense) observed by upholders of the cell chain theory.

DESCRIPTION OF PLATES.

GENERAL LIST OF ABBREVIATIONS.

c.f. Contractile fibrils.
d.r. Dorsal root.
g. Glycogen-containing outer portion of myoblast.
i.w. Inner wall of myotome.
l.vag. Lateral branch of vagus.
me. Mesenchyme.
m.n.r. Motor nerve root.
my. Myotome.
n. Notochord.
nu. Nucleus.
n.c. Neural canal.
n.s. Nuclei of protoplasmic sheath.
n.t. Nerve trunk.
o.w. Outer wall of myotome.
p.s. Protoplasmic sheath of nerve trunk.
s.c. Spinal cord.
v.r. Ventral root.
y. Yolk granules.

PLATE I.

[The figures on this plate are all from untouched negatives.†]

Fig. 1. Portion of spinal nerve of stage 34. (108·28 35.) *l.vag.*, *lateralis vagi*; *my.* myotome; *n.* notochord; *n.t.* nerve trunk; *n.s.* nuclei of protoplasmic sheath; *d.r.* dorsal root; *v.r.* ventral root.

Fig. 2. Portion of spinal nerve trunk of stage 29.* (93 C. 1435.) *my.* myotome; *n.* notochord; *n.t.* motor nerve trunk; *p.s.* protoplasmic sheath containing yolk granules (black), and large nuclei rich in chromatin.

* *Op. cit.*, p. 238.

† Much of the minute detail has unfortunately disappeared in the mechanical printing of these figures. I shall be glad therefore to show any specialists who are interested sun prints from the same negatives, in which the full detail is brought out.

Fig. 3. Part of transverse section at stage 27. (69 C. 1064.) Showing whole length of motor nerve trunk. *my.* myotome; *n.* notochord; *n.t.* motor nerve trunk; *p.s.* protoplasmic sheath now in form of a mass of richly yolked mesenchymatous protoplasm aggregated round the nerve trunk near its outer end. The greater part of the nerve trunk is still naked. At its outer end the nerve trunk passes out into conically arranged strands of protoplasm forming the inner ends of the muscle cells.

Fig. 4. Part of transverse section at stage 25. (79 A. 933.) *my.* myotome; *n.* notochord; *n.t.* motor nerve trunk already faintly fibrillated passing between spinal cord (*s.c.*) and myotome. The mesenchyme (*me.*) has not yet begun to concentrate round the nerve trunk.

Fig. 5. Part of transverse section at stage 24. (73 G. 986.) *my.* myotome; *n.* notochord; *n.t.* motor nerve trunk, at this time composed of granular protoplasm, and naked.

Fig. 6. Similar section to last, but taken from rather less advanced specimen. (Stage 24; 73 F. 688.) *my.* myotome; *n.t.* motor nerve trunk—a naked protoplasmic bridge connecting myotome and spinal cord (*s.c.*).

Plate II.

[All figures are camera drawings of single sections. Figs. 7, 8 and 9 represent the same sections as are photographed in figs. 2, 3 and 4, with the additional detail visible under Zeiss' 3 mm. apochromatic homogeneous immersion objective.]

Fig. 7 (=fig. 2). The motor nerve trunk (*n.t.*) is seen within its thick protoplasmic sheath (*p.s.*).

Fig. 8 (=fig. 3). The motor trunk is now naked except for the large mass of yolk-laden mesenchymatous protoplasm (*p.s.*) which has concentrated round it towards its outer end. The continuity of nerve trunk with protoplasm of muscle cell is seen.

Plate III.

Fig. 9 (=fig. 4). The nerve trunk is seen to be already fibrillar in structure. Mesenchyme (*me.*) has penetrated in between myotome and spinal cord, but has not yet begun to concentrate round the nerve trunk to form its sheath.

Fig. 10. Stage 24. The nerve has been torn away from the inner surface of the myotome so that its expanded outer end is seen.

Fig. 11. Part of transverse section, stage 30. (93 B. 2844.) The two walls of the myotome are seen—the outer one-layered, the inner composed of a layer of myoblasts in which are seen the first contractile fibrils (*c.f.*).

Fig. 12. Part of transverse section, stage 31. (103 D. 2463.) The outer wall of the myotome is beginning to show more than one layer of nuclei. In the myoblasts of the inner wall the contractile fibrils (*c.f.*) have greatly increased in number, most of the protoplasm of the inner half of the cell being converted into fibrils.

Plate IV.

Fig. 12A. Part of longitudinal horizontal section at stage 31 (103 C. 822), showing a single myotome. *i.w.* Inner wall cell; *c.f.* contractile fibrils; *g.* glycogen-containing outer part of cell; *nu.* nuclei of inner wall cell; *y.* yolk granules; *o.w.* outer wall of myotome, two of the nuclei undergoing mitosis; *me.* mesenchyme nuclei of septum between myotomes.

Fig. 13. Part of transverse section at stage 31+. (106 C. 1573.) The outer wall of the myotome is now several layers thick, and the cells of this wall have also developed contractile fibrils (*c.f.*).

Fig. 13A. Longitudinal horizontal section through a myotome of stage 31+. (106 A. 1811.) The nuclei of the inner wall myoblast are seen to have considerably increased in numbers. The contractile fibres of the outer wall (*c.f.*) are visible.

Fig. 14. A nucleus of one of the inner wall myoblasts during mitosis, showing the cell-like demarcation of the protoplasm immediately surrounding it.

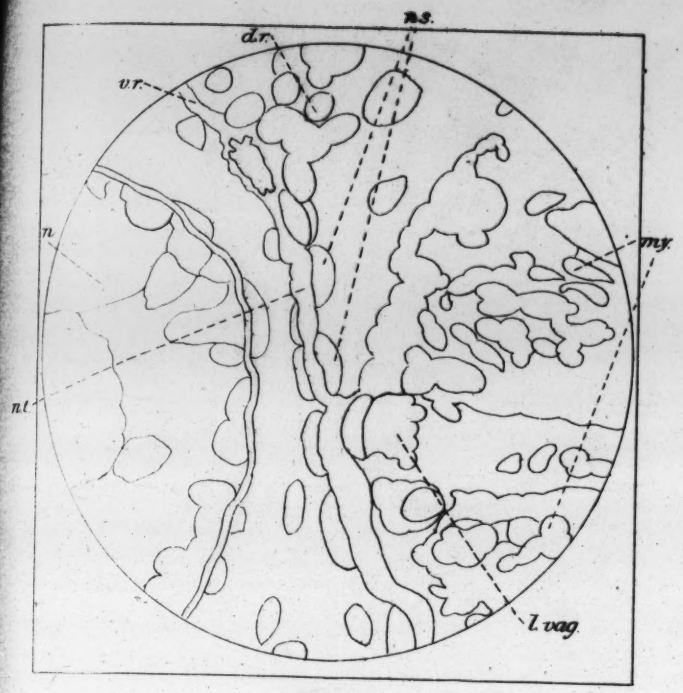


FIG. 1.

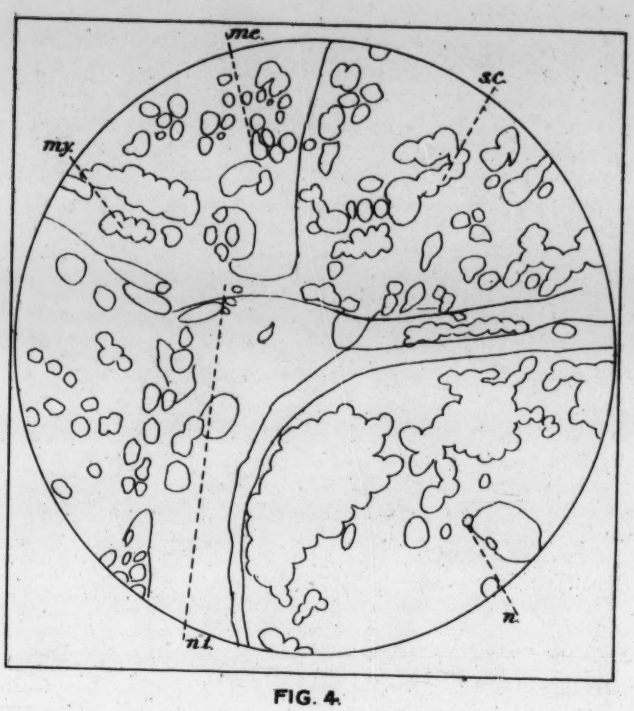


FIG. 4.

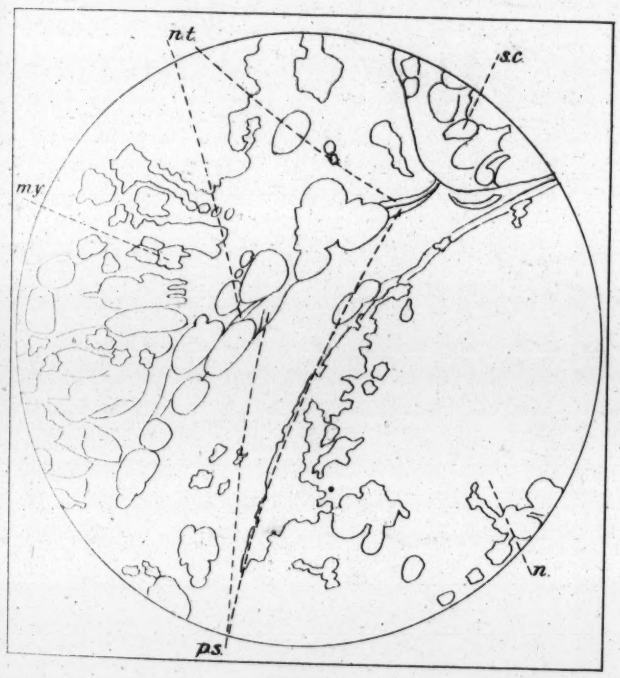


FIG. 2.

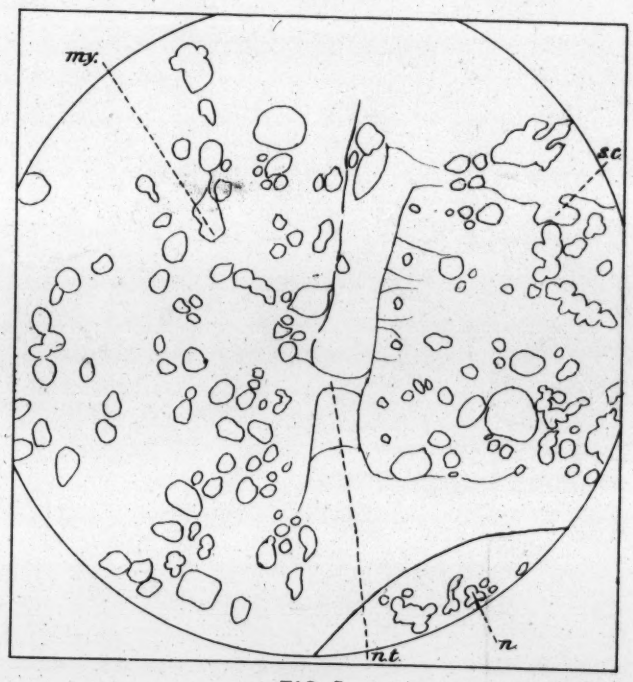


FIG. 5.

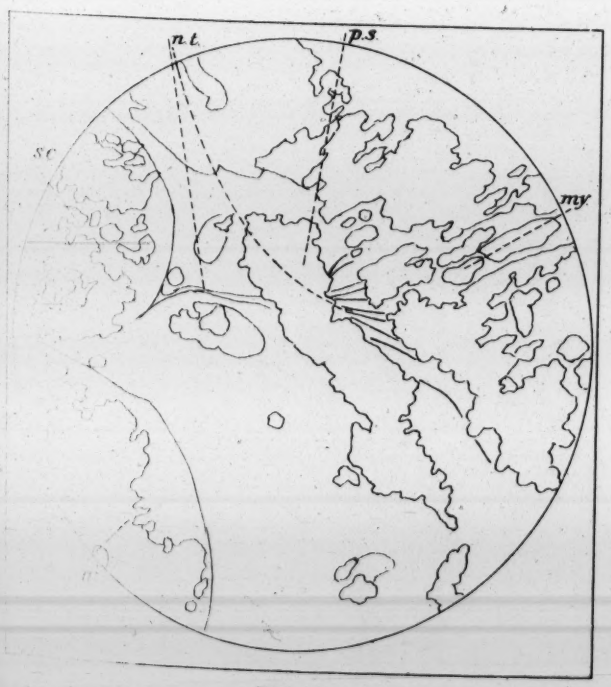


FIG. 3.

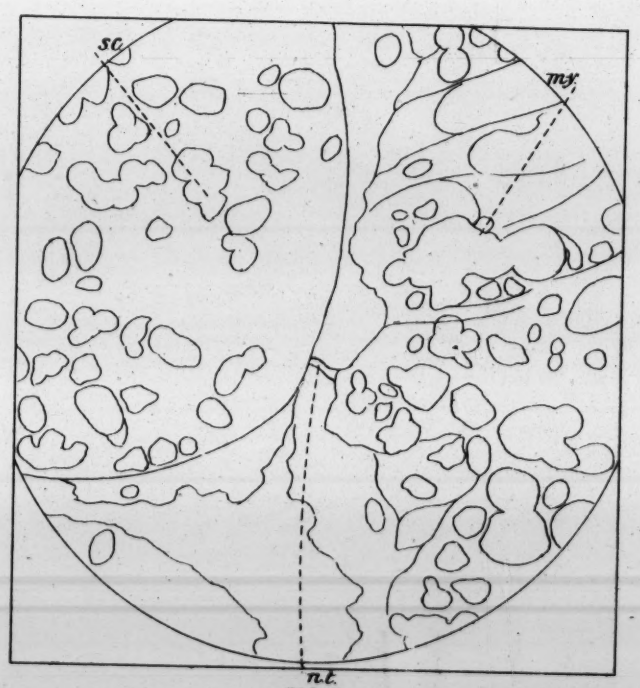
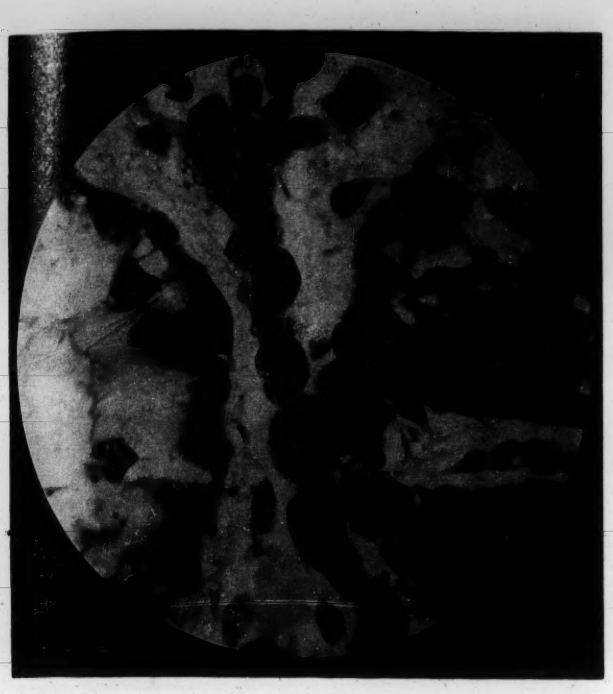
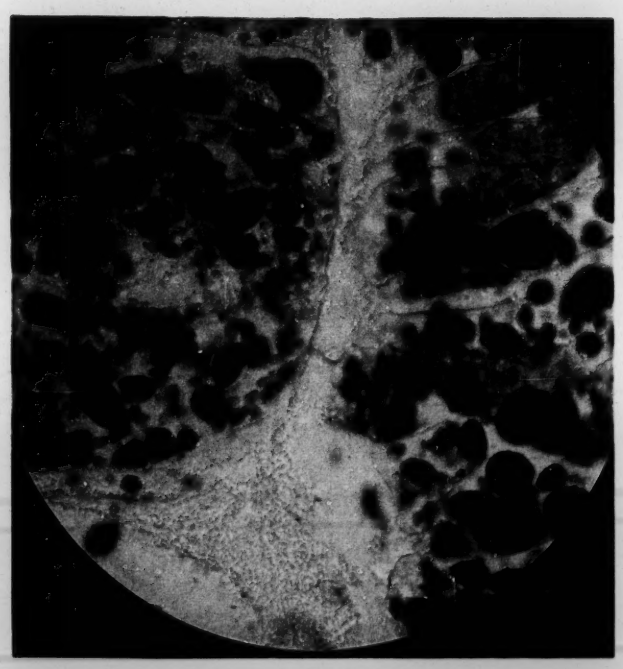


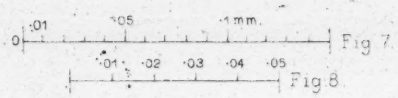
FIG. 6.

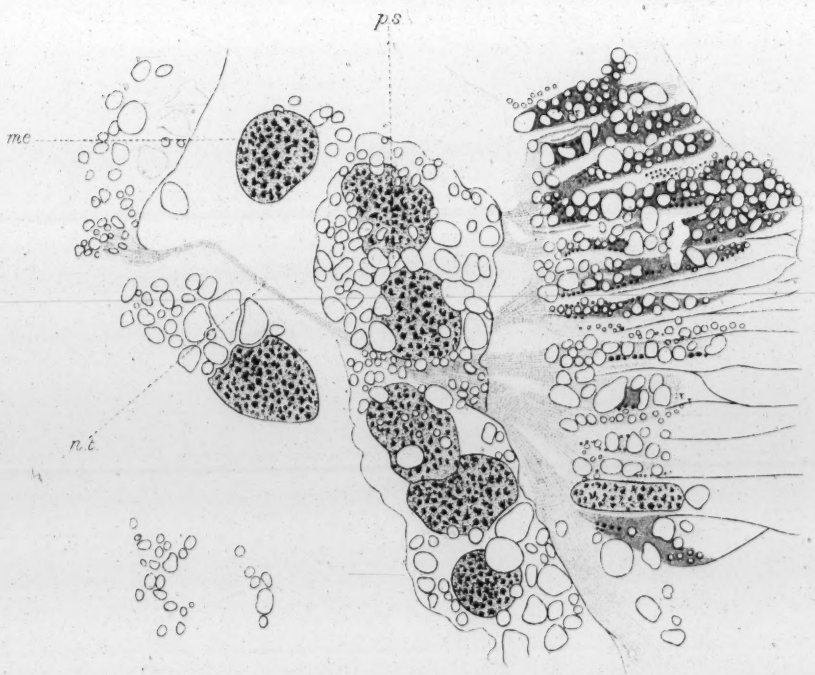




KERR: EARLY DEVELOPMENT OF MOTOR NERVE TRUNKS AND MYOTOMES IN LEPIDOSIREN PARADOXA. PLATE 2.

7.

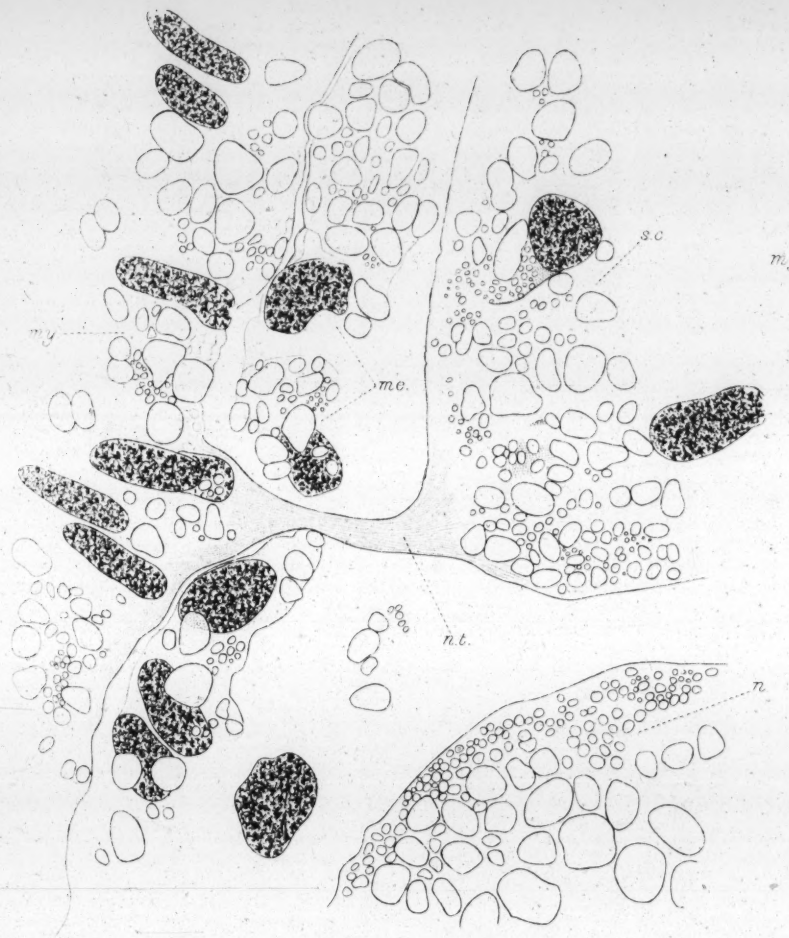




8.

E. Wilson, Cambridge.

KERR: EARLY DEVELOPMENT OF MOTOR NERVE TRUNKS AND MYOTOMES IN LEPIDOSIREN PARADOXA. PLATE 3.



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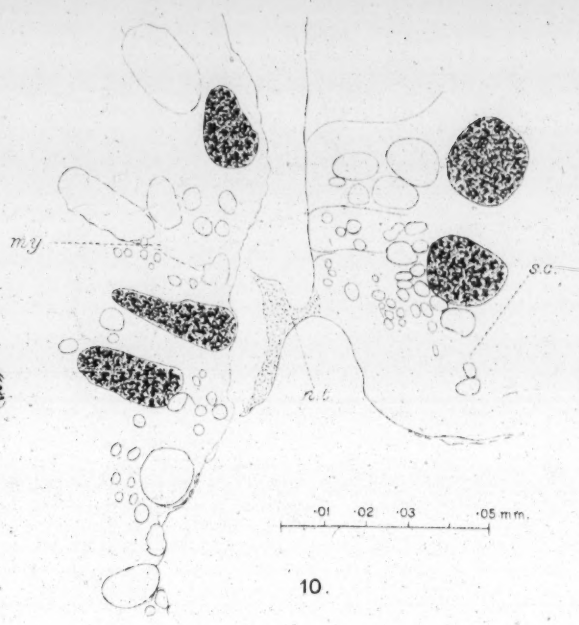

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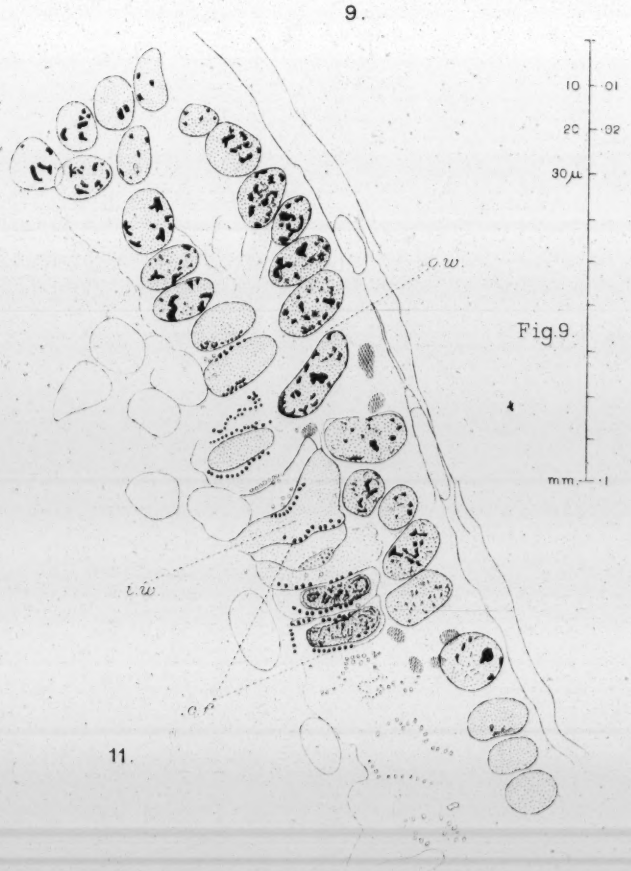
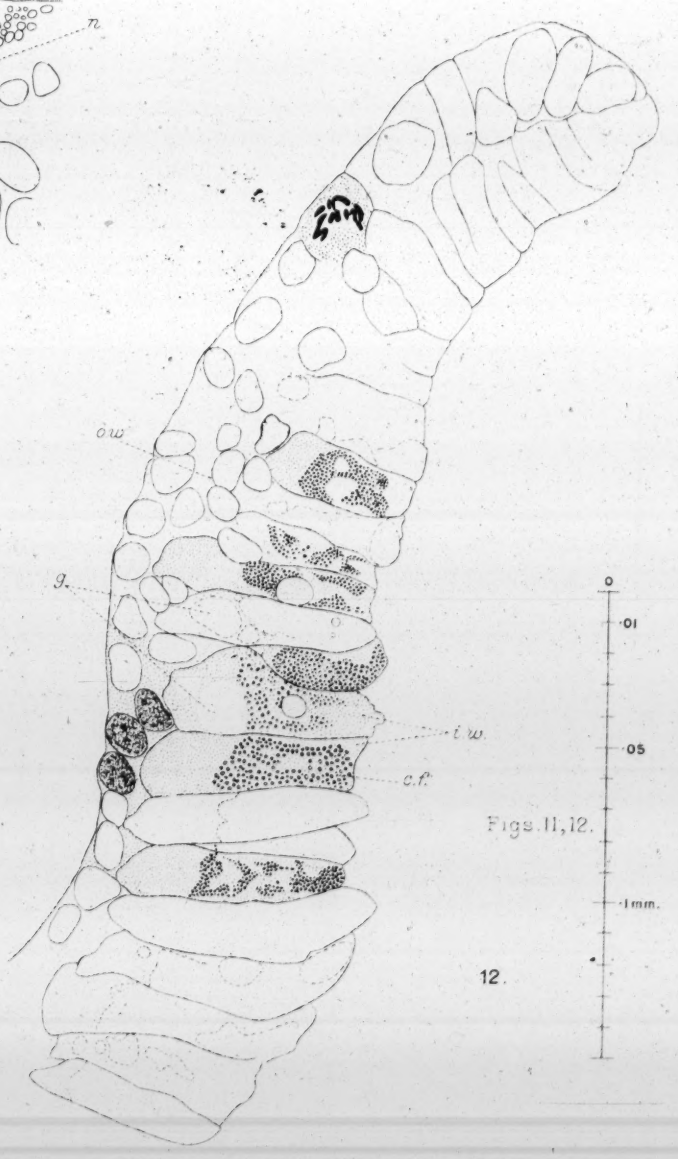


11.

Fig. 9.

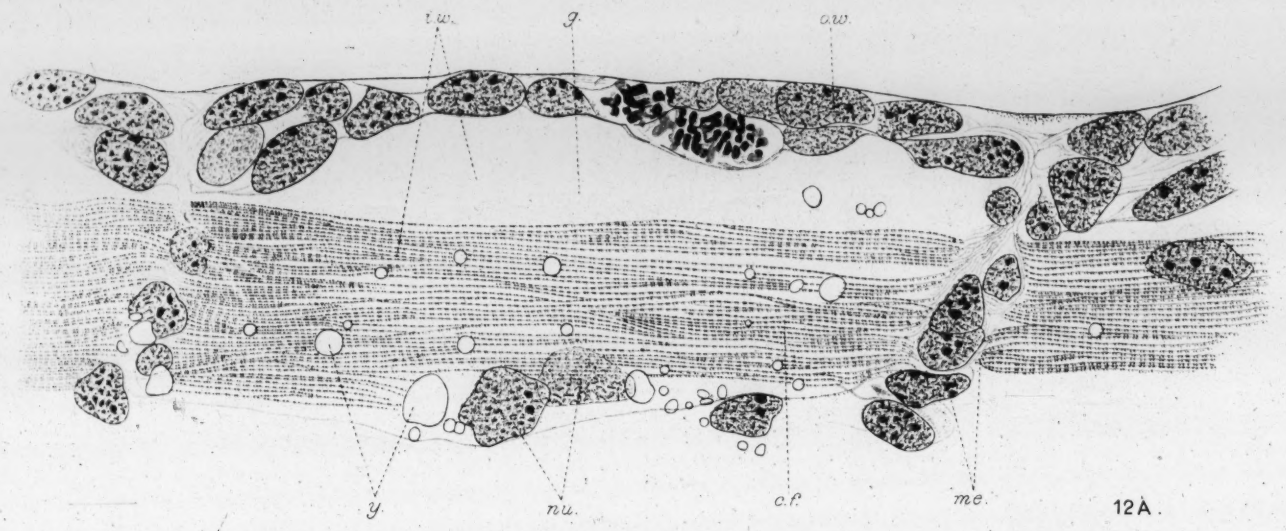


12.

Figs. 11, 12.

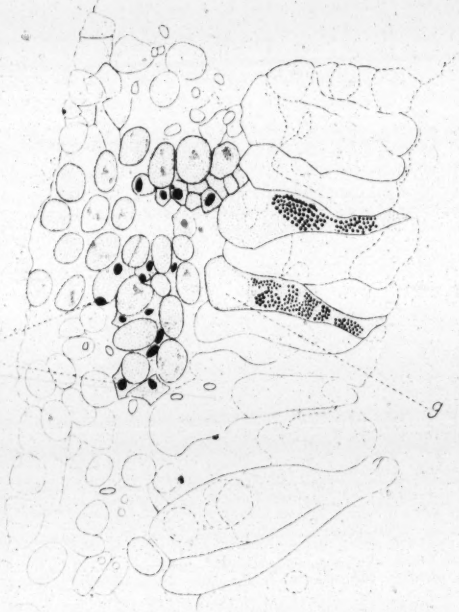
E. Wilson, Cambridge

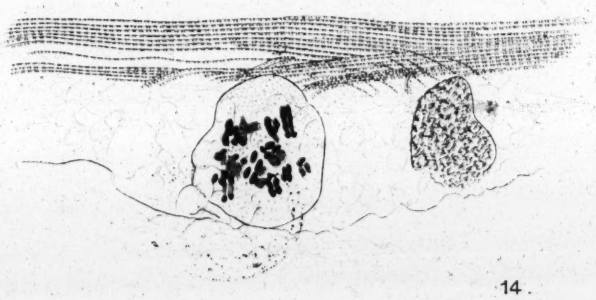
KERR: EARLY DEVELOPMENT OF MOTOR NERVE TRUNKS AND MYOTOMES IN LEPIDOSIREN PARADOXA. PLATE 4.



12A.

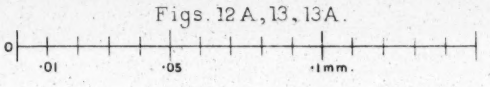
13.

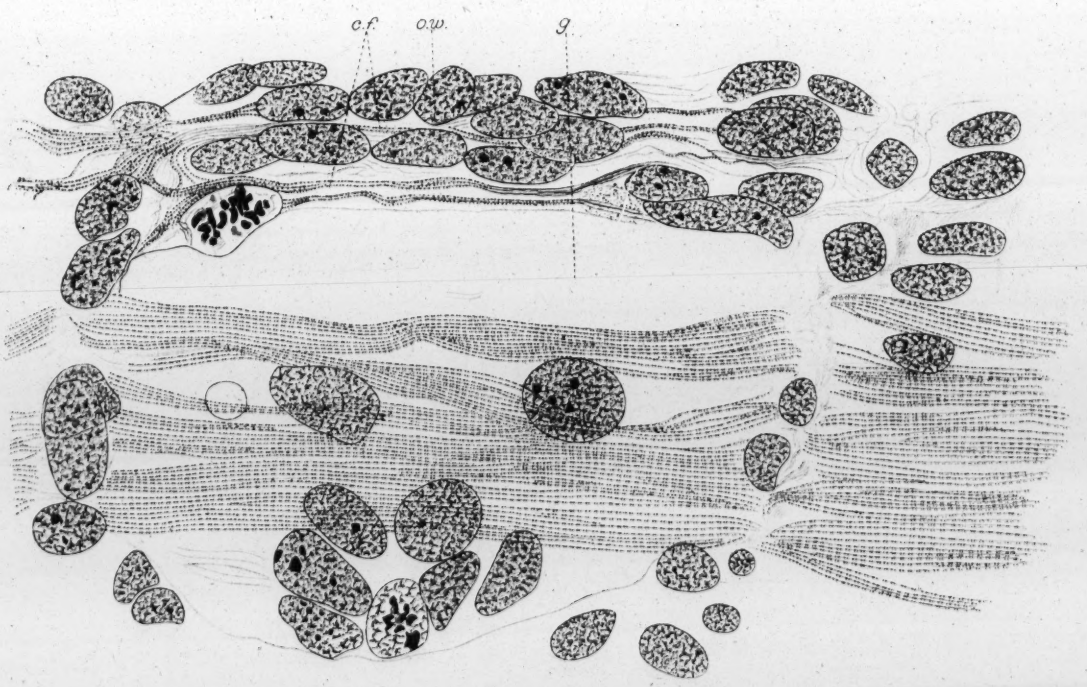




14.

Figs. 12A, 13, 13A.





13A.

E. Wilson, Cambridge.

Prof. J. GRAHAM KERR on "Some points in the Early Development of Motor Nerve Trunks and Myotomes in *Lepidosiren paradoxa*."—PLATE V.

DESCRIPTION OF TEXT FIGURES A–H.

Camera outlines of portions of transverse sections of the trunk region of young Lepidosirens at various stages, to show the topographical relations of the myotomes. Zeiss Obj. A, Oc. 2. *i.w.*, Myoblasts of inner wall of myotome; *l. vag.*, lateral branch of vagus.

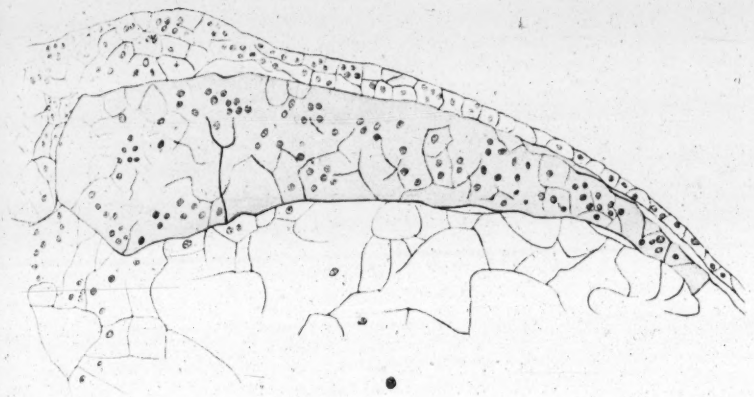


FIG. A.—Stage 16.

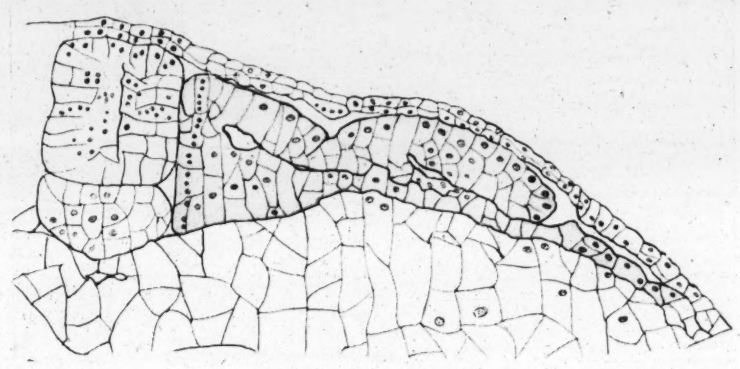


FIG. B.—Stage 21.

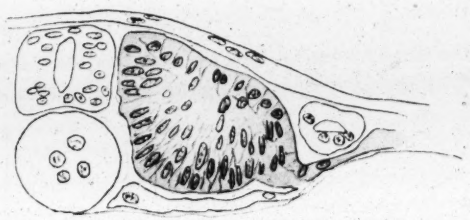


FIG. C.—Stage 24.

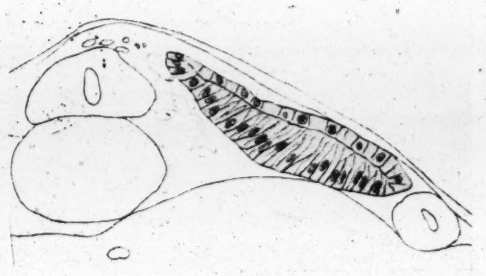


FIG. D.—Stage 27.

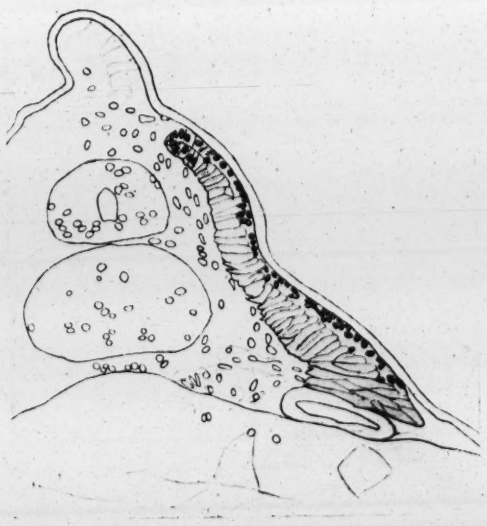


FIG. E.—Stage 31.

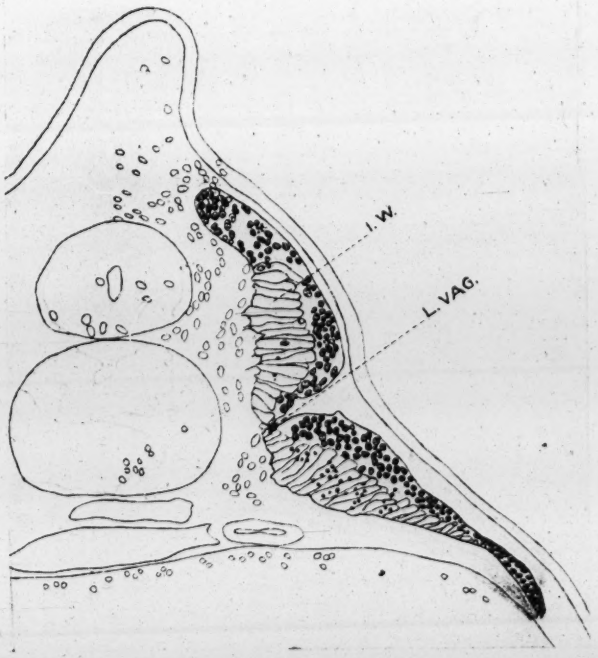


FIG. F.—Stage 31+.

VIII.—An Analytical Theory of the Equilibrium of an Isotropic Elastic Plate.

By John Dougall, M.A. *Communicated by* Professor G. A. Gibson.

(MS. received February 4, 1904. Read March 21, 1904. Issued separately August 5, 1904.)

The following paper contains a purely analytical discussion of the problem of the deformation of an isotropic elastic plate under given forces. The problem is an unusually interesting one. It was the first to be attacked (by Lamé and Clapeyron in 1828) after the establishment of the general equations by Navier. The solution of the problem of normal traction given by these authors, when reduced to its simplest form, involves double integrals of simple harmonic functions of the coordinates. The integrals are of complicated form, and practically impossible to interpret, a fact which, without doubt, has had much to do with the neglect of the problem in later times, and the almost complete absence of attempts to establish the approximate theory on the basis of an exact solution. An even more serious defect of Lamé and Clapeyron's solution is that the integrals, as they stand, do not converge. A flaw of this sort has often been treated lightly by physical writers, the non-convergence of an integral being regarded as due to the inclusion of an infinite but unimportant constant. In the present case, however, the infinite terms are not constant, but functions of the coordinates, and the modifications necessary to secure convergence, so far from being unimportant, lead directly to the most significant terms of the solution.

The next writer to deal with the exact problem was Sir W. Thomson, who, at the end of the memoir in which he solved the problem of a spherical shell, indicated the form which the solution would take in the limiting case of a plate. His method, if carried out, would lead to integrals of the same form as Lamé's.

Solutions of special problems have been given by other writers. Prof. Lamb has worked out the solution for an infinite solid subjected to normal pressure proportional to cos kx, and verified in this particular case some of the results of the approximate theory of thin plates (*Proc. Lond. Math. Soc.*, vol. xxi., 1889–90).

The history of the approximate theory is well known and easily accessible. It will be sufficient here to refer to—

(i) Todhunter and Pearson's *History of the Elasticity and Strength of Materials.*

(ii) Clebsch's standard treatise, *Théorie de l'élasticité des corps solides*, as translated by Saint Venant; in particular, Part I. chap. iii., and Saint Venant's brilliant note on § 73.

(iii) Prof. Love's *Treatise on the Theory of Elasticity*, 1892, — especially the historical introductions to both volumes.

The various forms of the approximate theory rest partly upon the general equations of equilibrium, partly upon auxiliary hypotheses or physical principles. These principles are recognised as contained in the general equations, but on account of the

analytical difficulties in the way of deducing them rigorously, they are either simply assumed, or else supported by reasoning plausible rather than demonstrative.

In the following pages the problem is treated as a purely mathematical one, and the approximate theory for a finite plate deduced from an exact solution for an infinite plate. The main features of the method are—

(i) The use of Bessel functions in place of the simple harmonic functions of previous writers. Only the symmetrical forms, or functions of order zero, are required.

(ii) Transformation of the definite integrals, in terms of which the solutions are in the first place obtained, into series, by means of Cauchy's theory of contour integration and residues. The series involve Bessel functions of the second kind with complex argument, and are so highly convergent that the principal features of the strain represented by the solution can be made out with the utmost ease. (The transformations belong to a class discussed systematically, apparently for the first time, in a paper "On the Determination of Green's Function by means of Cylindrical and Spherical Harmonics," *Proc. Edin. Math. Soc.*, vol. xviii.)

(iii) Detailed solutions of the problems of internal force with vanishing face traction. The usual method of dealing with a general problem in Elasticity is to find a particular solution for the bodily force, and then to treat the problem of surface tractions completely. This is theoretically sufficient, but leaves the result in a complicated form, which in the present case must be simplified before practical applications can be made.

(iv) Use of Betti's Theorem (LOVE, *Elasticity*, vol. i. § 140) to develop a method analogous to the method of Green's function in the Theory of the Potential, by which the properties of the solution for a finite plate can be deduced from the infinite plate solution. (Cf. *Proc. Edin. Math. Soc.*, vol. xvi., "On a general Method of Solving the Equations of Elasticity.")

The results of the ordinary theory are fully confirmed, and extended in various directions. The infinite solid solution gives, of course, an exact particular solution for internal force and traction on the plane faces of a finite plate. At the head of the solution appear the terms given by the approximate theory. In the case of flexure, the equation of Lagrange is obtained to a second approximation.

The problem of a finite plate under given edge tractions cannot be completely solved, but exact solutions are given of certain problems relating to a circular plate. For a thin plate, with edge of any shape, the conditions satisfied at the edge by the principal terms of the exact solution are found to a degree of approximation beyond the reach of any theory which rests merely on the "principle of the elastic equivalence of statically equipollent loads." For example, the celebrated boundary conditions given by KIRCHHOFF, in correction of POISSON, are verified, and extended by the inclusion of terms of higher order.

In conclusion, it may be mentioned that the methods given here are equally applicable to the problem of the vibrations of a plate, and to the problems of the equilibrium and vibration of a finite circular cylinder, or of an open spherical shell. Some account of these applications I hope to publish shortly.

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Introductory Analysis.

(*a*) The Bessel function J is defined by the series

$$J_m(z) = \frac{z^m}{2^m \Pi m}\left(1 - \frac{z^2}{2 \cdot 2m+2} + \frac{z^4}{2 \cdot 4 \cdot 2m+2 \cdot 2m+4} - \cdots\right)$$

For the function of the second kind we take as definition

$$G_m(z) = \frac{\pi}{2}\frac{J_{-m}(z) - e^{-im\pi}J_m(z)}{\sin m\pi}$$

This makes $G_m z$ an analytic function of m, the value of which, when m is a real integer, is

$$G_m(z) = \left(\log 2 - \gamma + \frac{\pi i}{2}\right)J_m(z) - Y_m(z)$$

where $Y_m(z)$ is Neumann's function.
In this case, therefore, $G_m z = -\log z J_m z$ + a uniform function of z.

In the following pages we are concerned chiefly with the function of order zero

$$G_0 z = \left(\log 2 - \gamma + \frac{\pi i}{2}\right)\left(1 - \frac{z^2}{2^2} + \frac{z^4}{2^2.4^2} - \cdots\right) - \log z\left(1 - \frac{z^2}{2^2} + \frac{z^4}{2^2.4^2} - \cdots\right) - \frac{z^2}{2^2} + \left(1 + \frac{1}{2}\right)\frac{z^4}{2^2.4^2} - \cdots$$

When mod z is very large, while the phase (argument) of z lies between $-\pi/2$ and $3\pi/2$, then approximately

$$G_m z = e^{\frac{-m\pi i}{2}} e^{i\left(z + \frac{\pi}{4}\right)} \sqrt{\frac{\pi}{2z}}.$$

Similarly, when the phase of z is between 0 and π (excluding those values)

$$J_m z = e^{\frac{m\pi i}{2}} e^{-i\left(z - \frac{\pi}{4}\right)} \sqrt{\frac{1}{2\pi z}}.$$

(*b*) If x, y, z and ρ, ω, z are the rectangular and cylindrical coordinates of a point in space, so that $x = \rho \cos \omega$, $y = \rho \sin \omega$, then the most important property of the Bessel Functions is that each of the eight functions

$$(e^{\kappa z} \text{ or } e^{-\kappa z})\ (J_m \kappa\rho \text{ or } G_m \kappa\rho)\ (\cos m\omega \text{ or } \sin m\omega)$$

satisfies Laplace's equation, or in other words is a potential function.
Hence

$$(\nabla^2 + \kappa^2) \cdot (J_m \kappa\rho \text{ or } G_m \kappa\rho)\ (\cos m\omega \text{ or } \sin m\omega) = 0.$$

Further, if

$$\begin{aligned} R^2 &= (x - x')^2 + (y - y')^2 \\ &= \rho^2 + \rho'^2 - 2\rho\rho' \cos(\omega - \omega') \end{aligned}$$

then

$$(\nabla^2 + \kappa^2) \cdot (J_0 \kappa R \text{ or } G_0 \kappa R) = 0.$$

Let now $I = \iint G_0 \kappa R f(x', y') dx' dy'$, the integral being taken over a finite area A.

Then $(\nabla^2 + \kappa^2) I = 0$, if (x, y) is without,

but $(\nabla^2+\kappa^2)\mathrm{I} = -2\pi f(x,y)$, if (x,y) is within this area; as easily follows from the theorem

$$\nabla^2 \iint \log \mathrm{R} f(x',y')dx'dy' = 2\pi f(x,y).$$

The differential equation satisfied by I, together with the conditions that I and its first derivatives $d\mathrm{I}/dx$, $d\mathrm{I}/dy$ are continuous throughout, define the value of the integral completely, and in many cases make its evaluation easy.

(*c*) For example, take $f(x,y) = \mathrm{J}_m\beta\rho\cos m\omega$, with m an integer, so that

$$\mathrm{I} = \iint \mathrm{G}_0\kappa\mathrm{R}\,\mathrm{J}_m\beta\rho'\cos m\omega'\rho'd\rho'd\omega'$$

and suppose the area of integration to be a circle of radius a, with centre at the origin. For convenience in the proof, let the imaginary part of κ be positive.

$$\begin{aligned}\text{Then}\quad \mathrm{I} &= \frac{2\pi}{\beta^2-\kappa^2}\mathrm{J}_m\beta\rho\cos m\omega + \mathrm{A}\mathrm{J}_m\kappa\rho\cos m\omega, \quad \text{when } \rho<a,\\ &= \mathrm{B}\mathrm{G}_m\kappa\rho\cos m\omega, \qquad\qquad\quad , \quad \text{when } \rho>a\end{aligned}$$

A, B are determined from the conditions that I and $d\mathrm{I}/d\rho$ are continuous at $\rho=a$. Thus we find

$$\begin{aligned}\mathrm{I} &= \frac{2\pi}{\beta^2-\kappa^2}\mathrm{J}_m\beta\rho\cos m\omega + \frac{2\pi}{\beta^2-\kappa^2}\mathrm{J}_m\kappa\rho\cos m\omega(\kappa a\mathrm{G}_m'\kappa a\mathrm{J}_m\beta a - \mathrm{G}_m\kappa a\beta a\mathrm{J}_m'\beta a); \quad (\rho<a)\\ &= \frac{2\pi}{\beta^2-\kappa^2}\mathrm{G}_m\kappa\rho\cos m\omega(\kappa a\mathrm{J}_m'\kappa a\mathrm{J}_m\beta a - \mathrm{J}_m\kappa a\beta a\mathrm{J}_m'\beta a); \qquad (\rho>a)\end{aligned}$$

By the principle of continuation in the Theory of Functions, the result is true whatever be the phase of κ. But when the phase of κ is diminished by 2π,

$$\mathrm{G}_m(\kappa c) \quad \text{is increased by} \quad 2\pi i\mathrm{J}_m(\kappa c)$$

and

$$\mathrm{G}_m'(\kappa c) \quad \text{by} \quad 2\pi i\mathrm{J}_m'(\kappa c);$$

hence, equating the corresponding changes in I and its value, we obtain

$$\begin{aligned}&\int_0^a\int_0^{2\pi}\mathrm{J}_0\kappa\mathrm{R}\,\mathrm{J}_m\beta\rho'\cos m\omega'\rho'd\rho'd\omega'\\ &= \frac{2\pi}{\beta^2-\kappa^2}\mathrm{J}_m\kappa\rho\cos m\omega(\kappa a\mathrm{J}_m'\kappa a\mathrm{J}_m\beta a - \mathrm{J}_m\kappa a\beta a\mathrm{J}_m'\beta a).\end{aligned}$$

From this again it easily follows that in I and its value we may replace the G functions by the Y functions.

(*d*) We have

$$\begin{aligned}\mathrm{Y}_0\kappa\rho &= \log\kappa\rho\mathrm{J}_0\kappa\rho + \tfrac{1}{4}\kappa^2\rho^2 - \ldots\\ &= \log\kappa\mathrm{J}_0\kappa\rho + \log\rho(1-\tfrac{1}{4}\kappa^2\rho^2\ldots) + \tfrac{1}{4}\kappa^2\rho^2\ldots\end{aligned}$$

Thus $\log\kappa\,\mathrm{J}_0\kappa\rho - \mathrm{Y}_0\kappa\rho$ is an integral function of κ, in which

$$\begin{aligned}\text{coefficient of } \kappa^0 \text{ is } &-\log\rho,\\ \text{and coefficient of } \kappa^2 \text{ is } &\tfrac{1}{4}\rho^2\log\rho - \tfrac{1}{4}\rho^2.\end{aligned}$$

The functions $\log \rho$ and $\frac{1}{4}\rho^2 \log \rho - \frac{1}{4}\rho^2$ are thus in a sense degenerate forms of the Bessel Functions, and any theorem relating to the G or Y functions will yield a corresponding theorem in these.

Thus by equating coefficients of κ^2 in the equation

$$(\nabla^2 + \kappa^2)(\log \kappa J_0 \kappa\rho - Y_0 \kappa\rho) = 0$$

we obtain

$$\nabla^2 \log \rho = 0$$
$$\nabla^2(\tfrac{1}{4}\rho^2 \log \rho - \tfrac{1}{4}\rho^2) = \log \rho$$

and therefore

$$\nabla^4(\tfrac{1}{4}\rho^2 \log \rho - \tfrac{1}{4}\rho^2) = 0 .$$

We deduce at once

$$\nabla^2 \iint (\tfrac{1}{4}R^2 \log R - \tfrac{1}{4}R^2) f(x', y') dx' dy' = \iint \log R f(x', y') dx' dy'$$

and

$$\nabla^4 \iint (\tfrac{1}{4}R^2 \log R - \tfrac{1}{4}R^2) f(x', y') dx' dy' = \nabla^2 \iint \log R f(x', y') dx' dy'$$
$$= 2\pi f(x, y) .$$

(*e*) Again, from the addition theorem

$$Y_0 \kappa R = Y_0 \kappa\rho J_0 \kappa\rho' + 2 \sum_{m=1} Y_m \kappa\rho J_m \kappa\rho' \cos m(\omega - \omega') ; \quad (\rho > \rho')$$

we deduce

$$\log R = \log \rho - \sum_{m=1} \frac{1}{m} \left(\frac{\rho'}{\rho}\right)^m \cos m(\omega - \omega') ; \quad \rho > \rho'$$

and

$$\tfrac{1}{4}R^2 \log R - \tfrac{1}{4}R^2 = (\tfrac{1}{4}\rho^2 \log \rho - \tfrac{1}{4}\rho^2) + \tfrac{1}{4}\rho'^2 \log \rho$$
$$+ \left\{ (\rho - 2\rho \log \rho) \frac{\rho'}{4} - \frac{\rho'^3}{8\rho} \right\} \cos(\omega - \omega')$$
$$+ \sum_{m=2} \frac{1}{4m} \left(\frac{\rho'}{\rho}\right)^m \left(\frac{\rho^2}{m-1} - \frac{\rho'^2}{m+1} \right) \cos m(\omega - \omega') ; \quad (\rho > \rho') .$$

(*f*) In the same way, from the results of (*c*), we may deduce the value of the integral

$$I_0 = \int_0^a \int_0^{2\pi} (\tfrac{1}{4}R^2 \log R - \tfrac{1}{4}R^2) J_m \beta\rho' \cos m\omega' \rho' d\rho' d\omega' .$$

The form of the result varies in the cases $m = 0$, $m = 1$, $m > 1$.

$m = 0$:

$$I_0 = \frac{2\pi}{\beta^4} J_0 \beta\rho + \frac{2\pi}{\beta^2} \left\{ \left(\frac{1}{\beta^2} - \frac{\rho^2}{4} \right) \left(\log a \beta a J_0' \beta a - J_0 \beta a \right) \right.$$
$$\left. + \frac{a^2}{4} \left(1 - \log a\right) \beta a J_0' \beta a + \frac{a^2}{4} \left(2 \log a - 1\right) J_0 \beta a \right\}, \quad \rho < a$$
$$= \frac{2\pi}{\beta^2} \left\{ \left(\frac{1}{4}\rho^2 \log \rho - \frac{1}{4}\rho^2 \right) \left(- \beta a J_0' \beta a \right) + \log \rho \left(\frac{1}{\beta^2} - \frac{a^2}{4} \beta a J_0' \beta a + \frac{a^2}{2} J_0 \beta a \right) \right\}, \quad \rho > a .$$

$m = 1$:

$$I_0 = \frac{2\pi}{\beta^4} J_1 \beta\rho \cos \omega - \frac{2\pi}{\beta^2} \frac{\rho \cos \omega}{2a} \left\{ \left(\frac{1}{\beta^2} - \frac{\rho^2}{8} \right) \left(J_1 \beta a + \beta a J_1' \beta a \right) \right.$$
$$\left. + \frac{a^2}{4} \left(1 + 2 \log a\right) J_1 \beta a + \frac{a^2}{4} \left(1 - 2 \log a\right) \beta a J_1' \beta a \right\}, \quad \rho < a .$$
$$= \frac{2\pi}{\beta^2} \frac{a \cos \omega}{2\rho} \left\{ \left(\frac{1}{\beta^2} + \frac{1}{4}\rho^2 - \frac{1}{2}\rho^2 \log \rho \right) \left(J_1 \beta a - \beta a J_1' \beta a \right) + \frac{a^2}{8} \left(\beta a J_1' \beta a - 3 J_1 \beta a \right) \right\}, \quad \rho > a .$$

$m > 1$:

$$I_0 = \frac{2\pi}{\beta^4} J_m\beta\rho \cos m\omega - \frac{2\pi}{\beta^2}\frac{\rho^m \cos m\omega}{2ma^m}\left\{\left(\frac{1}{\beta^2} - \frac{\rho^2}{4m+4}\right)\left(\beta a J_m'\beta a + m J_m\beta a\right)\right.$$
$$\left. + \frac{a^2}{4m-4}\left(\beta a J_m'\beta a + \overline{m-2} J_m\beta a\right)\right\}, \quad \rho < a$$
$$= \frac{2\pi}{\beta^2}\frac{a^m \cos m\omega}{2m\rho^m}\left\{\left(\frac{1}{\beta^2} + \frac{\rho^2}{4m-4}\right)\left(m J_n\beta a - \beta a J_m'\beta a\right) + \frac{a^2}{4m+4}\left(\beta a J_m'\beta a - m + 2 J_m\beta a\right)\right\}, \quad \rho > a.$$

The corresponding integrals with $\log R$ in place of $\frac{1}{4}R^2 \log R - \frac{1}{4}R^2$ may be obtained at once by taking ∇^2 of the above.

Also, all through we may write $\log(R/c)$, $\log(\rho/c)$, $\log(a/c)$, instead of $\log R$, $\log \rho$, $\log a$, this amounting merely to a change in the unit of length.

(g) By equating coefficients of like powers of β in the results of (c), (f) we can obtain

$$\int_0^a \int_0^{2\pi} G_0\kappa R \rho'^{m+2n} \cos m\omega' \rho' d\rho' d\omega'$$

and

$$\int_0^a \int_0^{2\pi} (\tfrac{1}{4}R^2 \log R - \tfrac{1}{4}R^2) \rho'^{m+2n} \cos m\omega' \rho' d\rho' d\omega'.$$

In the case when $m = n = 0$

$$\int_0^a \int_0^{2\pi} G_0\kappa R \rho' d\rho' d\omega' = -\frac{2\pi}{\kappa^2} - \frac{2\pi}{\kappa^2} J_0\kappa\rho\kappa a G_0'\kappa a, \quad \rho < a$$
$$= -\frac{2}{\kappa^2} G_0\kappa\rho\kappa a J_0'\kappa a, \quad \rho > a.$$

$$\int_0^a \int_0^{2\pi} \left(\frac{1}{4}R^2 \log R - \frac{1}{4}R^2\right)\rho' d\rho' d\omega' = \frac{\pi}{32}\left\{\rho^4 + 4\rho^2 a^2 (2\log a - 1) + a^4 (4\log a - 5)\right\}, \quad \rho < a$$
$$= \left(\frac{1}{4}\rho^2 \log\rho - \frac{1}{4}\rho^2\right)\pi a^2 + \log\rho \frac{\pi a^4}{8}, \quad \rho > a.$$

These results and those of (f) may easily be verified, or obtained, from the values of ∇^4 of the integral, with the conditions that I, $\frac{dI}{d\rho}$, $\nabla^2 I$, $\frac{d}{d\rho}\nabla^2 I$ are continuous at $\rho = a$.

(h) In certain problems a class of potential functions occurs, which may be deduced from the fundamental potential $1/r$, where $r^2 = x^2 + y^2 + z^2$, by successive integration with respect to z.

Writing

$$u_1 = \log(r+z)$$
$$u_2 = z\log(r+z) - r$$
$$u_3 = \tfrac{1}{2}(z^2 - \tfrac{1}{2}\rho^2)\log(r+z) - \tfrac{3}{4}rz + \tfrac{1}{4}\rho^2$$

we may easily verify that u_1, u_2, u_3 are potential functions, and that

$$\frac{du_2}{dz} = u_2$$
$$\frac{du_2}{dz} = u_1$$
$$\frac{du_1}{dz} = \frac{1}{r}$$

These z-integrals of $1/r$ may be expressed in the form of definite integrals involving the Bessel function J, analogous to the integral forms for r^{-1} and its z-derivatives,

$$r^{-1} = \int_0^\infty e^{-\kappa z} \mathrm{J}_0 \kappa\rho d\kappa$$

$$\frac{d}{dz} r^{-1} = \int_0^\infty (-\kappa) e^{-\kappa z} \mathrm{J}_0 \kappa\rho d\kappa, \text{ etc., where } z>0.$$

We may notice that the value of $\int_0^\infty e^{-\kappa z} \mathrm{J}_0 \kappa\rho d\kappa$ follows at once from the remark that it is a potential symmetrical about the axis of z, and taking on that axis the value $\int_0^\infty e^{-\kappa z} d\kappa = \frac{1}{z}$. We may use this idea to express u_1, u_2, u_3 in similar form.

For we have

$$\int_0^\infty \left(e^{-\kappa z} - e^{-\kappa c}\right) \frac{d\kappa}{\kappa} = -\log \frac{z}{c},$$

and

$$\int_0^\infty \left(e^{-\kappa z} - 1 + \kappa z e^{-\kappa c}\right) \frac{d\kappa}{\kappa^2} = z \log \frac{z}{c} - z$$

$$\int_0^\infty \left(e^{-\kappa z} - 1 + \kappa z - \frac{1}{2}\kappa^2 z^2 e^{-\kappa c}\right) \frac{d\kappa}{\kappa^3} = -\frac{1}{2} z^2 \log \frac{z}{c} + \frac{3}{4} z^2,$$

by integration with respect to z from 0 to z.

Hence

$$\int_0^\infty \left(e^{-\kappa z} \mathrm{J}_0 \kappa\rho - e^{-\kappa c}\right) \frac{d\kappa}{\kappa} = -\log \frac{r+z}{2c}$$

$$\int_0^\infty \left(e^{-\kappa z} \mathrm{J}_0 \kappa\rho - 1 + \kappa z e^{-\kappa c}\right) \frac{d\kappa}{\kappa^2} = z \log \frac{r+z}{2c} - r$$

$$\int_0^\infty \left(e^{-\kappa z} \mathrm{J}_0 \kappa\rho - 1 + \kappa z - \frac{1}{2}\kappa^2 z^2 - \frac{1}{2}\rho^2 e^{-\kappa c}\right) \frac{d\kappa}{\kappa^3} = -\frac{1}{2}\left(z^2 - \frac{1}{2}\rho^2\right) \log \frac{r+z}{2c} + \frac{3}{4} zr - \frac{1}{4}\rho^2$$

because in each case the functions equated are symmetrical potentials, taking the same value on the axis of symmetry.

By putting $z=0$ in the first and third of these we obtain two integrals, of great importance in the following analysis,

$$\int_0^\infty \left(\mathrm{J}_0 \kappa\rho - e^{-\kappa c}\right) \frac{d\kappa}{\kappa} = -\log \frac{\rho}{2c}$$

$$\int_0^\infty \left(\mathrm{J}_0 \kappa\rho - 1 + \frac{1}{4}\kappa^2 \rho^2 e^{-\kappa c}\right) \frac{d\kappa}{\kappa^3} = \frac{1}{4}\rho^2 \log \frac{\rho}{2c} - \frac{1}{4}\rho^2.$$

There is no difficulty in generalising the above results, but those given are all that we shall require.

(*i*) With a view to indicating the broad lines of the treatment of the elastic problem given in the succeeding pages, a discussion on similar lines may be given here of a simple problem in potential, in which the attention is not distracted from the principles of the method by any complexity in the calculations.

The problem is to find the flow from a source situated between two parallel planes $z = \pm h$, under the condition that there is no flow across these planes.

We require a potential V, becoming infinite as $1/r$ at (x', y', z'), but with no other singularity at a finite distance, and such that $\frac{dV}{dz} = 0$ when $z = \pm h$.

$$\text{If} \quad R = \sqrt{(x-x')^2 + (y-y')^2},$$

$$\text{then} \quad r^{-1} = \int_0^\infty e^{-\kappa(z-z')} J_0 \kappa R \, d\kappa, \quad \text{when} \quad z > z'$$

$$= \int_0^\infty e^{\kappa(z-z')} J_0 \kappa R \, d\kappa, \quad \text{when} \quad z < z'.$$

$$\frac{d}{dz} r^{-1} = \int_0^\infty (-\kappa) e^{-\kappa(z-z')} J_0 \kappa R \, d\kappa, \quad \text{when} \quad z > z'$$

$$= \int_0^\infty \kappa e^{\kappa(z-z')} J_0 \kappa R \, d\kappa, \quad \text{when} \quad z < z'.$$

We therefore begin by finding a potential

$$V_1 = (A \cosh \kappa z + B \sinh \kappa z) J_0 \kappa R$$

giving

$$\frac{dV_1}{dz} = \kappa e^{-\kappa(z-z')} J_0 \kappa R \quad \text{on} \quad z = h$$

$$= -\kappa e^{\kappa(z-z')} J_0 \kappa R \quad \text{on} \quad z = -h.$$

We obtain

$$A \sinh \kappa h + B \cosh \kappa h = e^{-\kappa(h-z')}$$

$$A \sinh \kappa h - B \cosh \kappa h = e^{-\kappa(h+z')}$$

$$A = e^{-\kappa h} \frac{\cosh \kappa z'}{\sinh \kappa h}$$

$$B = e^{-\kappa h} \frac{\sinh \kappa z'}{\cosh \kappa h}$$

$$V_1 = e^{-\kappa h} \left(\frac{\cosh \kappa z \cosh \kappa z'}{\sinh \kappa h} + \frac{\sinh \kappa z \sinh \kappa z'}{\cosh \kappa h} \right) J_0 \kappa R.$$

If this could be integrated with respect to κ from 0 to ∞, we should have a potential just balancing at the boundary the flow from the source.

But V_1 becomes infinite as $1/\kappa h$ at $\kappa = 0$, and the integration cannot be performed. We may, however, subtract from V_1 the (constant) potential $e^{-\kappa c}/\kappa h$, where c is an arbitrary positive quantity. This makes integration possible, without introducing any flow across the boundary.

A solution of the problem is then

$$V = \frac{1}{r} + \int_0^\infty \left\{ e^{-\kappa h} \left(\frac{\cosh \kappa z \cosh \kappa z'}{\sinh \kappa h} + \frac{\sinh \kappa z \sinh \kappa z'}{\cosh \kappa h} \right) J_0 \kappa R - \frac{e^{-\kappa c}}{\kappa h} \right\} d\kappa.$$

But this form of solution, while theoretically complete, is of little value because of the difficulty of interpretation. For example, it gives no indication of what on physical grounds we should expect to be the chief feature of the phenomenon, namely, the practically two-dimensional character of the flow at a moderate distance from the source.

The transformation to which we proceed brings this out as luminously as possible.

First, it is convenient to separate V_1 into its odd and even parts in κ, as is easily done by writing $\cosh \kappa h - \sinh \kappa h$ for $e^{-\kappa h}$.

Thus

$$V_1 = \left(-\cosh \kappa \overline{z-z'} + \frac{\cosh \kappa h}{\sinh \kappa h} \cosh \kappa z \cosh \kappa z' - \frac{\sinh \kappa h}{\cosh \kappa h} \sinh \kappa z \sinh \kappa z' \right) J_0 \kappa R .$$

Next we replace the term $1/r$ in V by the equivalent integral

$$\int_0^\infty e^{\mp \kappa (z-z')} J_0 \kappa R \, d\kappa .$$

Hence

$$V = \int_0^\infty \left\{ \left(\mp \sinh \kappa \overline{z-z'} + \frac{\cosh \kappa h}{\sinh \kappa h} \cosh \kappa z \cosh \kappa z' - \frac{\sinh \kappa h}{\cosh \kappa h} \sinh \kappa z \sinh \kappa z' \right) J_0 \kappa R - \frac{e^{-\kappa c}}{\kappa h} \right\} d\kappa ,$$

the upper or lower sign being taken in the ambiguous term according as z is $>$ or $<$ z'.

When $R>0$, this integral can be separated into the two

$$\int_0^\infty J_0 \kappa R \left(\mp \sinh \kappa \overline{z-z'} + \frac{\cosh \kappa h}{\sinh \kappa h} \cosh \kappa z \cosh \kappa z' - \frac{\sinh \kappa h}{\cosh \kappa h} \sinh \kappa z \sinh \kappa z' - \frac{1}{\kappa h} \right) d\kappa$$
$$+ \int_0^\infty \left(J_0 \kappa R - e^{-\kappa c} \right) \frac{d\kappa}{\kappa h} .$$

The value of the latter integral we have found to be $-\frac{1}{h} \log \frac{R}{2c}$.

The former integral is of the form $\int_0^\infty J_0 \kappa R F(\kappa) d\kappa$, where $F(\kappa)$ is an *odd* function of κ, vanishing for $\kappa = 0$. It may be expressed as a complex integral

$$\frac{1}{\pi i} \int G_0 \kappa R F(\kappa) d\kappa ,$$

the path being from west to east along the whole of the real axis in the κ plane, for

$$G_0(\kappa R) - G_0(\kappa e^{i\pi} R) = \pi i J_0 \kappa R .$$

Now, from the original form of V_1, and the integral forms of $1/r$, it is obvious that $F(\kappa)$ vanishes at infinity in the eastern half of the κ plane; being odd in κ it must vanish likewise in the western half.

Hence by Cauchy's Theorem, the integral $\frac{1}{\pi i} \int G_0 \kappa R F(\kappa) d\kappa$ is equal to twice the sum of the residues of the function $G_0 \kappa R F(\kappa)$ at its poles in the upper half of the κ plane, and

$$V = -\frac{1}{h} \log \frac{R}{2c}$$
$$+ \frac{2}{h} \sum_{n=1} \cos \frac{n\pi z}{h} \cos \frac{n\pi z'}{h} G_0 \frac{in\pi R}{h}$$
$$+ \frac{2}{h} \sum_{n=0} \sin \left(n + \frac{1}{2} \right) \frac{\pi z}{h} \sin \left(n + \frac{1}{2} \right) \frac{\pi z'}{h} G_0 \left(n + \frac{1}{2} \right) \frac{i\pi R}{h} .$$

(j) The solution indicates (i) a main current in two dimensions, defined by the potential $-\frac{1}{h} \log \frac{R}{2c}$, and (ii) an infinite series of local currents in three dimensions,

practically insensible when the distance from the source is a moderate multiple of the thickness of the plate. In the following pages we shall deduce analogous solutions for 'sources of strain' of the different types which may exist in an elastic solid, and develop these solutions in various directions. The corresponding development of the present solution is extremely easy, but would carry us too far. We merely mention that the 'main current' in the hydrodynamical problems corresponds to the 'principal modes of strain,' the determination of which is the object of the theory of thin plates. But there is one important distinction in the two cases. In the flow problems the exact conditions defining the 'main current' can always be found, and are indeed obvious; on the other hand, the analogous conditions in the strain problems can only be found by approximation.

(*k*) The following conventions seem to be very generally adopted, but to prevent any risk of ambiguity they may be stated explicitly here. Consider any continuous plane area A bounded externally by a closed curve C_0, and internally by one or more closed curves C_1, C_2, etc. At any point E of a bounding curve let Ex, Ey be drawn in the directions of the rectangular axes of coordinates. Let Ex, Ey be turned through an angle ϵ, which will be taken as positive when the rotation is counter-clockwise, until they coincide with Eξ, Eη, the direction of Eξ being that of the normal at E when drawn from within A towards the boundary. Eξ, Eη will be taken to be the positive directions of the normal and tangent at E, and if $f(x, y)$ be any function given within A, $\frac{df}{dn}$ and $\frac{df}{ds}$ will be used to denote the rates of variation of f per unit length in these positive directions.

The curvature at E is $\frac{d\epsilon}{ds}$ and is denoted by $1/\rho$. ρ is therefore positive when, in order to reach the centre of curvature from E, we have to proceed *into* the area A.

If we suppose the figure traced on level ground, a person proceeding along the boundary in the positive direction will have the area on his left, and the curvature will be positive when he is rotating about the vertical in the counter-clockwise sense.

The following formulæ relating to differentiation along the arc and normal will be much used in the later sections of the paper. Suppose the axes of x and y to coincide with the positive normal and tangent at a point O of the bounding curve. At a neighbouring point E (x, y) on this curve

$$\left.\begin{aligned} \frac{df}{dn} &= \cos\epsilon\,\frac{df}{dx} + \sin\epsilon\,\frac{df}{dy} \\ \frac{df}{ds} &= -\sin\epsilon\,\frac{df}{dx} + \cos\epsilon\,\frac{df}{dy} \end{aligned}\right\} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \text{(i)}$$

By putting x, y, ϵ equal to zero, we have at O

$$\frac{df}{dn} = \frac{df}{dx}\,;\ \frac{df}{ds} = \frac{df}{dy} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \text{(ii)}$$

Differentiate the first of equations (i) with respect to s. Thus

$$\frac{d}{ds}\left(\frac{df}{dn}\right)=\cos\epsilon\,\frac{d}{ds}\left(\frac{df}{dx}\right)+\sin\epsilon\,\frac{d}{ds}\left(\frac{df}{dy}\right)-\sin\epsilon\,\frac{d\epsilon}{ds}\frac{df}{dx}+\cos\epsilon\,\frac{d\epsilon}{ds}\frac{df}{dy}$$

$$=\cos\epsilon\left(-\sin\epsilon\,\frac{d^2f}{dx^2}+\cos\epsilon\,\frac{d^2f}{dx\,dy}\right)+\text{ etc.}$$

$$\text{and at O,}\quad \frac{d}{ds}\frac{df}{dn}=\frac{d^2f}{dx\,dy}+\frac{d\epsilon}{ds}\frac{df}{dy}$$

$$\text{or}\quad \frac{d^2f}{dx\,dy}=\frac{d}{ds}\frac{df}{dn}-\frac{1}{\rho}\frac{df}{ds} \qquad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

Similarly from the second of (i),

$$\text{at O,}\quad \frac{d^2f}{ds^2}=\frac{d^2f}{dy^2}-\frac{1}{\rho}\frac{df}{dx}$$

$$\text{or}\quad \frac{d^2f}{dy^2}=\frac{d^2f}{ds^2}+\frac{1}{\rho}\frac{df}{dn} \qquad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

Thus the values at O of $\frac{df}{dx}$, $\frac{df}{dy}$, $\frac{d^2f}{dx\,dy}$, $\frac{d^2f}{dy^2}$ are known when f and $\frac{df}{dn}$ are given along the boundary.

$\frac{d^2f}{dx^2}+\frac{d^2f}{dy^2}$ or $\nabla^2 f$ being an invariant for all systems of rectangular axes, we may also conveniently take

$$\frac{d^2f}{dx^2}=\nabla^2 f-\frac{1}{\rho}\frac{df}{dn}-\frac{d^2f}{ds^2} \qquad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

1. *Equations of equilibrium. Form of solution for a plate free from bodily force.*

The equations of equilibrium of a homogeneous isotropic elastic solid are of the form

$$\left.\begin{aligned}\frac{d\,\widehat{xx}}{dx}+\frac{d\,\widehat{xy}}{dy}+\frac{d\,\widehat{xz}}{dz}+\mathrm{X}&=0\\ \frac{d\,\widehat{xy}}{dx}+\frac{d\,\widehat{yy}}{dy}+\frac{d\,\widehat{yz}}{dz}+\mathrm{Y}&=0\\ \frac{d\,\widehat{xz}}{dx}+\frac{d\,\widehat{yz}}{dy}+\frac{d\,\widehat{zz}}{dz}+\mathrm{Z}&=0\end{aligned}\right\} \qquad . \quad . \quad . \quad . \quad \text{(1)}$$

where X, Y, Z are the components of the bodily force per unit volume, and $\widehat{xx}$, $\widehat{yy}$, $\widehat{zz}$, $\widehat{xy}$, $\widehat{xz}$, $\widehat{yz}$ are the components of stress, these being given in terms of the displacements u, v, w by the equations

$$\left.\begin{aligned}\widehat{xx}&=\lambda\Delta+2\mu\frac{du}{dx}\quad,\quad \widehat{yz}=\mu\left(\frac{dw}{dy}+\frac{dv}{dz}\right)\\ \widehat{yy}&=\lambda\Delta+2\mu\frac{dv}{dy}\quad,\quad \widehat{xz}=\mu\left(\frac{du}{dz}+\frac{dw}{dx}\right)\\ \widehat{zz}&=\lambda\Delta+2\mu\frac{dw}{dz}\quad,\quad \widehat{xy}=\mu\left(\frac{dv}{dx}+\frac{du}{dy}\right)\end{aligned}\right\} \qquad . \quad . \quad . \quad \text{(2)}$$

where

$$\Delta=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}.$$

In terms of the displacements, the equations of equilibrium are therefore

$$\left.\begin{aligned} \mu\nabla^2 u + (\lambda+\mu)\frac{d\Delta}{dx} + X &= 0 \\ \mu\nabla^2 v + (\lambda+\mu)\frac{d\Delta}{dy} + Y &= 0 \\ \mu\nabla^2 w + (\lambda+\mu)\frac{d\Delta}{dz} + Z &= 0 \end{aligned}\right\} \quad . \quad . \qquad (3)$$

When the bodily force is null, or $X = Y = Z = 0$, the following forms are easily shown to satisfy equations (3),

$$\text{(i)} \quad \left.\begin{aligned} u &= 2\frac{d\psi}{dy} \\ v &= -2\frac{d\psi}{dx} \end{aligned}\right\}$$

$$\text{(ii)} \quad \left.\begin{aligned} u &= \frac{d\theta}{dx} \\ v &= \frac{d\theta}{dy} \\ w &= \frac{d\theta}{dz} \end{aligned}\right\}$$

$$\text{(iii)} \quad \left.\begin{aligned} u &= \frac{\lambda+3\mu}{\lambda+\mu}\frac{d\phi}{dx} + 2z\frac{d^2\phi}{dz\,dx} \\ v &= \frac{\lambda+3\mu}{\lambda+\mu}\frac{d\phi}{dy} + 2z\frac{d^2\phi}{dz\,dy} \\ w &= -\frac{\lambda+3\mu}{\lambda+\mu}\frac{d\phi}{dz} + 2z\frac{d^2\phi}{dz^2} \end{aligned}\right\}$$

where ψ, θ, ϕ are potential functions, so that

$$\nabla^2\psi = 0, \quad \nabla^2\theta = 0, \quad \nabla^2\phi = 0.$$

These solutions have been used by BOUSSINESQ in his treatment of the problem of a solid bounded by a single plane $z = 0$. They are equally effective when the boundary consists of two parallel z-planes. Thus, as will explicitly appear in the sequel, and as might be proved at once, any solution of (3), with $X = Y = Z = 0$, in the space between the planes $z = \pm h$, can be expressed in the form

$$\left.\begin{aligned} u &= 2\frac{d\psi}{dy} + \frac{d\theta}{dx} + a\frac{d\phi}{dx} + 2z\frac{d^2\phi}{dz\,dx} \\ v &= -2\frac{d\psi}{dx} + \frac{d\theta}{dy} + a\frac{d\phi}{dy} + 2z\frac{d^2\phi}{dz\,dy} \\ w &= \frac{d\theta}{dz} - a\frac{d\phi}{dz} + 2z\frac{d^2\phi}{dz^2} \end{aligned}\right\} \quad . \quad . \quad . \qquad (4)$$

Here, and throughout the paper, the symbol a is used to denote the fraction $(\lambda + 3\mu)/(\lambda + \mu)$.

With these values of u, v, w the stresses across a z-plane, viz., $\widehat{zx}$, $\widehat{zy}$, $\widehat{zz}$, are given by

$$\left.\begin{aligned} \frac{\widehat{zx}}{2\mu} &= \frac{d^2\psi}{dy\,dz} + \frac{d^2\theta}{dx\,dz} + \frac{d^2\phi}{dx\,dz} + 2z\frac{d^3\phi}{dx\,dz^2} \\ \frac{\widehat{zy}}{2\mu} &= -\frac{d^2\psi}{dx\,dz} + \frac{d^2\theta}{dy\,dz} + \frac{d^2\phi}{dy\,dz} + 2z\frac{d^3\phi}{dy\,dz^2} \\ \frac{\widehat{zz}}{2\mu} &= \frac{d^2\theta}{dz^2} - \frac{d^2\phi}{dz^2} + 2z\frac{d^3\phi}{dz^3} \end{aligned}\right\} \quad . \quad . \quad . \qquad (5)$$

2. *Force applied at a single point.*

Returning to the solutions (i), (ii), (iii), we note that (i) and (ii) contribute nothing to the dilatation Δ, and (ii), (iii) nothing to the z-rotation $\omega_3 = \frac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right)$.

These properties can be used to resolve any given displacement into its ψ, θ, ϕ components, the bodily force being null.

An example of fundamental importance is the displacement in an infinite solid due to a single force applied at a given point. Thus for a unit force applied at the origin in the direction of the axis of z we have

$$\left.\begin{aligned} u &= \frac{xz}{r^3} \\ v &= \frac{yz}{r^3} \\ w &= \frac{z^2}{r^3} + \frac{a}{r} \end{aligned}\right\} \text{ each multiplied by } \frac{1}{8\pi\mu}\frac{\lambda+\mu}{\lambda+2\mu} = \frac{1}{4\pi\mu(a+1)}$$

where $r^2 = x^2 + y^2 + z^2$; or say, for a Z force of $4\pi\mu\,(a+1)$ units applied at (x', y', z') we have

$$\left.\begin{aligned} u &= (z'-z)\frac{dr^{-1}}{dx} \\ v &= (z'-z)\frac{dr^{-1}}{dy} \\ w &= (z'-z)\frac{dr^{-1}}{dz} + ar^{-1} \end{aligned}\right\} \qquad . \quad . \quad . \quad . \quad . \quad (6)$$

r^{-1} being written for $1/r$, where r is the distance from (x, y, z) to (x', y', z').
These give

$$\Delta = (a-1)\frac{dr^{-1}}{dz}\ ;\ \omega_3 = 0\,.$$

But in (4)

$$\Delta = 2(1-a)\frac{d^2\phi}{dz^2}\ ;\ \omega_3 = \frac{d^2\psi}{dz^2}\,.$$

Hence we take $\psi = 0$, and choose ϕ so that $\frac{d\phi}{dz} = -\frac{1}{2}r^{-1}$.

Now the functions $\log(r + \overline{z-z'})$ and $-\log(r - \overline{z-z'})$ are both potentials having r^{-1} for z-derivative; the former is without singular point in the region $z > z'$, the latter in the region $z < z'$. We may without confusion use a single symbol to denote either function indifferently, and define

$$\left.\begin{aligned} \frac{d^{-1}r^{-1}}{dz^{-1}} &= \log(r+z-z') \quad \text{when} \quad z > z' \\ &= -\log(r-z+z') \quad \text{when} \quad z < z' \end{aligned}\right\} \qquad . \quad . \quad . \quad . \quad (7)$$

We may therefore take

$$\phi = -\frac{1}{2}\frac{d^{-1}r^{-1}}{dz^{-1}}\,.$$

Comparison of the displacement w in (4) and (6) gives now

$$\frac{d\theta}{dz} = z'\frac{dr^{-1}}{dz} + \frac{a}{2}r^{-1}\,.$$

For a Z force of $4\pi\mu(a+1)$ units at (x', y', z') we have therefore

$$\left.\begin{aligned}\psi &= 0\\ \theta &= z'r^{-1} + \frac{a}{2}\frac{d^{-1}r^{-1}}{dz^{-1}}\\ \phi &= -\frac{1}{2}\frac{d^{-1}r^{-1}}{dz^{-1}}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

It is easy to verify that these values of ψ, θ, ϕ substituted in (4) do actually reproduce equations (6).

Similarly for an X force of $4\pi\mu(a+1)$ units at (x', y', z') we find

$$\left.\begin{aligned}\psi &= -\frac{a+1}{2}\frac{d}{dy}\frac{d^{-2}r^{-1}}{dz^{-2}}\\ \theta &= z'\frac{d}{dx}\frac{d^{-1}r^{-1}}{dz^{-1}} - \frac{a}{2}\frac{d}{dx}\frac{d^{-2}r^{-1}}{dz^{-2}}\\ \phi &= -\frac{1}{2}\frac{d}{dx}\frac{d^{-2}r^{-1}}{dz^{-2}}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9)$$

Here $\frac{d^{-2}r^{-1}}{dz^{-2}}$ denotes a potential function having $\frac{d^{-1}r^{-1}}{dz^{-1}}$ for z-derivative, and is defined by the equations

$$\left.\begin{aligned}\frac{d^{-2}r^{-1}}{dz^{-2}} &= (z-z')\log(r+z-z') - r \quad \text{when} \quad z>z'\\ &= -(z-z')\log(r-z+z') - r \quad \text{when} \quad z<z'\end{aligned}\right\} \quad . \quad . \quad . \quad (10)$$

It may be observed that the necessity for dealing separately with the two regions $z>z'$ and $z<z'$ in these cases is not inconsistent with the theorem of (4), which refers only to a displacement free from singularity in the space considered.

3. *Solution of the problem of normal traction.*

Coming now to the problems relating to a solid bounded by the two parallel planes $z=h$ and $z=-h$, we begin with the simplest of these, and seek a solution of the equations of equilibrium giving

$$X=Y=Z=0 \text{ throughout the body;}$$

the normal stress

$$\begin{aligned}\widehat{zz} &= f(x, y) \quad \text{on} \quad z=h,\\ &= 0 \quad \text{on} \quad z=-h;\end{aligned}$$

the tangential stresses $\widehat{zx}$, $\widehat{zy}=0$ on both faces $z=\pm h$. The arbitrary function $f(x, y)$, which we shall suppose to vanish at all points without a given finite area A, is expressed in a form amenable to analytical treatment in the familiar theorem

$$\underset{\epsilon=0}{\text{Limit}} \iint \frac{\epsilon f(x', y')dx'dy'}{\{(x-x')^2+(y-y')^2+\epsilon^2\}^{\frac{3}{2}}} = 2\pi f(x, y) \quad . \quad . \quad . \quad (11)$$

the integral being taken over the area A.

(If we imagine the plane $z=0$ to be covered with attracting matter of surface density $f(x, y)$, then the theorem expresses the well-known relation between the density at

$(x, y, 0)$ and the limiting value of the normal attraction at (x, y, ϵ) as ϵ approaches zero.)

As a preliminary to the general problem, we take therefore the special case in which $\widehat{zz}$ on $z=h$ is equal to $\epsilon/\{(x-x')^2+(y-y')^2+\epsilon^2\}^{\frac{3}{2}}$, or, in the form of a definite integral,

$$\int_0^\infty e^{-\kappa\epsilon}\kappa J_0(\kappa R)d\kappa,$$

where

$$R^2=(x-x')^2+(y-y')^2.$$

Making a further reduction, we begin by taking, in place of this integral, simply the function $\kappa J_0\kappa R$.

The function ψ is not required, and ϕ, θ are of the forms

$$\left.\begin{aligned}\phi&=(C_1\sinh\kappa z+C_2\cosh\kappa z)J_0\kappa R\\ \theta&=(C_3\sinh\kappa z+C_4\cosh\kappa z)J_0\kappa R\end{aligned}\right\}$$

In accordance with (5) these satisfy the conditions

$$\left.\begin{aligned}\frac{d\theta}{dz}+\frac{d\phi}{dz}+2z\frac{d^2\phi}{dz^2}&=0 \quad \text{on } z=\pm h\\ \frac{d^2\theta}{dz^2}-\frac{d^2\phi}{dz^2}+2z\frac{d^3\phi}{dz^3}&=0 \quad \text{on } z=-h\\ &=\kappa J_0\kappa R/2\mu \quad \text{on } z=h\end{aligned}\right\} \qquad . \quad . \quad . \quad (12)$$

Hence we easily find

$$\left.\begin{aligned}4\mu\phi=&-\frac{\cosh\kappa h}{\kappa(\sinh 2\kappa h-2\kappa h)}J_0\kappa R\sinh\kappa z\\ &-\frac{\sinh\kappa h}{\kappa(\sinh 2\kappa h+2\kappa h)}J_0\kappa R\cosh\kappa z\\ 4\mu\theta=&\frac{\cosh\kappa h+2\kappa h\sinh\kappa h}{\kappa(\sinh 2\kappa h-2\kappa h)}J_0\kappa R\sinh\kappa z\\ &+\frac{\sinh\kappa h+2\kappa h\cosh\kappa h}{\kappa(\sinh 2\kappa h+2\kappa h)}J_0\kappa R\cosh\kappa z\end{aligned}\right\} \qquad . \quad . \quad . \quad (13)$$

If these expressions, multiplied by $e^{-\kappa\epsilon}$, could be integrated with respect to κ from 0 to ∞, we should have at once a solution of the preliminary problem. But this integration is not possible, owing to the nature of the functions of κ near the lower limit $\kappa=0$. In fact, if the values of $4\mu\phi$, $4\mu\theta$ in (13) be expanded in ascending powers of κ, the expansions will contain terms in $1/\kappa^3$ and $1/\kappa$, so that near $\kappa=0$

$$\begin{aligned}4\mu\phi&=H/\kappa^3+K/\kappa+\text{terms of positive degree}\\ 4\mu\theta&=L/\kappa^3+M/\kappa+\quad\text{„}\qquad\text{„}\end{aligned}$$

These terms of negative degree are potentials contributing nothing to the stresses on $z=\pm h$, as we see from (12), since $\kappa J_0\kappa R$ contains no terms of negative degree. They might therefore be subtracted from the expressions (13) without affecting the satisfaction of the conditions in (12). This simple subtraction would, however, introduce terms not integrable right up to the upper limit, at least after ϵ is put equal to zero, as eventually it will be. The difficulty is met by subtracting from $4\mu\phi$, not $H/\kappa^3+K/\kappa$, but $H/\kappa^3+Ke^{-\kappa h}/\kappa$; and from $4\mu\theta$, not $L/\kappa^3+M/\kappa$, but $L/\kappa^3+Me^{-\kappa h}/\kappa$.

(There are, of course, any number of equally suitable modifications; instead of $e^{-\kappa h}$ we might take $e^{-\kappa\epsilon}$ or $1/(1+\kappa^2)$, for instance.) A solution of the preliminary problem of normal traction equal to $\epsilon/(R^2+\epsilon^2)^{\frac{3}{2}}$ on the face $z=h$ is thus obtained in the form

$$4\mu\phi = -\int_0^\infty e^{-\kappa\epsilon}\left\{\frac{\cosh\kappa h}{\kappa(\sinh 2\kappa h - 2\kappa h)}J_0\kappa R\sinh\kappa z + \frac{\sinh\kappa h}{\kappa(\sinh 2\kappa h + 2\kappa h)}J_0\kappa R\cosh\kappa z - \frac{H}{\kappa^3} - \frac{Ke^{-\kappa h}}{\kappa}\right\}d\kappa$$

$$4\mu\theta = \int_0^\infty e^{-\kappa\epsilon}\left\{\frac{\cosh\kappa h + 2\kappa h\sinh\kappa h}{\kappa(\sinh 2\kappa h - 2\kappa h)}J_0\kappa R\sinh\kappa z + \frac{\sinh\kappa h + 2\kappa h\cosh\kappa h}{\kappa(\sinh 2\kappa h + 2\kappa h)}J_0\kappa R\cosh\kappa z - \frac{L}{\kappa^3} - \frac{Me^{-\kappa h}}{\kappa}\right\}d\kappa \quad . \quad . \quad (14)$$

The solution of the original general problem is found by multiplying by $f(x',y')/2\pi$, integrating with respect to x', y' over the area A, and finally taking the limit for $\epsilon=0$. But a glance at the forms near $\kappa=\infty$ of the functions in (14) shows that the triple integrals are absolutely convergent, it being supposed that $-h<z<h$. Hence we may integrate with respect to x', y' first, and by a well-known theorem the limits for $\epsilon=0$ may then be found by simply putting $\epsilon=0$ in the integrands, provided the resulting integrals are convergent, as they manifestly are.

This gives the value of ϕ, for example, in the form

$$\int_0^\infty d\kappa\iint f(x',y')F(\kappa)dx'dy',$$

but, always provided $-h<z<h$, we may change this if we please into

$$\iint f(x',y')dx'dy'\int_0^\infty F(\kappa)d\kappa.$$

Finally, we may with great advantage confine our study in the first place to what is usually spoken of as a *unit element* of normal traction at $(x',\ y',\ h)$. The area A enclosing this point is diminished, and the intensity of traction increased without limit, so that $\iint f(x',y')dx'dy'$ remains equal to unity. The resulting solution is simply that of (14), but with ϵ put equal to zero within the integral signs.

As we have just seen, the solution for the general case can at any time be found from this elementary solution (15) by multiplying by $f(x',y')/2\pi$ and integrating over the area A.

4. *Flexural and extensional components of the strain. Disadvantages of the solution in definite integrals.*

In the elementary solution each of the potentials ϕ, θ may with advantage be decomposed into an odd and an even part in z. Thus, for an element of normal

traction of $8\pi\mu$ units at (x', y', h), a solution is given by

$$\phi = \phi_0 + \phi_e\,;\quad \theta = \theta_0 + \theta_e$$

where

$$\left.\begin{aligned}
\phi_0 &= \int_0^\infty \left\{ -\frac{\cosh \kappa h}{\kappa(\sinh 2\kappa h - 2\kappa h)} \mathrm{J}_0 \kappa \mathrm{R} \sinh \kappa z + \frac{3z}{4\kappa^3 h^3} + \frac{e^{-\kappa h}}{\kappa}\left(\frac{2z^3 - 3\mathrm{R}^2 z}{16 h^3} + \frac{9}{40}\frac{z}{h}\right)\right\} d\kappa \\
\theta_0 &= \int_0^\infty \left\{ \frac{\cosh \kappa h + 2\kappa h \sinh \kappa h}{\kappa(\sinh 2\kappa h - 2\kappa h)} \mathrm{J}_0 \kappa \mathrm{R} \sinh \kappa z - \frac{3z}{4\kappa^3 h^3} - \frac{e^{-\kappa h}}{\kappa}\left(\frac{2z^3 - 3\mathrm{R}^2 z}{16 h^3} + \frac{9}{40}\frac{z}{h} + \frac{3}{2}\frac{z}{h}\right)\right\} d\kappa
\end{aligned}\right\} \quad . \quad (16)$$

$$\left.\begin{aligned}
\phi_e &= \int_0^\infty \left\{ -\frac{\sinh \kappa h}{\kappa(\sinh 2\kappa h + 2\kappa h)} \mathrm{J}_0 \kappa \mathrm{R} \cosh \kappa z + \frac{e^{-\kappa h}}{4\kappa}\right\} d\kappa \\
\theta_e &= \int_0^\infty \left\{ \frac{\sinh \kappa h + 2\kappa h \cosh \kappa h}{\kappa(\sinh 2\kappa h + 2\kappa h)} \mathrm{J}_0 \kappa \mathrm{R} \cosh \kappa z - \frac{3e^{-\kappa h}}{4\kappa}\right\} d\kappa
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The conditions satisfied at the faces by the partial solutions (16) and (17) are easily made out. For when ϕ, θ are both odd functions of z, then $\widehat{zx}$, $\widehat{zy}$ are even and $\widehat{zz}$ odd; but when ϕ, θ are even functions of z, then $\widehat{zx}$, $\widehat{zy}$ are odd, $\widehat{zz}$ even, as is obvious from (5). Hence (16) gives equal values of opposite sign for $\widehat{zz}$ at corresponding points on $z = \pm h$; (17) gives equal values of the same sign.

It follows that (16) is the solution for elements of normal force of $4\pi\mu$ units at each of the points (x', y', h), $(x', y', -h)$, the force being in the positive direction of Oz in each case, and therefore a traction on $z = h$, but a pressure on $z = -h$; in (17) the only difference is that the force is a traction on both planes.

Hence, also, (16) subtracted from (17) will give the solution for traction on $z = -h$ alone.

Each of the integrals in (16), (17) defines a potential function without singularity at a finite distance in the space between the planes $z = \pm h$, and all the successive derivatives with respect to x, y, z of any of these functions may be calculated by differentiation within the sign of integration, provided we are dealing with a point actually within the solid, so that $-h < z < h$.

The solutions defined by these integrals are therefore formally satisfactory. It is, however, a serious objection to them that they do not lend themselves readily to interpretation, and it is not easy to make out from them any of the simple laws which the ordinary approximate theory leads us to anticipate.

In particular, the solutions in their present form throw no light on the question of the behaviour of the functions and their derivatives at points the distance of which from the sources of strain is great in comparison with the thickness of the plate, a question of great importance for the application to the thin-plate theory.

The analytical transformations to which we now proceed reduce the solutions to a form entirely free from these objections. Each of the integrals is shown to be composed of two parts of very different character. The first part represents a function the value of which diminishes with great rapidity as the distance from the source increases, while the remaining part is a function of very simple form. Each solution is thus resolved into a permanent or persistent element and a local, transitory, or decaying element, the latter being insignificant beyond the immediate vicinity of the source.

5. *Transformation of the definite integrals into series by means of Cauchy's Theorem.*

The integral ϕ_0 of (16) can be written as the sum of three integrals, namely—

$$\left.\begin{aligned}\phi_0 = \int_0^\infty J_0\kappa R \left\{ -\frac{\cosh \kappa h \sinh \kappa z}{\kappa(\sinh 2\kappa h - 2\kappa h)} + \frac{3z}{4\kappa^3 h^3} + \frac{1}{\kappa}\left(\frac{z^3}{8h^3} + \frac{9}{40}\frac{z}{h}\right)\right\} d\kappa \\ - \frac{3z}{4h^3}\int_0^\infty (J_0\kappa R - 1 + \frac{1}{4}\kappa^2 R^2 e^{-\kappa h})\frac{d\kappa}{\kappa^3} \\ - \left(\frac{z^3}{8h^3} + \frac{9}{40}\frac{z}{h}\right)\int_0^\infty (J_0\kappa R - e^{-\kappa h})\frac{d\kappa}{\kappa}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (18)$$

It should be observed that the first and third of these integrals cease to converge when $R = 0$. Hence the transformation does not apply to points on the line $x = x'$, $y = y'$, the normal to the plate through the sources.

Consider now the first integral in (18). The function of κ multiplying $J_0\kappa R$ within the integral sign is an odd function of κ vanishing for $\kappa = 0$. Hence, as in (*i*), the integral is equivalent to the complex integral

$$\int \frac{1}{\pi i} G_0\kappa R \left\{ -\frac{\cosh \kappa h \sinh \kappa z}{\kappa(\sinh 2\kappa h - 2\kappa h)} + \frac{3z}{4\kappa^3 h^3} + \frac{1}{\kappa}\left(\frac{z^3}{8h^3} + \frac{9}{40}\frac{z}{h}\right)\right\} d\kappa,$$

the path of integration running from west to east along the whole of the real axis, and just avoiding the origin, which is a singular point of $G_0\kappa R$, on the north or upper side.

On this path take points E, W at distances $n\pi/2h$ to the right and left of the origin, and on E W as side describe a square E W A B in the upper part of the plane. The integral over each of the sides W A, A B, B E is easily proved to have zero for limit when n tends to infinity through positive integral values.

Hence, by Cauchy's fundamental theorem, the integral over the path W E is equal to the sum of the residues of the integrand at its poles in the upper half of the κ plane, multiplied by $2\pi i$, that is, to the series

$$\sum_\kappa G_0\kappa R(-)\frac{\cosh \kappa h \sinh \kappa z}{\kappa h(\cosh 2\kappa h - 1)},$$

the summation extending over the zeroes of the function $\sinh 2\kappa h - 2\kappa h$ in the upper half of the κ plane, in the order of their moduli.

If ζ_n is a zero of the function $\sinh \zeta - \zeta$, the corresponding zero of $\sinh 2\kappa h - 2\kappa h$ is $\kappa_n = \zeta_n/2h$, and

$$G_0\kappa_n R = G_0\frac{\zeta_n R}{2h} = \sqrt{\frac{\pi h}{\zeta_n R}}\, e^{\frac{i\pi}{4}}\, e^{\frac{i\zeta_n R}{2h}} \text{ approximately,}$$

and we see that this part of ϕ_0, with its successive derivatives, is practically insensible when R is a very moderate multiple of $2h$. (*Cf.* § 7, *infra.*)

As to the other two integrals which occur in (18), we have proved in (h) that

$$\int_0^\infty (J_0\kappa R - 1 + \frac{1}{4}\kappa^2 R^2 e^{-\kappa h})\frac{d\kappa}{\kappa^3} = \frac{1}{4}R^2 \log\frac{R}{2h} - \frac{1}{4}R^2,$$

and

$$\int_0^\infty (J_0\kappa R - e^{-\kappa h})\frac{d\kappa}{\kappa} = -\log\frac{R}{2h}.$$

These functions will occur so often that it will be convenient to reserve an invariable symbol for the former of them, say

$$\left.\begin{aligned} \chi(R), \text{ or simply } \chi, &= \frac{1}{4}R^2 \log\frac{R}{2h} - \frac{1}{4}R^2 \\ \nabla^2\chi &= \log\frac{R}{2h} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (19)$$

and then

The persistent part of ϕ_0 is therefore

$$-\frac{3z}{4h^3}\chi + \left(\frac{z^3}{8h^3} + \frac{9}{40}\frac{z}{h}\right)\nabla^2\chi,$$

which is the sum of two potential functions,

$$-\frac{3}{4h^3}\left(z\chi - \frac{1}{6}z^3\nabla^2\chi\right) \quad \text{and} \quad \frac{9}{40h}z\nabla^2\chi.$$

6. *Types of the particular solutions composing the general solution.*

A glance at the relation between the results just obtained and the form of ϕ_0 in (16) enables us to write down at once the corresponding transformations of θ_0, ϕ_e, θ_e.

Collecting the results, we find

$$\left.\begin{aligned} \phi_0 &= \sum_\kappa G_0\kappa R(-)\frac{\cosh\kappa h \sinh\kappa z}{\kappa h(\cosh 2\kappa h - 1)} - \frac{3}{4h^3}\left(z\chi - \frac{1}{6}z^3\nabla^2\chi\right) + \frac{9}{40h}z\nabla^2\chi \\ \theta_0 &= \sum_\kappa G_0\kappa R\frac{(\cosh\kappa h + 2\kappa h \sinh\kappa h)\sinh\kappa z}{\kappa h(\cosh 2\kappa h - 1)} + \frac{3}{4h^3}\left(z\chi - \frac{1}{6}z^3\nabla^2\chi - 2h^2 z\nabla^2\chi\right) - \frac{9}{40h}z\nabla^2\chi \end{aligned}\right\} \quad . \quad . \quad (20)$$

where κ is a zero of $\sinh 2\kappa h - 2\kappa h$, with positive imaginary part.

$$\left.\begin{aligned} \phi_e &= \sum_\kappa G_0\kappa R(-)\frac{\sinh\kappa h \cosh\kappa z}{\kappa h(\cosh 2\kappa h + 1)} + \frac{1}{4}\nabla^2\chi \\ \theta_e &= \sum_\kappa G_0\kappa R\frac{(\sinh\kappa h + 2\kappa h \cosh\kappa h)\cosh\kappa z}{\kappa h(\cosh 2\kappa h + 1)} - \frac{3}{4}\nabla^2\chi \end{aligned}\right\} \quad . \quad . \quad (21)$$

where κ is a zero of $\sinh 2\kappa h + 2\kappa h$, with positive imaginary part.

The solution must give $\widehat{zx}$, $\widehat{zy}$, $\widehat{zz}$ all equal to zero at the two plane faces of the plate, except when $R = 0$, and we are thus prepared to find that the strain defined by the terms corresponding to any one root κ gives zero stress across $z = \pm h$. Thus in (20) $\sum_\kappa$ contains a series of particular solutions of the type

(i)
$$\left.\begin{aligned} \phi &= -\cosh\kappa h \sinh\kappa z\, F(x, y) \\ \theta &= (\cosh\kappa h + 2\kappa h \sinh\kappa h)\sinh\kappa z\, F(x, y) \end{aligned}\right\} \left\{\begin{aligned} &\text{where } (\nabla^2 + \kappa^2)F = 0 \\ &\sinh 2\kappa h - 2\kappa h = 0 \end{aligned}\right\} \quad . \quad . \quad (22)$$

Calculating the stresses by means of (5), we find

$$\frac{d\theta}{dz}+\frac{d\phi}{dz}+2z\frac{d^2\phi}{dz^2}=(2\kappa^2 h\sinh\kappa h\cosh\kappa z-2\kappa^2 z\cosh\kappa h\sinh\kappa z)\mathrm{F}$$

$$\frac{d^2\theta}{dz^2}-\frac{d^2\phi}{dz^2}+2z\frac{d^3\phi}{dz^3}=2\kappa^2(\cosh\kappa h+\kappa h\sinh\kappa h)\sinh\kappa z\cdot\mathrm{F}-2\kappa^3 z\cosh\kappa h\cosh\kappa z\cdot\mathrm{F}$$

both of which vanish when $z=\pm h$.

We have further in (20) a solution of the type

(ii) $$\left.\begin{aligned}\phi&=-z\mathrm{F}+\tfrac{1}{6}z^3\nabla^2\mathrm{F}\\ \theta&=\ z\mathrm{F}-\tfrac{1}{6}z^3\nabla^2\mathrm{F}-2h^2z\nabla^2\mathrm{F}\end{aligned}\right\}\ \mathrm{F}\text{ a function of }x,\ y\text{ with }\nabla^4\mathrm{F}=0.$$

From this, by (4), (5),

$$\left.\begin{aligned}u&=\frac{d}{dx}\left\{-(\alpha+1)(z\mathrm{F}-\tfrac{1}{6}z^3\nabla^2\mathrm{F})+2(\tfrac{1}{3}z^3-h^2z)\nabla^2\mathrm{F}\right\}\\ v&=\frac{d}{dy}\left\{\text{ditto}\right\}\\ w&=(\alpha+1)(\mathrm{F}-\tfrac{1}{2}z^2\nabla^2\mathrm{F})+2(z^2-h^2)\nabla^2\mathrm{F}\end{aligned}\right\}\quad\ldots\quad(23)$$

$$\left.\begin{aligned}\widehat{zx}&=4\mu(z^2-h^2)\frac{d}{dx}\nabla^2\mathrm{F}\\ \widehat{zy}&=4\mu(z^2-h^2)\frac{d}{dy}\nabla^2\mathrm{F}\\ \widehat{zz}&=0\end{aligned}\right\}$$

The solution for unit normal traction on $z=h$ contains a strain of this type, with $\mathrm{F}=\chi(\mathrm{R})\cdot 3/32\pi\mu h^3$.

Lastly, in (20) there is a solution of the type

(iii) $$\left.\begin{aligned}\phi&=-z\mathrm{F}\\ \theta&=\ z\mathrm{F}\end{aligned}\right\}\ \mathrm{F}\text{ a function of }x,\ y\text{ with }\nabla^2\mathrm{F}=0,$$

giving

$$\left.\begin{aligned}u&=-(\alpha+1)z\frac{d\mathrm{F}}{dx}\\ v&=-(\alpha+1)z\frac{d\mathrm{F}}{dy}\\ w&=\ (\alpha+1)\mathrm{F}\end{aligned}\right\}\widehat{zx}=\widehat{zy}=\widehat{zz}=0\quad\ldots\quad(24)$$

Obviously this is merely a degenerate case of (ii).

Again in (21) we have a series of solutions of the type

(iv) $$\left.\begin{aligned}\phi&=-\sinh\kappa h\cosh\kappa z\ \mathrm{F}(x,y)\\ \theta&=(\sinh\kappa h+2\kappa h\cosh\kappa h)\cosh\kappa z\ \mathrm{F}(x,y)\end{aligned}\right\}\ \left\{\begin{aligned}&\text{where }(\nabla^2+\kappa^2)\mathrm{F}=0\\ &\sinh 2\kappa h+2\kappa h=0\end{aligned}\right\}\quad\ldots\quad(25)$$

In this, as in (i), $\widehat{zx}=\widehat{zy}=\widehat{zz}=0$ on $z=\pm h$.

(v) The persistent part of (21) is of the type

$$\left.\begin{aligned}&\phi=\mathrm{F}(x,y);\ \theta=-3\mathrm{F}(x,y),\text{ where }\nabla^2\mathrm{F}=0\\ &u=(\alpha-3)\frac{d\mathrm{F}}{dx},\ v=(\alpha-3)\frac{d\mathrm{F}}{dy};\ \widehat{zx}=\widehat{zy}=\widehat{zz}=0\end{aligned}\right\}\quad\ldots\quad(26)$$

giving

7. *Position of the zeroes of the functions* $\sinh\zeta \pm \zeta$.

The nature of the infinite series occurring in the above solution will be made more intelligible by a short discussion of the position of the roots of the functions $\sinh 2\kappa h \pm 2\kappa h$. These are obviously found from the corresponding roots of $\sinh\zeta \pm \zeta$ by dividing by $2h$.

(i) $\sinh\zeta - \zeta = 0$.

$\zeta = 0$ is a triple root, and the remaining roots are all complex, falling into sets of 4 of the form $\pm p \pm iq$, where p, q are real. If, then, $\zeta = \xi + i\eta$, ξ and η being real, we need only consider the case of ξ and η both positive. We have then

$$\sinh\xi\cos\eta = \xi \quad \text{and} \quad \cosh\xi\sin\eta = \eta .$$

Cos η and $\sin\eta$ are therefore both positive, and η must lie between $2n\pi$ and $2n\pi + \pi/2$. It is easy to prove that there is no root between 0 and $\pi/2$. For $\xi > \tanh\xi$, or $\xi/\sinh\xi > 1/\cosh\xi$, so that

$$\cos\eta > \sin\eta/\eta \quad \text{or} \quad \eta > \tan\eta \quad \text{or} \quad \eta > \pi/2 .$$

For every positive integral value of n, however, beginning with $n = 1$, there is one root, and one root only, with η between $2n\pi$ and $2n\pi + \pi/2$. This will be readily seen on roughly tracing the graphs $\sinh\xi\cos\eta = \xi$ and $\cosh\xi\sin\eta = \eta$, or it may be proved by an elementary application of the Theory of Functions. Thus, if we make the variable ζ describe the contour of the rectangle formed by the four lines

$$\eta = 2n\pi , \quad \xi = \mathrm{N} , \quad \eta = 2n\pi + \pi/2 , \quad \xi = 0 ,$$

where N is a large positive number, it will be found that the function $\nu = \sinh\zeta - \zeta$ describes once a contour enclosing the point $\nu = 0$ in the ν plane. There is therefore just one point within the rectangle at which ν becomes zero.

For the large roots $\cos\eta$ must be small, or

$$\eta = 2n\pi + \pi/2 - \epsilon, \text{ where } \epsilon \text{ is small.}$$

Hence

$$\cosh\xi = \eta = 2n\pi + \pi/2 ,$$
$$\text{or } \xi = \log(\overline{4n+1}\,\pi) \text{ approximately.}$$

Then

$$\epsilon = \xi/\sinh\xi = 2\log(\overline{4n+1}\,\pi)/(\overline{4n+1}\,\pi).$$

By successive approximation we may now find the roots as nearly as we wish, but exact values are not at all necessary, the first approximation being quite sufficient for our purpose,

$$\zeta_n = \log(4n+1\,\pi) + (2n+\tfrac{1}{2})\pi i \qquad . \quad . \quad . \quad . \quad (27)$$

(ii) $\sinh\zeta + \zeta = 0$.

In this case $\zeta = 0$ is a simple root, and the rest of the roots are complex. If $\zeta = \xi + i\eta$, we have

$$\sinh\xi\cos\eta + \xi = 0 \quad \text{and} \quad \cosh\xi\sin\eta + \eta = 0 .$$

Cos η, $\sin\eta$ are both negative when ξ and η are positive; hence η lies between $(2n-1)\pi$

and $(2n-1)\pi+\pi/2$, and it may be shown, as in the previous case, that there is actually one root with η between these limits for all positive integral values of n.

Also

$$\eta=(2n-1)\pi+\pi/2-\epsilon\,;\ \cosh\xi=2n\pi-\pi/2$$
$$\epsilon=2\log(\overline{4n-1}\,\pi)/(\overline{4n-1}\,\pi)\,.$$

To a first approximation

$$\zeta_n=\log(\overline{4n-1}\,\pi)+(2n-\tfrac{1}{2})\pi i \qquad . \quad . \quad . \quad . \quad (28)$$

In addition to the roots of (27), (28) we have of course a corresponding series in the second quadrant, the images of these in the axis of imaginaries.

8. *Approximate forms of the* nth *terms of the infinite series, when* n *is large.*

It may be useful to give in terms of n approximate forms for the general terms of (20), (21) corresponding to the n^{th} roots in the first quadrant.

(i) ϕ_0 and θ_0.

$$\kappa h=\tfrac{1}{2}\log 4n\pi+(n+\tfrac{1}{4})\pi i$$
$$\sinh\kappa h=\tfrac{1}{2}e^{\kappa h}=\tfrac{1}{2}(4n\pi)^{\frac{1}{2}}e^{(n+\frac{1}{4})\pi i}$$
$$\frac{\cosh\kappa h}{\kappa h(\cosh 2\kappa h-1)}=\frac{1}{\sinh\kappa h\cdot 2\sinh^2\kappa h}=\frac{4}{(4n\pi)^{\frac{3}{2}}}e^{-(n+\frac{1}{4})\pi i}(-i)$$
$$\sinh\kappa z=\tfrac{1}{2}(e^{\kappa h})^{\frac{z}{h}}-\tfrac{1}{2}(e^{\kappa h})^{-\frac{z}{h}}$$
$$=\tfrac{1}{2}\left\{(4n\pi)^{\frac{z}{2h}}e^{(n+\frac{1}{4})\frac{\pi iz}{h}}-(4n\pi)^{-\frac{z}{2h}}e^{-(n+\frac{1}{4})\frac{\pi iz}{h}}\right\}$$

Hence in ϕ_0 the general term

$$=\frac{i}{2n\pi}\mathrm{G}_0\kappa\mathrm{R}\left\{(4n\pi)^{\frac{z-h}{2h}}e^{(n+\frac{1}{4})\frac{\pi i(z-h)}{h}}-(4n\pi)^{-\frac{z+h}{2h}}e^{-(n+\frac{1}{4})\frac{\pi i(z+h)}{h}}\right\}\,;$$

in θ_0, the same as this, with the factor $i/2n\pi$ omitted.

In both

$$\mathrm{G}_0\kappa\mathrm{R}=\sqrt{\frac{h}{2n\mathrm{R}}}e^{-(n+\frac{1}{4})\frac{\pi\mathrm{R}}{h}}e^{\frac{i\mathrm{R}\log 4n\pi}{2h}}$$

(ii) ϕ_e and θ_e.

$$\kappa h=\tfrac{1}{2}\log 4n\pi+(n-\tfrac{1}{4})\pi i$$
$$\cosh\kappa h=\tfrac{1}{2}(4n\pi)^{\frac{1}{2}}e^{(n-\frac{1}{4})\pi i}$$

In ϕ_e the general term

$$\frac{i}{2n\pi}\mathrm{G}_0\kappa\mathrm{R}\left\{(4n\pi)^{\frac{z-h}{2h}}e^{(n-\frac{1}{4})\frac{\pi i(z-h)}{h}}+(4n\pi)^{-\frac{z+h}{2h}}e^{-(n-\frac{1}{4})\frac{\pi i(z+h)}{h}}\right\}$$

In θ_0 the same, with the factor $i/2n\pi$ omitted.

$$\mathrm{G}_0\kappa\mathrm{R}=\sqrt{\frac{h}{2n\mathrm{R}}}e^{-(n-\frac{1}{4})\frac{\pi\mathrm{R}}{h}}e^{\frac{i\mathrm{R}\log 4n\pi}{2h}}\,.$$

9. *The solution for arbitrary normal traction. Questions for discussion.*

The solution of the general problem of given normal traction requires the multiplication of the functions in (20), (21) by $f(x', y')$ and integration with respect to x', y' over a finite area A. There is no difficulty in showing that these integrations can be performed term by term, and that the resulting series converge absolutely.

When the solution of the general problem has thus been obtained in terms of series of surface integrals, several questions present themselves for treatment, among which may be specially mentioned

(i) For how many times in succession may these series be differentiated term by term with respect to the coordinates x, y, z?

(ii) When the thickness of the plate is infinitesimal, but $f(x, y)$ does not vary as h tends to zero, what are the orders of the various parts of the solution, and of the related physical quantities?

(iii) How are the answers to these two questions affected by discontinuity in the applied traction, or its x, y derivatives?

A perfectly general discussion of these questions would be tedious and difficult, and it will probably be more useful to consider the points suggested in the light of a special case, in which the integrations required can be performed, and the outstanding features of the solution can be grasped with comparative ease.

10. *Detailed solution of a special case. Term by term differentiations.*

The solution we propose to work out is to satisfy the following boundary conditions :—

$$\left.\begin{aligned} \widehat{zz} &= 4\pi\mu \mathrm{J}_m(\beta\rho)\cos m\omega, \text{ on } z=h \\ &= -4\pi\mu \mathrm{J}_m(\beta\rho)\cos m\omega, \text{ on } z=-h \end{aligned}\right\} \text{when } \rho<a$$

$$\widehat{zz} = 0 \quad \text{on } z=\pm h, \text{ when } \rho>a$$

$$\widehat{zx} = \widehat{zy} = 0, \quad \text{on } z\pm h.$$

ρ, ω, z are the cylindrical coordinates of the point (x, y, z), so that $x=\rho\cos\omega$, $y=\rho\sin\omega$.

β is any constant, and m is an integer.

The solution, obtained from (20) by integration, is

$$\phi = \phi_1 + \phi_2; \quad \theta = \theta_1 + \theta_2, \quad \text{where}$$

$$\phi_1 = -\frac{3}{4h^3}\left(z\mathrm{F} - \frac{1}{6}z^3\nabla^2\mathrm{F}\right) + \frac{9}{40h}z\nabla^2\mathrm{F}$$

$$\theta_1 = \frac{3}{4h^3}\left(z\mathrm{F} - \frac{1}{6}z^3\nabla^2\mathrm{F} - 2h^2 z\nabla^2\mathrm{F}\right) - \frac{9}{40h}z\nabla^2\mathrm{F} \qquad (29)$$

$$\text{with} \quad \mathrm{F} = \iint \chi(\mathrm{R})\mathrm{J}_m(\beta\rho')\cos m\omega'\rho' d\rho' d\omega'$$

$$\left.\begin{aligned} \phi_2 &= \sum_\kappa (-)\frac{\cosh\kappa h \sinh\kappa z}{\kappa h(\cosh 2\kappa h - 1)}\iint \mathrm{G}_0\kappa\mathrm{R}\mathrm{J}_m\beta\rho'\cos m\omega'\rho\, d\rho' d\omega' \\ \theta_2 &= \sum_\kappa \frac{(\cosh\kappa h + 2\kappa h\sinh\kappa h)\sinh\kappa z}{\kappa h(\cosh 2\kappa h - 1)} \quad \text{(same integral)} \end{aligned}\right\} \qquad (30)$$

the integrals being taken over the circle of radius a.

Consider in the first place the part of the solution defined by ϕ_2, θ_2.

The value of the surface integral in (30) takes different forms when $\rho >$ and $< a$.

As proved in (*c*)

$$\text{when} \quad \rho<a, \text{ the integral} = \frac{2\pi}{\beta^2-\kappa^2}\mathrm{J}_m\beta\rho\cos m\omega + \frac{2\pi}{\beta^2-\kappa^2}\mathrm{J}_m\kappa\rho\cos m\omega(\kappa a\mathrm{G}_m'\kappa a\mathrm{J}_m\beta a - \mathrm{G}_m\kappa a\beta a\mathrm{J}_m'\beta a)$$

$$\text{,,} \quad \rho>a, \quad \text{,,} = \frac{2\pi}{\beta^2-\kappa^2}\mathrm{G}_m\kappa\rho\cos m\omega(\kappa a\mathrm{J}_m'\kappa a\mathrm{J}_m\beta a - \mathrm{J}_m\kappa a\beta a\mathrm{J}_m'\beta a).$$

Now when $\kappa\rho$, κa are both large,

$$\mathrm{J}_m\kappa a\mathrm{G}_m\kappa\rho = \frac{i}{2\kappa\sqrt{a\rho}}e^{i\kappa(\rho-a)},$$

which, with its derivatives, is very small when $(\rho-a)/2h$ is even moderately large. Thus, in the space without the cylinder $\rho=a$, the part of the strain given by ϕ_2, θ_2 is, when h is small, insensible except in the immediate neighbourhood of that cylinder. The same remark applies to the strain within the cylinder, so far as it is given by the parts of ϕ_2, θ_2 arising from the second term in the value of the surface integral.

We naturally inquire, how do these rapidly decaying parts of the solution behave, and what is the order of magnitude of the corresponding displacements and stresses, at points actually on the surface $\rho=a$? Now, taking for example the value of ϕ_2 in the external region, namely,

$$\phi_2 = \sum_\kappa (-)\frac{\cosh\kappa h\sinh\kappa z}{\kappa h(\cosh 2\kappa h-1)}\frac{2\pi}{\beta^2-\kappa^2}\mathrm{G}_m\kappa\rho\cos m\omega\left(\kappa a\mathrm{J}_m'\kappa a\mathrm{J}_m\beta a - \mathrm{J}_m\kappa a\beta a\mathrm{J}_m'\beta a\right) \quad . \quad . \quad (30')$$

we see from § 8 (i) that when $\rho=a$, the general term has the approximate form

$$\frac{\mathrm{A}}{n^3}\left\{n^{\frac{z-h}{2h}}e^{(n+\frac{1}{4})\frac{\pi i(z-h)}{h}} - n^{-\frac{z+h}{2h}}e^{-(n+\frac{1}{4})\frac{\pi i(z+h)}{h}}\right\},$$

A being independent of n. Moreover, each differentiation of ϕ_2 with respect to ρ or z will remove a factor $1/n$ from the general term. Hence *three* such differentiations, but no more, are permissible, if $-h<z<h$. But from (4) it is clear that none of the displacements requires more than two, and none of the stresses more than three of these differentiations for their calculation. As for θ_2, the general term is of one order higher in n than the corresponding term in ϕ_2, but in compensation for this only two differentiations are required to find the stresses. Hence, so far as the decaying part of the solution is concerned, displacements and stresses at $\rho=a$ may be found by means of term by term differentiation, and subsequent substitution of a for ρ.

Again, considering the order of these various quantities in h, regarded as infinitesimal, and remembering that κh and κz are of order zero in h, we see that the expression for ϕ_2 in (30′) and the corresponding expression for θ_2 are of order h^2 when $\rho=a$, and each differentiation with respect to ρ or z diminishes the order by one. Hence the displacements at $\rho=a$ are of order h and the stresses of order zero, so far as they arise from the decaying part of the solution ϕ_2, θ_2.

11. *The same special problem. Summation of two infinite series.*

An important part of the strain given by ϕ_2, θ_2 remains to be considered, namely, that which arises from the term $2\pi/(\beta^2-\kappa^2)$. $J_m\beta\rho\cos m\omega$ in the value of the surface integral for the case when the point (ρ, ω) is within the cylinder $\rho=a$. Denoting these parts of ϕ_2, θ_2 by ϕ_3, θ_3, we have

$$\left.\begin{aligned}\phi_3 &= 2\pi J_m\beta\rho\cos m\omega \sum_\kappa (-)\frac{\cosh\kappa h\sinh\kappa z}{(\beta^2-\kappa^2)\kappa h(\cosh 2\kappa h-1)}\\ \theta_3 &= 2\pi J_m\beta\rho\cos m\omega \sum_\kappa \frac{(\cosh\kappa h+2\kappa h\sinh\kappa h)\sinh\kappa z}{(\beta^2-\kappa^2)\kappa h(\cosh 2\kappa h-1)}\end{aligned}\right\} \quad \ldots\ldots \quad (31)$$

We note in the first place that ϕ_3 admits of three, and θ_3 of two term by term differentiations with respect to z when $-h<z<h$, while x, y differentiations can be performed without restriction. Combining this result with those already obtained, we see that in the complete solution in terms of surface integrals, all the differentiations necessary to give the displacements and strains or stresses at any point in the body of the plate can be performed on the series term by term.

When h is small, ϕ_3 and θ_3 are of order h^2, and a z-differentiation lowers the order by one. This can be seen from the series, or otherwise, for, as we shall now show, the value of the series can be found in finite terms.

Consider the function of κ,

$$\frac{\cosh\kappa h\sinh\kappa z}{(\beta^2-\kappa^2)\kappa(\sinh 2\kappa h-2\kappa h)}.$$

This function, multiplied by κ, vanishes at infinity at all points of the path E W A B described in § 5; hence the sum of its residues vanishes. The function being odd in κ, the residues at the poles $\kappa=\pm\kappa'$ are equal. Thus

$$\begin{aligned}&2\,(\text{series of residues at zeroes of } \sinh 2\kappa h-2\kappa h \text{ in upper part of plane})\\ &+2\,(\text{residue at } \kappa=\beta)+(\text{residue at } \kappa=0)=0.\end{aligned}$$

The residue at $\kappa=\beta$ is

$$(-)\frac{\cosh\beta h\sinh\beta z}{2\beta^2(\sinh 2\beta h-2\beta h)}.$$

Also if

$$\frac{\cosh\kappa h\sinh\kappa z}{\kappa(\sinh 2\kappa h-2\kappa h)}=\frac{A}{\kappa^3}+\frac{B}{\kappa}+\cdots\cdots, \quad \text{near } \kappa=0,$$

then the residue at $\kappa=0$ is

$$\frac{A}{\beta^4}+\frac{B}{\beta^2}.$$

Hence

$$\sum_\kappa(-)\frac{\cosh\kappa h\sinh\kappa z}{(\beta^2-\kappa^2)\kappa h(\cosh 2\kappa h-1)}=-\frac{\cosh\beta h\sinh\beta z}{\beta^2(\sinh 2\beta h-2\beta h)}+\frac{A}{\beta^4}+\frac{B}{\beta^2}.$$

It may be noted that $A/\beta^4+B/\beta^2$ are simply the terms of negative degree in the expansion of

$$\frac{\cosh\beta h\sinh\beta z}{\beta^2(\sinh 2\beta h-2\beta h)}$$

in ascending powers of β.

Hence, putting in the values of A and B from (16), (18),

$$\left.\begin{aligned}\phi_3 &= 2\pi J_m\beta\rho\cos m\omega\left\{-\frac{\cosh\beta h\sinh\beta z}{\beta^2(\sinh 2\beta h-2\beta h)}+\frac{3z}{4\beta^4h^3}+\frac{1}{\beta^2}\left(\frac{z^3}{8h^3}+\frac{9}{40}\frac{z}{h}\right)\right\}\\ \text{and similarly}\\ \theta_3 &= 2\pi J_m\beta\rho\cos m\omega\left\{\frac{(\cosh\beta h+2\beta h\sinh\beta h)\sinh\beta z}{\beta^2(\sinh 2\beta h-2\beta h)}-\frac{3z}{4\beta^4h^3}-\frac{1}{\beta^2}\left(\frac{z^3}{8h^3}+\frac{69}{40}\frac{z}{h}\right)\right\}\end{aligned}\right\} \quad . \quad (32)$$

12. *The same problem. Final form of the solution.*

We come lastly to ϕ_1, θ_1 of § 10.

The function F requires separate formulæ for its expression in the three cases $m=0$, $m=1$, $m>1$, but in all cases

$$\left.\begin{aligned}F &= (2\pi/\beta^4)J_m\beta\rho\cos m\omega+F_1, \quad \text{when } \rho<a\\ F &= F_2, \quad \text{when } \rho>a\end{aligned}\right\}$$

where F_1 and F_2 satisfy the equations $\nabla^4F_1=0$, $\nabla^4F_2=0$. The values of F_1, F_2 for the various cases are given in (f).

When $\rho<a$, the term $(2\pi/\beta^4)J_m\beta\rho\cos m\omega$ of F, taken by itself in ϕ_1 and θ_1, would give

$$\left.\begin{aligned}\phi_1 &= -\frac{3}{4h^3}\frac{z}{\beta^4}-\frac{1}{\beta^2}\left(\frac{z^3}{8h^3}+\frac{9}{40}\frac{z}{h}\right)\\ \theta_1 &= \frac{3}{4h^3}\frac{z}{\beta^4}+\frac{1}{\beta^2}\left(\frac{z^3}{8h^3}+\frac{69}{40}\frac{z}{h}\right)\end{aligned}\right\}\cdot 2\pi J_m\beta\rho\cos m\omega.$$

These are precisely the terms of negative degree (both in β and in h) with signs changed, in the expressions for ϕ_3 and θ_3 given at the end of § 11.

If, then, we take this part of ϕ_1, θ_1 along with ϕ_3, θ_3 we have the complete solution in the form

$$\begin{aligned}\phi &= -2\pi J_m\beta\rho\cos m\omega\frac{\cosh\beta h\sinh\beta z}{\beta^2(\sinh 2\beta h-2\beta h)}\\ &+2\pi\sum_\kappa(-)\frac{\cosh\kappa h\sinh\kappa z}{(\beta^2-\kappa^2)\kappa h(\cosh 2\kappa h-1)}J_m\kappa\rho\cos m\omega(\kappa aG_m'\kappa aJ_m\beta a-G_m\kappa a\beta aJ_m'\beta a)\\ &-\frac{3}{4h^3}\left(zF_1-\frac{1}{6}z^3\nabla^2F_1\right)+\frac{9}{40h}z\nabla^2F_1\\ \theta &= 2\pi J_m\beta\rho\cos m\omega\frac{(\cosh\beta h+2\beta h\sinh\beta h)\sinh\beta z}{\beta^2(\sinh 2\beta h-2\beta h)}\\ &+2\pi\sum_\kappa\frac{(\cosh\kappa h+2\kappa h\sinh\kappa h)\sinh\kappa z}{(\beta^2-\kappa^2)\kappa h(\cosh 2\kappa h-1)}J_m\kappa\rho\cos m\omega(\kappa aG_m'\kappa aJ_m\beta a-G_m\kappa a\,\beta aJ_m'\beta a)\\ &+\frac{3}{4h^3}\left(zF_1-\frac{1}{6}z^3\nabla^2F_1-2h^2z\nabla^2F_1\right)-\frac{9}{40h}z\nabla^2F_1\end{aligned}$$

when $\rho<a$ . . . . . (33)

and

$$\begin{aligned}\phi &= 2\pi\sum_\kappa(-)\frac{\cosh\kappa h\sinh\kappa z}{(\beta^2-\kappa^2)\kappa h(\cosh 2\kappa h-1)}G_m\kappa\rho\cos m\omega(\kappa aJ_m'\kappa aJ_m\beta a-J_m\kappa a\beta a\,J_m'\beta a)\\ &-\frac{3}{4h^3}\left(zF_2-\frac{1}{6}z^3\nabla^2F_2\right)+\frac{9}{40h}z\nabla^2F_2\\ \theta &= 2\pi\sum_\kappa\frac{(\cosh\kappa h+2\kappa h\sinh\kappa h)\sinh\kappa z}{(\beta^2-\kappa^2)\kappa h(\cosh 2\kappa h-1)}G_m\kappa\rho\cos m\omega(\kappa aJ_m'\kappa aJ_m\beta a-J_m\kappa a\,\beta aJ_m'\beta a)\\ &+\frac{3}{4h^3}\left(zF_2-\frac{1}{6}z^3\nabla^2F_2-2h^2z\nabla^2F_2\right)-\frac{9}{40h}z\nabla^2F_2,\end{aligned}$$

when $\rho>a$ . . . . . (34)

In (33) and (34) each line represents a potential function; in (33) the first lines define a particular solution giving the proper values of the tractions at the surface, as may be seen from (13); the partial solutions given by the κ series give zero surface tractions, and represent a strain insensible except in the vicinity of $\rho = a$; and the solutions defined by the last lines, being of the form (23), give zero surface tractions.

From these remarks it follows immediately that the solution (33), (34) satisfies all the conditions of the problem in the two regions $\rho < a$, $\rho > a$, taken separately. To verify the solution completely, it would be necessary to show in addition that certain conditions are fulfilled at the surface $\rho = a$, namely,

(i) that the displacements and strains are continuous at this surface, and

(ii) that the integral value of the stresses $\widehat{zx}$, $\widehat{zy}$, $\widehat{zz}$ over any small area lying partly within and partly without the cylinder $\rho = a$, on either of the plane faces of the plate, tends to zero when the area is indefinitely diminished.

The condition (i) ensures the 'synexis' of the solution across the surface $\rho = a$, and can be proved by showing, as may easily be done by means of summations similar to those of § 11, that ϕ, θ, $\frac{d\phi}{d\rho}$ and $\frac{d\theta}{d\rho}$ are continuous at that surface. For by the Theory of the Potential this carries with it the continuity of all the derivatives of ϕ and θ, and therefore of the displacements and stresses, as well as of all their derivatives, under the proviso, of course, that $-h < z < h$.

The condition (ii), or some equivalent, is required in order to exclude the possibility of stresses with finite resultant passing into the solid through the lines $z = \pm h$, $\rho = a$; or, in other words, in order to ensure that the solution is not partly due to linear elements of traction at these lines.

13. *Order of the various parts of the solution, when* h *is small.*

The final form of the solution, as exhibited in (33), (34) was obtained by combining parts of ϕ_1, θ_1 with ϕ_2, θ_2, and until this was done, it was not immediately evident that ϕ and θ were potentials. Thus the part of the solution arising from the imaginary values of κ, or from any one of them, is not, within the region of applied traction, a potential by itself, and the same is true of the ϕ_1, θ_1 part, which may be considered as coming from the zero values of κ. This has sometimes to be taken into account in calculating the stresses; the formula for $\widehat{zz}$, for example, in (5) requires additional terms if, while u, v, w are still given by (4), ϕ and θ are not potentials.

On the other hand, the separation of the solution into the two parts (29), (30) has this very marked advantage that, when h is very small, the first part gives the terms of the two lowest orders in h of ϕ, θ, namely those of orders h^{-2} and h^0, while the second part, as we have already seen, contains no terms of lower order than h^2. When, however, we come to calculate displacements and stresses, the separation is less simple, mainly in consequence of the fact that x, y differentiations do not change the order of ϕ_1, θ_1, but diminish the order of ϕ_2, θ_2 by one for each differentiation.

The following table, which may be deduced immediately from the results of §§ 10, 11, shows the order in h of displacements and stresses arising from ϕ_1, θ_1 and ϕ_2, θ_2 respectively.

	ϕ_1, θ_1	ϕ_2, θ_2	
		$\rho \gtrless a$	$\rho = a$
u, v	$-2, 0$	2	1
w	$-3, -1$	1	1
$\widehat{xx}, \widehat{xy}, \widehat{yy}$	$-2, 0$	0	0
$\widehat{zx}, \widehat{zy}$	-1	1	0
$\widehat{zz}$	0	0	0

It thus appears that the first part of the solution gives all the displacements to a second approximation, and all the stresses but $\widehat{zz}$ to a first approximation. With regard to $\widehat{zz}$, it should be observed that the solutions depending on F_1, F_2 contribute nothing to it, so that, within $\rho = a$, its value comes altogether from the particular solution, and without $\rho = a$, its value is zero beyond the immediate vicinity of that surface.

14. *Methods and results of the special case extended to the general problem of arbitrary normal traction.*

One feature of the solution expressed by equations (33), (34) we have already found useful, especially in the important case when h is small, in such a way that βh and h/a are small fractions. We refer to the explicit separation in the solution of a purely local element, entirely negligible except within a certain strip of breadth comparable with the thickness of the plate, from an element of a persistent or permanent character, with an area of influence not affected by the indefinite diminution of h.

Another advantage of the form of solution in (33), (34) is that the particular solution for the space within which the traction is applied is found in such a form that it can be readily expanded in powers of h, so as to give the terms of positive order in the infinitesimal h, as well as those of negative order which were already separated in (29). Thus in the particular solution, or first line of ϕ in (33), the factor

$$\frac{\cosh \beta h \sinh \beta z}{\beta^2(\sinh 2\beta h - 2\beta h)}$$

can be expanded in ascending powers of β, the series converging if $|2\beta h| < |\zeta_1|$, where ζ_1 is the complex root of $\sinh \zeta - \zeta = 0$ with smallest modulus. Since z is of the same order as h, and we are supposing β independent of h, it is clear that the terms of the series will be of ascending order in h.

We shall now show how the solution for the general case when the given normal traction is a function of x, y of unspecified form may be transformed, under certain restrictions, so as to yield the advantages to which we have been referring as pertaining to the solution (33), (34).

The problem we suppose to be the same as that stated at the beginning of § 10,

but with $f(x, y)$ instead of $J_m\beta\rho\cos m\omega$, and with any continuous area A instead of the circle within $\rho = a$.

The solution will then be of the form defined in (29), (30), but the integral of (29) will now be

$$F = \iint \chi(R) f(x', y') dx' dy'$$

and the integral of (30)

$$I = \iint G_0(\kappa R) f(x', y') dx' dy'.$$

Since $\nabla^2\chi(R) = \log(R/2h)$ we have

also
$$\left.\begin{aligned} \nabla^4 F &= \nabla^2(\nabla^2 F) = 2\pi f(x, y) \\ (\nabla^2 + \kappa^2) I &= -2\pi f(x, y) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

If $f(x, y)$ and its derivatives of the first two orders are finite within A, we may transform I by Green's Theorem. Thus, excluding from the area A an infinitesimal circle about (x, y) as centre, and writing ∇'^2 for $\frac{d^2}{dx'^2} + \frac{d^2}{dy'^2}$,

$$I = -\frac{1}{\kappa^2}\iint f(x', y') \nabla'^2 G_0 \kappa R dx' dy$$

or

$$\left.\begin{aligned} I = &-\frac{1}{\kappa^2}\iint G_0 \kappa R \nabla'^2 f(x', y') dx' dy' \\ &-\frac{1}{\kappa^2}\int \left\{ f(x', y') \frac{d}{dn} G_0 \kappa R - G_0 \kappa R \frac{d}{dn} f(x', y') \right\} ds \\ &-\frac{2\pi}{\kappa^2} f(x, y) \end{aligned}\right\} \quad . \quad . \quad . \quad (36)$$

the line integral being taken round the boundary of A.

If this three-termed equivalent of the integral I be substituted in the series for ϕ_2 and θ_2, each of these series may be subdivided into three, ϕ_2 for instance into

(i) $$\sum_\kappa \frac{1}{\kappa^2} \frac{\cosh \kappa h \sinh \kappa z}{\kappa h(\cosh 2\kappa h - 1)} \iint G_0 \kappa R \, \nabla'^2 f(x', y') dx' dy',$$

a series of the same general form as the original series, but at once more convergent, and of two orders higher in h;

(ii) $$\sum_\kappa \frac{1}{\kappa^2} \frac{\cosh \kappa h \sinh \kappa z}{\kappa h(\cosh 2\kappa h - 1)} \int \left\{ f(x', y') \frac{d}{dn} G_0 \kappa R - G_0 \kappa R \frac{d}{dn} f(x', y') \right\} ds,$$

which corresponds to a strain local to the boundary of A;

(iii) $2\pi f(x, y) \sum_\kappa \frac{1}{\kappa^2} \frac{\cosh \kappa h \sinh \kappa z}{\kappa h(\cosh 2\kappa h - 1)}$, a series which can be summed in the same way as (31), being in fact simply the first series of (31) with $\beta = 0$. The sum is therefore

$$2\pi f(x, y) \cdot \text{coefficient of } \kappa^0 \text{ in } \frac{-\cosh \kappa h \sinh \kappa z}{\kappa^2(\sinh 2\kappa h - 2\kappa h)}.$$

We may now, by repetition of the same transformation, obtain a similar threefold

equivalent for the series (i), and continue the process as far as the continuity of the derivatives of $f(x, y)$ will permit. We should thus obtain, in place of ϕ_2,

(i) $$(-)^n \sum_\kappa (-) \frac{1}{\kappa^{2n}} \frac{\cosh \kappa h \sinh \kappa z}{\kappa h(\cosh 2\kappa h - 1)} \iint G_0 \kappa R \nabla'^{2n} f(x', y') dx' dy',$$

a function of order h^{2n+2}.

(ii) A series of line-integrals which we need not write down, corresponding to a local perturbation at the edge of A, and giving the edge values of the relative part of ϕ up to terms in h^{2n}.

(iii) A series of n terms

$$2\pi(c_0 f - c_2 \nabla^2 f + c_4 \nabla^4 f - \cdots\cdots + (-)^{n-1} c_{2n-2} \nabla^{2n-2} f),$$

where c_{2r} is the coefficient of κ^{2r} in the ascending power expansion of $(-) \frac{\cosh \kappa h \sinh \kappa z}{\kappa^2(\sinh 2\kappa h - 2\kappa h)}$, and is obviously a rational integral function of z and h of degree $2r+2$. If the function $f(x, y)$ has its derivatives of every finite order continuous throughout the area A, the process can be carried to as high a value of n as we please, and we can thus obtain the values of ϕ_2, θ_2 to any required order in h. It should be noticed, however, that the series (iii) is not necessarily convergent when continued to infinity, as we may see by taking as an example $f = \cos \alpha x$, when the series would become $2\pi \cos \alpha x (c_0 + c_2 a^2 + c_4 a^4 + \cdots)$, which is the expansion (without the terms of negative degree) of

$$-2\pi \cos \alpha x \frac{\cosh ah \sinh az}{a^2(\sinh 2ah - 2ah)},$$

and is therefore divergent if $|2ah| > |\zeta_1|$, ζ_1 being the complex root of $\sinh \zeta - \zeta = 0$ with smallest modulus. The form of the condition suggests that in ordinary cases the series will be convergent if h is small enough; and when this is so, this part of ϕ_2, θ_2 taken along with ϕ_1, θ_1 will define an exact particular solution within A, giving the proper values of the surface tractions, and arranged in terms of ascending order in h.

As a special case, the series will terminate if, for some finite value of n, $\nabla^{2n} f = 0$, and in particular if f be a rational integral function of x, y. (It may be noted here that the solution for $f = \rho^{m+2p} \cos m\omega$ might be obtained from the solution for $f = J_m \beta\rho \cos m\omega$ by expanding in powers of β, and equating coefficients of β^{m+2p} in conditions and solution.)

Looking back now to the ϕ_1, θ_1 part of the solution, and having regard to (18), (29), (35), we see that we may write symbolically

$$F = 2\pi \nabla^{-4} f; \quad \nabla^2 F = 2\pi \nabla^{-2} f, \quad \text{and}$$
$$\phi_1 = 2\pi(c_{-4} \nabla^{-4} f - c_{-2} \nabla^{-2} f).$$

The particular solution to any order in h is then given by

$$\phi = 2\pi(c_{-4} \nabla^{-4} f - c_{-2} \nabla^{-2} f + c_0 f - c_2 \nabla^2 f + \cdots\cdots)$$

where c_{2r} is the coefficient of κ^{2r} in the expansion of $(-) \frac{\cosh \kappa h \sinh \kappa z}{\kappa^2(\sinh 2\kappa h - 2\kappa h)}$, for negative as well as positive values of r; or, as we may put it, this particular part of ϕ is given by expanding $(-2\pi) \frac{\cosh \kappa h \sinh \kappa z}{\kappa^2(\sinh 2\kappa h - 2\kappa h)}$, writing $-\nabla^2$ for κ^2, and operating on $f(x, y)$.

15. *Independent symbolical solution of the general problem.*

The form of the last result suggests a method of dealing with the problem from the beginning, which, though not easy to develop independently with thorough rigour, has the advantage of conciseness, and will therefore be useful in giving a rapid account both of the foregoing solution and of those to be obtained in the following pages.

We begin by observing that

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\sinh\kappa z f(x,y) = \sinh\kappa z\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\kappa^2\right)f(x,y)$$
$$= \sinh\kappa z(\nabla^2+\kappa^2)f(x,y).$$

Hence $\sinh\kappa z f(x,y)$ is a potential function, provided we regard κ as an operator such that $\kappa^2=-\nabla^2$. We may, if we please, take $\kappa=i\nabla$, but it will not be necessary to interpret odd powers of the operator ∇.

On this understanding, it is obvious from (12), (13) that we obtain a solution giving

$$\left.\begin{aligned} \widehat{zx}=\widehat{zy}&=0 \quad \text{on } z=\pm h\\ zz &=0 \quad \text{on } z=-h\\ zz &=f(x,y) \text{ on } z=h, \text{ within the area A,}\end{aligned}\right\}$$

by taking

$$\left.\begin{aligned} 4\mu\phi &= -\frac{\cosh\kappa h}{\kappa^2(\sinh 2\kappa h-2\kappa h)}\sinh\kappa z\cdot f-\frac{\sinh\kappa h}{\kappa^2(\sinh 2\kappa h+2\kappa h)}\cosh\kappa z\cdot f\\ 4\mu\theta &= \frac{\cosh\kappa h+2\kappa h\sinh\kappa h}{\kappa^2(\sinh 2\kappa h-2\kappa h)}\sinh\kappa z\cdot f+\frac{\sinh\kappa h+2\kappa h\cosh\kappa h}{\kappa^2(\sinh 2\kappa h+2\kappa h)}\cosh\kappa z\cdot f\end{aligned}\right\} \quad . \quad . \quad . \quad (37)$$

Now, taking as a specimen the first term of $4\mu\phi$, we observe that the function of κ

$$(-)\frac{\cosh\kappa h}{\kappa^2(\sinh 2\kappa h-2\kappa h)}\sinh\kappa z$$

vanishes at infinity round the path W A B E of § 5.

Hence the function is represented by the sum of its polar elements. (If κ_1 be a simple pole of the function, and if in the vicinity of this pole the function $=A_1/(\kappa-\kappa_1)+$finite, then $A_1/(\kappa-\kappa_1)$ is the polar element at this pole, and A_1 is the residue there. The point $\kappa=0$ is a multiple pole, and the polar element there has the form $A/\kappa^4+B/\kappa^2$, these being the terms of negative degree in the expansion of the function near $\kappa=0$).

Taking the elements belonging to $\pm\kappa_1$ together we obtain

$$(-)\frac{\cosh\kappa h\sinh\kappa z}{\kappa^2(\sinh 2\kappa h-2\kappa h)} = \frac{A}{\kappa^4}+\frac{B}{\kappa^2}+A_1\left(\frac{1}{\kappa-\kappa_1}-\frac{1}{\kappa+\kappa_1}\right)+\cdots\cdots$$
$$= \frac{A}{\kappa^4}+\frac{B}{\kappa^2}+\frac{2A_1\kappa_1}{\kappa^2-\kappa_1^2}+\cdots\cdots$$
$$= \frac{A}{\kappa^4}+\frac{B}{\kappa^2}+(-)\frac{\cosh\kappa_1 h\sinh\kappa_1 z}{\kappa_1 h(\cosh 2\kappa_1 h-1)}\frac{1}{\kappa^2-\kappa_1^2}+\cdots\cdots$$

the series extending over the poles with positive imaginary part. When we put $\kappa^2=-\nabla^2$, this part of $4\mu\phi$ becomes

$$A\nabla^{-4}f-B\nabla^{-2}f+\sum_{\kappa}\frac{\cosh\kappa h\sinh\kappa z}{\kappa h(\cosh 2\kappa h-1)}\frac{1}{\nabla^2+\kappa^2}\cdot f,$$

where κ is no longer an operator, but simply a root of $\sinh 2\kappa h-2\kappa h=0$.

Now since

$$(\nabla^2+\kappa^2)\iint G_0\kappa R f(x',y')dx'dy' = -2\pi f(x,y),$$

one value of $\dfrac{1}{\nabla^2+\kappa^2}\cdot f$ is $-\dfrac{1}{2\pi}\iint G_0\kappa R f(x',y')dx'dy'$.

Similarly one value of $\nabla^{-2}f$ is $\dfrac{1}{2\pi}\iint \log(R/2h) f(x',y')dx\,dy$

and of $\nabla^{-4}f$ $\dfrac{1}{2\pi}\iint\left(\dfrac{1}{4}R^2\log\dfrac{R}{2h}-\dfrac{1}{4}R^2\right)f(x',y')dx'dy'$ . . (38)

Hence for the first part of $4\mu\phi$ we obtain

$$\left.\begin{aligned} &-\frac{1}{2\pi}\frac{3z}{4h^3}\iint\chi(R)f(x',y')dx'dy' + \frac{1}{2\pi}\left(\frac{z^3}{8h^3}+\frac{9}{40}\frac{z}{h}\right)\iint\nabla^2\chi(R)f(x',y')dx'dy' \\ &\quad+\frac{1}{2\pi}\sum_\kappa(-)\frac{\cosh\kappa h\sinh\kappa z}{\kappa h(\cosh 2\kappa h-1)}\iint G_0\kappa R f(x',y')dx'dy' \end{aligned}\right\} \quad . \quad . \quad (39)$$

which agrees with our previous solution, the element of which is given in (20).

Further, the results obtained at the end of § 14 clearly agree with what we should get by expanding the function of κ in (37) in ascending power series and interpreting.

As an example of this use of equations (37) we may find to a first approximation the value of $\widehat{zz}$ at points not very close to the edge of A.

From (5)

$$\begin{aligned} 2(\widehat{zz}) &= \frac{d^2}{dz^2}\left(4\mu\theta\right)-\frac{d^2}{dz^2}\left(4\mu\phi\right)+2z\frac{d^3}{dz^3}\left(4\mu\phi\right) \\ &= \frac{(2\cosh\kappa h+2\kappa h\sinh\kappa h)\sinh\kappa z-2\kappa z\cosh\kappa h\cosh\kappa z}{\sinh 2\kappa h-2\kappa h}\cdot f \\ &\quad+\frac{(2\sinh\kappa h+2\kappa h\cosh\kappa h)\cosh\kappa z-2\kappa z\sinh\kappa h\sinh\kappa z}{\sinh 2\kappa h+2\kappa h}\cdot f \\ &= (3h^2-z^2)zf/2h^3+f. \end{aligned}$$

Thus

$$\widehat{zz}=\{(3h^2-z^2)z/4h^3+1/2\}f(x,y), \quad . \quad . \quad . \quad . \quad (40)$$

and we verify at a glance that this gives the proper values at the faces.

16. *The problem of tangential face traction. Solution for an element of traction.*

We will now pass to the problem in which the given surface traction is tangential. Taking the direction of the traction parallel to the axis of x, we may take for conditions

$$\left.\begin{aligned} \widehat{zx} &= f(x,y) \quad \text{on} \quad z=+h \\ \widehat{zx} &= 0 \quad \text{on} \quad z=-h \\ \widehat{zy}=\widehat{zz} &= 0 \quad \text{on} \quad z=\pm h \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (41)$$

According to the method explained in § 3, we begin with the function $\kappa J_0(\kappa R)$ in place of $f(x,y)$, and determine potentials ψ, θ, ϕ giving

$$\left.\begin{aligned} \frac{d^2\psi}{dy\,dz}+\frac{d^2\theta}{dx\,dz}+\frac{d^2\phi}{dx\,dz}+2z\frac{d^3\phi}{dz^2dx} &= \frac{1}{2\mu}\kappa J_0\kappa R \quad \text{on} \quad z=h \\ &= 0 \quad \text{on} \quad z=-h \\ -\frac{d^2\psi}{dx\,dz}+\frac{d^2\theta}{dy\,dx}+\frac{d^2\phi}{dy\,dz}+2z\frac{d^3\phi}{dz^2dy} &= 0 \quad \text{on} \quad z=\pm h \\ \frac{d^2\theta}{dz^2}-\frac{d^2\phi}{dz^2}+2z\frac{d^3\phi}{dz^3} &= 0 \quad \text{on} \quad z=\pm h \end{aligned}\right\} \quad . \quad . \quad (42) \quad .$$

Since $J_0\kappa R = -\frac{1}{\kappa^2}\frac{d^2}{dx^2}J_0\kappa R - \frac{1}{\kappa^2}\frac{d^2}{dy^2}J_0\kappa R$, it is clear that these equations will all be satisfied if we take

$$\psi = \frac{d\psi'}{dy}, \theta = \frac{d\theta'}{dx}, \phi = \frac{d\phi'}{dx} \quad . \quad . \quad . \quad . \quad . \quad (43)$$

where

$$\left.\begin{aligned} \frac{d\psi'}{dz} &= -\frac{1}{2\mu}\frac{1}{\kappa}J_0\kappa R \quad \text{on} \quad z=h \\ &= 0 \qquad\quad \text{on} \quad z=-h \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (44)$$

$$\left.\begin{aligned} \frac{d\theta'}{dz} + \frac{d\phi'}{dz} + 2z\frac{d^2\phi'}{dz^2} &= -\frac{1}{2\mu}\frac{1}{\kappa}J_0\kappa R \quad \text{on} \quad z=h \\ &= 0 \qquad\quad \text{on} \quad z=-h \\ \frac{d^2\theta'}{dz^2} - \frac{d^2\phi'}{dz^2} + 2z\frac{d^3\phi'}{dz^3} &= 0 \qquad\quad \text{on} \quad z=\pm h \end{aligned}\right\} \quad . \quad . \quad . \quad (45)$$

From (44),

$$4\mu\psi' = -\frac{2}{\kappa^2}\frac{\cosh\kappa(z+h)}{\sinh 2\kappa h}J_0\kappa R,$$

or, separating the odd and even parts in z,

$$\left.\begin{aligned} 4\pi\psi' &= -\frac{1}{\kappa^2}\frac{\cosh\kappa z}{\sinh\kappa h}J_0\kappa R - \frac{1}{\kappa^2}\frac{\sinh\kappa z}{\cosh\kappa h}J_0\kappa R \\ 4\pi\phi' &= -\frac{\sinh\kappa h}{\kappa^2(\sinh 2\kappa h - 2\kappa h)}J_0\kappa R\sinh\kappa z \\ &\quad -\frac{\cosh\kappa h}{\kappa^2(\sinh 2\kappa h + 2\kappa h)}J_0\kappa R\cosh\kappa z \\ 4\mu\theta' &= \frac{2\kappa h\cosh\kappa h - \sinh\kappa h}{\kappa^2(\sinh 2\kappa h - 2\kappa h)}J_0\kappa R\sinh\kappa z \\ &\quad +\frac{2\kappa h\sinh\kappa h - \cosh\kappa h}{\kappa^2(\sinh 2\kappa h + 2\kappa h)}J_0\kappa R\cosh\kappa z \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (46)$$

We also find easily

Treating these expressions as in § 3, we find a solution for an element of X-traction at (x', y', h) of $8\pi\mu$ units given by (43) with

$$\begin{aligned} \psi' = &\int_0^\infty\left(-\frac{\sinh\kappa z}{\kappa^2\cosh\kappa h}J_0\kappa R + e^{-\kappa h}\frac{z}{\kappa}\right)d\kappa \\ &+\int_0^\infty\left\{-\frac{\cosh\kappa z}{\kappa^2\sinh\kappa h}J_0\kappa R + \frac{1}{\kappa^3 h} + \frac{e^{-\kappa h}}{\kappa h}\left(\frac{1}{2}z^2 - \frac{1}{6}h^2 - \frac{1}{4}R^2\right)\right\}d\kappa \\ \phi' = &\int_0^\infty\left\{-\frac{\sinh\kappa h\sinh\kappa z}{\kappa^2(\sinh 2\kappa h - 2\kappa h)}J_0\kappa R + \frac{3z}{4\kappa^3 h^2} + \frac{e^{-\kappa h}}{8\kappa h^2}\left(z^3 - \frac{3}{2}R^2 z - \frac{1}{5}h^2 z\right)\right\}d\kappa \\ &+\int_0^\infty\left\{-\frac{\cosh\kappa h\cosh\kappa z}{\kappa^2(\sinh 2\kappa h + 2\kappa h)}J_0\kappa R + \frac{1}{4\kappa^3 h} + \frac{e^{-\kappa h}}{8\kappa h}\left(z^2 - \frac{1}{2}R^2 + \frac{1}{3}h^2\right)\right\}d\kappa \\ \theta' = &\int_0^\infty\left\{\frac{(2\kappa h\cosh\kappa h - \sinh\kappa h)\sinh\kappa z}{\kappa^2(\sinh 2\kappa h - 2\kappa h)}J_0\kappa R - \frac{3z}{4\kappa^3 h^2} - \frac{e^{-\kappa h}}{8\kappa h^2}\left(z^3 - \frac{3}{2}R^2 z + \frac{19}{5}h^2 z\right)\right\}d\kappa \\ &+\int_0^\infty\left\{\frac{(2\kappa h\sinh\kappa h - \cosh\kappa h)\cosh\kappa z}{\kappa^2(\sinh 2\kappa h + 2\kappa h)}J_0\kappa R + \frac{1}{4\kappa^3 h} + \frac{e^{-\kappa h}}{8\kappa h}\left(z^2 - \frac{1}{2}R^2 - \frac{11}{3}h^2\right)\right\}d\kappa \end{aligned}$$

. . . . . (47)

These expressions may be transformed by the method of § 5, and a slight inspection of the relations between (16), (17) and (20), (21) will enable us to write down the

results at once. These are, if we separate the parts odd and even in z,

$$\psi_0' = \sum_\kappa (-2)\frac{\sinh \kappa z}{\kappa^2 h \sinh \kappa h} G_0(\kappa R) + z\nabla^2\chi \Big\} \quad . \quad . \quad . \quad . \quad (48)$$

where κ is a positive imaginary root of $\cosh \kappa h$,

$$\left.\begin{aligned} \phi_0' &= \sum_\kappa (-)\frac{\sinh \kappa h \sinh \kappa z}{\kappa^2 h(\cosh 2\kappa h - 1)} G_0 \kappa R - \frac{3}{4h^2}\Big(z\chi - \frac{1}{6}z^3\nabla^2\chi\Big) - \frac{1}{40}z\nabla^2\chi \\ \theta_0' &= \sum_\kappa \frac{(2\kappa h \cosh \kappa h - \sinh \kappa h)\sinh \kappa z}{\kappa^2 h(\cosh 2\kappa h - 1)} G_0(\kappa R) + \frac{3}{4h^2}\Big(z\chi - \frac{1}{6}z^3\nabla^2\chi - 2h^2 z\nabla^2\chi\Big) + \frac{1}{40}z\nabla^2\chi + z\nabla^2\chi \end{aligned}\right\} \quad . \quad (49)$$

where κ is a zero of $\sinh 2\kappa h - 2\kappa h$, with positive imaginary part.

$$\psi_e' = \sum_\kappa (-2)\frac{\cosh \kappa z}{\kappa^2 h \cosh \kappa h} G_0(\kappa R) - \frac{1}{h}\Big(\chi - \frac{1}{2}z^2\nabla^2\Big) - \frac{h}{6}\nabla^2\chi \Big\} \quad . \quad . \quad (50)$$

where κ is a positive imaginary root of $\sinh \kappa h$.

$$\left.\begin{aligned} \phi_e' &= \sum_\kappa (-)\frac{\cosh \kappa h \cosh \kappa z}{\kappa^2 h(\cosh 2\kappa h + 1)} G_0 \kappa R - \frac{1}{4h}\Big(\chi - \frac{1}{2}z^2\nabla^2\chi\Big) + \frac{h}{24}\nabla^2\chi \\ \theta_e' &= \sum_\kappa \frac{(2\kappa h \sinh \kappa h - \cosh \kappa h)\cosh \kappa z}{\kappa^2 h(\cosh 2\kappa h + 1)} G_0(\kappa R) - \frac{1}{4h}\Big(\chi - \frac{1}{2}z^2\nabla^2\chi\Big) - \frac{11h}{24}\nabla^2\chi \end{aligned}\right\} \quad . \quad (51)$$

where κ is a zero of $\sinh 2\kappa h + 2\kappa h$, with positive imaginary part.

The solution is defined by these equations with

$$\psi = \frac{d\psi'}{dy}, \quad \theta = \frac{d\theta'}{dx}, \quad \phi = \frac{d\phi'}{dx}.$$

17. *Composition of the solution.*

On examining the composition of the solution, we observe in the decaying parts of ϕ, θ, solutions of the class already obtained in (22), (25), and in the corresponding part of ψ, solutions of the type

$$\psi = \sin(2n+1)\frac{\pi z}{2h} F(x, y)$$

$$\psi = \cos\frac{n\pi z}{h} F(x, y),$$

each giving $\frac{d\psi}{dz} = 0$ on $z = \pm h$, and therefore zero tractions at the surface. As for the permanent terms, they may be arranged in the following groups, in each of which the surface stresses vanish.

(i)
$$\left.\begin{aligned} \phi &= -\frac{3}{4h^2}\frac{d}{dx}\Big(z\chi - \frac{1}{6}z^3\nabla^2\chi\Big) \\ \theta &= +\frac{3}{4h^2}\frac{d}{dx}\Big(z\chi - \frac{1}{6}z^3\nabla^2\chi - 2h^2 z\nabla^2\chi\Big) \end{aligned}\right\}$$

(ii)
$$\left.\begin{aligned} \phi &= -\frac{1}{40}z\frac{d}{dx}\nabla^2\chi \\ \theta &= \frac{1}{40}z\frac{d}{dx}\nabla^2\chi \end{aligned}\right\}$$

These are of the types (23), 24).

(iii)
$$\left.\begin{aligned}\psi &= z\frac{d}{dy}\nabla^2\chi\\ &= z\frac{d}{dx}\nabla^2\chi\end{aligned}\right\}$$

This gives, by (4), since $\nabla^4\chi = 0$,

$$\left.\begin{aligned}u &= z\frac{d^2}{dy^2}\nabla^2\chi = -z\frac{d^2}{dx^2}\nabla^2\chi\\ v &= -z\frac{d^2}{dxdy}\nabla^2\chi\\ w &= \frac{d}{dx}\nabla^2\chi\end{aligned}\right\}$$

and may therefore be considered as of the type (24). It is important to note, however, both here and in the cases immediately following, that the transformation on the value of u will not hold after the elementary solution has been integrated for the purposes of the general problem in (41).

(iv)
$$\left.\begin{aligned}\psi &= -\frac{1}{h}\frac{d}{dy}\left(\chi - \frac{1}{2}z^2\nabla^2\chi\right)\\ \phi &= -\frac{1}{4h}\frac{d}{dx}\left(\chi - \frac{1}{2}z^2\nabla^2\chi\right)\\ &= -\frac{1}{4h}\frac{d}{dx}\left(\chi - \frac{1}{2}z^2\nabla^2\chi\right)\end{aligned}\right\}$$

Now F being any function of x, y, satisfying $\nabla^4 F = 0$, the solution

$$\left.\begin{aligned}\psi &= 4\frac{d}{dy}\left(F - \frac{1}{2}z^2\nabla^2 F\right)\\ \phi = \theta &= \frac{d}{dx}\left(F - \frac{1}{2}z^2\nabla^2 F\right)\end{aligned}\right\} \qquad (52)$$

leads to

$$\left.\begin{aligned}u &= 8\frac{d^2F}{dy^2} + (\alpha+1)\frac{d^2F}{dx^2} + \frac{3-\alpha}{2}z^2\frac{d^2}{dx^2}\nabla^2 F\\ v &= -8\frac{d^2F}{dx\,dy} + (\alpha+1)\frac{d^2F}{dx\,dy} + \frac{3-\alpha}{2}z^2\frac{d^2}{dx\,dy}\nabla^2 F\\ w &= (\alpha-3)z\frac{d}{dx}\nabla^2 F\end{aligned}\right\} \qquad (53)$$

and $\widehat{zx} = \widehat{zy} = \widehat{zz} = 0$.

(v)
$$\psi = -\frac{h}{6}\frac{d}{dy}\nabla^2\chi,\ \phi = \frac{h}{24}\frac{d}{dx}\nabla^2\chi,\ \theta = -\frac{11h}{24}\frac{d}{dx}\nabla^2\chi$$

which may be further decomposed into

$$\psi = 0,\ \phi = \frac{h}{24}\frac{d}{dx}\nabla^2\chi,\ \theta = -\frac{h}{8}\frac{d}{dx}\nabla^2\chi,$$

of the form (26), and

$$\psi = -\frac{h}{6}\frac{d}{dy}\nabla^2\chi,\ \phi = 0,\ \theta = -\frac{h}{3}\frac{d}{dx}\nabla^2\chi,$$

the displacements corresponding to which vanish.

18. *General solution. Comparison with the solution for normal traction.*

The solution of the general problem of (41) may be found by multiplying the expressions for ψ', θ', ϕ' in (48) . . . . (51) by $\frac{1}{8\pi\mu} f(x', y')$ and integrating term by term over the area A within which f is finite. As in the case of the problem of given normal traction, term by term differentiations of the resulting series will be legitimate just so far as the derivatives are required for the calculation of displacements and stresses. In order to see this, it should be noticed that, while an extra differentiation as to x or y will be required in virtue of (43), the series for ψ', θ', ϕ' have general terms of one order higher in $1/\kappa$ as compared with those of (20), (21). One effect, however, of this additional differentiation will be to increase the relative importance at the edge of the area A of that part of the displacement and stress which arises from the local perturbation, such displacement being of order h, and stress of order zero, as in the former problem, whereas the displacement and stress as a whole is of higher order in h than before.

The functions ψ', θ', ϕ' being symmetrical about the axis $R = 0$, it is clear that the solution for an element of traction of $8\pi\mu$ units at (x', y', h) parallel to the axis of y is given by

$$\psi = -\frac{d\psi'}{dx}, \quad \theta = \frac{d\theta'}{dy}, \quad \phi = \frac{d\phi'}{dy}, \qquad . \quad . \quad . \quad . \quad (54)$$

with ψ', θ', ϕ' as in (48) . . . . (51).

It will be seen presently that surface traction may be regarded as a special case of force applied in the body of the plate. We may therefore postpone any more extended development of the above solution, and in particular any more explicit comparison of the results with those of the accepted approximate theory of thin plates, until we have obtained the solutions of the problems relative to sources of strain situated in the interior of the solid.

19. *Normal force applied at a single internal point. Solution in definite integrals.*

We take first the case of a single force, say for convenience of $4\pi\mu(a+1)$ units, applied at (x', y', z') parallel to Oz, the faces of the plate being free from stress. Referring to (6), we see that the conditions of the problem may be taken to be

(i)
$$\left.\begin{aligned} u &= (z'-z)\frac{dr^{-1}}{dx} + \mathrm{U} \\ v &= (z'-z)\frac{dr^{-1}}{dy} + \mathrm{V} \\ w &= (z'-z)\frac{dr^{-1}}{dz} + ar^{-1} + \mathrm{W} \end{aligned}\right\} \qquad . \quad . \quad . \quad . \quad (55)$$

(ii) U, V, W, along with their derivatives as to (x, y, z) of the first order, are finite and continuous at every point of the solid at a finite distance, and have derivatives of

the second order satisfying

$$\mu\nabla^2 U + (\lambda+\mu)\frac{d\Delta}{dx} = 0\,;\ \mu\nabla^2 V + (\lambda+\mu)\frac{d\Delta}{dy} = 0\,;\ \mu\nabla^2 W + (\lambda+\mu)\frac{d\Delta}{dz} = 0\,;$$

where

$$\Delta = \frac{dU}{dx} + \frac{dV}{dy} + \frac{dW}{dz}.$$

(iii) $\widehat{zx} = \widehat{zy} = \widehat{zz} = 0$ on $z = \pm h$

It is clear that these conditions do not completely define the solution, seeing that no condition to be satisfied at infinity is mentioned. But instead of laying down any such condition at infinity, it is simpler in the first instance to be content with *any* solution fulfilling (i), (ii), and (iii). The most general solution can then be obtained without difficulty, and with this before us, conditions at infinity can be discussed to much greater advantage than at present.

The problem is solved when U, V, W are found in the form (4), so as to give the same tractions on $z = \pm h$ as those due to (6), but reversed. These reversed tractions, as follows very readily from (5), (8), are given by

$$\frac{\widehat{zx}}{2\mu} = \frac{d}{dx}\left\{\frac{1-\alpha}{2}r^{-1} + (z-z')\frac{dr^{-1}}{dz}\right\}$$

$$\frac{\widehat{zy}}{2\mu} = \frac{d}{dy}\left\{\frac{1-\alpha}{2}r^{-1} + (z-z')\frac{dr^{-1}}{dz}\right\}$$

$$\frac{\widehat{zz}}{2\mu} = -\frac{1+\alpha}{2}\frac{dr^{-1}}{dz} + (z-z')\frac{d^2r^{-1}}{dz^2}$$

Now when $z > z'$,

$$r^{-1} = \int_0^\infty e^{-\kappa(z-z')} J_0\kappa R\, d\kappa$$

$$\frac{dr^{-1}}{dz} = \int_0^\infty (-\kappa) e^{-\kappa(z-z')} J_0\kappa R\, d\kappa$$

$$\frac{d^2r^{-1}}{dz^2} = \int_0^\infty \kappa^2 e^{-\kappa(z-z')} J_0\kappa R\, d\kappa$$

but when $z < z'$,

$$r^{-1} = \int_0^\infty e^{\kappa(z-z')} J_0\kappa R\, d\kappa$$

$$\frac{dr^{-1}}{dz} = \int_0^\infty \kappa e^{\kappa(z-z')} J_0\kappa R\, d\kappa$$

$$\frac{d^2r^{-1}}{dz^2} = \int_0^\infty \kappa^2 e^{\kappa(z-z')} J_0\kappa R\, d\kappa \qquad (56)$$

Hence if U, V, W be defined as in (4), the function ψ is not required, and the conditions to be satisfied by θ, ϕ are, if in the first instance we take integrands instead of integrals,

$$\frac{d\theta}{dz} + \frac{d\phi}{dz} + 2z\frac{d^2\phi}{dz^2} = \left\{\frac{1-\alpha}{2} - \kappa(h-z')\right\} e^{-\kappa(h-z')} J_0\kappa R, \text{ on } z = h$$

$$= \left\{\frac{1-\alpha}{2} - \kappa(h+z')\right\} e^{-\kappa(h+z')} J_0\kappa R, \text{ on } z = -h$$

$$\frac{d^2\theta}{dz^2} - \frac{d^2\phi}{dz^2} + 2z\frac{d^3\phi}{dz^3} = \left\{\frac{1+\alpha}{2}\kappa + \kappa^2(h-z')\right\} e^{-\kappa(h-z')} J_0\kappa R, \text{ on } z = h$$

$$= \left\{-\frac{1+\alpha}{2}\kappa - \kappa^2(h+z')\right\} e^{-\kappa(h+z')} J_0\kappa R, \text{ on } z = -h \qquad (57)$$

Assuming

$$\left.\begin{aligned}\phi &= \mathrm{A}\sinh\kappa z + \mathrm{B}\cosh\kappa z\\ \theta &= \mathrm{C}\sinh\kappa z + \mathrm{D}\cosh\kappa z\end{aligned}\right\}\mathrm{J}_0\kappa\mathrm{R}$$

(57) give four equations to determine A, B, C, D. By addition and subtraction these are resolved into two equations for A, C, and other two for B, D. Thus we find

$$\begin{aligned}\phi = {}& \frac{\sinh\kappa z\,\mathrm{J}_0\kappa\mathrm{R}}{\kappa(\sinh 2\kappa h - 2\kappa h)}\left\{\kappa z'\sinh\kappa z' - \tfrac{1}{2}\cosh\kappa z'(e^{-2\kappa h} + a + 2\kappa h)\right\}\\ &+ \frac{\cosh\kappa z\,\mathrm{J}_0\kappa\mathrm{R}}{\kappa(\sinh 2\kappa h + 2\kappa h)}\left\{\kappa z'\cosh\kappa z' + \tfrac{1}{2}\sinh\kappa z'(e^{-2\kappa h} - a - 2\kappa h)\right\}\\ \theta = {}& \frac{\sinh\kappa z\,\mathrm{J}_0\kappa\mathrm{R}}{\kappa(\sinh 2\kappa h - 2\kappa h)}\left\{-\kappa z'\sinh\kappa z'(e^{-2\kappa h} + 2\kappa h) + \cosh\kappa z'\left(\frac{a}{2}e^{-2\kappa h} + \frac{1}{2} + a\kappa h + 2\kappa^2h^2\right)\right\}\\ &+ \frac{\cosh\kappa z\,\mathrm{J}_0\kappa\mathrm{R}}{\kappa(\sinh 2\kappa h + 2\kappa h)}\left\{\kappa z'\cosh\kappa z'(e^{-2\kappa h} - 2\kappa h) + \sinh\kappa z'\left(-\frac{a}{2}e^{-2\kappa h} + \frac{1}{2} + a\kappa h + 2\kappa^2h^2\right)\right\}\end{aligned} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

If these expressions could be integrated with respect to κ from 0 to ∞, the balancing displacements U, V, W of (55) would be determined. Near the upper limit the functions converge to zero exponentialwise, since both z and z' lie between $-h$ and $+h$. But for $\kappa = 0$ both functions are infinite, and their expansions in ascending powers of κ contain terms of negative degree which must be removed after the manner of § 3. The integrals are then convergent, but a further modification of a different sort is necessary before they can be transformed into series as in §§ 5, 16. The possibility of this transformation in the former cases was intimately related to the fact that the functions in (13), (46) were odd in κ, which the functions in (58) obviously are not. However, when the odd and even parts are separated, the latter are found to have a very simple form, free from the denominators $\sinh 2\kappa h \pm 2\kappa h$, for we find

$$\begin{aligned}\phi = {}& \frac{1}{2\kappa}\sinh\kappa(z - z')\mathrm{J}_0\kappa\mathrm{R}\\ &+ \frac{\sinh\kappa z\,\mathrm{J}_0\kappa\mathrm{R}}{\kappa(\sinh 2\kappa h - 2\kappa h)}\left\{\kappa z'\sinh\kappa z' - \tfrac{1}{2}(a + \cosh 2\kappa h)\cosh\kappa z'\right\}\\ &+ \frac{\cosh\kappa z\,\mathrm{J}_0\kappa\mathrm{R}}{\kappa(\sinh 2\kappa h + 2\kappa h)}\left\{\kappa z'\cosh\kappa z' + \tfrac{1}{2}(\cosh 2\kappa h - a)\sinh\kappa z'\right\}\\ \theta = {}& -\left\{\frac{a}{2\kappa}\sinh\kappa(z - z') + z'\cosh\kappa(z - z')\right\}\mathrm{J}_0\kappa\mathrm{R}\\ &+ \frac{\sinh\kappa z\,\mathrm{J}_0\kappa\mathrm{R}}{\kappa(\sinh 2\kappa h - 2\kappa h)}\left\{-\kappa z'\cosh 2\kappa h\sinh\kappa z' + \left(\frac{a}{2}\cosh 2\kappa h + \frac{1}{2} + 2\kappa^2h^2\right)\cosh\kappa z'\right\}\\ &+ \frac{\cosh\kappa z\,\mathrm{J}_0\kappa\mathrm{R}}{\kappa(\sinh 2\kappa h + 2\kappa h)}\left\{\kappa z'\cosh 2\kappa h\cosh\kappa z' + \left(-\frac{a}{2}\cosh 2\kappa h + \frac{1}{2} + 2\kappa^2h^2\right)\cosh\kappa z'\right\}\end{aligned} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

The even terms in κ can be eliminated from these expressions by including the

values of ϕ, θ which define the source, given in (8). For these are

$$\begin{aligned}
\phi = &\ \frac{1}{2}\int_0^\infty \left\{ e^{-\kappa(z-z')}\mathrm{J}_0\kappa\mathrm{R} - e^{-\kappa h} \right\} \frac{d\kappa}{\kappa}, \text{ if } z>z' \\
= &-\frac{1}{2}\int_0^\infty \left\{ e^{\kappa(z-z')}\mathrm{J}_0\kappa\mathrm{R} - e^{-\kappa h} \right\} \frac{d\kappa}{\kappa}, \text{ if } z<z'. \\
\theta = &-\frac{a}{2}\int_0^\infty \left\{ e^{-\kappa(z-z')}\mathrm{J}_0\kappa\mathrm{R} - e^{-\kappa h} \right\} \frac{d\kappa}{\kappa} + z'\int_0^\infty e^{-\kappa(z-z')}\mathrm{J}_0\kappa\mathrm{R}\,d\kappa, \text{ if } z>z' \\
= &\ \frac{a}{2}\int_0^\infty \left\{ e^{\kappa(z-z')}\mathrm{J}_0\kappa\mathrm{R} - e^{-\kappa h} \right\} \frac{d\kappa}{\kappa} + z'\int_0^\infty e^{\kappa(z-z')}\mathrm{J}_0\kappa\mathrm{R}\,d\kappa, \text{ if } z<z'.
\end{aligned} \qquad \ldots\ldots\ (60)$$

These will obviously be reproduced, after preparation by removal of terms of negative degree and integration, from

$$\begin{aligned}
\phi = &\ \frac{1}{2\kappa} e^{-\kappa(z-z')}\mathrm{J}_0\kappa\mathrm{R}, \text{ if } z>z' \\
= &-\frac{1}{2\kappa} e^{\kappa(z-z')}\mathrm{J}_0\kappa\mathrm{R}, \text{ if } z<z'. \\
\theta = &\left(-\frac{a}{2\kappa} + z' \right) e^{-\kappa(z-z')}\mathrm{J}_0\kappa\mathrm{R}, \text{ if } z>z' \\
= &\left(\frac{a}{2\kappa} + z' \right) e^{\kappa(z-z')}\mathrm{J}_0\kappa\mathrm{R}, \text{ if } z<z'.
\end{aligned}$$

When these last terms are taken in, the first lines of ϕ, θ in (59) become

$$\phi = \pm\frac{1}{2\kappa}\cosh\kappa(z-z')\mathrm{J}_0\kappa\mathrm{R}$$

$$\theta = \mp\frac{a}{2\kappa}\cosh\kappa(z-z')\mathrm{J}_0\kappa\mathrm{R} \mp z\,\sinh\kappa(z-z')\mathrm{J}_0\kappa\mathrm{R},$$

the upper or lower sign being taken in the ambiguities according as $z>$ or $<z'$. Hence, when the source is taken in, the following are the unprepared and unintegrated forms of ϕ, θ :—

$$\begin{aligned}
\phi = &\pm\frac{1}{2\kappa}\cosh\kappa(z-z')\mathrm{J}_0\kappa\mathrm{R} \\
&+\frac{\sinh\kappa z\mathrm{J}_0\kappa\mathrm{R}}{\kappa(\sinh 2\kappa h - 2\kappa h)}\left\{ \kappa z'\sinh\kappa z' - \tfrac{1}{2}(a+\cosh 2\kappa h)\cosh\kappa z' \right\} \\
&+\frac{\cosh\kappa z\mathrm{J}_0\kappa\mathrm{R}}{\kappa(\sinh 2\kappa h + 2\kappa h)}\left\{ \kappa z'\cosh\kappa z' + \tfrac{1}{2}(\cosh 2\kappa h - a)\sinh\kappa z' \right\} \\
\theta = &\mp\frac{a}{2\kappa}\cosh\kappa(z-z')\mathrm{J}_0\kappa\mathrm{R} \mp z'\sinh\kappa(z-z')\mathrm{J}_0\kappa\mathrm{R} \\
&+\frac{\sinh\kappa z\mathrm{J}_0\kappa\mathrm{R}}{\kappa(\sinh 2\kappa h - 2\kappa h)}\left\{ -\kappa z'\cosh 2\kappa h\sinh\kappa z' + \left(\frac{a}{2}\cosh 2\kappa h + \frac{1}{2} + 2\kappa^2h^2\right)\cosh\kappa z' \right\} \\
&+\frac{\cosh\kappa z\mathrm{J}_0\kappa\mathrm{R}}{\kappa(\sinh 2\kappa h + 2\kappa h)}\left\{ \kappa z'\cosh 2\kappa h\cosh\kappa z' + \left(-\frac{a}{2}\cosh 2\kappa h + \frac{1}{2} + 2\kappa^2h^2\right)\sinh\kappa z' \right\}
\end{aligned} \qquad \ldots\ldots\ (61)$$

In (61) the terms of negative degree in κ are of the forms :—in ϕ, $\mathrm{H}/\kappa^3 + \mathrm{K}/\kappa$; in θ, $\mathrm{L}/\kappa^3 + \mathrm{M}/\kappa$; and these terms, as in § 3, give $\widehat{zx} = \widehat{zy} = \widehat{zz} = 0$ at the faces of the plate.

Hence if from ϕ in (61) we subtract $H/\kappa^3 + Ke^{-\kappa h}/\kappa$, and from θ, $L/\kappa^3 + Me^{-\kappa h}/\kappa$, the resulting expressions, integrated with respect to κ from 0 to ∞, will define a solution of the problem stated at the beginning of this article.

20. *Normal force applied at a single internal point. Solution in series.*

To the integrals thus obtained we can apply the transformation of § 5, but one remark should be made. From the synthesis which gave (61), it is sufficiently obvious, in view of the forms in (58) and (60), that the expressions of (61), with $G_0\kappa R$ substituted for $J_0(\kappa R)$, vanish effectively at infinity in the first quadrant of the κ plane; that they similarly vanish in the second quadrant follows at once from the fact that the functions of (61) are odd functions of κ.

A glance at the relation between (16), (17) and (20), (21) will again save us the necessity of writing down the details. Thus, let the values of H, K, L, M, when R is put equal to zero, be denoted by H_0, K_0, L_0, M_0. Then the persistent part of the transformed solution is given by

$$\left.\begin{aligned} \phi &= H_0\chi(R) - K_0\nabla^2\chi(R) \\ \theta &= L_0\chi(R) - M_0\nabla^2\chi(R) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

The decaying part is given by

$$\left.\begin{aligned} \phi &= \sum_\kappa \frac{G_0\kappa R \sinh \kappa z}{\kappa h(\cosh 2\kappa h - 1)} \left\{ \kappa z' \sinh \kappa z' - \frac{1}{2}(a + \cosh 2\kappa h)\cosh \kappa z' \right\} \\ \theta &= \sum_\kappa \frac{G_0\kappa R \sinh \kappa z}{\kappa h(\cosh 2\kappa h - 1)} \left\{ -\kappa z' \cosh 2\kappa h \sinh \kappa z' + \left(\frac{a}{2}\cosh 2\kappa h + \frac{1}{2} + 2\kappa^2 h^2\right)\cosh \kappa z' \right\} \\ &= \sum_\kappa \frac{G_0\kappa R \sinh \kappa z}{\kappa h(\cosh 2\kappa h - 1)} \left\{ \kappa z' \sinh \kappa z' - \frac{1}{2}(a + \cosh 2\kappa h)\cosh \kappa z' \right\} (-\cosh 2\kappa h) \end{aligned}\right\} \quad . \quad (63)$$

where κ is a zero of $\sinh 2\kappa h - 2\kappa h$, with positive imaginary part;
with

$$\left.\begin{aligned} \phi &= \sum_\kappa \frac{G_0\kappa R \cosh \kappa z}{\kappa h(\cosh 2\kappa h + 1)} \left\{ \kappa z' \cosh \kappa z' + \frac{1}{2}(\cosh 2\kappa h - a)\sinh \kappa z' \right\} \\ \theta &= \sum_\kappa \frac{G_0\kappa R \cosh \kappa z}{\kappa h(\cosh 2\kappa h + 1)} \left\{ \kappa z' \cosh \kappa z' + \frac{1}{2}(\cosh 2\kappa h - a)\sinh \kappa z' \right\} \cosh 2\kappa h \end{aligned}\right\} \quad . \quad . \quad (64)$$

where κ is a zero of $\sinh 2\kappa h + 2\kappa h$, with positive imaginary part.

When the values of H_0, K_0, L_0, M_0 are obtained from (61), it will be found that (62) may be decomposed as follows:—

(i) $$\phi = -\frac{3(a+1)}{8h^3}\left(z\chi - \frac{1}{6}z^3\nabla^2\chi\right)$$
$$\theta = \frac{3(a+1)}{8h^3}\left(z\chi - \frac{1}{6}z^3\nabla^2\chi - 2h^2 z\nabla^2\chi\right)$$

(ii) $$\left.\begin{aligned} \phi &= -z\nabla^2\chi \\ \theta &= z\nabla^2\chi \end{aligned}\right\} \text{ each multiplied by } \frac{3}{4h^3}(z'^2 - h^2) + \frac{3(a+1)}{8h^3}\left(\frac{1}{5}h^2 - \frac{1}{2}z'^2\right)$$

(iii) $$\phi = \frac{z'(a-3)}{8h}\nabla^2\chi = \theta$$

(iv) $\phi = \mp \frac{1}{2}\nabla^2\chi$; $\theta = \pm\frac{1}{2}a\nabla^2\chi$, with upper or lower signs, as z is greater or less than z'.

. . . . . (65)

There being no discontinuity of displacement or strain at the plane $z=z'$, except at the point where the force is applied, we are prepared to find that (iv) give no displacement at all if $R>0$. So long, however, as we keep to the specification of the strain by the ϕ, θ functions, it is convenient to retain the terms in (iv). By so doing, we of course make the ϕ of the space above the plane $z=z'$ and the ϕ of the space below that plane two distinct potential functions, but we preserve the non-singular character of each of these functions at the axis $R=0$.

If we take the limit of the above solution for $z'=h$, which obviously may be done by putting $z'=h$ in each term and using the lower signs in (iv), we obtain simply the solution of (20), (21) multiplied by $\frac{1}{2}(a+1)$. Since the present solution is for a force of $4\pi\mu(a+1)$ units, and the other for an element of traction of $8\pi\mu$ units, it follows that a unit element of traction may be regarded as simply the limiting case of a unit force, the point of application of which approaches indefinitely near the surface.

21. *Solution of a special problem of internal areal normal force.*

When the displacements due to a unit Z force at (x', y', z'), with the surface free, are known, the corresponding displacements for a body distribution of force, of amount $Z(x', y', z')$ per unit volume at (x', y', z'), can be found by multiplying by $Z(x', y', z')dx'\,dy'\,dz'$ and integrating through the space in which Z is finite. Certain peculiarities in the form of the solution given in § 20 make it convenient to take the integration with respect to z' last, or, as comes to the same thing, to begin by considering the solution for an *areal* distribution of force on the plane $z=z'$, of magnitude $Z(x', y', z')$ per unit area.

We take first a special problem analogous to that worked out in § 10, and suppose the Z force to be distributed over the area of a circle of radius a in the plane $z=z'$, with centre on Oz, the intensity per unit area being $4\pi\mu(a+1)J_m\beta\rho\cos m\omega$. It will be sufficient to attend to the value of ϕ, for when that is known, the corresponding value of θ can be written down at once.

The series deduced by integration from (63), (64), say ϕ_2, fall naturally into two parts as in § 10, viz., (i) series defining a local perturbation at the cylinder $\rho=a$,

$$\phi = \sum_\kappa \frac{\sinh \kappa z}{\kappa h(\cosh 2\kappa h - 1)} \left\{ \kappa z' \sinh \kappa z' - \frac{1}{2}(a+\cosh 2\kappa h)\cosh \kappa z' \right\} \cdot \frac{2\pi}{\beta^2-\kappa^2} P_\kappa \quad (\text{where } \sinh 2\kappa h - 2\kappa h = 0)$$
$$+ \sum_\kappa \frac{\cosh \kappa z}{\kappa h(\cosh 2\kappa h + 1)} \left\{ \kappa z' \cosh \kappa z' + \frac{1}{2}(\cosh 2\kappa h - a)\sinh \kappa z' \right\} \cdot \frac{2\pi}{\beta^2-\kappa^2} P_\kappa \quad (\text{where } \sinh 2\kappa h + 2\kappa h = 0)$$

with

$$\begin{aligned} P_\kappa &= J_m\kappa\rho\cos m\omega(\kappa a G_m'\kappa a J_m\beta a - G_m\kappa a\beta a J_m'\beta a), \quad \text{if } \rho<a \\ &= G_m\kappa\rho\cos m\omega(\kappa a J_m'\kappa a J_m\beta a - J_m\kappa a\cdot\beta a J_m'\beta a), \quad \text{if } \rho>a \end{aligned} \qquad (66)$$

(ii) When $\rho<a$, series which can be summed in finite terms,

$$\phi = 2\pi J_m\beta\rho\cos m\omega \left[\sum_\kappa \frac{\sinh \kappa z}{(\beta^2-\kappa^2)\kappa h(\cosh 2\kappa h - 1)} \left\{ \kappa z' \sinh \kappa z' - \frac{1}{2}(a+\cosh 2\kappa h)\cosh \kappa z' \right\} (\text{where } \sinh 2\kappa h - 2\kappa h = 0) \right.$$
$$\left. + \sum_\kappa \frac{\cosh \kappa z}{(\beta^2-\kappa^2)\kappa h(\cosh 2\kappa h + 1)} \left\{ \kappa z' \cosh \kappa z' + \frac{1}{2}(\cosh 2\kappa h - a)\sinh \kappa z' \right\} (\text{where } \sinh 2\kappa h + 2\kappa h = 0) \right. \qquad (67)$$

In order to sum these last series, consider the function of κ,

$$\begin{aligned}
&\pm && \frac{\cosh \kappa(z-z')}{2\kappa(\beta^2-\kappa^2)} && \left(\pm \text{ according as } z> \text{ or } <z'\right)\\
&+ && \frac{\sinh \kappa z}{(\beta^2-\kappa^2)\kappa(\sinh 2\kappa h - 2\kappa h)} && \left\{\kappa z' \sinh \kappa z' - \tfrac{1}{2}(a+\cosh 2\kappa h)\cosh \kappa z'\right\}\\
&+ && \frac{\cosh \kappa z}{(\beta^2-\kappa^2)\kappa(\sinh 2\kappa h + 2\kappa h)} && \left\{\kappa z' \cosh \kappa z' + \tfrac{1}{2}(\cosh 2\kappa h - a)\sinh \kappa z'\right\} \quad . \quad . \quad . \quad (68)
\end{aligned}$$

Looking back at (61) we assure ourselves that this function vanishes at infinity in such a way as to make the sum of its residues zero. Also, since the function is odd in κ, the residues at $\kappa = \pm \kappa_1$ are equal, and therefore the sum of the residues at the zeroes of $\sinh 2\kappa h \pm 2\kappa h$ is simply the coefficient of $2\pi J_m \beta\rho \cos m\omega$ in (67). The sum of the residues at $\kappa = \pm \beta$ is

$$\begin{aligned}
&\mp && \frac{\cosh \beta(z-z')}{2\beta^2} && \\
&- && \frac{\sinh \beta z}{\beta^2(\sinh 2\beta h - 2\beta h)} && \left\{\beta z' \sinh \beta z' - \tfrac{1}{2}(a+\cosh 2\beta h)\cosh \beta z'\right\}\\
&- && \frac{\cosh \beta z}{\beta^2(\sinh 2\beta h + 2\beta h)} && \left\{\beta z' \cosh \beta z' + \tfrac{1}{2}(\cosh 2\beta h - a)\sinh \beta z'\right\} \quad . \quad . \quad . \quad (69)
\end{aligned}$$

If this last expression near $\beta = 0$ be of the form $\frac{A}{\beta^4} + \frac{B}{\beta^2} + \ldots$, the residue at $\kappa = 0$ of (68) is simply $-\frac{A}{\beta^4} - \frac{B}{\beta^2}$.

Hence the coefficient of $2\pi J_m \beta\rho \cos m\omega$ in (67) is simply (69) with sign changed and the terms of negative degree in β subtracted. These terms of negative degree, just as in § 12, are added on again when we take in the part of the solution coming from (65), which is obtained by writing F for χ in (65) where

$$\begin{aligned}
F &= \iint \chi(R) J_m \beta\rho' \cos m\omega' \, \rho' \, d\rho' \, d\omega'.\\
&= \frac{2\pi}{\beta^4} J_m \beta\rho \cos m\omega + F_1. \quad \text{(Introd. } (f)\text{.)}
\end{aligned}$$

The term $\frac{2\pi}{\beta^4} J_m \beta\rho \cos m\omega$ being taken in for the purpose just mentioned, we are left with F_1 instead of χ in (65). Since $\nabla^4 F_1 = 0$, these equations now define a combination of deformations of the persistent or permanent type, under no body force and no surface traction.

The solution therefore resolves itself into

(i) this free deformation of the permanent mode;

(ii) a local perturbation;

(iii) a particular solution giving the proper discontinuity of stress corresponding to the applied areal force.

The particular solution is

$$\phi = 2\pi J_m \beta\rho \cos m\omega \left\{ \begin{array}{rl} \pm & \frac{1}{2\beta^2}\cosh\beta(z-z') \\ + & \frac{\sinh\beta z}{\beta^2(\sinh 2\beta h - 2\beta h)}(\beta z' \sinh\beta z' - \tfrac{1}{2}\overline{a+\cosh 2\beta h}\cosh\beta z') \\ + & \frac{\cosh\beta z}{\beta^2(\sinh 2\beta h + 2\beta h)}(\beta z' \cosh\beta z' + \tfrac{1}{2}\overline{\cosh 2\beta h - a}\sinh\beta z') \end{array} \right\} \quad . \quad (70)$$

with a corresponding expression for θ, obtainable from (61) by changing κ into β and then replacing $J_0\beta R$ by $\frac{1}{\beta}2\pi J_m\beta\rho\cos m\omega$.

It is easy to verify that this is actually a particular solution. Consider in the first place the analogous forms of ϕ, θ in (61), and for greater generality, suppose $J_0\kappa R$ replaced by $f(x, y)$ where $(\nabla^2+\kappa^2)f=0$.

Then, from the method by which (61) were found, it is obvious that they give no stress across the planes $z=\pm h$. Let us examine the effect of the discontinuity in the forms of ϕ, θ at the plane $z=z'$, on the displacements and stresses as given in (4), (5).

If we take simply

$$\begin{aligned} \phi &= \frac{1}{2\kappa}\cosh\kappa(z-z')f \\ \theta &= \frac{-a}{2\kappa}\cosh\kappa(z-z')f - z'\sinh\kappa(z-z')f \end{aligned}$$

then we find at $z=z'$,

$$\begin{aligned} u = v = w &= 0 \\ \widehat{zx} = \widehat{zy} &= 0 \\ \widehat{zz} &= -\mu(a+1)\kappa f. \end{aligned}$$

Thus with the complete expression (61), the displacements are continuous, as also the stresses $\widehat{zz}$, $\widehat{zy}$, but $\widehat{zz}$ $(z=z'+)$ exceeds $\widehat{zz}$ $(z=z'-)$ by $-2\mu(a+1)\kappa f$. We thus see that in (70) the corresponding discontinuity in $\widehat{zz}$ will be $-4\pi\mu(a+1)J_m\beta\rho\cos m\omega$. This continuity of displacement, and discontinuity in $\widehat{zz}$, are precisely as demanded by the conditions of equilibrium of the plate.

If we take (61) with $J_0\kappa R$ unaltered, prepare them for integration as in § 19, multiply by $e^{-\kappa\epsilon}$ and integrate with respect to κ from 0 to ∞, the discontinuity in $\widehat{zz}$ at $z=z'$ will become

$$-2\mu(a+1)\int_0^\infty e^{-\kappa\epsilon}\kappa J_0\kappa R\,d\kappa .$$

If further we multiply this by $Z(x', y', z')dx'\,dy'$ and integrate with respect to x', y', and then take the limit for $\epsilon=0$, the discontinuity becomes, in virtue of (11),

$$-4\pi\mu(a+1)Z(x, y, z').$$

We have thus a proof of the solution for an areal distribution of Z force, independent of the infinite solid solution (6), which might itself be found from the beginning by this method.

22. *The general problem of internal normal force. Approximate forms of displacements and stresses.*

The developments given in §§ 14, 15 may obviously be applied in the present case also.

Thus if in (61) we divide by $\kappa J_0\kappa R$, expand in ascending powers of κ, put $-\nabla^2$, that is $-\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)$ for κ^2, and operate on Z (x, y, z'), we obtain a form of solution which, with the interpretation of $\nabla^{-4}Z$ and $\nabla^{-2}Z$ given in (38), is simply the foregoing general solution for areal force of intensity $2\mu(\alpha+1)Z$, arranged in terms of ascending order in h^2. The solution in this form fails if at (x, y, z), Z or any of its successive derivatives become discontinuous, but it has been shown in § 14 how the local perturbation in the neighbourhood of any surface of discontinuity may be calculated.

For the case when Z vanishes outside an area A, the principal part of the perturbation at the edge of A, when h is small, is found by substituting for $G_0\kappa R$ in (63), (64),

$$-\frac{1}{\kappa^2}\int\left\{Z(x', y')\frac{d}{dn}G_0\kappa R - G_0(\kappa R)\frac{d}{dn}Z(x', y')\right\}ds,$$

where differentiations and integrations have reference to the accented coordinates.

Since the solution for the case when there are any finite number of surfaces at which Z or its derivatives become discontinuous can be found from this elementary case by simple summation, we see that discontinuity in the force itself gives rise to values of ϕ, θ in the perturbation terms of order h^2 at the surface, discontinuity in $\frac{dZ}{dn}$ to terms of order h^3 if Z itself is continuous. The next term is of order h^4 and depends on discontinuity of $\nabla^2 Z$, that is, of the second derivatives of Z, and so on.

The symbolical solution for Z force distributed on the plane $z=z'$ with intensity $2\mu(\alpha+1)Z(x, y, z')$ per unit area at (x, y, z') is given by

$$\begin{aligned}
\phi &= \pm\frac{1}{2\kappa^2}\cosh\kappa(z-z') \\
&\quad + \frac{\sinh\kappa z}{\kappa^2(\sinh 2\kappa h - 2\kappa h)}\left(\kappa z'\sinh\kappa z' - \tfrac{1}{2}\overline{\alpha+\cosh 2\kappa h}\cosh\kappa z'\right) \\
&\quad + \frac{\cosh\kappa z}{\kappa^2(\sinh 2\kappa h + 2\kappa h}\left(\kappa z'\cosh\kappa z' + \tfrac{1}{2}\overline{\cosh 2\kappa h - \alpha}\sinh\kappa z'\right) \\
\theta &= \mp\frac{\alpha}{2\kappa^2}\cosh\kappa(z-z') \mp \frac{z'}{\kappa}\sinh\kappa(z-z') \\
&\quad + \frac{\sinh\kappa h}{\kappa^2(\sinh 2\kappa h - 2\kappa h}\left(-\kappa z'\cosh 2\kappa h\sinh\kappa z' + \overline{\frac{\alpha}{2}\cosh 2\kappa h + \tfrac{1}{2} + 2\kappa^2h^2}\cosh\kappa z'\right) \\
&\quad + \frac{\cosh\kappa z}{\kappa^2(\sinh 2\kappa h + 2\kappa h)}\left(\kappa z'\cosh 2\kappa h\cosh\kappa z' + \overline{\tfrac{1}{2} - \frac{\alpha}{2}\cosh 2\kappa h + 2\kappa^2h^2}\sinh\kappa z'\right)
\end{aligned}$$

with $\kappa^2 = -\nabla^2$, operating on $Z(x, y, z')$ . . . . (71)

The approximate solution is obtained by expanding in ascending powers of κ^2. By retaining only the terms of negative degree in κ^2, each of the displacements will be

given to a second approximation, and each of the stresses, except $\widehat{zz}$, to a first approximation. The result is obviously the same as that found by integrating the permanent terms (65) of the original source solution.

If we write

$$\mathrm{F} = \iint \mathrm{Z}(x', y', z')\, \chi(\mathrm{R})\, dx'\, dy',$$

then the displacements for Z force of intensity $\mathrm{Z}(x, y, z')$ per unit area on $z = z'$ are

$$\left.\begin{aligned} u &= \frac{d}{dx} \\ v &= \frac{d}{dy} \end{aligned}\right\} \frac{3}{32\pi\mu h^3} \left\{ -(\alpha+1)z\mathrm{F} + \nabla^2\mathrm{F}\left(\frac{\alpha+5}{6}z^3 + \frac{\alpha-3}{2}z'^2 z - \frac{+1}{5}h^2 z + \frac{\alpha-3}{3}h^2 z'\right)\right\}$$

$$w = \frac{3}{32\pi\mu h^3}\left\{ (\alpha+1)\mathrm{F} + \nabla^2\mathrm{F}\left(\frac{3-\alpha}{2}z^2 + \frac{3-\alpha}{2}z'^2 - 4h^2 + \frac{\alpha+1}{5}h^2\right)\right\} \quad . \quad . \quad . \quad (72)$$

The corresponding results for a volume distribution of force, $\mathrm{Z}(x, y, z)$ per unit volume, are found by integrating these with respect to z' from $-h$ to $+h$.

In order to calculate the stress $\widehat{zz}$ from displacements, we should need the value of w to a third approximation. It is therefore easier to find $\widehat{zz}$ directly from (71) and (5). On dividing by $2\mu(\alpha+1)$, we find, corresponding to (72),

$$\widehat{zz} = \mp\tfrac{1}{2}\mathrm{Z} + \frac{3h^2 z - z^3}{4h^3}\,\mathrm{Z} \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

When the force is $\mathrm{Z}(x, y, z)$ per unit volume, this leads to

$$\widehat{zz} = \frac{3h^2 z - z^3}{4h^3}\int_{-h}^{+h}\mathrm{Z}(x, y, z')dz' - \frac{1}{2}\int_{-h}^{z}\mathrm{Z}(x, y, z')dz' + \frac{1}{2}\int_{z}^{h}\mathrm{Z}(x, y, z')dz' \quad . \quad . \quad . \quad . \quad (74)$$

We can now find the stresses $\widehat{xx}$, $\widehat{xy}$, $\widehat{yy}$ to a second approximation.

For

$$\widehat{xx} = (\lambda+2\mu)\frac{du}{dx} + \lambda\frac{dv}{dy} + \lambda\frac{dw}{dz}$$

$$\widehat{zz} = \lambda\frac{du}{dx} + \lambda\frac{dv}{dy} + (\lambda+2\mu)\frac{dw}{dz},$$

whence, eliminating $\frac{dw}{dz}$,

$$\widehat{xx} = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\left(\frac{du}{dx} + \sigma\frac{dv}{dy}\right) + \frac{\lambda}{\lambda+2\mu}\widehat{zz},$$

and similarly

$$\widehat{yy} = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\left(\sigma\frac{du}{dx} + \frac{dv}{dy}\right) + \frac{\lambda}{\lambda+2\mu}\widehat{zz}, \quad \text{where} \quad \sigma = \frac{1}{2}\lambda/(\lambda+\mu).$$

Also

$$\widehat{xy} = \mu\left(\frac{du}{dy} + \frac{dv}{dx}\right).$$

We have now only to put in the values of u, v from (72) and the value of $\widehat{zz}$ from (73).

If we denote $\frac{3}{32\pi\mu h^3}(a+1)\mathrm{F}$, the principal term in w in (72), by W, we have

$$\begin{aligned}\nabla^4\mathrm{W} &= \frac{3}{32\pi\mu h^3}(a+1)\nabla^4\mathrm{F}\\ &= \frac{3}{16\mu h^3}\cdot(a+1)\mathrm{Z}\\ &= \frac{3}{8\mu h^3}\,\frac{\lambda+2\mu}{\lambda+\mu}\mathrm{Z}\end{aligned}$$

or
$$\mathrm{C}\nabla^4\mathrm{W}=\mathrm{Z},\quad \text{where}\quad \mathrm{C}=\tfrac{8}{3}\mu h^3(\lambda+\mu/(\lambda+2\mu)\quad .\quad .\quad .\quad (75)$$

In this notation, to a first approximation

$$\begin{aligned}\widehat{xx} &= -\frac{3\mathrm{C}}{2h^3}z\left(\frac{d^2\mathrm{W}}{dx^2}+\sigma\frac{d^2\mathrm{W}}{dy^2}\right)\\ \widehat{yy} &= -\frac{3\mathrm{C}}{2h^3}z\left(\sigma\frac{d^2\mathrm{W}}{dx^2}+\frac{d^2\mathrm{W}}{dy^2}\right)\\ \widehat{xy} &= -\frac{3\mathrm{C}}{2h^3}(1-\sigma)z\frac{d^2\mathrm{W}}{dx\,dy}\quad .\quad .\quad .\quad .\quad (76)\end{aligned}$$

Again from (72),

$$\begin{aligned}\widehat{zx} &= \mu\left(\frac{du}{dz}+\frac{dw}{dx}\right)=\frac{3\mu}{32\pi\mu h^3}\cdot 4(z^2-h^2)\frac{d}{dx}\nabla^2\mathrm{F}\\ &= \frac{3}{4h^3}\mathrm{C}(z^2-h^2)\frac{d}{dx}\nabla^2\mathrm{W}\quad .\quad .\quad .\quad .\quad (77)\end{aligned}$$

23. *Normal force a function of z only.*

It may be useful to put down here the next term in the development of w, of which the two principal terms are given in (72). This is

$$\begin{aligned}&\frac{3}{32\pi\mu h^3}\nabla^4\mathrm{F}\left\{\begin{aligned}&\frac{a-7}{24}(z^4+z'^4)+\frac{(a-3)^2}{4(a+1)}z^2z'^2+\left(\frac{3-a}{10}+\frac{2a-2}{a+1}\right)h^2(z^2+z'^2)+\left(\frac{4}{3a+3}-\frac{4}{5}+\frac{11a+11}{3\cdot5^2\cdot7}\right)h^4\\ &\qquad+\frac{(3-a)^2}{3a+3}h^2zz'\end{aligned}\right\}\quad .\quad (78)\\ &\pm(1-a)(z-z')\ \mathrm{Z}(x,y,z')\,.\end{aligned}$$

In this, of course,

$$\nabla^4\mathrm{F}=2\pi\mathrm{Z}(x,y,z')\,.$$

The terms which have to be added to (72), (78) in order to give the complete particular values of u, v, w, all contain x, y derivatives of $\nabla^4\mathrm{F}$ or Z. Hence, if $\mathrm{Z}(x,y,z')$ is a function of z' alone, (72) and (78) give a complete particular solution of the problem. Further, Z may have one constant value in one region of the plane $z=z'$, and another constant value in another region of that plane. (72), (78) will still give a particular solution in each of those regions taken separately, or rather in the cylindrical spaces of which these regions are sections, but it ought to be carefully noticed that it is not in general an exact solution when the two regions are considered together as part of one body. The point of failure is, it need scarcely be said, the condition of synexis;

the two particular solutions do not fit, that is, they do not give the same values for displacements and strains on the two sides of the cylindrical surface or surfaces of discontinuity.

On the other hand, the supplementary terms required in order to make the solution synectic belong to what we have called the decaying type. They give rise to displacements and strains of infinitely high order, if we may so speak, in the small quantity h, except very near the surfaces of discontinuity. This being so, we need not be surprised to find that the solution (72), (78) is not necessarily the simplest particular solution in any one region within which Z is continuously constant.

Thus, for example, if we pick out the terms which contain z'^2 as a factor, we find displacements proportional to

$$\left.\begin{aligned} u &= \frac{d}{dx} \\ v &= \frac{d}{dy} \end{aligned}\right| \; -z\nabla^2\mathrm{F}$$

$$w = \nabla^2\mathrm{F} + \nabla^4\mathrm{F}\left(\frac{3-a}{a+1}\,\frac{1}{2}z^2 + \mathrm{C}_1h^2\right)$$

which belong to the type (23), and contribute nothing to body force or face tractions. These terms might therefore be omitted in any problem where the condition of synexis is irrelevant, and in particular when the object is merely to obtain a particular solution for body force and face traction in a problem relating to a *finite* solid.

24. *Internal force parallel to the faces.*

We will now go on to consider the problem of force applied to the body in a direction parallel to the faces of the plate.

A force of $4\pi\mu(a+1)$ units applied at (x', y', z') in the direction of Ox gives in an infinite solid displacements defined, according to (9), by

$$\left.\begin{aligned} \psi &= -\frac{a+1}{2}\,\frac{d}{dy}\,\frac{d^{-2}r^{+1}}{dz^{-2}} \\ \theta &= -\frac{a}{2}\,\frac{d}{dx}\,\frac{d^{-2}r^{-1}}{dz^{-2}} + z'\frac{d}{dx}\,\frac{d^{-1}r^{-1}}{dz^{-1}} \\ \phi &= -\frac{1}{2}\,\frac{d}{dx}\,\frac{d^{-2}r^{-1}}{dz^{-2}} \end{aligned}\right\}$$

Hence the tractions which such a force produces on $z = \pm h$ will be neutralised by a system ψ, θ, ϕ for which

$$\left.\begin{aligned} \frac{d\psi}{dz} &= \frac{d}{dy}\left(\frac{a+1}{2}\,\frac{d^{-1}r^{-1}}{dz^{-1}}\right) \\ \frac{d\theta}{dz} + \frac{d\phi}{dz} + 2z\frac{d^2\phi}{dz^2} &= \frac{d}{dx}\left(\frac{a+1}{2}\,\frac{d^{-1}r^{-1}}{dz^{-1}} + \overline{z-z'}\;r^{-1}\right) \\ \frac{d^2\theta}{dz^2} - \frac{d^2\phi}{dz^2} + 2z\frac{d^3\phi}{dz^3} &= \frac{d}{dx}\left(\frac{a-1}{2}\;r^{-1} + \overline{z-z'}\;\frac{dr^{-1}}{dz}\right) \end{aligned}\right\} \text{on } z = \pm h\,.$$

These conditions are satisfied if

$$\psi=\frac{d\psi'}{dy},\quad \theta=\frac{d\theta'}{dx},\quad \phi=\frac{d\phi'}{dx}\quad .\quad .\quad .\quad .\quad .\quad (79)$$

and

$$\left.\begin{aligned}
\frac{d\psi}{dz}&=\frac{a+1}{2}\,\frac{d^{-1}r^{-1}}{dz^{-1}}\\
\frac{d\theta'}{dz}+\frac{d\phi'}{dz}+2z\frac{d^2\phi'}{dz^2}&=\frac{a+1}{2}\,\frac{d^{-1}r^{-1}}{dz^{-1}}+\overline{z-z'}\;r^{-1}\\
\frac{d^2\theta'}{dz^2}-\frac{d^2\phi'}{dz^2}+2z\frac{d^3\phi'}{dz^3}&=\frac{a-1}{2}\;r^{-1}+\overline{z-z'}\;\frac{dr^{-1}}{dz}
\end{aligned}\right\}\text{on } z=\pm h.$$

We may take

$$\frac{d^{-2}r^{-1}}{dz^{-2}}=\int_0^\infty\left\{e^{\mp\kappa(z-z')}\mathrm{J}_0\kappa\mathrm{R}-1\pm\kappa(z-z')e^{-\kappa h}\right\}\frac{d\kappa}{\kappa^2}$$

$$\frac{d^{-1}r^{-1}}{dz^{-1}}=\int_0^\infty(\mp)\left\{e^{\mp\kappa(z-z')}\mathrm{J}_0\kappa\mathrm{R}-e^{-\kappa h}\right\}\frac{d\kappa}{\kappa}$$

$$r^{-1}=\int_0^\infty e^{\mp\kappa(z-z')}\mathrm{J}_0\kappa\mathrm{R}\,d\kappa$$

upper or lower signs being taken, as all along, according as $z>z'$ or $z<z'$. We therefore determine provisional values of ψ', θ', ϕ', such that

$$\left.\begin{aligned}
\frac{d\psi'}{dz}&=\mp\frac{a+1}{2\kappa}e^{\mp\kappa(z-z')}\mathrm{J}_0\kappa\mathrm{R}\\
\frac{d\theta'}{dz}+\frac{d\phi'}{dz}+2z\frac{d^2\phi'}{dz^2}&=\left(\mp\frac{a+1}{2\kappa}+z-z'\right)e^{\mp\kappa(z-z')}\mathrm{J}_0\kappa\mathrm{R}\\
\frac{d^2\theta'}{dz^2}-\frac{d^2\phi'}{dz^2}+2z\frac{d^3\phi'}{dz^3}&=\left(\frac{a-1}{2}\mp\kappa\overline{z-z'}\right)e^{\mp\kappa(z-z')}\mathrm{J}_0\kappa\mathrm{R}
\end{aligned}\right\}\text{on } z=\pm h$$

These provisional values are easily found to be

$$\left.\begin{aligned}
\psi'=&\frac{a+1}{2\kappa^2}\cosh\kappa(z-z')\\
&+\frac{a+1}{2\kappa^2\cosh\kappa h}\sinh\kappa h\sinh\kappa z'\sinh\kappa z\\
&-\frac{a+1}{2\kappa^2\sinh\kappa h}\cosh\kappa h\cosh\kappa z'\cosh\kappa z
\end{aligned}\right\}\mathrm{J}_0\kappa\mathrm{R}$$

$$\left.\begin{aligned}
\phi'=&\frac{1}{2\kappa^2}\cosh\kappa(z-z')\\
&+\frac{\sinh\kappa z}{\kappa^2(\sinh 2\kappa h-2\kappa h)}\left\{-\kappa z'\cosh\kappa z'+\frac{1}{2}(\cosh 2\kappa h-a)\sinh\kappa z'\right\}\\
&-\frac{\cosh\kappa z}{\kappa^2(\sinh 2\kappa h+2\kappa h)}\left\{\kappa z'\sinh\kappa z'+\frac{1}{2}(\cosh 2\kappa h+a)\cosh\kappa z'\right\}
\end{aligned}\right\}\mathrm{J}_0\kappa\mathrm{R}$$

$$\left.\begin{aligned}
\theta'=&\frac{a}{2\kappa^2}\cosh\kappa(z-z')-\frac{z'}{\kappa}\sinh\kappa(z-z')\\
&+\frac{\sinh\kappa z}{\kappa^2(\sinh 2\kappa h-2\kappa h)}\left\{\kappa z'\cosh 2\kappa h\cosh\kappa z'+\frac{1}{2}(a\cosh 2\kappa h-1-4\kappa^2h^2)\sinh\kappa z'\right\}\\
&-\frac{\cosh\kappa z}{\kappa^2(\sinh 2\kappa h+2\kappa h)}\left\{\kappa z'\cosh 2\kappa h\sinh\kappa z'+\frac{1}{2}(a\cosh 2\kappa h+1+4\kappa^2h^2)\cosh\kappa z'\right\}
\end{aligned}\right\}\mathrm{J}_0\kappa\mathrm{R}$$

. . . . . (80)

The source itself is similarly given by the temporary values

$$\left.\begin{aligned} \psi' &= -\frac{a+1}{2\kappa^2} \\ \theta' &= -\frac{a}{2\kappa^2} \mp \frac{z'}{\kappa} \\ \phi' &= -\frac{1}{2\kappa^2} \end{aligned}\right\} \epsilon^{\mp\kappa(z-z')} J_0\kappa R$$

Thus, when the source is included, the provisional values of ψ', θ', ϕ' are as in (80), but with the first lines altered,

$$\left.\begin{aligned} &\text{in } \psi' \text{ to } \pm\frac{a+1}{2\kappa^2}\sinh\kappa(z-z') \\ &\text{,, } \phi' \text{ ,, } \pm\frac{1}{2\kappa^2}\sinh\kappa(z-z') \\ &\text{,, } \theta' \text{ ,, } \pm\frac{a}{2\kappa^2}\sinh\kappa(z-z') \mp \frac{z'}{\kappa}\cosh\kappa(z-z') \end{aligned}\right\} J_0\kappa R \quad . \quad . \quad . \quad (81)$$

25. *Solution of the problem of internal force parallel to the faces.*

From these expressions the solution in the form of definite integrals, and finally of series, is obtained as in the previous cases. After the explanations already given, it will be sufficient to write down the final results. For the transitory part of the solution,

$$\psi' = \sum_\kappa (a+1)\frac{1}{\kappa^2 h}\sinh\kappa z'\sinh\kappa z G_0\kappa R, \text{ (}\kappa \text{ a pos. imag. root of } \cosh\kappa h\text{)}$$

$$-\sum_\kappa (a+1)\frac{1}{\kappa^2 h}\cosh\kappa z'\cosh\kappa z G_0\kappa R, \text{ (}\kappa \text{ a pos. imag. root of } \sinh\kappa h\text{)}.$$

$$\left.\begin{aligned} \phi' &= \sum_\kappa \frac{\sinh\kappa z\, G_0\kappa R}{\kappa^2 h(\cosh 2\kappa h - 1)}\left\{ -\kappa z'\cosh\kappa z' + \frac{1}{2}(\cosh 2\kappa h - a)\sinh\kappa z' \right\} \\ \theta' &= \sum_\kappa \text{ same as previous line multiplied by } (-\cosh 2\kappa h) \end{aligned}\right]$$

where κ is a zero of $\sinh 2\kappa h - 2\kappa h$, with pos. imag. part.
With

$$\left.\begin{aligned} \phi' &= \sum_\kappa (-)\frac{\cosh\kappa z G_0\kappa R}{\kappa^2 h(\cosh 2\kappa h + 1)}\left\{ \kappa z'\sinh\kappa z' + \frac{1}{2}(\cosh 2\kappa h + a)\cosh\kappa z' \right\} \\ \theta' &= \sum_\kappa \text{ same as previous line multiplied by } \cosh 2\kappa h \end{aligned}\right]$$

where κ is a zero of $\sinh 2\kappa h + 2\kappa h$, with pos. imag. part

. . . . . (82)

We may recall the method of obtaining the permanent terms. Taking any one of the functions of (80), altered as in (81), we omit the factor $J_0\kappa R$, and then find its expansion near $\kappa = 0$ to contain terms of negative degree in κ, say $A/\kappa^3 + B/\kappa$. The

permanent part of this function is then $A\chi(R) - B\nabla^2\chi(R)$. We thus find for this part of the solution

$$\psi' = -\frac{\alpha+1}{2h}\left(\chi - \frac{1}{2}z^2\nabla^2\chi\right) + \frac{\alpha+1}{2h}\left(\frac{1}{2}z'^2 + \frac{1}{3}h^2\right)\nabla^2\chi \mp \frac{\alpha+1}{2}(z-z')\nabla^2\chi$$

$$\phi' = -\frac{\alpha+1}{8h}\left(\chi - \frac{1}{2}z^2\nabla^2\chi\right) + \frac{1}{4h}\left(\frac{\alpha+5}{4}z'^2 + \frac{5-\alpha}{6}h^2\right)\nabla^2\chi$$
$$-\frac{3(\alpha+1)}{8h^3}z'\left(z\chi - \frac{1}{6}z^3\nabla^2\chi\right) + \frac{3}{4h^3}\left(\frac{\alpha+5}{12}z'^3 - h^2z' - \frac{\alpha+1}{10}h^2z'\right)z\nabla^2\chi$$
$$\mp\frac{1}{2}(z-z')\nabla^2\chi$$

$$\theta' = -\frac{\alpha+1}{8h}\left(\chi - \frac{1}{2}z^2\nabla^2\chi\right) + \frac{1}{4h}\left(\frac{\alpha+5}{4}z'^2 + \frac{5-\alpha}{6}h^2 + \alpha + 1h^2\right)\nabla\chi$$
$$+\frac{3(\alpha+1)}{8h^3}z'\left(z\chi - \frac{1}{6}z^3\nabla^2\chi - 2h^2z\nabla^2\chi\right) - \frac{3}{4h^3}\left(\frac{\alpha+5}{12}z'^3 - h^2z' - \frac{\alpha+1}{10}h^2z'\right)z\nabla\chi$$
$$\mp\frac{\alpha}{2}(z-z')\nabla^2\chi \pm z'\nabla^2\chi \qquad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

When z' is put equal to h in the above values of ψ', ϕ', θ' it will be found, with very little trouble, that they reduce to those of (48) . . . . (51), multiplied by $\frac{1}{2}(\alpha+1)$. (*Cf.* § 20.) As in § 20, the displacements due to the ambiguous terms in (83) are null if $R > 0$. But there is this difference in the present case, that they do not continue to vanish in the corresponding solution for an areal distribution of force on $z = z'$.

If the intensity of the distribution is $X(x, y, z')$ per unit area at (x, y), this solution is defined as in (79), ψ', θ', ϕ' being obtained from (82), (83) by multiplying by

$$\frac{1}{4\pi\mu(\alpha+1)}X(x', y', z')dx'\,dy',$$

and integrating over the area within which X is finite.

When this is done we find that the ambiguous terms lead to

$$\left.\begin{aligned} u &= \mp\frac{1}{2\mu}(z-z')X \\ v &= 0 \\ w &= 0 \end{aligned}\right| \qquad . \quad . \quad . \quad . \quad . \quad (84)$$

In verification, we observe that these displacements are continuous above and below the plane $z = z'$, and that the corresponding stresses are also continuous with the exception of $\widehat{zx}$, the value of which just below $z = z'$ exceeds its value just above by X. The value of $\widehat{zx}$ being $\mp\frac{1}{2}X$, we have for the contribution of (84) to the resultant $\int_{-h}^{+h}\widehat{zx}\,dz$,

$$\frac{1}{2}X\left(\int_{-h}^{z'}dz - \int_{z'}^{h}dz\right) = z'X \qquad . \quad . \quad . \quad . \quad . \quad (85)$$

26. *Approximate values of the displacements. Lagrange's equation for flexure to a second approximation.*

The unambiguous terms in (83), as in (82), fall naturally into two classes, in the first of which ψ', θ', ϕ' are odd functions of z, while in the second they are even. Of the displacements derived from the first class, u and v are odd, and w even in z, and the strain may be described as *flexural*. In the other class u, v are even, and w odd in z, and the strain may be described as *extensional*. A force X at (x', y', z') acting along with a parallel but oppositely directed force X at $(x', y', -z')$ would give rise to flexural strain only; equal and similarly directed X forces at these two points to extensional strain only. This follows at once from the fact that the terms of ψ', θ', ϕ', which are odd in z, are also odd in z', and *vice versa*.

The distribution of force being $X(x, y, z')$ per unit area at (x, y) on $z = z'$, let

$$F = \frac{d}{dx}\int\int X(x', y', z')\chi(R)dx'\,dy'.$$

Then from the flexural part of (83),

$$\left.\begin{aligned} \phi &= -\frac{3}{32\pi\mu h^3} z'\left(zF - \frac{1}{6}z^3\nabla^2F\right) \\ \theta &= \frac{3}{32\pi\mu h^3} z'\left(zF - \frac{1}{6}z^3\nabla^2F - 2h^2z\nabla^2F\right) \end{aligned}\right\} \text{with}$$

$$\left.\begin{aligned} \phi &= -z\nabla^2F \\ \theta &= z\nabla^2F \end{aligned}\right\} \text{ each multiplied by } \frac{3}{32\pi\mu h^3}\cdot\frac{2}{a+1}\left(\frac{a+11}{10}h^2z' - \frac{a+5}{12}z'^3\right)$$

These lead to

$$\left.\begin{aligned} u &= \frac{d}{dx} \\ v &= \frac{d}{dy} \end{aligned}\right| \frac{3}{32\pi\mu h^3}\left\{-(a+1)z'zF + \nabla^2F\left(\frac{a+5}{6}z'z^3 + \frac{a+5}{6}z'^3z - \frac{a+21}{5}h^2z'z\right)\right\}$$

$$w = \frac{3}{32\pi\mu h^3}\left\{(a+1)z'F + \nabla^2F\left(\frac{3-a}{2}z'z^2 + \frac{a+1}{5}h^2z' - \frac{a+5}{6}z'^3\right)\right\}. \qquad (86)$$

For Y force the same expressions hold if we take $F = \frac{d}{dy}\iint Y(x', y', z')\,\chi(R)dx'dy'$.

These formulæ, with all of (72) but the last terms of u, v, and with the odd parts in z arising from the ambiguous terms, give to a second approximation the displacements of the flexural mode under any forces. The differential equation satisfied by $\bar{w}$, the normal displacement of the mid plane, or value of w, for z equal to zero, is important in the history of the approximate theory. We can now write it down to a second approximation, namely, with C as in (75),

$$CV^4\bar{w} = Z + z'\left(\frac{dX}{dx} + \frac{dY}{dy}\right)$$
$$+ \left(\frac{a-19}{a+1}\frac{h^2}{5} + \frac{3-a}{a+1}\frac{z'^2}{2}\right)\nabla^2Z + \left(\tfrac{1}{5}h^2z' - \frac{a+5}{a+1}\frac{z'^3}{6}\right)\nabla^2\left(\frac{dX}{dx} + \frac{dY}{dy}\right). \qquad (87)$$

This equation gives the result for an areal distribution X , Y , Z on $z=z'$. For traction on $z=\pm h$ replace z' by $\pm h$; for a volume distribution X , Y , Z replace

$$\text{Z by } \int_{-h}^{h} \mathrm{Z}(x, y, z')dz'$$

$$z'\left(\frac{d\mathrm{X}}{dx}+\frac{d\mathrm{Y}}{dy}\right) \text{ by } \frac{d}{dx}\int_{-h}^{h} z'\mathrm{X}(x, y, z')dz' + \frac{d}{dy}\int_{-h}^{h} z'\mathrm{Y}(x, y, z')dz' ,$$

and so on.

27. *Extensional strain. Differential equations of the principal mode.*

The unambiguous extensional terms of (83) remain to be considered.

Write $\mathrm{E}=\dfrac{1}{32\pi\mu h}\iint \mathrm{X}(x', y', z')\chi(\mathrm{R})dx'dy'.$

Then for an areal distribution X , these terms are

$$\left.\begin{aligned}
\psi' &= -4(\mathrm{E}-\tfrac{1}{2}z^2\nabla^2\mathrm{E})+4(\tfrac{1}{2}z'^2+\tfrac{1}{3}h^2)\nabla^2\mathrm{E}\\
\theta' &= -(\mathrm{E}-\tfrac{1}{2}z^2\nabla^2\mathrm{E})+\frac{2}{\alpha+1}\left(\frac{\alpha+5}{4}z'^2+\frac{5-\alpha}{6}h^2+\overline{\alpha+1}h^2\right)\nabla^2\mathrm{E}\\
\phi' &= -(\mathrm{E}-\tfrac{1}{2}z^2\nabla^2\mathrm{E})+\frac{2}{\alpha+1}\left(\frac{\alpha+5}{4}z'^2+\frac{5-\alpha}{6}h^2\right)\nabla^2\mathrm{E}
\end{aligned}\right\}$$

The second parts of these expressions give

$$\left.\begin{aligned}
u &= (\alpha-3)\,(\tfrac{1}{2}z'^2-\tfrac{1}{3}h^2)\,\frac{d^2}{dx^2}\nabla^2\mathrm{E}+4(z'^2+\tfrac{2}{3}h^2)\nabla^4\mathrm{E}\\
v &= (\alpha-3)\,(\tfrac{1}{2}z'^2-\tfrac{1}{3}h^2)\,\frac{d^2}{dx\,dy}\nabla^2\mathrm{E}\\
w &= 0
\end{aligned}\right\}$$

The first parts give

$$\left.\begin{aligned}
u &= -(\alpha+1)\frac{d^2\mathrm{E}}{dx^2}-8\frac{d^2\mathrm{E}}{dy^2}+z^2\left(\frac{\alpha-3}{2}\frac{d^2}{dx^2}\nabla^2\mathrm{E}+4\nabla^4\mathrm{E}\right)\\
v &= -(\alpha+1)\frac{d^2\mathrm{E}}{dx\,dy}+8\frac{d^2\mathrm{E}}{dx\,dy}+z^2\,\frac{\alpha-3}{2}\frac{d^2}{dx\,dy}\nabla^2\mathrm{E}\\
w &= (3-\alpha)\,z\frac{d}{dx}\nabla^2\mathrm{E}
\end{aligned}\right\} .$$

If now further we write $\mathrm{K}=\dfrac{1}{32\pi\mu h}\iint \mathrm{Y}(x', y', z')\chi(\mathrm{R})dx'dy'$

the corresponding displacements for a distribution of Y force on $z=z'$ can at once be written down from symmetry. The results for X and Y force combined cannot conveniently be expressed in terms of one function, as in the case of the flexural mode, and the best plan is probably to put everything in terms of the principal values of u , v , namely,

$$\left.\begin{aligned}
\mathrm{U} &= -(\alpha+1)\frac{d^2\mathrm{E}}{dx^2}-8\frac{d^2\mathrm{E}}{dy^2}-(\alpha+1)\frac{d^2\mathrm{K}}{dx\,dy}+8\frac{d^2\mathrm{K}}{dx\,dy}\\
\mathrm{V} &= -(\alpha+1)\frac{d^2\mathrm{E}}{dx\,dy}+8\frac{d^2\mathrm{E}}{dx\,dy}-(\alpha+1)\frac{d^2\mathrm{K}}{dy^2}-8\frac{d^2\mathrm{K}}{dx^2}
\end{aligned}\right\} . \qquad (88)$$

We have then

$$\left.\begin{aligned} u &= \mathrm{U} + \frac{3-a}{a+1}(\tfrac{1}{2}z^2 + \tfrac{1}{2}z'^2 - \tfrac{1}{3}h^2)\frac{d}{dx}\Big(\frac{d\mathrm{U}}{dx} + \frac{d\mathrm{V}}{dy}\Big) + (\tfrac{1}{2}z^2 + \tfrac{1}{2}z'^2 + \tfrac{1}{3}h^2)\frac{\mathrm{X}}{2\mu h} \\ v &= \mathrm{V} + \frac{3-a}{a+1}(\tfrac{1}{2}z^2 + \tfrac{1}{2}z'^2 - \tfrac{1}{3}h^2)\frac{d}{dy}\Big(\frac{d\mathrm{U}}{dx} + \frac{d\mathrm{V}}{dy}\Big) + (\tfrac{1}{2}z^2 + \tfrac{1}{2}z'^2 + \tfrac{1}{3}h^2)\frac{\mathrm{Y}}{2\mu h} \\ w &= \frac{a-3}{a+1}\, z\Big(\frac{d\mathrm{U}}{dx} + \frac{d\mathrm{V}}{dy}\Big) \end{aligned}\right\} \quad . \quad . \quad (89)$$

The ordinary approximate theory obtains differential equations to determine U, V. These are easily found by eliminating E and K in turn from (88).

Thus

$$\begin{aligned} \frac{d\mathrm{U}}{dx} + \frac{d\mathrm{V}}{dy} &= -(a+1)\Big(\frac{d}{dx}\nabla^2\mathrm{E} + \frac{d}{dy}\nabla^2\mathrm{K}\Big) \\ \frac{d\mathrm{U}}{dy} - \frac{d\mathrm{V}}{dx} &= \quad -8\Big(\frac{d}{dy}\nabla^2\mathrm{E} - \frac{d}{dx}\nabla^2\mathrm{K}\Big) \end{aligned}$$

and

$$\left.\begin{aligned} \frac{1}{a+1}\,\frac{d}{dx}\Big(\frac{d\mathrm{U}}{dx} + \frac{d\mathrm{V}}{dy}\Big) + \tfrac{1}{8}\frac{d}{dy}\Big(\frac{d\mathrm{U}}{dy} - \frac{d\mathrm{V}}{dx}\Big) &= -\frac{\mathrm{X}}{16\mu h} \\ \frac{1}{a+1}\,\frac{d}{dy}\Big(\frac{d\mathrm{U}}{dx} + \frac{d\mathrm{V}}{dy}\Big) - \tfrac{1}{8}\frac{d}{dx}\Big(\frac{d\mathrm{U}}{dy} - \frac{d\mathrm{V}}{dx}\Big) &= -\frac{\mathrm{Y}}{16\mu h} \end{aligned}\right\} \quad . \quad . \quad . \quad (90)$$

The principal parts of the contribution of Z force to extensional displacements appear in (72), (78). In the notation of those formulæ

$$\left.\begin{aligned} u &= \frac{d}{dx}\nabla^2\mathrm{F} \\ v &= \frac{d}{dy}\nabla^2\mathrm{F} \\ w &= \frac{a-3}{a+1}z\nabla^4\mathrm{F} \end{aligned}\right\} \quad . \quad \frac{a-3}{32\pi\mu h}z'$$

with in addition, w = the odd part in z of the ambiguous term in (78).

If these last values of u, v be included in the principal values U, V, then the right-hand members of (90) will become respectively

$$\frac{1}{16\mu h}\Big(-\mathrm{X} + \frac{a-3}{a+1}z'\frac{d\mathrm{Z}}{dx}\Big)\ ,\ \frac{1}{16\mu h}\Big(-\mathrm{Y} + \frac{a-3}{a+1}z'\frac{d\mathrm{Z}}{dy}\Big) \quad . \quad . \quad . \quad . \quad . \quad (91)$$

28. *Approximate values of the stresses across a plane parallel to the faces.*

For any distribution of force parallel to the faces of the plate, the formulæ of §§ 26, 27 give the terms of the two lowest orders in the values of u, v, and the term of lowest order in w.* From these terms we can calculate all the stresses but $\widehat{zz}$ to a first approximation, and as in § 22, when the first term of $\widehat{zz}$ is known, we can find two terms of $\widehat{xx}$, $\widehat{xy}$ and $\widehat{yy}$. This first term of $\widehat{zz}$ we may get very easily from the symbolical form of the solution corresponding to (80), (81). Thus for areal force X

$$\widehat{zz} = \pm\tfrac{1}{2}(z-z')\frac{d\mathrm{X}}{dx} + \frac{3}{4h^3}z'(h^2z - \tfrac{1}{3}z^3)\frac{d\mathrm{X}}{dx} - \frac{1}{4h}(z^2+h^2)\frac{d\mathrm{X}}{dx} \quad . \quad . \quad . \quad . \quad . \quad (92)$$

* There should be added from (84) the terms

$$u = \mp(z-z')\mathrm{X}/2\mu\,, \qquad v = \mp(z-z')\mathrm{Y}/2\mu\,, \qquad w = 0\,.$$

From (84), (86), (89) for areal force X

$$\left.\begin{aligned} \widehat{zx} &= \mp \tfrac{1}{2}\mathrm{X} + \frac{3}{4h^3}z'(z'^2-h^2)\frac{1}{2\pi}\frac{d^2}{dx^2}\iint \mathrm{X}(x',y',z')\nabla^2\chi(\mathrm{R})dx'dy' + \frac{z}{2h}\mathrm{X} \\ \widehat{zy} &= \frac{3}{4h^3}z'(z'^2-h^2)\frac{1}{2\pi}\frac{d^2}{dx\,dy}\iint \mathrm{X}(x',y',z')\nabla^2\chi(\mathrm{R})dx'dy' \end{aligned}\right\} \quad . \quad . \quad . \quad (93)$$

It may be verified that these give zero stress on $z = \pm h$, and $\frac{d\widehat{zx}}{dx} + \frac{d\widehat{zy}}{dy} + \frac{d\widehat{zz}}{dz} = 0$.

From the formulæ we have given, it is of course merely a matter of the simplest algebra to calculate any of the stresses to whatever order of approximation may be required, but it may be worth while to remark here that the fundamental equations of equilibrium (1) may be used with great advantage in obtaining the principal results. If, for example, we know only the first terms of $\widehat{xx}$, $\widehat{xy}$, $\widehat{yy}$, the two first of these equations would give the first terms of $\widehat{zx}$, $\widehat{zy}$ by a simple integration with respect to z, and then the last equation would give the first term of $\widehat{zz}$. Similarly, when the first two terms of $\widehat{xx}$, $\widehat{xy}$, $\widehat{yy}$ are known (as above), we may find the first two terms of the other stresses.

29. *Transmission of force to a distance. Expansions in polar coordinates.*

We have up to this point been considering mainly the particular solution to which our general source solutions lead for any given distribution of force; or, as we may say, we have been investigating the effect of any given force system on that part of the solid to which the force is applied. But it is also of great interest to inquire what is the effect of this force at points of the solid remote from its region of application. It is obvious that we obtain a sufficient answer to this question by retaining only the permanent terms in the source solutions, those terms, namely, which are given in (65) and (83).

For force applied only at points on a given normal to the plate, these formulæ are all that we require. They show at a glance that the distant effect depends chiefly on *resultant* forces and couples, but not entirely, since z' and z'^2 occur in the formulæ for Z force, and z'^2, z'^3 in those for X force. When the force is not confined to a line, but is distributed over a finite volume of the solid, the result is obtained in more intelligible form if before integration the function χ is suitably expanded so as to yield a series of solutions in which accented and unaccented coordinates are explicitly separated. The most convenient expansion of χ is in terms of polar coordinates as given in (*e*) of the introductory section.

Suppose, then, a single force applied at the point (x_1, y_1, z_1) or (ρ_1, ω_1, z_1), the components of the force being X_1, Y_1, Z_1, parallel to the rectangular axes, or P_1, Ω_1, Z_1 parallel to radius vector, transverse, and axis of z. We have to find the displacements at (ρ, ω, z) where we suppose $\rho > \rho_1$.

For an X force, the value of ψ is $\frac{d\psi'}{dy}$ with ψ' given in (83), the coefficient depending on the magnitude of X being for the moment suppressed.

This is the same as $-\frac{d\psi'}{dy}$ or $(-)$ rate of variation of ψ' in the direction perpendicular to the force at its point of application.

Similarly $\phi = -\frac{d\phi'}{dx} = (-)$ rate of variation of ϕ' in direction of force.

Hence for P_1, $\psi = -\frac{1}{\rho_1}\frac{d\psi'}{d\omega_1}$, $\theta = -\frac{d\theta'}{d\rho_1}$, $\phi = -\frac{d\phi'}{d\rho_1}$; and for Ω_1, $\psi = \frac{d\psi'}{d\rho_1}$, $\theta = -\frac{1}{\rho_1}\frac{d\theta'}{d\omega_1}$, $\phi = -\frac{1}{\rho_1}\frac{d\phi'}{d\omega_1}$.

We shall take separately the extensional and flexural parts of the solution.
Also in the following u, v are the displacements along radius vector and transverse.

I. Extensional terms.

The following solutions occur.

(i) $$u = \left(\frac{7-\alpha}{4} - \frac{9+\alpha}{2}\log\frac{\rho}{2h} + \frac{3-\alpha}{2}z^2\rho^{-2}\right)\cos\omega$$
$$v = \left(\frac{7-\alpha}{4} + \frac{9+\alpha}{2}\log\frac{\rho}{2h} + \frac{3-\alpha}{2}z^2\rho^{-2}\right)\sin\omega$$
$$w = (3-\alpha)\,z\rho^{-1}\cos\omega$$

(ii) Same as (i) with cos ω changed into sin ω, and sin ω into $-\cos\omega$

(iii) When $m>1$,
$$u = \left\{\frac{8m-(m-2)(\alpha+1)}{4(m-1)}\rho^{-m+1} + \frac{3-\alpha}{2}mz^2\rho^{-m-1}\right\}\cos m\omega$$
$$v = \left\{\frac{8(m-2)-m(\alpha+1)}{4(m-1)}\rho^{-m+1} + \frac{3-\alpha}{2}mz^2\rho^{-m-1}\right\}\sin m\omega$$
$$w = (3-\alpha)z\rho^{-m}\cos m\omega$$

(iv) Same as (iii) with cos $m\omega$ changed into sin $m\omega$, and sin $m\omega$ into $-\cos m\omega$.

(v) $u = \rho^{-m-1}\cos m\omega$, $v = \rho^{-m-1}\sin m\omega$

(vi) $u = \rho^{-m-1}\sin m\omega$, $v = -\rho^{-m-1}\cos m\omega$

For the force with components P_1, Ω_1, Z_1, the coefficients of the above solutions are the following, in each case divided by $32\pi\mu h$.

(i) $P_1\cos\omega_1 - \Omega_1\sin\omega_1 = X_1$

(ii) $P_1\sin\omega_1 + \Omega_1\cos\omega_1 = Y_1$

(iii) $\rho_1^{m-1}\cos m\omega_1 P_1 - \rho_1^{m-1}\sin m\omega_1\Omega_1$

(iv) $\rho_1^{m-1}\sin m\omega_1 P_1 + \rho_1^{m-1}\cos m\omega_1\Omega_1$

(v) $$\left\{\frac{-8m+(m+2)(\alpha+1)}{4(m+1)}\rho_1^{m+1} + (3-\alpha)\left(\tfrac{1}{2}z_1^2 - \tfrac{1}{3}h^2\right)m\rho_1^{m-1}\right\}\cos m\omega_1 P_1$$
$$+ \left\{\frac{8(m+2)-m(\alpha+1)}{4(m+1)}\rho_1^{m+1} - (3-\alpha)\left(\tfrac{1}{2}z_1^2 - \tfrac{1}{3}h^2\right)m\rho_1^{m-1}\right\}\sin m\omega_1\Omega_1$$
$$+ \quad (\alpha-3)\rho_1^m z_1\cos m\omega_1 Z_1$$

(vi) Same as (v) with cos $m\omega_1$ changed into sin $m\omega_1$, and sin $m\omega_1$ into $-\cos m\omega_1$.

II. Flexural terms.

The flexural solutions are of the form given in (23), or in polar coordinates

$$\left.\begin{aligned} u &= \frac{d}{d\rho} \\ v &= \frac{1}{\rho}\frac{d}{d\omega}\Big[-(a+1)(zF - \tfrac{1}{6}z^3\nabla^2 F) + 2(\tfrac{1}{3}z^3 - h^2 z)\nabla^2 F \Big] \\ w &= \qquad (a+1)(F - \tfrac{1}{2}z^2\nabla^2 F) + 2(z^2 - h^2)\nabla^2 F \end{aligned}\right\} \quad . \quad . \quad . \quad (94)$$

(i) $F = \chi(\rho) = \frac{1}{4}\rho^2 \log\frac{\rho}{2h} - \frac{1}{4}\rho^2$

(ii) $F = \frac{1}{4}\left(\rho - 2\rho\log\frac{\rho}{2h}\right)\cos\omega$

(iii) $F = \frac{1}{4}\left(\rho - 2\rho\log\frac{\rho}{2h}\right)\sin\omega$

(iv) $F = \dfrac{1}{4m(m-1)}\rho^{-m+2}\cos m\omega$

(v) $F = \dfrac{1}{4m(m-1)}\rho^{-m+2}\sin m\omega$ $\Big\}\ m>1.$

(vi) $F = \log\dfrac{\rho}{2h}$

(vii) $F = \rho^{-m}\cos m\omega$

(viii) $F = \rho^{-m}\sin m\omega$ $\Big\}\ m>0.$

For the force with components P_1, Ω_1, Z_1 the coefficients of the above solutions are the following, in each case divided by $\frac{32}{3}\pi\mu h^3$.

(i) Z_1

(ii) $-z_1\cos\omega_1 P_1 + z_1\sin\omega_1\Omega_1 + Z_1\rho_1\cos\omega_1 = -X_1 z_1 + Z_1 x_1$

(iii) $-z_1\sin\omega_1 P_1 - z_1\cos\omega_1\Omega_1 + Z_1\rho_1\sin\omega_1 = -Y_1 z_1 + Z_1 y_1$

(iv) $-z_1 m\rho_1^{m-1}\cos m\omega_1 P_1 + z_1 m\rho_1^{m-1}\sin m\omega_1\Omega_1 + \rho_1^m\cos m\omega_1 Z_1$

(v) Same as (iv) with $\cos m\omega_1$ changed to $\sin m\omega_1$, and $\sin m\omega_1$ to $-\cos m\omega_1$

(vi) $-\frac{1}{2}z_1\rho_1 P_1 + \left\{\frac{1}{4}\rho_1^2 + \frac{1}{5}h^2 - \frac{1}{2}z_1^2 + \dfrac{2}{a+1}(z_1^2 - h^2)\right\} Z_1$

(vii) $\left[z_1\dfrac{m+2}{4m(m+1)}\rho_1^{m+1} - \left\{\frac{1}{6}z_1^3 - \frac{1}{5}h^2 z_1 + \dfrac{2}{a+1}(\tfrac{1}{3}z_1^3 - h^2 z_1)\right\}\rho_1^{m-1}\right]\cos m\omega_1 P$

$-\left[z_1\dfrac{1}{4(m+1)}\rho_1^{m+1} - \left\{\quad\text{"}\quad\right\}\rho_1^{m-1}\right]\sin m\omega_1\Omega_1$

$-\left[\dfrac{1}{4m(m+1)}\rho_1^{m+2} + \left\{\frac{1}{5}h^2 - \frac{1}{2}z_1^2 + \dfrac{2}{a+1}(z_1^2 - h^2)\right\}\dfrac{1}{m}\rho_1^m\right]\cos m\omega_1 Z_1$

(viii) Same as (vii) with $\cos m\omega_1$ changed to $\sin m\omega_1$ and $\sin m\omega_1$ to $-\cos m\omega_1$.

30. *Types of deformation conveying a given resultant stress.*

In these formulæ we remark at once a striking relation between the forms of the displacements u, v, w in the various solutions, and the multipliers of P_1, Ω_1, Z_1 in the coefficients of the solutions.

In I. (iii), *e.g.*, these multipliers are $\rho_1^{m-1}\cos m\omega_1, -\rho_1^{m-1}\sin m\omega_1, 0$, which are simply the displacements of I. (v) with sign of m changed, and consequently suitable for space containing the origin.

Similarly in I. (v) the multipliers of P_1, Ω_1, Z_1 are displacements compounded of the types (iii), (v), with sign of m changed, and so on.

The full explanation of this peculiarity will be given presently, when it will be seen that an independent verification of all the results may be obtained by means of the important principle known as Betti's Theorem.

In the meantime we may examine the scheme of solutions from another very important point of view.

With reference to any individual solution, the following questions are obviously of prime importance :—

(1) What is the resultant stress transmitted ?

(2) Is the whole potential energy of the part of the solid bounded internally by a given cylindrical surface, finite or infinite ?

Now, in order to single out those solutions which convey a finite resultant stress across any cylinder (or other surface) surrounding the origin, we have merely to look at the table of coefficients. Thus, for instance, I. (i) appears with coefficient $X_1/32\pi\mu h$, from which we may infer (as verified below) that this solution conveys a stress with resultant a force of $32\pi\mu h$ units parallel to the axis of x, and passing through the origin.

In this way we find that the six solutions, corresponding to the six elements which specify the resultant of a force system, are

I. (i), (ii), (vi) with $m=0$; II. (i), (ii), (iii).

For these we shall write down the values of the stresses $\widehat{\rho\rho}, \widehat{\rho\omega}, \widehat{\rho z}$, the components of the stress across the cylinder $\rho =$ constant.

In all, of course, we have $\widehat{zz}=0$, and in I. in addition $\widehat{z\rho}=\widehat{z\omega}=0$.

I. (i)
$$\left.\begin{aligned}\frac{\widehat{\rho\rho}}{2\mu} &= \left(\frac{a-15}{2}\rho^{-1}+\overline{a-3}\,z^2\rho^{-3}\right)\cos\omega\\ \frac{\widehat{\rho\omega}}{2\mu} &= \left(\frac{a+1}{2}\rho^{-1}+\overline{a-3}\,z^2\rho^{-3}\right)\sin\omega\end{aligned}\right\}$$

The resultant is a force along Ox, of magnitude

$$\iint(\widehat{\rho\rho}\cos\omega-\widehat{\rho\omega}\sin\omega)\rho d\omega dz, \qquad \text{taken over the cylinder } \rho,$$
$$=\left(\frac{a-15}{2}-\frac{a+1}{2}\right)\pi\cdot 2h\cdot 2\mu = -32\pi\mu h.$$

I. (ii)
$$\left.\begin{aligned}\frac{\widehat{\rho\rho}}{2\mu} &= \left(\frac{a-15}{2}\rho^{-1}+\overline{a-3}\,z^2\rho^{-3}\right)\sin\omega\\ \frac{\widehat{\rho\omega}}{2\mu} &= -\left(\frac{a+1}{2}\rho^{-1}+\overline{a-3}\,z^2\rho^{-3}\right)\cos\omega\end{aligned}\right\}$$

The resultant is a force along Oy of magnitude $-32\pi\mu h$.

I. (vi), $m=0$.
$$\left.\begin{aligned}u &= 0\\ v &= -\rho^{-1}\\ w &= 0\end{aligned}\right\}. \qquad \left.\begin{aligned}\widehat{\rho\rho} &= 0\\ \widehat{\rho\omega} &= 2\mu\rho^{-2}.\end{aligned}\right\}$$

The resultant is a couple in the plane xy, of magnitude $8\pi\mu h$, and we observe that the solution occurs with coefficient $-\Omega_1\rho_1/8\pi\mu h$.

II. The stresses in the general flexural solution (94) are

$$\left.\begin{aligned} \frac{\widehat{\rho\rho}}{2\mu} &= (a-3)z\nabla^2\mathrm{F} + \frac{du}{d\rho} \\ \frac{\widehat{\rho\omega}}{2\mu} &= \frac{dv}{d\rho} \\ \frac{\widehat{\rho z}}{2\mu} &= 2(z^2-h^2)\frac{d}{d\rho}\nabla^2\mathrm{F}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (95)$$

II. (i)
$$\left.\begin{aligned} \frac{\widehat{\rho\rho}}{2\mu} &= \frac{a-7}{2}z\log\frac{\rho}{2h} - \frac{a+1}{4}z - \left(\frac{a+5}{6}z^3 - 2h^2z\right)\rho^{-2} \\ \widehat{\rho\omega} &= 0 \\ \frac{\widehat{\rho z}}{2\mu} &= 2(z^2-h^2)\rho^{-1} \end{aligned}\right\}$$

The resultant is a force along Oz, of magnitude $-\frac{32}{3}\pi\mu h^3$.

II. (ii)
$$\left.\begin{aligned} \frac{\widehat{\rho\rho}}{2\mu} &= \frac{7-a}{2}z\rho^{-1}\cos\omega - 2\left(\frac{a+5}{6}z^3 - 2h^2z\right)\rho^{-3}\cos\omega \\ \frac{\widehat{\rho\omega}}{2\mu} &= -\frac{a+1}{2}z\rho^{-1}\sin\omega - 2\left(\frac{a+5}{6}z^3 - 2h^2z\right)\rho^{-3}\sin\omega \\ \frac{\widehat{\rho z}}{2\mu} &= 2(z^2-h^2)\rho^{-2}\cos\omega \end{aligned}\right\}$$

The resultant is a couple about Oy, of magnitude

$$\iint\{z(\widehat{\rho\rho}\cos\omega - \widehat{\rho\omega}\sin\omega) - \rho\cos\widehat{\omega\rho z}\}\rho d\omega dz, \quad \text{taken over the cylinder } \rho,$$
$$= \frac{32}{3}\pi\mu h^3.$$

II. (iii)
$$\begin{aligned} \frac{\widehat{\rho\rho}}{2\mu} &= \frac{7-a}{2}z\rho^{-1}\sin\omega - 2\left(\frac{a+5}{6}z^3 - 2h^2z\right)\rho^{-3}\sin\omega \\ \frac{\widehat{\rho\omega}}{2\mu} &= \frac{a+1}{2}z\rho^{-1}\cos\omega + 2\left(\frac{a+5}{6}z^3 - 2h^2z\right)\rho^{-3}\cos\omega \\ \frac{\widehat{\rho z}}{2\mu} &= 2(z^2-h^2)\rho^{-2}\sin\omega. \end{aligned}$$

The resultant is a couple about Ox, of magnitude

$$\iint\{-z(\widehat{\rho\rho}\sin\omega + \widehat{\rho\omega}\cos\omega) + \rho\sin\omega\cdot\widehat{\rho z}\}\rho\, d\omega\, dz = -\frac{32}{3}\pi\mu h^3.$$

31. *Conditions for the existence of a solution with finite potential energy. Elastic equivalence of statically equipollent loads.*

The corresponding results for any distribution of body force, or of traction on the faces of the plate, may be deduced at once from the above by integration with respect to ρ_1, ω_1, z_1 or ρ_1, ω_1 with $z_1 = \pm h$.

If the region within which the force is applied be entirely enclosed by a cylinder $\rho = a$, the results are valid for all points exterior to this cylinder.

For a distribution of force of finite intensity per unit area or per unit volume, the potential energy of that part of the solid within the cylinder is clearly finite. The energy of the remaining part of the solid can be determined from the forms of § 29.

Now, the energy between the cylinders $\rho = a$, $\rho = p$ is the integral of

$$\tfrac{1}{2}(u\,\widehat{\rho\rho} + v\,\widehat{\rho\omega} + w\,\widehat{\rho z})$$

taken over the belt of the cylinder $\rho = p$ cut off by the plate, diminished by the corresponding integral for $\rho = a$. Hence the condition of finiteness of the whole potential energy is simply that the value of the integral for the surface $\rho = p$ tends to zero as p tends to infinity. This condition is obviously satisfied by all the partial solutions of § 29, except those which have been already singled out as conveying a finite resultant stress. It is also satisfied by one of the latter class, namely, that which conveys a couple in the plane of the plate.

Hence, when force is applied to a circumscribed portion of the solid, a solution giving finite potential energy will exist provided the force either constitutes an equilibrating system, or reduces to a couple in the plane of the plate. It does not follow, however, even for an equilibrating application of force, that a solution will exist giving vanishing displacements at infinity. We need only point to the solutions of § 29, II. (vi) and (iv), (v) with $m = 2$. This being so, it may be of interest to write down a few more details of those solutions which rank in importance next to the solutions of finite resultant stress.

I. (iii) with $m = 2$.

$$\left.\begin{aligned} u &= \{\, 4\rho^{-1} + (3-a)z^2\rho^{-3}\}\cos 2\omega \\ v &= \left\{\, -\frac{a+1}{2}\rho^{-1} + (3-a)z^2\rho^{-3} \,\right\}\sin 2\omega \\ w &= (3-a)z\rho^{-2}\cos 2\omega \end{aligned}\right| \qquad \left.\begin{aligned} \frac{\widehat{\rho\rho}}{2\mu} &= \left\{\, (a-7)\rho^{-2} + (a-3)3z^2\rho^{-4} \,\right\}\cos 2\omega \\ \frac{\widehat{\rho\omega}}{2\mu} &= \left\{\, \frac{a-7}{2}\,\rho^{-2} + (a-3)3z^2\rho^{-4} \,\right\}\sin 2\omega \end{aligned}\right|$$

This solution occurs with coefficient $(X_1x_1 - Y_1y_1)/32\pi\mu h$.

I. (iv) with $m = 2$ is obtained by writing $\sin 2\omega, -\cos 2\omega$ for $\cos 2\omega, \sin 2\omega$ in the preceding, and the coefficient is $(X_1y_1 + Y_1x_1)/32\pi\mu h$.

I. (v) with $m = 0$.

$$\left.\begin{aligned} u &= \rho^{-1} \\ v &= 0 \\ w &= 0 \end{aligned}\right| \qquad \left.\begin{aligned} \widehat{\rho\rho} &= -2\mu\rho^{-2} \\ \widehat{\rho\omega} &= 0 \end{aligned}\right| \quad \text{Coefficient} = \left\{\, \frac{a+1}{2}(X_1x_1 + Y_1y_1) + (a-3)z_1Z_1 \,\right\} \Big/ 32\pi\mu h.$$

II. (iv) with $m = 2$. $\quad F = \frac{1}{8}\cos 2\omega$.

$$\left.\begin{aligned} u &= \left(\frac{a+5}{6}z^3 - 2h^2z\right)\rho^{-3}\cos 2\omega \\ v &= \frac{a+1}{4}z\rho^{-1}\sin 2\omega + \left(\frac{a+5}{6}z^3 - 2h^2z\right)\rho^{-3}\sin 2\omega \\ w &= \frac{a+1}{8}\cos 2\omega + \left(\frac{a-3}{4}z^2 + h^2\right)\rho^{-2}\cos 2\omega \end{aligned}\right| \qquad \left.\begin{aligned} \frac{\widehat{\rho\rho}}{2\mu} &= \frac{3-a}{2}z\rho^{-2}\cos 2\omega + \text{a term in } \rho^{-4} \\ \frac{\widehat{\rho\omega}}{2\mu} &= -\frac{a+1}{4}z\rho^{-2}\sin 2\omega + \;\cdot\;\cdot \\ \frac{\widehat{\rho z}}{2\mu} &= 2(z^2 - h^2)\rho^{-3}\cos 2\omega\,. \end{aligned}\right|$$

The coefficient is

$$\frac{3}{32\pi\mu h^3}\left\{ -2z_1x_1X_1+2z_1y_1Y_1+(x_1^2-y_1^2)Z_1 \right\}$$

II. (v) with $m=2$ is the above with $\sin 2\omega, -\cos 2\omega$ for $\cos 2\omega, \sin 2\omega$ and coefficient

$$\frac{3}{32\pi\mu h^3}\left\{ -2z_1y_1X_1-2z_1x_1Y_1+2x_1y_1Z_1 \right\}$$

II. (vi). $F=\log\frac{\rho}{2h}$.

$$\left.\begin{aligned} u &= -(a+1)z\rho^{-1} \\ v &= 0 \\ w &= (a+1)\log\frac{\rho}{2h} \end{aligned}\right\} \quad \left.\begin{aligned} \widehat{\rho\rho}/2\mu &= (a+1)z\rho^{-2} \\ \widehat{\rho\omega} = \widehat{\rho z} &= 0\,. \end{aligned}\right\}$$

Coefficient is

$$\left\{ -\tfrac{1}{2}z_1(x_1X_1+y_1Y_1)+\left(\frac{x_1^2+y_1^2}{4}+\frac{h^2}{5}-\frac{z_1^2}{2}+\frac{2}{a+1}\overline{z_1^2-h^2}\right)Z_1 \right\} 3/32\pi\mu h^3.$$

For all the remaining solutions, the stresses are of the third or higher order in $1/\rho$. The results of this and the preceding article bear directly upon a principle of fundamental importance in theories of approximation, generally referred to as the principle of the *elastic equivalence of statically equipollent systems of load*, and a study of these results will be found of service in imparting precision and definiteness to one's view of the principle in its application to the theory of plates.

It may be noted here, with reference to the occurrence of the function $\log(\rho/2h)$ in some of the principal solutions of § 30, that it would make no essential difference if this function were replaced throughout by $\log(\rho/c)$, c being any length whatever, the unit of length for example. The change would be equivalent to adding a solution of the permanent type, giving no body force or traction on the faces, and it will be observed that the addition would disappear altogether when the applied forces are in equilibrium.

We have here, in fact, an instance of the indeterminateness that of necessity arises in the absence of conditions at infinity, and we are thus brought to the question, what is the exact extent of this indeterminateness? or, as it may be put, given one solution of a problem satisfying the conditions at a finite distance, what is the most general solution satisfying such conditions?

For the investigation of this question we have at hand a powerful instrument in *Betti's Theorem*, which occupies in the theory of elastic solids the place held by Green's Theorem in the Theory of the Potential.

32. *Betti's reciprocal theorem. Verification of preceding solutions.*

Betti's Theorem may be thus stated:—Given two sets of displacements of an elastic solid, with the two corresponding sets of forces maintaining these displacements (including body forces, surface tractions, and kinetic reactions), then the work done by the forces of the first set acting on the displacements of the second set is

equal to the work done by the forces of the second set acting over the displacements of the first.

In potential theory one of the chief applications of Green's Theorem is to the case when one of the potential systems includes a mass concentrated at a single point, and in the present subject Betti's Theorem finds an application of like importance when one of the displacement systems contains a finite force applied at one point, or, in analytical language, includes a point singularity of the first order, that is to say, of one of the three forms indicated in (6).

Thus, let us suppose the solid to be bounded by a surface S, and in the first set let the displacements be u, v, w; the components of body force per unit volume X, Y, Z; and the components of the traction on S, F, G, H; in the second set let the displacements be u', v', w'; the only internal force a force X′, Y′, Z′ at (x', y', z'), and the tractions on S, F′, G′, H′.

We may apply Betti's Theorem to the space bounded by S and a sphere S′ of radius ϵ drawn round (x', y', z') as centre. Thus we have

$$\iiint(\mathrm{X}u'+\mathrm{Y}v'+\mathrm{Z}w')d\mathrm{V}+\iint(\mathrm{F}u'+\mathrm{G}v'+\mathrm{H}w')d\mathrm{S}+\iint(\mathrm{F}u'+\mathrm{G}v'+\mathrm{H}w')d\mathrm{S}'$$
$$=\iint(\mathrm{F}'u+\mathrm{G}'v+\mathrm{H}'w)d\mathrm{S}+\iint(\mathrm{F}'u+\mathrm{G}'v+\mathrm{H}'w)d\mathrm{S}'.$$

Now take the limits of both members of this equality for $\epsilon=0$.

Since near the centre of the sphere S′, u', v', w' are of order $1/\epsilon$, F′, G′, H′ of order $1/\epsilon^2$, and $d\mathrm{S}'$ of order ϵ^2, the effect on the volume integral is simply to extend it to the whole volume within S; the surface integral $\iint(\mathrm{F}u'+\mathrm{G}v'+\mathrm{H}w')d\mathrm{S}'$ vanishes, and the surface integral $\iint(\mathrm{F}'u+\mathrm{G}'v+\mathrm{H}'w)d\mathrm{S}'$ has the same limit as

$$u(x', y', z')\iint\mathrm{F}'d\mathrm{S}'+v(x', y', z')\iint\mathrm{G}'d\mathrm{S}'+w(x', y', z')\iint\mathrm{H}'d\mathrm{S}',$$

namely,

$$u(x', y', z')\mathrm{X}'+v(x', y', z')\mathrm{Y}'+w(x', y', z')\mathrm{Z}',$$

the tractions F′, G′, H′ on S′ being statically equivalent to the force X′, Y′, Z′ at its centre.

It is thus apparent, and might indeed have been anticipated, that Betti's Theorem may legitimately be applied when one of the systems contains a force acting at a single point, provided the work done by this force on the other system of displacements be taken into account.

The theorem thus becomes

$$\iiint(\mathrm{X}u'+\mathrm{Y}v'+\mathrm{Z}w')d\mathrm{V}+\iint(\mathrm{F}u'+\mathrm{G}v'+\mathrm{H}w')d\mathrm{S}-\iint(\mathrm{F}'u+\mathrm{G}'v+\mathrm{H}'w)d\mathrm{S}$$
$$=u(x', y', z')\mathrm{X}'+v(x', y', z')\mathrm{Y}'+w(x', y', z')\mathrm{Z}' \quad . \quad . \quad . \quad (96)$$

In order to apply the theorem to the plate problems under discussion, take for the solid a portion of the plate bounded externally by any orthogonal cylinder. Let us

also suppose that the system u, v, w is maintained solely by tractions on the cylindrical edge, and the system u', v', w' by such tractions along with the force at (x', y', z'). Further, it will be convenient to decompose the latter system, and take u_1, v_1, w_1 as due to a unit X force, u_2, v_2, w_2 to a unit Y force, and u_3, v_3, w_3 to a unit Z force. The corresponding tractions on the edge we will denote by X, Y, Z; X_1, Y_1, Z_1; X_2, Y_2, Z_2; X_3, Y_3, Z_3. The theorem (96) then gives

$$\left.\begin{aligned} u(x', y', z') &= \iint(Xu_1 + Yv_1 + Zw_1 - X_1u - Y_1v - Z_1w)dS \\ v(x', y', z') &= \iint(Xu_2 + Yv_2 + Zw_2 - X_2u - Y_2v - Z_2w)dS \\ w(x', y', z') &= \iint(Xu_3 + Yv_3 + Zw_3 - X_3u - Y_3v - Z_3w)dS \end{aligned}\right\} . \quad . \quad . \quad (97)$$

the integrals being taken over the edge.

As one application of these forms, we may indicate briefly how they can be used to verify the single force solutions already obtained.

Take, for example, the case of a Z force, and let u_3, v_3, w_3 have the values defined in (63), (64), (65). Also let the edge be the cylinder R = constant.

(i) The coefficient of the principal flexural term, in which, with the notation of (94) $F \infty \chi(R)$, is determined from the condition that the resultant of the stress $\widehat{zR}$ must balance the applied force.

It is interesting to note that the conditions of equilibrium of applied forces and surface tractions may be regarded as special cases of Betti's Theorem. We have only to take for auxiliary systems the rigid body displacements $u = 0$, $v = 0$, $w = 1$; $u = y$, $v = -x$, $w = 0$, etc.

(ii) In the third of equations (97) take for u, v, w the values of (94) with $F = R^2$. Only the two flexural terms of (65) contribute to the surface integral; the contribution from the particular solution $\phi = G_0\kappa R \sinh \kappa z$, $\theta = -\cosh 2\kappa h \cdot \phi$ must vanish, as we see by pushing the edge to infinity.

This, with the result of (i), gives the coefficient of the second flexural term of (65).

(iii) The principal extensional term is verified by taking

$$u = \frac{\alpha+1}{2}(x - x'), \quad v = \frac{\alpha+1}{2}(y - y'), \quad w = (\alpha - 3)z.$$

(iv) The coefficient of the particular solution $\phi = G_0\kappa R \sinh \kappa z$, $\theta = -\cosh 2\kappa h \cdot \phi$ in (63) is verified by taking for u, v, w the values defined by $\phi = J_0\kappa R \sinh \kappa z$, $\theta = -\cosh 2\kappa h \cdot \phi$.

None of the solutions corresponding to the other roots of $\sinh 2\kappa h - 2\kappa h$ contribute to the surface integral. In fact, the partial contribution from a root κ' being independent of the radius of the cylinder, must vanish identically, since the Bessel Functions supply a factor tending to zero or infinity when R is made infinite, according as κ' is a higher or lower root than κ.

(v) The coefficient of the particular solution $\phi = G_0\kappa R \cosh \kappa z$, $\theta = \cosh 2\kappa h \cdot \phi$, may be verified in the same way.

It is now easy to see the significance of the forms of the coefficients in the solutions of § 29 and the confirmation of the values there given would obviously present no difficulties.

33. *Finite plate under edge tractions. Form of the solution deduced by means of Betti's Theorem.*

We pass, however, to a more important application of the theorems (97). The system u, v, w we still suppose maintained by edge tractions alone, but in addition to the external edge the solid may now be bounded by one or more internal edges. For u_1, v_1, w_1, etc., we take the definite values defined in (79), (82), (83), and in (63), (64), (65).

Thus in (97) u_1, v_1, w_1, X_1, Y_1, Z_1, and the other displacements and tractions marked with suffixes, are known functions of x', y', z', and the equations give explicitly the values of the displacements at any internal point in terms of the displacement and stress at the edge or edges.

The ideal solution would give the internal displacement in terms of edge displacement alone, or of edge stress alone, but the analytical difficulties are such that we are unable to solve the problem thus completely even for the simplest case, that of a single infinite plane edge. Meantime, however, we may derive valuable information from the expressions of (97), and in the first place as to the *form* into which any solution due to edge tractions alone may be thrown.

Just as in the case of the original source solutions, we find that the solution, in which, of course, the accented letters are now the variables, may be decomposed into an extensional and a flexural part, while in each of those parts we may separate a permanent mode from an infinite series of transitory or decaying modes of two types, the ψ type, characterised by no dilatation or normal displacement, and the θ, ϕ type, in which there is no molecular rotation in the plane of the plate.

In the following analysis integrals of the same form as those in (97) occur frequently; the system u, v, w appearing in each case, but associated with various other systems. For conciseness we shall refer to the first integral of (97) as the *work difference from* u_1, v_1, w_1, and similarly in other cases.

I. Extensional part of the solution.

(i) Permanent mode.

In u_1, v_1, w_1, the terms which relate to this mode are the unambiguous terms, even in z, of (83), after these have been divided by $4\pi\mu(a+1)$. These, as may be seen from a glance at the beginning of § 27, are equivalent to

$$\left.\begin{aligned} \psi &= \frac{1}{8\pi\mu h}\,\frac{d}{dy'}\left(\chi - \frac{1}{2}z^2\nabla^2\chi\right) \\ \theta = \phi &= \frac{1}{32\pi\mu h}\left\{\frac{d}{dx'}\left(\chi - \frac{1}{2}z^2\nabla^2\chi\right) + \frac{3-a}{a+1}\left(\frac{1}{2}z'^2 - \frac{1}{3}h^2\right)\frac{d}{dx'}\nabla'^2\chi\right\} \end{aligned}\right|$$

∇'^2 standing for $\dfrac{d^2}{dx'^2} + \dfrac{d^2}{dy'^2}$.

Now let the work difference from the system $\psi=\frac{1}{8\pi\mu h}\left(\chi-\frac{1}{2}z^2\nabla^2\chi\right)$ be denoted by E_1, and that from the system $\theta=\phi=\frac{1}{32\pi\mu h}\left(\chi-\frac{1}{2}z^2\nabla^2\chi\right)$ by E_2; then obviously the work differences from the two systems immediately preceding are respectively

$$\frac{d\mathrm{E}_1}{dy'} \quad \text{and} \quad \frac{d\mathrm{E}_2}{dx'}+\frac{3-\alpha}{\alpha+1}\left(\frac{1}{2}z'^2-\frac{1}{3}h^2\right)\frac{d}{dx'}\nabla'^2\mathrm{E}_2.$$

Hence (97) gives

$$u(x',y',z') = \frac{d\mathrm{E}_1}{dy'}+\frac{d\mathrm{E}_2}{dx'}+\frac{3-\alpha}{\alpha+1}\left(\frac{1}{2}z'^2-\frac{1}{3}h^2\right)\frac{d}{dx'}\nabla'^2\mathrm{E}_2$$

In the same way from u_2, v_2, w_2 and u_3, v_3, w_3 we obtain

$$\left.\begin{aligned} v(x',y',z') &= -\frac{d\mathrm{E}_1}{dx'}+\frac{d\mathrm{E}_2}{dy'}+\frac{3-\alpha}{\alpha+1}\left(\frac{1}{2}z'^2-\frac{1}{3}h^2\right)\frac{d}{dy'}\nabla'^2\mathrm{E}_2 \\ w(x'\,y',z') &= \frac{\alpha-3}{\alpha+1}z'\nabla'^2\mathrm{E}_2 \end{aligned}\right\}$$

Moreover, it can be seen in a moment that the displacements due to

$$\psi=\frac{d}{dy'}\nabla'^2\chi \quad \text{and to} \quad \theta=\phi=-\frac{2}{\alpha+1}\frac{d}{dx'}\nabla'^2\chi$$

are in reality the same; as also those due to

$$\psi=\frac{d}{dx'}\nabla'^2\chi \quad \text{and to} \quad \theta=\phi=\frac{2}{\alpha+1}\frac{d}{dy'}\nabla'^2\chi.$$

It follows that

$$\left.\begin{aligned} &\frac{d}{dy'}\nabla'^2\mathrm{E}_1+\frac{8}{\alpha+1}\frac{d}{dx'}\nabla'^2\mathrm{E}_2=0 \\ \text{and} \quad &\frac{d}{dx'}\nabla'^2\mathrm{E}_1-\frac{8}{\alpha+1}\frac{d}{dy'}\nabla'^2\mathrm{E}_2=0 \end{aligned}\right\}$$

If we write U for $\quad \frac{d\mathrm{E}_1}{dy'}+\frac{d\mathrm{E}_2}{dx'}-\frac{3-\alpha}{\alpha+1}\frac{1}{3}h^2\frac{d}{dx'}\nabla'^2\mathrm{E}_2,$

and V for $\quad -\frac{d\mathrm{E}_1}{dx'}+\frac{d\mathrm{E}_2}{dy'}-\frac{3-\alpha}{\alpha+1}\frac{1}{3}h^2\frac{d}{dy'}\nabla'^2\mathrm{E}_2,$

we obtain the form which it is convenient to take as the standard for this kind of strain, namely,

$$\left.\begin{aligned} u(x',y',z') &= \mathrm{U}+\frac{3-\alpha}{\alpha+1}\frac{1}{2}z'^2\frac{d}{dx'}\left(\frac{d\mathrm{U}}{dx'}+\frac{d\mathrm{V}}{dy'}\right) \\ v(x',y',z') &= \mathrm{V}+\frac{3-\alpha}{\alpha+1}\frac{1}{2}z'^2\frac{d}{dy'}\left(\frac{d\mathrm{U}}{dx'}+\frac{d\mathrm{V}}{dy'}\right) \\ w(x',y',z') &= \frac{\alpha-3}{\alpha+1}z'\left(\frac{d\mathrm{U}}{dx'}+\frac{d\mathrm{V}}{dy'}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (98)$$

with

$$\left.\begin{aligned} &\frac{d}{dy'}\left(\frac{d\mathrm{U}}{dy'}-\frac{d\mathrm{V}}{dx'}\right)+\frac{8}{\alpha+1}\frac{d}{dx'}\left(\frac{d\mathrm{U}}{dx'}+\frac{d\mathrm{V}}{dy'}\right)=0 \\ &\frac{d}{dx'}\left(\frac{d\mathrm{U}}{dy'}-\frac{d\mathrm{V}}{dx}\right)-\frac{8}{\alpha+1}\frac{d}{dy}\left(\frac{d\mathrm{U}}{dx'}+\frac{d\mathrm{V}}{dy'}\right)=0 \end{aligned}\right\}$$

(ii) Transitory modes, ψ or rotational type.

Referring to the expression for ψ' in (82), put

$$\mathrm{E}_3 \equiv \text{work difference from the system } \psi = \frac{1}{8\pi\mu\kappa^2 h}\cosh \kappa z \mathrm{G}_0 \kappa \mathrm{R}.$$

Then for this part of the solution

$$\left.\begin{aligned} u(x', y', z') &= \sum_\kappa 2\frac{d\mathrm{E}_3}{dy'}\cosh \kappa z' \\ v(x', y', z') &= \sum_\kappa (-2)\frac{d\mathrm{E}_3}{dx'}\cosh \kappa z' \\ w(x', y', z') &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (99)$$

where κ is a pos. imag. root of $\sinh \kappa h$, and $\frac{d^2\mathrm{E}_3}{dx'^2}+\frac{d^2\mathrm{E}_3}{dy'^2}+\kappa^2\mathrm{E}_3=0$.

The solutions here are obviously of the type $\psi = \cosh \kappa z' \mathrm{E}_3(x', y')$.

(iii) Transitory modes, θ-ϕ or dilatational type.

Looking to (64), (82), put

$\mathrm{E}_4 \equiv$ work difference from the system $\left\{\begin{aligned} \phi &= \frac{\cosh \kappa z \mathrm{G}_0 \kappa \mathrm{R}}{8\pi\mu(\alpha+1)\kappa^2 h(\cosh 2\kappa h + 1)} \\ \theta &= \cosh 2\kappa h \cdot \phi \end{aligned}\right.$

Then

$$\left.\begin{aligned} u(x', y', z') &= \sum_\kappa \frac{d\mathrm{E}_4}{dx'}\Big\{2\kappa z' \sinh \kappa z' + (\cosh 2\kappa h + \alpha)\cosh \kappa z'\Big\} \\ v(x', y', z') &= \sum_\kappa \frac{d\mathrm{E}_4}{dy'}\Big\{\qquad\qquad\qquad\qquad\qquad\qquad\Big\} \\ w(x', y', z') &= \sum_\kappa \kappa \mathrm{E}_4\{2\kappa z' \cosh \kappa z' + (\cosh 2\kappa h - \alpha)\sinh \kappa z'\} \end{aligned}\right\} \quad . \quad . \quad (100)$$

where κ is a zero of $\sinh 2\kappa h + 2\kappa h$ with pos. imag. part, and $\frac{d^2\mathrm{E}_4}{dx'^2}+\frac{d^2\mathrm{E}_4}{dy'^2}+\kappa^2\mathrm{E}_4=0$.

The solutions are of the type $\phi = \cosh \kappa z' \mathrm{E}_4(x', y')$, $\theta = \cosh 2\kappa h \cdot \phi$.

II. Flexural part of the solution.

(i) Permanent mode.

Let $\mathrm{F}_1 \equiv$ work difference from the system $\left\{\begin{aligned} \phi &= -\frac{3}{32\pi\mu h^3}\left(z\dot{\chi} - \frac{1}{6}z^3\nabla^2\chi\right) \\ \theta &= \frac{3}{32\pi\mu h^3}\left(z\chi - \frac{1}{6}z^3\nabla^2\chi - 2h^2 z\nabla^2\chi\right) \end{aligned}\right.$

Then $\nabla'^2\mathrm{F}_1 =$ work difference from $\phi = -\frac{3}{32\pi\mu h^3} z\nabla^2\chi = -\theta$,

and

$$\left.\begin{aligned} u(x', y', z') &= -z'\frac{d\mathrm{F}_1}{dx'} + \frac{2}{\alpha+1}\left(\frac{\alpha+5}{12}z'^3 - \frac{\alpha+11}{10}h^2 z'\right)\frac{d}{dx'}\nabla'^2\mathrm{F}_1 \\ v(x', y', z') &= -z'\frac{d\mathrm{F}_1}{dy'} + \frac{2}{\alpha+1}\left(\frac{\alpha+5}{12}z'^3 - \frac{\alpha+11}{10}h^2 z'\right)\frac{d}{dy'}\nabla'^2\mathrm{F}_1 \\ w(x', y', z') &= \quad \mathrm{F}_1 \quad + \left\{\tfrac{1}{5}h^2 - \tfrac{1}{2}z'^2 + \frac{2}{\alpha+1}(z'^2 - h^2)\right\}\nabla'^2\mathrm{F}_1 \end{aligned}\right\} \quad . \quad . \quad (101)$$

Here $\nabla'^4 F_1 = 0$, and if we write $(\alpha+1)F$ for $F_1 + \frac{1}{5}h'\nabla'^2 F_1$, these expressions reduce to the form which we have taken throughout for this kind of strain, namely,

$$\left.\begin{aligned} u(x', y', z') &= \frac{d}{dx'}\left\{ -(\alpha+1)(z'F - \tfrac{1}{6}z'^3\nabla'^2 F) + 2(\tfrac{1}{3}z'^3 - h^2 z')\nabla' F \right\} \\ v(x', y', z') &= \frac{d}{dy'}\left\{ \right\} \\ w(x', y', z') &= (\alpha+1)(F - \tfrac{1}{2}z'^2\nabla'^2 F) + 2(z'^2 - h^2)\nabla'^2 F \end{aligned}\right| \quad . \quad . \quad (101')$$

where $\nabla'^4 F = 0$.

(ii) Transitory modes, ψ or rotational type.

Put $F_2 \equiv$ work difference from the system $\psi = -\frac{1}{8\pi\mu\kappa^2 h}\sinh \kappa z G_0 \kappa R$.

Then

$$\left.\begin{aligned} u(x', y', z') &= \sum_\kappa 2 \frac{dF_2}{dy'}\sinh \kappa z' \\ v(x', y', z') &= \sum_\kappa (-2)\frac{dF_2}{dx'}\sinh \kappa z' \\ w(x', y', z') &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (102)$$

where κ is a pos. imag. root of $\cosh \kappa h$, and $\frac{d^2F_2}{dx'^2} + \frac{d^2F_2}{dy'^2} + \kappa^2 F_2 = 0$.

The solutions are of the type $\psi = \sinh \kappa z' F_2(x', y')$.

(iii) Transitory modes, θ-ϕ or dilatational type.

Put $F_3 =$ work difference from the system $\begin{cases} \phi = \dfrac{\sinh \kappa z G_0 \kappa R}{8\pi\mu(\alpha+1)\kappa^2 h(\cosh 2\kappa h - 1)} \\ \theta = -\cosh 2\kappa h \cdot \phi \end{cases}$

Then

$$\left.\begin{aligned} u(x', y', z') &= \sum_\kappa \frac{dF_3}{dx'}\left\{ 2\kappa z' \cosh \kappa z' + (\alpha - \cosh 2\kappa h)\sinh \kappa z' \right\} \\ v(x', y', z') &= \sum_\kappa \frac{dF_3}{dy'}\left\{ \right\} \\ w(x', y', z') &= \sum_\kappa \kappa F_3 \left\{ 2\kappa z' \sinh \kappa z' - (\alpha + \cosh 2\kappa h)\cosh \kappa z' \right\} \end{aligned}\right| \quad . \quad . \quad (103)$$

where κ is a zero of $\sinh 2\kappa h - 2\kappa h$ with pos. imag. part, and $\frac{d^2F_3}{dx'^2} + \frac{d^2F_3}{dy'^2} + \kappa^2 F_3 = 0$.

The solutions are of the type $\phi = \sinh \kappa z' F_3(x', y')$, $\theta = -\cosh 2\kappa h \cdot \phi$.

34. *Form of the solution for edge tractions deduced by another method.*

We have thus shown that the most general deformation of a finite plate under edge tractions only is compounded of the types specified in (98) . . . . (103). The deformation is of the same form as that given by our infinite plate solutions for any part of the solid free from body force or surface traction, and it may be of advantage to show in a direct manner why this should be so.

Suppose, then, that we have given a displacement (u, v, w) of a finite plate bounded by an external edge S and one or more internal edges S′, the only applied forces being tractions on the edges. Imagine the plate continued inwards and outwards so as to

form a complete infinite plate. By the general existence theorem of the subject, there exist values of u, v, w in the space within an internal edge, continuous at the edge with the values of the displacements of the original solid, and produced by edge tractions alone. Similarly, if we take any surface S″, within the infinite plate, but completely enclosing the edge S, there exist values of u, v, w continuous with the original u, v, w at the external edge, and becoming zero on S″; these also being produced by edge tractions only, namely, on S and S″.

If, then, we take u, v, w to be zero outside S″, we obtain altogether a system of displacements continuous throughout the infinite solid. The forces required to maintain this system are given directly by the general equations of equilibrium. These forces form areal distributions on S, S′, S″, and are measured by the discontinuity of stress at these surfaces. Further, on the whole they make up an equilibrating system. But we have shown in the preceding pages how to find a solution for such a system of force, this solution giving displacements of order log R at most, and stresses of order R^{-2} at most, at a great distance. Only one solution fulfilling these conditions being possible, our solution is *the* solution.

Hence, finally, any displacement of a finite plate under edge tractions only is of the same form as that given by our infinite solid solutions for a certain system of areal force, distributed partly over the edges, and partly over an arbitrary external surface. This is what we proposed to prove.

35. *General solution for an infinite solid under any forces.*

It is now easy to determine the most general form of displacement of an infinite solid, under null body force and face traction, and free from singularity at a finite distance. For if u, v, w be any such displacement, then within any surface S, however distant, we have proved that u, v, w are given by the absolutely convergent series (98) . . . . (103).

If we take a right circular cylinder for the surface S, the functions F which satisfy equations of the form $\frac{d^2F}{dx^2} + \frac{d^2F}{dy^2} + \kappa^2 F = 0$ can be expressed in series of the form

$$\sum_m J_m \kappa\rho (A_m \cos m\omega + B_m \sin m\omega),$$

and the only restriction on the coefficients A_m, B_m is that they must make the double series in which the complete solution is thus expressed absolutely convergent for all values of ρ, however great.

The most general solution for any system of force applied at a finite distance is of course obtained by adding to this complete free solution the particular solution already investigated. It may be observed that this final result might have been obtained in one step by the process of § 33, if in that article we had taken for u, v, w any displacements under given body force and surface traction, instead of under edge traction only. The identity of the results of the two methods will be seen to depend essentially on the

fact that in the solution for a single force in any direction, the component displacement in that direction is *symmetrical in the accented and unaccented coordinates*, a theorem analogous to a well-known property of Green's function in Potential Theory.

It is interesting to observe that, in the process suggested in the last sentence but one, we only need to know the comparatively simple source solution for a single Z force in order to deduce the w displacement for any system of forces and face tractions whatever.

36. *Application of Betti's Theorem to the problem of given edge tractions.*

In the remaining pages, we shall be occupied almost exclusively with deformations of a finite plate under edge tractions only. For brevity we may refer to such deformations as *free*.

The formulæ (97) express the internal displacements in terms of the edge displacements and edge tractions. We may indicate here the general lines along which we naturally proceed in the attempt to reduce these formulæ to expressions in terms of displacements alone or of tractions alone.

Taking the first equation of (97), for example, if we wish a formula containing edge displacements only, we look for free displacements in the form of functions u_1', v_1', w_1' of x, y, z, such that u_1+u_1', v_1+v_1', w_1+w_1' shall be equal to zero at the edge.

If X_1', Y_1', Z_1' be the edge tractions in the system u_1', v_1', w_1', then by Betti's Theorem

$$\iint\left(Xu_1'+Yv_1'+Zw_1'-X_1'u-Y_1'v-Z_1'w\right)dS=0,$$

and by addition of this equation to (97),

$$u(x', y', z')=-\iint\left\{u(X_1+X_1')+v(Y_1+Y_1')+w(Z_1+Z_1')\right\}dS.$$

The problem of arbitrary edge displacements is thus reduced to a problem in which these displacements have a comparatively simple form.

When we attempt to find a formula in terms of edge tractions only, the procedure is not quite so simple, in consequence of the fact that the tractions X_1, Y_1, Z_1 are not equilibrating, but equivalent to a negative unit X force through (x', y', z'). From various methods of meeting this difficulty we select the following as the most convenient in the present case.

We have seen in § 30 that the system u_1, v_1, w_1 can be decomposed into four systems. The first system, say U_1, V_1, W_1 conveys no resultant stress; the second system conveys a stress equivalent to a unit X force through the origin, and the displacements are independent of x', y', z'; the third system conveys a couple z' in the plane zOx, the displacements contain z' as a factor, but are otherwise independent of x', y', z'; the fourth system conveys a couple $-y'$ in the plane xOy, the displacements involving x', y', z' only in the form of the factor y'.

The displacements u_2, v_2, w_2 and u_3, v_3, w_3 are similarly decomposable into equilibrating systems U_2, V_2, W_2 and U_3, V_3, W_3 with other systems conveying resultant forces and couples. The contributions to $u(x', y', z')$, $v(x', y', z')$, $w(x', y', z')$ in (97) from the various systems conveying forces and couples amount on the whole merely to a rigid body displacement of the plate. If this be neglected, then the value of $u(x', y', z')$, for instance, becomes simply the work difference from the system U_1, V_1, W_1, the edge tractions due to which are equilibrating, and can be balanced by a free system U_1', V_1', W_1'. We then obtain from (97)

$$u(x', y', z') = \iint \left\{ X(U_1 + U_1') + Y(V_1 + V_1') + Z(W_1 + W_1') \right\} dS$$

and similarly for v, w.

37. *Exact solutions of special problems for a circular plate.*

As already stated, we are not at present in a position to complete the solution of the problem of arbitrary edge tractions, even for the simplest form of edge. The method just indicated may be used, however, whatever be the form of the edge, to obtain approximately the boundary conditions which define the permanent part of the solution. But before entering on this important application, we shall consider a few special problems which admit of exact solution. All of these have reference to a plate bounded by a right circular cylinder, with or without a concentric circular aperture, and to systems of displacement symmetrical about the axis.

The radius of the external edge is a, of the internal edge b; and the axis of z coincides with the axis of the cylinder. u, v, w are the displacements in the directions in which the coordinates ρ, ω, z increase.

Problem 1. *Symmetrical transverse displacement.*

The displacement v, in the most general case, is given by a series involving cosines and sines of multiples of ω. We can determine the symmetrical term of the series. This constitutes the whole solution when the plate is subjected only to symmetrical torsional force.

For a transverse force Ω_1 applied at the point (ρ_1, ω_1, z_1) we have seen in article 29 that the solution is

$$\left.\begin{aligned} \psi &= \frac{d\psi'}{d\rho_1} \\ \theta &= -\frac{1}{\rho_1}\frac{d\theta'}{d\omega_1} \\ \phi &= -\frac{1}{\rho_1}\frac{d\phi'}{d\omega_1} \end{aligned}\right\} \cdot \frac{\Omega_1}{4\pi\mu(a+1)}$$

The solution for a constant linear distribution of transverse force on the circle $\rho = \rho_1$, $z = z_1$, of intensity $\Omega_1/2\pi\rho_1$ per unit length, is found by integrating this with respect to ω_1 from 0 to 2π, and dividing by 2π. The result of the integration is simply to

eliminate all but the symmetrical part of ψ, and to eliminate θ, ϕ altogether. The solution is therefore,

when $\rho > \rho_1$,

$$\left.\begin{aligned} v = & \frac{1}{8\pi\mu h}\frac{\rho_1}{\rho} \\ & + \frac{1}{2\pi\mu h}\sum_{\kappa} \cosh \kappa z_1 \cosh \kappa z \mathrm{J}_0'\kappa\rho_1 \mathrm{G}_0'\kappa\rho\,, \ (\kappa \text{ a pos. imag. root of } \sinh \kappa h) \\ & - \frac{1}{2\pi\mu h}\sum_{\kappa} \sinh \kappa z_1 \sinh \kappa z \mathrm{J}_0'\kappa\rho_1 \mathrm{G}_0'\kappa\rho\,, \ (\kappa \text{ a pos. imag. root of } \cosh \kappa h) \end{aligned}\right\} \text{(i)}$$

when $\rho < \rho_1$, ρ and ρ_1 have simply to be interchanged.

The only stress across a cylinder ρ is

$$\widehat{\rho\omega} = \mu\left(\frac{dv}{d\rho} - \frac{v}{\rho}\right).$$

Hence if Ω_a, Ω_b be the transverse components of the traction on $\rho = a$, $\rho = b$, and v_1 the transverse displacement at (ρ_1, ω_1, z_1) produced by this traction, Betti's Theorem gives, as in (97), for the permanent part of v_1,

$$\frac{1}{2\pi}\int_0^{2\pi} v_1\, d\omega_1 = \frac{1}{8\pi\mu h}\iint\left(\Omega_a\frac{\rho_1}{a} + v_a\frac{2\mu\rho_1}{a^2}\right)d\mathrm{S}_a + \frac{1}{8\pi\mu h}\iint\Omega_b\frac{b}{\rho_1}d\mathrm{S}_b\,.$$

Also, since the distributions Ω_a, Ω_b have equal moments about Oz,

$$a\iint\Omega_a d\mathrm{S}_a + b\iint\Omega_b d\mathrm{S}_b = 0\,.$$

In the case of a symmetrical deformation, we have therefore

$$v_1 = \frac{b^2}{4\mu h}\left(\frac{1}{\rho_1} - \frac{\rho_1}{a^2}\right)\int_{-h}^{h}\Omega_b dz + \frac{\rho_1}{a}v_a\,.$$

The displacement $v_1 = \rho_1$ being merely a rigid body rotation about Oz, the permanent solution is practically

$$v_1 = \frac{b^2}{4\mu h\rho_1}\int_{-h}^{h}\Omega_b dz = -\frac{a^2}{4\mu h\rho_1}\int_{-h}^{h}\Omega_a dz \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

This might have been got at once by omitting the term $v = \rho_1/8\pi\mu h\rho$ from the source solution, in accordance with the method explained at the end of the last article.

For a uniform system of couple about the edge-normal $\int\Omega_b dz$ vanishes, and the permanent displacement is rigorously null.

For the transitory part of the solution we will, to simplify the algebra, suppose the cylinder solid.

Further, this not being a case where the separation into odd and even parts in z is of much consequence, we may shorten the formulæ by combining the two κ series into one.

Thus $\cosh\kappa\,(h+z)\,\cosh\kappa\,(h+z_1)$ being obviously equal to $\cosh\kappa z\,\cosh\kappa z_1$ when

$\sinh \kappa h = 0$, but equal to $-\sinh \kappa z \sinh \kappa z_1$ when $\cosh \kappa h = 0$, we may transform (i) if we put $h+z=\zeta$, $h+z_1=\zeta_1$, into

$$v=\frac{1}{8\pi\mu h}\frac{\rho_1}{\rho}+\frac{1}{2\pi\mu h}\sum_{\kappa}\cosh\kappa\zeta_1\cosh\kappa\zeta \mathrm{J_0}'\kappa\rho_1\mathrm{G_0}'\kappa\rho, \ (\kappa \text{ a pos. imag. root of } \sinh 2\kappa h) . \quad . \quad \text{(iii)}$$

For

$$v=\mathrm{G_0}'\kappa\rho, \text{ we have when } \rho=a, \ \widehat{\rho\omega}=-\frac{\mu}{a}(2\mathrm{G_0}'\kappa a+\kappa a\mathrm{G_0}\kappa a).$$

Hence, when the transitory part of v in (iii) is balanced, the solution becomes $(\rho>\rho_1)$

$$v=\frac{1}{8\pi\mu h}\frac{\rho_1}{\rho}+\frac{1}{2\pi\mu h}\sum_{\kappa}\cosh\kappa\zeta_1\cosh\kappa\zeta \mathrm{J_0}'\kappa\rho_1\left(\mathrm{G_0}'\kappa\rho-\frac{2\mathrm{G_0}'\kappa a+\kappa a\mathrm{G_0}\kappa a}{2\mathrm{J_0}'\kappa a+\kappa a\mathrm{J_0}\kappa a}\mathrm{J_0}'\kappa\rho\right) \quad . \quad \text{(iv)}$$

This gives at $\rho=a$,

$$v=\frac{1}{8\pi\mu h}\frac{\rho_1}{a}+\frac{1}{2\pi\mu h}\sum_{\kappa}\cosh\kappa\zeta_1\cosh\kappa\zeta \mathrm{J_0}'\kappa\rho_1\left(-\frac{1}{2\mathrm{J_0}'\kappa a+\kappa a\mathrm{J_0}\kappa a}\right)$$

Hence, for the free displacement at (ρ_1, z_1) under symmetrical transverse traction Ω_a on $\rho=a$, Betti's Theorem gives (omitting the rigid body rotation)

$$v_1=\frac{a}{\mu h}\sum_{\kappa}\mathrm{J_0}'\kappa\rho_1\cosh\kappa\zeta_1\left(-\frac{1}{2\mathrm{J_0}'\kappa a+\kappa a\mathrm{J_0}\kappa a}\right)\int_0^{2h}\cosh\kappa\zeta\Omega_a d\zeta \quad . \quad . \quad . \quad \text{(v)}$$

From this

$$(\rho\omega)_1=\frac{1}{h}\sum_{\kappa}\frac{2\mathrm{J_0}'\kappa\rho+\kappa\rho\mathrm{J_0}\kappa\rho}{2\mathrm{J_0}'\kappa a+\kappa a\mathrm{J_0}\kappa a}\cosh\kappa\zeta_1\int_0^{2h}\cosh\kappa\zeta\Omega_a d\zeta$$

The series passes continuously, as ρ increases to a, into the limit

$$\frac{1}{h}\sum_{\kappa}\cosh\kappa\zeta_1\int_0^{2h}\cosh\kappa\zeta\Omega_a d\zeta, \text{ provided this latter series converges.}$$

By Fourier's Theorem we know that it does, namely, to the value Ω_a, it being noted that $\int_0^{2h}\Omega_a d\zeta=0$. The solution is thus verified. Of course, it could easily be obtained by the Fourier method *ab initio*.

The series (iv) converges very rapidly unless ρ and ρ_1 are nearly equal. By an application of the Residue Calculus, it may be transformed into a series in which the functions of ρ, ρ_1 are the fluctuating functions, and the functions of ζ, ζ_1 the convergence factors. For consider the function of κ,

$$\frac{1}{\pi\mu}\frac{\cosh\kappa(2h-\zeta)\cosh\kappa\zeta_1}{\sinh 2\kappa h}\mathrm{J_0}'\kappa\rho_1\left(\mathrm{G_0}'\kappa\rho-\frac{2\mathrm{G_0}'\kappa a+\kappa a\mathrm{G_0}\kappa a}{2\mathrm{J_0}'\kappa a+\kappa a\mathrm{J_0}\kappa a}\mathrm{J_0}'\kappa\rho\right).$$

It is easy to see that $\log\kappa$ disappears from the last factor, and that the whole function is a *uniform*, *odd* function of κ; also that if $\zeta>\zeta_1, \rho>\rho_1$, the function vanishes at infinity in such a way as to make the total sum of its residues equal to zero.

The poles of the function are $\kappa=0$, the (pure imaginary) zeroes of $\sinh 2\kappa h$, and the (real) zeroes of $2\mathrm{J_0}'\kappa a+\kappa a\mathrm{J_0}\kappa a$. The function being odd, we have (series of residues at pos. imag. roots of $\sinh 2\kappa h$) + (series of residues at pos. roots of $2\mathrm{J_0}'\kappa a+\kappa a\mathrm{J_0}\kappa a$) $+\frac{1}{2}$ residue at $(\kappa=0)$, equal to zero.

The first series of residues is the function v of (iv). Thus we obtain, $(\zeta > \zeta_1)$

$$\left.\begin{aligned} v = \frac{4}{\pi\mu a}\sum_{\kappa} \frac{\cosh\kappa(2h-\zeta)\cosh\kappa\zeta_1}{\sinh 2\kappa h}\frac{1}{\kappa^3 a^3(\mathrm{J}_0\kappa a)^2}\mathrm{J}_0'\kappa\rho_1\mathrm{J}_0'\kappa\rho \\ -\frac{2}{\pi\mu a^4}\rho_1\rho\zeta + \frac{\rho\rho_1}{3\pi\mu a^4 h}(a^2+4h^2) - \frac{\rho_1}{8\pi\mu a^4 h}(\rho^3 - 4\rho\zeta^2) - \frac{\rho}{8\pi\mu a^4 h}(\rho_1{}^3 - 4\rho_1\zeta_1{}^2) \end{aligned}\right\} \quad . \quad . \quad \text{(vi)}$$

When $\zeta > \zeta_1$, we have merely to interchange ζ and ζ_1 in this formula.

We may verify in a moment from this, as from the perfectly equivalent form (iv), that the internal couple is balanced by the stress at the cylindrical boundary, and that there is no stress across the plane ends. But if we remove the last three terms from (vi), we make no change in the internal singularity, these terms being the same whether ζ or ζ_1 be the greater.

We thus obtain the displacement when the internal couple is balanced at the plane ends, namely, $(\zeta > \zeta_1)$

$$v' = -\frac{2}{\pi\mu a^4}\rho_1\rho\zeta + \frac{4}{\pi\mu a}\sum_{\kappa}\frac{\cosh\kappa(2h-\zeta)\cosh\kappa\zeta_1}{\sinh 2\kappa h}\frac{1}{\kappa^3 a^3(\mathrm{J}_0\kappa a)^2}\mathrm{J}_0'\kappa\rho_1\mathrm{J}_0'\kappa\rho \quad . \quad . \quad \text{(vii)}$$

Here, as in (vi), the summation extends over the positive roots of $2\mathrm{J}_0'\kappa a + \kappa a\mathrm{J}_0\kappa a$. The solution for symmetric transverse traction Ω_{2h}, Ω_0 on the ends, which might be obtained in an abnormal form from (iv) with the cognate formula for $\rho > \rho_1$, is given in normal form by a direct application of Betti's Theorem to (vii).

Thus

$$\left.\begin{aligned} v(\rho_1,\zeta_1) = \frac{8}{\mu a}\sum_{\kappa}\frac{\cosh\kappa\zeta_1\mathrm{J}_0'\kappa\rho_1}{\sinh 2\kappa h}\frac{1}{\kappa^3 a^3(\mathrm{J}_0\kappa a)^2}\int_0^a \Omega_{2h}\rho\mathrm{J}_0'\kappa\rho d\rho \\ + \frac{8}{\mu a}\sum_{\kappa}\frac{\cosh\kappa(2h-\zeta_1)\mathrm{J}_0'\kappa\rho_1}{\sinh 2\kappa h}\frac{1}{\kappa^3 a^3(\mathrm{J}_0\kappa a)^2}\int_0^a \Omega_0\rho\mathrm{J}_0'\kappa\rho d\rho \\ - \frac{4}{\mu a^4}\rho_1\zeta_1\int_0^a \Omega_0\rho^2 d\rho \end{aligned}\right\} \quad . \quad . \quad \text{(viii)}$$

The result belongs rather to the theory of a long rod than to that of a thin plate. The permanent term depends only on the integral couple, and coincides with that given by SAINT VENANT's theory of Torsion.*

38. *Problem* 2. *Boundary values of the normal displacement* u, *and the shearing stress normal to the plate* $\widehat{z\rho}$, *are given functions symmetrical about the axis; the displacement* v, *or the shearing stress* $\widehat{\rho\omega}$, *vanishes.*

We begin with the case of a solid cylinder.

(i) *Permanent extensional mode.*

Referring to § 33, I. (i), we see that under the conditions proposed the function E_1 must vanish, and the solution in cylindrics is

$$u(\rho_1, z_1) = \frac{d\mathrm{E}_2}{d\rho_1} + \frac{3-a}{a+1}\left(\tfrac{1}{2}z_1{}^2 - \tfrac{1}{3}h^2\right)\frac{d}{d\rho_1}\nabla_1{}^2\mathrm{E}_2$$

$$w(\rho_1, z_1) = \frac{a-3}{a+1}z_1\nabla_1{}^2\mathrm{E}_2$$

* The writer hopes to publish shortly a solution of the problem of equilibrium of an infinite circular cylinder, in which the celebrated solutions of SAINT VENANT will appear as the leading terms. It will be shown that in a finite cylinder the permanent modes are given *exactly* by SAINT VENANT's theory. In the theory of thin plates, the permanent modes can only, in general, be found *approximately*.

where $E_2 = \frac{1}{32\pi\mu h}$. work diffce. from $\theta=\phi=\chi(\rho)-\frac{1}{2}z^2\log\rho+\frac{1}{4}\rho_1{}^2\log\rho$,

or $E_2=\frac{\rho_1{}^2}{128\pi\mu h}$. work diffce. from $\theta=\phi=\log\rho$, the part omitted being merely a constant.

Now in the system $\theta=\phi=\log\rho$

$$\begin{array}{l|l} u=(\alpha+1)/\rho & \widehat{\rho\rho}=-2\mu(\alpha+1)/\rho^2 \\ w=0 & \widehat{\rho z}=0 \end{array}$$

Hence the solution

$$\begin{array}{l|l} u=-(\alpha+1)\rho/a^2 & \widehat{\rho\rho}=2\mu(\alpha-7)/a^2 \\ w=-2(\alpha-3)z/a^2 & \widehat{\rho z}=0 \end{array}$$

will, taken along with $\theta=\phi=\log\rho$, give u and $\widehat{\rho z}=0$ at $\rho=a$. The balanced solution gives $\widehat{\rho\rho}=-16\mu/a^2$, $w=2(3-\alpha)z/a^2$, at the edge.
Hence

$$E_2=\frac{\rho_1{}^2}{16ah}\int_{-h}^{h}\left\{\frac{Z_a}{2\mu}(3-\alpha)z+4u_a\right\}dz$$

and

$$\left.\begin{array}{l} u(\rho_1,z_1)=\rho_1 \\ w(\rho_1,z_1)=2\frac{\alpha-3}{\alpha+1}z_1 \end{array}\right\} \cdot \frac{1}{8ah}\int_{-h}^{h}\left\{\frac{Z_a}{2\mu}(3-\alpha)z+4u_a\right\}dz .$$

(ii) *Transitory extensional modes.*

The solution is given by (100) with

$$E_4=\frac{J_0\kappa\rho_1}{8\pi\mu(\alpha+1)\kappa^2h(\cosh 2\kappa h+1)}\cdot \text{ work diffce. from the system } \left\{\begin{array}{l}\phi=G_0\kappa\rho\cosh\kappa z \\ \theta=\cosh 2\kappa h\cdot\phi\end{array}\right.$$

In the system mentioned

$$\begin{aligned} u &= \kappa G_0{}'\kappa\rho\{(\cosh 2\kappa h+\alpha)\cosh\kappa z+2\kappa z\sinh\kappa z\} \\ w &= \kappa G_0\kappa\rho\{(\cosh 2\kappa h-\alpha)\sinh\kappa z+2\kappa z\cosh\kappa z\} \\ \frac{1}{\kappa^2}\frac{\widehat{z\rho}}{2\mu} &= G_0{}'\kappa\rho\{(\cosh 2\kappa h+1)\sinh\kappa z+2\kappa z\cosh\kappa z\} \\ \frac{1}{\kappa^2}\frac{\widehat{\rho\rho}}{2\mu} &= -G_0\kappa\rho\{(\cosh 2\kappa h+3)\cosh\kappa z+2\kappa z\sinh\kappa z\} \\ &\quad -\frac{1}{\kappa\rho}G_0{}'\kappa\rho\{(\cosh 2\kappa h+\alpha)\cosh\kappa z+2\kappa z\sinh\kappa z\} . \end{aligned}$$

The balancing system for u and $\widehat{z\rho}$ at $\rho=a$ is therefore

$$\phi=-\frac{G_0{}'\kappa a}{J_0{}'\kappa a}J_0\kappa\rho\cosh\kappa z,\ \theta=\cosh 2\kappa h\phi .$$

In the balanced system, at the edge

$$\begin{aligned} w &= \frac{1}{aJ_0{}'\kappa a}\left\{(\cosh 2\kappa h-\alpha)\sinh\kappa z+2\kappa z\cosh\kappa z\right\} \\ \frac{\widehat{\rho\rho}}{2\mu} &= -\frac{\kappa}{aJ_0{}'\kappa a}\left\{(\cosh 2\kappa h+3)\cosh\kappa z+2\kappa z\sinh\kappa z\right\} \end{aligned}$$

Hence for the free solution with edge values $u=u_a$, $\widehat{z\rho}=Z_a$,

$$E_4=\frac{1}{2(\alpha+1)\kappa^2h(\cosh 2\kappa h+1)}\frac{J_0\kappa\rho_1}{J_0{}'\kappa a}\int_{-h}^{h}\left\{\begin{array}{l}\frac{Z_a}{2\mu}\overline{(\cosh 2\kappa h-\alpha}\sinh\kappa z+2\kappa z\cosh\kappa z) \\ +\kappa u_a\overline{(\cosh 2\kappa h+3}\cosh\kappa z+2\kappa z\sinh\kappa z\end{array}\right\}dz$$

(iii) *Permanent flexural mode.*

The solution is given by (101) with

$$F_1 = \frac{3}{32\pi\mu h^3} \cdot \frac{1}{4}\rho_1^2 \cdot \quad \text{work diffce. from} \quad \theta = -\phi = z\log\rho .$$

In the system

$$\theta = -\phi = z\log\rho , \quad \begin{array}{l|l} u = -(\alpha+1)z/\rho & \widehat{z\rho} = 0 \\ w = (\alpha+1)\log\rho & \widehat{\rho\rho} = 2\mu(\alpha+1)z/\rho^2 \end{array}$$

Hence the solution balancing u and $\widehat{z\rho}$ at $\rho = a$ is

$$\begin{array}{l|l} u = (\alpha+1)z\rho/a^2 & \widehat{z\rho} = 0 \\ w = -(\alpha+1)\rho^2/2a^2 + (\alpha-3)z^2/a^2 & \widehat{\rho\rho} = 2\mu(7-\alpha)z/a^2 \end{array}$$

In the balanced solution, at the edge

$$w = (\alpha+1)(\log a - \tfrac{1}{2}) + (\alpha-3)z^2/a^2 ; \quad \widehat{\rho\rho} = 16\mu z/a^2$$

The constant term in the value of w will disappear since $\int_{-h}^{h} \widehat{z\rho}\, dz = 0$. Thus

$$F_1 = \frac{3}{32ah^3}\rho_1^2 \int_{-h}^{h} \left\{ \frac{Z_a}{2\mu}(\alpha-3)z^2 - 8zu_a \right\} dz$$

and

$$\begin{array}{l} u(\rho_1, z_1) = -z_1\rho_1 \\ w(\rho_1, z_1) = \frac{1}{2}\rho_1^2 + \frac{3-\alpha}{\alpha+1}z_1^2 \end{array} \Bigg\} + \frac{3}{16ah^3}\int_{-h}^{h} \left\{ \frac{Z_a}{2\mu}(\alpha-3)z^2 - 8zu_a \right\} dz .$$

(iv) *Transitory flexural modes.*

The solution is given by (103) with

$$F_3 = \frac{J_0\kappa\rho_1}{8\pi\mu(\alpha+1)\kappa^2 h(\cosh 2\kappa h - 1)} \cdot \quad \text{work diffce. from the system} \quad \begin{cases} \phi = G_0\kappa\rho \sinh\kappa z \\ \theta = -\cosh 2\kappa h \cdot \phi \end{cases}$$

In this system

$$\begin{aligned} u &= \kappa G_0'\kappa\rho\{(\alpha - \cosh 2\kappa h)\sinh\kappa z + 2\kappa z\cosh\kappa z\} \\ w &= \kappa G_0\kappa\rho\{-(\alpha+\cosh 2\kappa h)\cosh\kappa z + 2\kappa z\sinh\kappa z\} \\ \frac{1}{\kappa^2}\frac{\widehat{z\rho}}{2\mu} &= G_0'\kappa\rho\{(1-\cosh 2\kappa h)\cosh\kappa z + 2\kappa z\sinh\kappa z\} \\ \frac{1}{\kappa^2}\frac{\widehat{\rho\rho}}{2\mu} &= G_0\kappa\rho\{(\cosh 2\kappa h - 3)\sinh\kappa z - 2\kappa z\cosh\kappa z\} \\ &\quad - \frac{1}{\kappa\rho}G_0'\kappa\rho\{(\alpha - \cosh 2\kappa h)\sinh\kappa z + 2\kappa z\cosh\kappa z\} \end{aligned}$$

The system balancing u and $\widehat{z\rho}$ at $\rho = a$ is

$$\phi = -\frac{G_0'\kappa a}{J_0'\kappa a} J_0\kappa\rho\sinh\kappa z , \quad \theta = -\cosh 2\kappa h \cdot \phi .$$

In the balanced system, at the edge

$$\begin{aligned} w &= \frac{1}{aJ_0'\kappa a}\left\{ -(\alpha+\cosh 2\kappa h)\cosh\kappa z + 2\kappa z\sinh\kappa z \right\} \\ \frac{\widehat{\rho\rho}}{2\mu} &= \frac{\kappa}{aJ_0'\kappa a}\left\{ (\cosh 2\kappa h - 3)\sinh\kappa z - 2\kappa z\cosh\kappa z \right\} \end{aligned}$$

Hence for the free solution with edge values $u = u_a$, $\widehat{z\rho} = Z_a$,

$$F_3 = \frac{1}{2(\alpha+1)\kappa^2 h(\cosh 2\kappa h - 1)}\frac{J_0\kappa\rho_1}{J_0'\kappa a}\int_{-h}^{h} \left\{ \begin{array}{l} \frac{Z_a}{2\mu}(-\alpha + \cosh 2\kappa h\cosh\kappa z + 2\kappa z\sinh\kappa z) \\ + \kappa u_a(3 - \cosh 2\kappa h\sinh\kappa z + 2\kappa z\cosh\kappa z) \end{array} \right\} dz .$$

If the given values of Z_a, u_a are the same as the edge values of $\widehat{z\rho}$, u, in one of the particular solutions, then clearly this particular solution by itself is *the* solution, and the

integrals which define the coefficients of the other particular solutions must vanish, while the integral corresponding to the solution left has its value determined. These results are easily verified by actual integration.

This remark may be used to find the solution for a hollow cylinder, which of course might also be obtained directly by the above process. We shall illustrate the method by finding the value of F_3 corresponding to any given root κ of $\sinh 2\kappa h - 2\kappa h$, when we have given u_a, u_b, Z_a, Z_b.

This value of F_3 we know is of the form $AJ_0\kappa\rho_1 + BG_0\kappa\rho_1$.

The complete values of u and of $\widehat{z\rho}/2\mu$ for $\rho = \rho_1$ are given by series which manifestly converge uniformly *so long as* $b < \rho_1 < a$.

Multiply the series for $\widehat{z\rho}/2\mu$ by $\quad -(a + \cosh 2\kappa h)\cosh \kappa z + 2\kappa z \sinh \kappa z$,

the series for u by $\quad \kappa(3 - \cosh 2\kappa h \sinh \kappa z + 2\kappa z \cosh \kappa z)$,

add, and integrate with respect to z from $-h$ to h. All the terms disappear except that associated with the given root κ. We thus find

$$(AJ_0'\kappa\rho_1 + BG_0'\kappa\rho_1)2(a+1)\kappa^2 h(\cosh 2\kappa h - 1)$$
$$= \int_{-h}^{h} \left\{ \frac{\widehat{z\rho}}{2\mu}(\rho=\rho_1)\left(-\overline{a+\cosh 2\kappa h}\cosh \kappa z + 2\kappa z \sinh \kappa z\right) + \kappa u(\rho=\rho_1)\left(\overline{3-\cosh 2\kappa h}\sinh \kappa z + 2\kappa z \cosh \kappa z\right) \right\} dz .$$

This is proved for the case $b < \rho_1 < a$. Now take the limits of both sides for $\rho_1 = a$. The limit of the integral is found simply by replacing $\widehat{z\rho}$ and $u\ (\rho = \rho_1)$ in the integrand by $\widehat{z\rho}$ and $u\ (\rho = a)$, *provided the resulting integral has a meaning*, which will be the case if $\widehat{z\rho}$ and $u\ (\rho = a)$ are integrable functions of z. Similarly we may take the limit for $\rho_1 = b$, and thus obtain two equations to determine A and B.

It will be observed that by this method we avoid two difficulties which in problems of this kind are often introduced unnecessarily by physical writers, namely, (i) the difficulty as to the convergence of the series for $\widehat{z\rho}$ and u, when the value $\rho_1 = a$ or b is substituted term by term, and (ii) the allied difficulty as to the continuity of the series right up to $\rho = a$ or b, even when it is known to converge. Judging from analogy, we may feel reasonably certain that the series will in fact converge at the limits, at least in the case of ordinary functions; but it is worth while noting that, whether they converge or not, the Fourier method of assuming the continuity and convergence, and determining the coefficients by integration, does give the correct values of these coefficients.

On the other hand, while our 'Green's function' method proves definitely that *any possible solution* has the form given above, it does not prove that a solution *is* possible for arbitrary edge values of $\widehat{z\rho}$ and u. The investigation might be completed by verifying that the solution obtained does actually satisfy the conditions, which would not be difficult in the present case. Alternatively, we may rely upon physical considerations, or upon a general analytical existence theorem. The proofs of theorems of this type in other branches of physical mathematics have been considerably improved within recent years by Poincaré and others, and their methods are equally applicable to the elastic equations.

39. *Problem* 3. *To determine the permanent modes, having given the symmetrical edge tractions* $\widehat{z\rho}$ *and* $\widehat{\rho\rho}$.

Supposing the cylinder solid, we have only to make a slight modification in the process of (i), (iii) in last article.

Extensional mode.

In (i) the balancing solution must now be taken as

$$u=\frac{(a+1)^2}{7-a}\frac{\rho}{a^2},\quad w=-\frac{2(3-a)(a+1)}{7-a}\frac{z}{a^2}.$$

This, along with $\theta=\phi=\log\rho$, gives $\widehat{\rho\rho}$ and $\widehat{\rho z}=0$ at $\rho=a$.
The balanced solution gives at the edge

$$u=\frac{8}{a}\frac{a+1}{7-a},\quad w=-\frac{2(3-a)(a+1)}{7-a}\frac{z}{a^2}.$$

Hence

$$\left.\begin{aligned} u(\rho_1,z_1)&=\rho_1\\ w(\rho_1,z_1)&=2\frac{a-3}{a+1}z_1\end{aligned}\right\}\ \frac{a+1}{7-a}\frac{1}{16\mu ah}\int_{-h}^{h}(4aP_a+\overline{a-3}\,zZ_a)dz.$$

This gives the permanent mode *exactly*. The ordinary approximate theory omits the term in Z_a from the integral.

In CHREE's solution of the problem of a rotating disc, the stress $\widehat{z\rho}$ vanishes identically at the edge, while $\int_{-h}^{h}P_a dz=0$. His solution is therefore *exact*, so far as the fundamental mode is concerned.

As in last article, we infer from the form of the above solution that for any other than the permanent mode $\int_{-h}^{h}(4\rho\,\widehat{\rho\rho}+\overline{a-3}\,z\,\widehat{z\rho})=0$.

This may be verified by actual integration. Further, in the case of a hollow cylinder the solution is of the form

$$\left.\begin{aligned} u(\rho_1,z_1)&=A\rho_1+B/\rho_1\\ w(\rho_1,z_1)&=2A(a-3)z_1/(a+1)\end{aligned}\right\}$$

and the coefficients A, B are found from the conditions that

$\int_{-h}^{h}(4\rho\cdot\widehat{\rho\rho}+\overline{a-3}\,z\cdot\widehat{z\rho})\,dz$ must, for $\rho=a$ and $\rho=b$

have the same value for the assumed form and for the given tractions.

Flexural mode.

Referring to (iii) of last article, the solution balancing

$\theta=\phi=z\log\rho$ at $\rho=a$ is now

$$u=-\frac{(a+1)^2}{7-a}\frac{z\rho}{a^2};\quad w=\frac{(a+1)^2}{7-a}\frac{\rho^2}{2a^2}+\frac{(3-a)(a+1)}{7-a}\frac{z^2}{a^2}.$$

The balanced solution gives at the edge

$$u=-8\frac{a+1}{7-a}\frac{z}{a};\quad w=\text{const}+\frac{(3-a)(a+1)}{7-a}\frac{z^2}{a^2}.$$

Hence

$$u(\rho_1, z_1) = -z_1\rho_1$$
$$w(\rho_1, z_1) = \tfrac{1}{2}\rho_1^2 + \frac{3-a}{a+1}\, z_1^2 \Big\} \frac{3}{32\mu a h^3} \frac{a+1}{7-a} \int_{-h}^{h} (-8az\mathrm{P}_a + 3\overline{-a}\, z^2 \mathrm{Z}_a)\, dz .$$

The ordinary approximate theory takes account of the term in $z\mathrm{P}_a$ only.

The solution for a hollow cylinder can be obtained by this method, or by taking, in the notation of (94),

$$\mathrm{F} = \mathrm{A}\rho_1^2 + \mathrm{B}\log\rho_1 + \mathrm{C}\,(\tfrac{1}{4}\rho_1^2 \log\rho_1 - \tfrac{1}{4}\rho_1^2),$$

determining C from the value of $\int_{-h}^{h} \widehat{z\rho}\, dz$ at either edge, and then determining A and B from the conditions that

$$\int_{-h}^{h} (-8\rho z\widehat{\rho\rho} + \overline{3-a}\, z^2 \widehat{z\rho})\, dz \text{ must for } \rho = a \text{ or } \rho = b$$

have the same value in the assumed form as in the actual displacement.

40. *Expansions of arbitrary functions.*

When we attempt to apply the method of last article to the determination of the modes corresponding to the various roots of $\sinh 2\kappa h \pm 2\kappa h$, we are at once confronted with an apparently insuperable difficulty. The determination of any one mode is reduced by the application of Betti's Theorem to the special problem of balancing the particular source solution involving a given root κ. Now in similar investigations connected with Laplace's equation, the equation of conduction of heat, and other partial differential equations of the second order which occur in physical mathematics, the analogous balancing problem can be solved without difficulty for certain simple forms of edge, and the balancing solution is of the same type as the particular source solution, that is, involves only the same root κ. In the present problem, however, the balancing solution will in general involve particular solutions of all types, as will be seen below.

Various theorems relating to the expansion of arbitrary functions may be found, similar to the theorem suggested at the end of § 38, but these do not help us, at all events immediately, to the general solution sought. One way of obtaining these expansion theorems may be indicated here; the method is of very wide application.

On a circular cylinder $\rho = a$, within the infinite plate, let areal force be distributed, the components of its intensity per unit area being $\mathrm{P}\cos m\omega$, $\Omega \sin m\omega$, $\mathrm{Z}\cos m\omega$, where P, Ω, Z are functions of z.

The infinite plate solution for this distribution of force can be written down, and the components of stress $\widehat{\rho\rho}$, $\widehat{\rho\omega}$, $\widehat{\rho z}$, calculated. These are given in different analytical forms according as ρ is greater or less than a. The expansion theorems are derived from the conditions of equilibrium

$$\underset{\epsilon=0}{\mathrm{Lt}} \left\{ \widehat{\rho\rho}(\rho = a - \epsilon) - \widehat{\rho\rho}(\rho = a + \epsilon) \right\} = \mathrm{P}$$

with two similar equations for Ω, Z.

There are other expansions which we know must exist, but the coefficients of which we cannot determine. The following examples are of special importance, and will be found useful immediately. They refer to the case of a plate bounded by a single infinite edge, filling, say, the region $x>0$, and the displacements considered are such that u, w are functions of x, z only, while v vanishes.

Extensional modes.

The permanent mode is of the form $u=x$, $w=\frac{a-3}{a+1}z$.

In a transitory mode with $\phi=e^{i\kappa x}\cosh\kappa z$, $\theta=\cosh 2\kappa h\cdot\phi$, we have

$$\frac{\widehat{xx}}{2\mu i\kappa^2}=ie^{i\kappa x}(\overline{\cosh 2\kappa h+3}\cosh\kappa x+2\kappa z\sinh\kappa z)$$

$$\frac{\widehat{xz}}{2\mu i\kappa^2}=e^{i\kappa x}(\overline{\cosh 2\kappa h+1}\sinh\kappa z+2\kappa z\cosh\kappa z)$$

As a special case of the results of last article, it follows that the coefficient of the permanent mode is determined from the given value of $\int_{-h}^{h}\widehat{xx}\,dz$ at the edge; and this integral normal stress is zero for each of the transitory modes.

Hence if $\mathrm{P}(z)$ be any even function of z, with $\int_{-h}^{h}\mathrm{P}(z)\,dz=0$, and $\mathrm{Z}(z)$ be any odd function of z, coefficients C_κ exist such that at the same time

$$\left.\begin{aligned}&\operatorname*{Lt}_{x=0}\sum_\kappa \mathrm{C}_\kappa ie^{i\kappa x}(\overline{\cosh 2\kappa h+3}\cosh\kappa z+2\kappa z\sinh\kappa z)=\mathrm{P}(z)\\ \text{and}\quad&\operatorname*{Lt}_{x=0}\sum_\kappa \mathrm{C}_\kappa e^{i\kappa x}(\overline{\cosh 2\kappa h+1}\sinh\kappa z+2\kappa z\cosh\kappa z)=\mathrm{Z}(z)\end{aligned}\right\}\quad\ldots\quad\text{(i)}$$

Flexural modes.

In the permanent mode F of equation (95) is of the form Ax^2+Bx^3. A and B are found from the edge values of $\int_{-h}^{h}z\cdot\widehat{xx}\,dz$ and $\int_{-h}^{h}\widehat{xz}\,dz$. These integrals vanish for a transitory mode, in which, with $\phi=e^{i\kappa x}\sinh\kappa z$, $\theta=-\cosh 2\kappa h\cdot\phi$,

$$\frac{\widehat{xx}}{2\mu i\kappa^2}z=ie^{i\kappa x}(\overline{3-\cosh 2\kappa h}\sinh\kappa z+2\kappa z\cosh\kappa z)$$

$$\frac{\widehat{xz}}{2\mu i\kappa^2}=e^{i\kappa x}(\overline{1-\cosh 2\kappa h}\cosh\kappa z+2\kappa z\sinh\kappa z).$$

We infer that if $\mathrm{P}(z)$ be any odd function of z, with $\int_{-h}^{h}z\,\mathrm{P}(z)\,dz=0$, and $\mathrm{Z}(z)$ any even function of z, with $\int_{-h}^{h}\mathrm{Z}(z)dz=0$, values of C_κ exist such that simultaneously

$$\left.\begin{aligned}&\operatorname*{Lt}_{x=0}\sum_\kappa \mathrm{C}_\kappa ie^{i\kappa x}(\overline{3-\cosh 2\kappa h}\sinh\kappa z+2\kappa z\cosh\kappa z)=\mathrm{P}(z)\\ &\operatorname*{Lt}_{x=0}\sum_\kappa \mathrm{C}_\kappa e^{i\kappa x}(\overline{1-\cosh 2\kappa h}\cosh\kappa z+2\kappa z\sinh\kappa z)=\mathrm{Z}(z)\end{aligned}\right\}\quad\ldots\quad\text{(ii)}$$

The limit for $x=0$ may be taken term by term, provided the resulting series converge. In the following analysis we shall assume that they do so, but this is merely in order to avoid lengthy forms of statement; the argument could be put, if necessary, in a form independent of this assumption.

41. *The problem of given edge tractions for a thin plate.*

The *form* of the complete solution is exactly known, and the three boundary conditions in their exact forms could, therefore, at once be written down. The whole strain is compounded of an infinite number of modes of equilibrium of known types, and it is obviously suggested as the method of attack that we should try to disentangle from the general boundary conditions those special conditions by which each mode is separately defined. When the plate is thin we find that within certain limits this can be done, and, in particular, the conditions defining the permanent modes, which in the case supposed are incomparably the most important, can be found with considerable exactness.

We shall understand that the edge traction, or any component of the edge traction, is given as a function of $x, y, z/h$ or of $s, z/h$, where s is the arc of the edge line, so that if ζ be put for z/h the form of this function is completely independent of h. The theory may be applied to cases in which the proviso is not fulfilled, but before such application the given traction is to be separated into parts of ascending order in h, say, for example, $f_0(x, y, \zeta) + h\ f_1(x, y, \zeta) + h^2\ f_2(x, y, \zeta) + \text{etc.}$; then for a first approximation we deal only with $f_0(x, y, \zeta)$. The theory does *not* contemplate such a distribution of traction, as, for example, $\sin(ms/h)$, m being a number, where the rate at which the traction varies along the arc is of a lower order in h than the traction itself.

The trace of the cylindrical edge on the middle plane of the plate is the *edge line*; the outward normal, and the tangent, to the edge line will be referred to as the *normal*, and *tangent* simply; the generator of the cylindrical edge at right angles to these at their point of intersection may be called the *perpendicular*.

Let l, m be the direction cosines of the normal,
then $-m$, l are those of the tangent.

The normal displacement is $p = lu + mv$
and the tangential displacement $q = -mu + lv$.

The tractions on the edge in the directions of normal, tangent, and perpendicular, are $\widehat{nn}, \widehat{ns}, \widehat{nz}$ or $\mathrm{N}, \mathrm{S}, \mathrm{Z}$.

42. *Extensional strain.*

In this case N, S are even functions, and Z an odd function of z.

It will be advantageous to express as far as possible the displacements and tractions at an edge in the various types of solution in terms of derivatives along the tangent and normal.

Alongside the symbol α we shall use the more familiar σ, the relation between the two being given by $\alpha + 1 = 4(1 - \sigma)$; $3 - \alpha = 4\sigma$.

(i) *Permanent mode.*

$$\left.\begin{aligned} u &= \mathrm{U} + \frac{\sigma}{1-\sigma}\frac{z^2}{2}\frac{d}{dx}\left(\frac{d\mathrm{U}}{dx}+\frac{d\mathrm{V}}{dy}\right) \\ v &= \mathrm{V} + \frac{\sigma}{1-\sigma}\frac{z^2}{2}\frac{d}{dy}\left(\frac{d\mathrm{U}}{dx}+\frac{d\mathrm{V}}{dy}\right) \\ w &= -\frac{\sigma}{1-\sigma}z\left(\frac{d\mathrm{U}}{dx}+\frac{d\mathrm{V}}{dy}\right) \end{aligned}\right\}$$

where

$$\left.\begin{aligned} 2\frac{d}{dx}\left(\frac{d\mathrm{U}}{dx}+\frac{d\mathrm{V}}{dy}\right)+(1-\sigma)\frac{d}{dy}\left(\frac{d\mathrm{U}}{dy}-\frac{d\mathrm{V}}{dx}\right)=0 \\ 2\frac{d}{dy}\left(\frac{d\mathrm{U}}{dx}+\frac{d\mathrm{V}}{dy}\right)-(1-\sigma)\frac{d}{dx}\left(\frac{d\mathrm{U}}{dy}-\frac{d\mathrm{V}}{dx}\right)=0 \end{aligned}\right|$$

Put

$$\left.\begin{aligned} \Delta &= \frac{1}{1-\sigma}\left(\frac{d\mathrm{U}}{dx}+\frac{d\mathrm{V}}{dy}\right)=\frac{1}{1-\sigma}\left(\frac{du}{dx}+\frac{dv}{dy}\right) \\ \Pi &= \tfrac{1}{2}\left(\frac{d\mathrm{U}}{dy}-\frac{d\mathrm{V}}{dx}\right) \quad =\tfrac{1}{2}\left(\frac{du}{dy}-\frac{dv}{dx}\right) \end{aligned}\right|$$

Then

$$\left.\begin{aligned} \frac{d\Delta}{dx}+\frac{d\Pi}{dy}=0 \\ \frac{d\Delta}{dy}-\frac{d\Pi}{dx}=0 \end{aligned}\right| \text{ and at an edge } \left.\begin{aligned} \frac{d\Delta}{dn}+\frac{d\Pi}{ds}=0 \\ \frac{d\Delta}{ds}-\frac{d\Pi}{dn}=0 \end{aligned}\right|$$

Also

$$\left.\begin{aligned} \widehat{xx} &= \frac{2\mu}{1-\sigma}\left(\frac{du}{dx}+\sigma\frac{dv}{dy}\right) \\ \widehat{yy} &= \frac{2\mu}{1-\sigma}\left(\sigma\frac{du}{dx}+\frac{dv}{dy}\right) \\ \widehat{xy} &= \frac{2\mu}{1-\sigma}\cdot\frac{1-\sigma}{2}\left(\frac{du}{dy}+\frac{dv}{dx}\right) \end{aligned}\right\}$$

The components of traction parallel to the axes are

$$\mathrm{X} = l\widehat{xx} + m\widehat{xy}\,;\ \mathrm{Y} = l\widehat{xy} + m\widehat{yy}\,.$$

These are easily transformed into

$$\mathrm{X} = 2\mu\left(l\Delta + m\Pi - \frac{dv}{ds}\right);\ \mathrm{Y} = 2\mu\left(-l\Pi + m\Delta + \frac{du}{ds}\right).$$

Hence

$$\left.\begin{aligned} \widehat{nn} &= l\mathrm{X} + m\mathrm{Y} = 2\mu\left(\Delta + m\frac{du}{ds} - l\frac{dv}{ds}\right) \\ \widehat{ns} &= -m\mathrm{X} + l\mathrm{Y} = 2\mu\left(-\Pi + l\frac{du}{ds} + m\frac{dv}{ds}\right) \end{aligned}\right\}$$

also $\widehat{nz} = 0$

or

$$\left.\begin{aligned} \widehat{nn} &= 2\mu\left(\Delta - \frac{dq}{ds} - \frac{p}{\rho}\right) \\ \widehat{ns} &= 2\mu\left(-\Pi + \frac{dp}{ds} - \frac{q}{\rho}\right) \end{aligned}\right|$$

If P, Q be the values of p, q when $z=0$, then

$$p=\mathrm{P}-\tfrac{1}{2}\sigma z^2\frac{d\Pi}{ds}\,;\quad q=\mathrm{Q}+\tfrac{1}{2}\sigma z^2\frac{d\Delta}{ds}$$

$$\left.\begin{aligned}\frac{\widehat{nn}}{2\mu}&=\Delta-\frac{d\mathrm{Q}}{ds}-\frac{\mathrm{P}}{\rho}-\frac{1}{2}\sigma z^2\left(\frac{d^2\Delta}{ds^2}-\frac{1}{\rho}\frac{d\Pi}{ds}\right)\\ \frac{\widehat{ns}}{2\mu}&=-\Pi+\frac{d\mathrm{P}}{ds}-\frac{\mathrm{Q}}{\rho}-\frac{1}{2}\sigma z^2\left(\frac{d^2\Pi}{ds^2}+\frac{1}{\rho}\frac{d\Delta}{ds}\right)\end{aligned}\right\}$$

(ii) *Rotational Transitory Modes.*

$$\left.\begin{aligned}u&=2\frac{d\psi}{dy}\\ v&=-2\frac{d\psi}{dx}\end{aligned}\right\}\qquad\left.\begin{aligned}\frac{\widehat{xx}}{2\mu}&=2\frac{d^2\psi}{dx\,dy}\\ \frac{\widehat{xy}}{2\mu}&=\frac{d^2\psi}{dy^2}-\frac{d^2\psi}{dx^2}\\ \frac{\widehat{xz}}{2\mu}&=\frac{d^2\psi}{dy\,dz}\end{aligned}\right\}$$

Taking the axes of x and y for a moment along the normal and tangent, these give at once by means of (k)

$$\left.\begin{aligned}p&=2\frac{d\psi}{ds}\\ q&=-2\frac{d\psi}{dn}\end{aligned}\right\}\qquad\left.\begin{aligned}\frac{\widehat{nn}}{2\mu}&=2\frac{d}{ds}\frac{d\psi}{dn}-\frac{2}{\rho}\frac{d\psi}{ds}\\ \frac{\widehat{ns}}{2\mu}&=\frac{d^2\psi}{dz^2}+\frac{2}{\rho}\frac{d\psi}{dn}+2\frac{d^2\psi}{ds^2}\\ \frac{\widehat{nz}}{2\mu}&=\frac{d^2\psi}{ds\,dz}\end{aligned}\right\}$$

The function ψ can be expressed as a series of terms of the form

$$\psi_n(x,y)\cos\frac{n\pi z}{h},\quad\text{where }\nabla^2\psi_n-\frac{n^2\pi^2}{h^2}\psi_n=0\,.$$

Hence in cases where the values of ψ along an edge are given independently of h, or generally, when the rate of variation of ψ along an edge is of the same order in h as ψ itself, say order zero, terms of various orders occur in the expressions for the displacements and tractions.
Thus

$$\left.\begin{aligned}\frac{d\psi}{ds},\ \frac{d^2\psi}{ds^2}&\quad\text{are of order}\quad 0\\ \frac{d\psi}{dn},\ \frac{d}{ds}\frac{d\psi}{dn},\ \frac{d}{ds}\frac{d\psi}{dz}&\quad\ldots\ldots\quad -1\\ \frac{d^2\psi}{dz^2}&\quad\text{is}\quad\ldots\ldots\quad -2\end{aligned}\right\}$$

It follows that in such cases this type of strain contributes mainly to the *tangential* displacement and traction at an edge.

We see also that the principal part of the displacement is of one order higher in h than the principal part of the traction.

(iii) *Dilatational transitory modes.*

There would be some advantage in working with the functional symbols θ, ϕ, as with ψ in the last case, but on the whole it seems clearer to deal with a typical solution corresponding to a single root κ of $\sinh 2\kappa h + 2\kappa h$.

$$\phi = \cosh \kappa z f(x, y)\,; \quad \theta = \cosh 2\kappa h \cdot \phi$$

$$\left.\begin{aligned} u &= \frac{df}{dx}\Big\{ (\cosh 2\kappa h + \alpha)\cosh \kappa z + 2\kappa z \sinh \kappa z \Big\} \\ v &= \frac{df}{dy}\Big\{ \\ w &= \kappa f \{ (\cosh 2\kappa h - \alpha)\sinh \kappa z + 2\kappa z \cosh \kappa z \} \end{aligned}\right]$$

$$\left.\begin{aligned} \frac{xx}{2\mu} &= -\kappa^2 f \Big\{ (\cosh 2\kappa h + 3)\cosh \kappa z + 2\kappa z \sinh \kappa z \Big\} \\ &\quad - \frac{d^2 f}{dy^2}\Big\{ (\cosh 2\kappa h + \alpha)\cosh \kappa z + 2\kappa z \sinh \kappa z \Big\} \\ \frac{\widehat{xy}}{2\mu} &= \frac{d^2 f}{dx\,dy}\Big\{ (\cosh 2\kappa h + \alpha)\cosh \kappa z + 2\kappa z \sinh \kappa z \Big\} \\ \frac{xz}{2\mu} &= \kappa \frac{df}{dx}\Big\{ (1 + \cosh 2\kappa h)\sinh \kappa z + 2\kappa z \cosh \kappa z \Big\} \end{aligned}\right]$$

Hence

$$\left.\begin{aligned} \frac{\widehat{nn}}{2\mu} &= -\kappa^2 f \Big\{ (\cosh 2\kappa h + 3)\cosh \kappa z + 2\kappa z \sinh \kappa z \Big\} \\ &\quad - \Big(\frac{1}{\rho}\frac{df}{dn} + \frac{d^2 f}{ds^2} \Big) \Big\{ (\cosh 2\kappa h + \alpha)\cosh \kappa z + 2\kappa z \sinh \kappa z \Big\} \\ \frac{\widehat{ns}}{2\mu} &= \Big(\frac{d}{ds}\frac{df}{dn} - \frac{1}{\rho}\frac{df}{ds} \Big) \Big\{ (\cosh 2\kappa h + \alpha)\cosh \kappa z + 2\kappa z \sinh \kappa z \Big\} \\ \frac{\widehat{nz}}{2\mu} &= \kappa \frac{df}{dn}\Big\{ (1 + \cosh 2\kappa h)\sinh \kappa z + 2\kappa z \cosh \kappa z \Big\} \end{aligned}\right]$$

Thus at an edge where the rate of variation of f is of the same order in h as f itself, say order zero, the *normal* and *perpendicular* displacements are of order -1, while the tangential displacement is of an order one higher; the *normal* and *perpendicular* tractions are of order -2, the tangential traction being of order -1, or again one higher.

Hence this type of strain contributes most to those components of displacement and traction to which the ψ type contributes least, at an edge.

(iv) It is now possible to assign approximately to each of the three types of strain the portion which it carries of any given distribution of edge traction. Let this distribution be N, S, Z, functions of z, s, of order zero in h. We can satisfy the conditions to the first order by a solution in which the *principal* part of the traction

is of order zero for each type of strain. For, taking account only of these principal parts, the equations to be satisfied at the edge are on this supposition

$$\left.\begin{aligned} \mathrm{N} &= \mathrm{N}_p + \mathrm{N}_d \\ \mathrm{S} &= \mathrm{S}_p + \mathrm{S}_r \\ \mathrm{Z} &= \mathrm{Z}_d \end{aligned}\right\}$$

the suffixes referring to the permanent, rotational, and dilatational types respectively. Now

$$\mathrm{S}_r = 2\mu \frac{d^2\psi}{dz^2} \quad \text{and} \quad \int_{-h}^{h} \mathrm{S}_r dz = 2\mu \left| \frac{d\psi}{dz} \right|_{-h}^{h} = 0 .$$

Also

$$\mathrm{N}_d = -2\mu \sum_{\kappa} \kappa^2 f_\kappa \left\{ (\cosh 2\kappa h + 3) \cosh \kappa z + 2\kappa z \sinh \kappa z) \right\},$$

and, as in art. 40,

$$\int_{-h}^{h} \mathrm{N}_d \, dz = 0 .$$

Hence the above equations give

$$\left.\begin{aligned} \mathrm{N}_p &= \frac{1}{2h} \int_{-h}^{h} \mathrm{N} dz \\ \mathrm{S}_p &= \frac{1}{2h} \int_{-h}^{h} \mathrm{S} dz \end{aligned}\right\}$$

and these conditions determine the permanent mode.

ψ can now be found from the boundary condition

$$2\mu \frac{d^2\psi}{dz^2} = \mathrm{S} - \frac{1}{2h} \int_{-h}^{h} \mathrm{S} dz .$$

For, taking

$$\psi = \sum_{n=1} \psi_n (x, y) \cos \frac{n\pi z}{h},$$

the condition is

$$\sum_{n=1} \psi_n \cdot n^2 \cos \frac{n\pi z}{h} = -\frac{h^2}{2\mu\pi^2} \left\{ \mathrm{S} - \frac{1}{2h} \int_{-h}^{h} \mathrm{S} dz \right\} .$$

Now the right-hand member here is a function of s, z, even in z, the z-integral of which from $-h$ to h is zero for all values of s. It can therefore be expanded by Fourier's Theorem in the form, valid from $z = -h$ to $z = h$,

$$\sum_{n=1} \mathrm{A}_n \cos \frac{n\pi z}{h} .$$

ψ_n is then determined as satisfying $\nabla^2 \psi_n + \frac{n^2\pi^2}{h^2} \psi_n = 0$ throughout the plate, and taking the value A_n / n^2 at the boundary.

Lastly, the equations to determine the dilatational mode are (since at the edge $\frac{df}{dn} = -i\kappa f$ to the first order),

$$\left.\begin{aligned} \sum_{\kappa} f_\kappa \cdot \kappa^2 \left\{ (\cosh 2\kappa h + 3) \cosh \kappa z + 2\kappa z \sinh \kappa z \right\} &= -\frac{1}{2\mu} \left\{ \mathrm{N} - \frac{1}{2h} \int_{-h}^{h} \mathrm{N} dz \right\} \\ \sum_{\kappa} f_\kappa \cdot i\kappa^2 \left\{ (1 + \cosh 2\kappa h) \sinh \kappa z + 2\kappa z \cosh \kappa z \right\} &= -\frac{1}{2\mu} \mathrm{Z} \end{aligned}\right]$$

By the same method as in the case of ψ, but using the theorem (i) of § 40, instead of Fourier's Theorem, we see that functions $f_\kappa(x, y)$ exist, solutions of $(\nabla^2+\kappa^2)f=0$ and satisfying the above boundary equations.

Thus the apportionment proposed for the edge tractions does actually satisfy the conditions to a first approximation. The solution found gives tractions of which the principal parts are the tractions actually assigned in the problem. The residual traction given by the solution is of the first and higher orders; and a second approximation to the problem will be obtained by subtracting a solution giving the residual tractions of the first order, such solution being found by the method used in the first approximation.

This process would be tedious, and the way would be blocked at an early stage by our ignorance of the coefficients of the expansion (i) of § 40.

We therefore pass at once to the consideration of the powerful method furnished by Betti's Theorem for the determination of the permanent mode.

43. *Extensional strain. The Green's Function method for the permanent mode.*

The method has already been explained (§ 36). If we wish the permanent displacement at (x', y', z') in any direction (say the displacement u), we take the permanent part of the solution for a unit force in that direction (a unit X force), modify it by removing the terms which convey resultant stress, and then try to *balance* it at the edge by adding a solution, without internal singularity, which shall neutralise its edge tractions.

The displacements at the edge in the balanced solution, *i.e.* in the solution obtained as the sum of the source and balancing solutions, being u', v', w', or p', q', w', and the given tractions X, Y, Z, or N, S, Z, we have

$$u(x', y', z') = \iint(\mathrm{X}u' + \mathrm{Y}v' + \mathrm{Z}w')ds\,dz$$
$$= \iint(\mathrm{N}p' + \mathrm{S}q' + \mathrm{Z}w')ds\,dz$$

the integral being taken over the cylindrical boundary.

The thickness $2h$ being supposed infinitesimal, the object of the method is to determine a few of the terms of p', q', w' of lowest order in h.

An alternative method would be to determine the functions $\mathrm{E}_1, \mathrm{E}_2$ of § 33, I. (i), in terms of edge tractions; but the least confusing method of all is perhaps to determine

$$\mathrm{U}' \equiv \frac{d\mathrm{E}_1}{dy'} + \frac{d\mathrm{E}_2}{dx'} \text{ and } \mathrm{V}' \equiv -\frac{d\mathrm{E}_1}{dx'} + \frac{d\mathrm{E}_2}{dx'}.$$

These do not contain z', but when they are known the complete solution can obviously be written down. We begin with U′, and in fact it will not be necessary to determine V′ separately.

Now U′ is the work difference from the system

$$\left.\begin{aligned}\psi &= \frac{1}{8\pi\mu h}\frac{d}{dy'}(\chi - \tfrac{1}{2}z^2\nabla^2\chi)\\ \theta = \phi &= \frac{1}{32\pi\mu h}\frac{d}{dx'}(\chi - \tfrac{1}{2}z^2\nabla^2\chi)\end{aligned}\right\}.$$

From this system let those terms be removed which transmit the resultant stress (equivalent to a unit X force through x', y', z'). Then the remaining displacements have still the forms discussed in § 42 (i), and we shall use for them, and for the various quantities related to them, the notation given there, modified by the addition of a suffix 0 in each case.

The problem is now to balance the edge tractions due to the system u_0, v_0, w_0.

The principal parts of these tractions, which in this case are simply the terms independent of z, are balanced by a solution of the permanent type (which we shall distinguish by the suffix 1) such that at the edge

$$\left.\begin{aligned}\left(\Delta_0 - \frac{dQ_0}{ds} - \frac{P_0}{\rho}\right) + \left(\Delta_1 - \frac{dQ_1}{ds} - \frac{P_1}{\rho}\right) &= 0\\ \left(-\Pi_0 + \frac{dP_0}{ds} - \frac{Q_0}{\rho}\right) + \left(-\Pi_1 + \frac{dP_1}{ds} - \frac{Q_1}{\rho}\right) &= 0\end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

These conditions define the solution with suffix 1.

The residual tractions from the compound solution u_0+u_1, v_0+v_1, w_0+w_1 contain the factor z^2 and are of order h^2 as compared with those already balanced. To balance these residual tractions, solutions of all types are required, but, as in § 42 (iv), the permanent solution (which will be marked with suffix 2) is determined from the integral residual tractions; the permanent displacements are of order h, while those from the transitory solutions are of order h^2, those from the source solution being of order h^{-1}.

The displacements of the balanced solution are therefore to terms of order h inclusive in the notation of § 42 (i),

$$\left.\begin{aligned}p' &= P_0 + P_1 && + P_2 - \tfrac{1}{2}\sigma z^2\frac{d}{ds}(\Pi_0+\Pi_1)\\ q' &= Q_0 + Q_1 && + Q_2 + \tfrac{1}{2}\sigma z^2\frac{d}{ds}(\Delta_0+\Delta_1)\\ w' &= && -\sigma z(\Delta_0+\Delta_1)\end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

and with these values of p', q', w'

$$U'(x', y', z') = \iint (Np' + Sq' + Zw')\,ds\,dz \quad . \quad . \quad . \quad . \quad (3)$$

All the steps of the above process can actually be carried out in the case of a circular plate, and the final formula gives a perfectly definite solution provided merely that N, S, Z are functions integrable over the edge. It should be specially noted that, in this form of the solution, discontinuity of the applied traction gives rise to no difficulty whatever.

On the other hand, the formula does not give a ready answer to such an important question as "What are the relations between the tractions actually applied, and the

tractions required to maintain the permanent solution alone?" or the practically equivalent question, "What conditions must the edge stress satisfy in order that the permanent mode may be absent from the resulting strain?"

The expression found for U′ may be transformed so as to supply answers to these questions.

In the first place, we note that the values of p', q', w' will still be correct to the order stated (but will contain superfluous terms) if in the expressions just given for them we write

$$\Pi_0 + \Pi_1 + \Pi_2 \quad \text{for} \quad \Pi_0 + \Pi_1$$

and

$$\Delta_0 + \Delta_1 + \Delta_2 \quad \text{for} \quad \Delta_0 + \Delta_1 .$$

Write also

$$\Pi \quad \text{for} \quad \Pi_0 + \Pi_1 + \Pi_2, \quad \Delta \quad \text{for} \quad \Delta_0 + \Delta_1 + \Delta_2 ,$$

and similarly for the other quantities.

Hence

$$\left.\begin{aligned} p' &= \mathrm{P} - \tfrac{1}{2}\sigma z^2 \frac{d\Pi}{ds} \\ q' &= \mathrm{Q} + \tfrac{1}{2}\sigma z^2 \frac{d\Delta}{ds} \\ w' &= -\sigma z \Delta \end{aligned}\right\} \text{the error in each case being of order } h^2$$

and

$$\mathrm{U}'(x', y', z') = \iint \left\{ \mathrm{N}\left(\mathrm{P} - \tfrac{1}{2}\sigma z^2 \frac{d\Pi}{ds}\right) + \mathrm{S}\left(\mathrm{Q} + \tfrac{1}{2}\sigma z^2 \frac{d\Delta}{ds}\right) - \mathrm{Z}\sigma z \Delta \right\} ds\, dz \quad . \quad . \quad (4)$$

Also from (1) $\Delta - \dfrac{d\mathrm{Q}}{ds} - \dfrac{\mathrm{P}}{\rho}$, $-\Pi + \dfrac{d\mathrm{P}}{ds} - \dfrac{\mathrm{Q}}{\rho}$ are of order h, and the formula for U′ will therefore be correct to the same order as before if we substitute $\dfrac{d\mathrm{Q}}{ds} + \dfrac{\mathrm{P}}{\rho}$ for Δ, and $\dfrac{d\mathrm{P}}{ds} - \dfrac{\mathrm{Q}}{\rho}$ for Π. We shall also write

$$\int_{-h}^{h} \mathrm{N}\,dz = \mathrm{N}_0, \quad \int_{-h}^{h} z^2 \mathrm{N}\,dz = \mathrm{N}_2$$

$$\int_{-h}^{h} \mathrm{S}\,dz = \mathrm{S}_0, \quad \int_{-h}^{h} z^2 \mathrm{S}\,dz = \mathrm{S}_2$$

$$\int_{-h}^{h} z\mathrm{Z}\,dz = \mathrm{Z}_1 .$$

Thus

$$\mathrm{U}'(x', y', z') = \int ds \left\{ \mathrm{N}_0\mathrm{P} - \tfrac{1}{2}\sigma \mathrm{N}_2 \frac{d}{ds}\left(\frac{d\mathrm{P}}{ds} - \frac{\mathrm{Q}}{\rho}\right) + \mathrm{S}_0\mathrm{Q} + \tfrac{1}{2}\sigma \mathrm{S}_2 \frac{d}{ds}\left(\frac{d\mathrm{Q}}{ds} + \frac{\mathrm{P}}{\rho}\right) - \sigma \mathrm{Z}_1 \left(\frac{d\mathrm{Q}}{ds} + \frac{\mathrm{P}}{\rho}\right) \right\} \quad . \quad (5)$$

If $\mathrm{N}_2, \mathrm{S}_2, \mathrm{Z}_1, \dfrac{d\mathrm{N}_2}{ds}, \dfrac{d\mathrm{S}_2}{ds}$ *are continuous over each edge line*, integration by parts transforms this into

$$\int ds \left\{ \mathrm{P}\left(\mathrm{N}_0 - \frac{\sigma}{\rho}\mathrm{Z}_1 - \tfrac{1}{2}\sigma \frac{d^2\mathrm{N}_2}{ds^2} - \tfrac{1}{2}\frac{\sigma}{\rho}\frac{d\mathrm{S}_2}{ds}\right) + \mathrm{Q}\left(\mathrm{S}_0 + \sigma \frac{d\mathrm{Z}_1}{ds} - \tfrac{1}{2}\frac{\sigma}{\rho}\frac{d\mathrm{N}_2}{ds} + \tfrac{1}{2}\sigma \frac{d^2\mathrm{S}_2}{ds^2}\right) \right\} \quad . \quad . \quad (6)$$

Hence, in order that the permanent mode should be absent from the strain due to N, S, Z the following two conditions are *sufficient* :—

$$\left.\begin{aligned} N_0 - \frac{\sigma}{\rho} Z_1 - \tfrac{1}{2}\sigma \frac{d^2 N_2}{ds^2} - \tfrac{1}{2}\frac{\sigma}{\rho}\frac{dS_2}{ds} = 0 \\ S_0 + \sigma \frac{dZ_1}{ds} \quad \tfrac{1}{2}\frac{\sigma}{\rho}\frac{dN_2}{ds} + \tfrac{1}{2}\sigma \frac{d^2 S_2}{ds^2} = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

at every point of the edge line or lines.

Further, all systems of traction for which the left-hand members of (7) have given values at every point of an edge will produce the same permanent mode. *Now as one such system of traction we may take the traction due to, or producing, the permanent mode by itself*, as given in § 42 (i). This gives at once the boundary conditions satisfied by the functions U, V of that section, and these boundary conditions, with the internal equations

$$\frac{d\Delta}{dx} + \frac{d\Pi}{dy} = 0 \, ; \; \frac{d\Delta}{dx} - \frac{d\Pi}{dy} = 0 \, ,$$

completely define U, V which are thus determined, to a third approximation in general. The defining differential equations and surface conditions being practically of the same form as in the familiar first approximation, we need not detail the proof that U, V are actually determinate from the conditions, but pass at once to the important conclusion, an immediate consequence of this determinateness, that the permanent strain will *not* be absent unless (7) are satisfied, or, in other words, that these conditions are *necessary*, as well as *sufficient*. From this again it follows that these conditions are fulfilled, to the order stated, by each of the transitory modes ; and this remark is valuable, because, once it has been verified by direct integration, it obviously leads, by an extension of the process of § 42 (iv), to a completely independent method of dealing with the whole problem. The method is noticeable for its simplicity and directness, but a somewhat serious defect is the difficulty of adapting it to the case when the edge stress is discontinuous.

This leads us to consider the correction that must be applied to the integral (6) when the conditions of continuity stated in connection with it are not fulfilled. It will be sufficient to take a case in which breach of continuity occurs at only one point E of the edge line.

We have defined the positive direction of an edge line in (*k*) ; let the excess of the value of $f(s)$ just on the negative side of E over its value just on the positive side be denoted by $[f(s)]$. Then if $\phi(s)$ be continuous

$$\int f(s) \frac{d}{ds}\phi(s)ds = \phi(s)[f(s)] - \int \phi(s) \frac{d}{ds} f(s) ds \, ,$$

the integrals, we may suppose, being taken round the edge line from the positive to the negative side of E, and the value of $\phi(s)$ in the integrated term being taken as at E.

Then to (6) we have in general to add

$$\left.\begin{aligned} &-\tfrac{1}{2}\sigma\left(\frac{d\mathrm{P}}{ds}-\frac{\mathrm{Q}}{\rho}\right)[\mathrm{N}_2]+\tfrac{1}{2}\sigma\left(\frac{d\mathrm{Q}}{ds}+\frac{\mathrm{P}}{\rho}\right)[\mathrm{S}_2]-\sigma\mathrm{Q}[\mathrm{Z}_1] \\ &+\tfrac{1}{2}\sigma\mathrm{P}\left[\frac{d\mathrm{N}_2}{ds}\right]-\tfrac{1}{2}\sigma\mathrm{Q}\left[\frac{d\mathrm{S}_2}{ds}\right] \end{aligned}\right\}$$

or, otherwise arranged,

$$\left.\begin{aligned} &\mathrm{P}\left\{\tfrac{1}{2}\frac{\sigma}{\rho}[\mathrm{S}_2]+\tfrac{1}{2}\sigma\left[\frac{d\mathrm{N}_2}{ds}\right]\right\}+\mathrm{Q}\left\{-\sigma[\mathrm{Z}_1]+\tfrac{1}{2}\frac{\sigma}{\rho}[\mathrm{N}_2]-\tfrac{1}{2}\sigma\left[\frac{d\mathrm{S}_2}{ds}\right]\right\} \\ &-\frac{d\mathrm{P}}{ds}\tfrac{1}{2}\sigma[\mathrm{N}_2]+\frac{d\mathrm{Q}}{ds}\tfrac{1}{2}\sigma[\mathrm{S}_2] \end{aligned}\right\}$$

The various terms of this expression may be interpreted with the help of the conceptions of sources and doublets. Thus, to go back to (3), we see that the part of U′ arising from an element $\mathrm{N}_0 ds$ of normal traction at E has $(\mathrm{P}_0+\mathrm{P}_1)\mathrm{N}_0 ds$ for principal term. $\mathrm{P}_0+\mathrm{P}_1$ is therefore (principal term of the) value of U′ due to a unit element of normal force at E. (Since this unit element can only exist in any actual deformation as part of an equilibrating system, the phrase *due to* in the last sentence must be taken under reservation. The solution of which $\mathrm{P}_0+\mathrm{P}_1$ is the x-displacement at $(x', y', 0)$ is in fact maintained by a unit element of normal traction at E, acting along with a continuous system of force in equilibrium with this element, and distributed over the edge in a manner depending only on the statical value of the element, and not at all on the position of E. For any equilibrating combination of elements, the aggregate of these continuous systems will disappear.)

Now the first of the above integrated terms is equivalent to $(\mathrm{P}_0+\mathrm{P}_1)\tfrac{1}{2}\frac{\sigma}{\rho}[\mathrm{S}_2]$.

Hence the discontinuity in S_2 at E has the same effect at a distance from the edge as would have an element of normal traction distributed over the perpendicular at E so as to give a resultant $\tfrac{1}{2}\frac{\sigma}{\rho}[\mathrm{S}_2]$.

Again an element $-\mathrm{A}$ of normal traction at E, combined with an element A at E′, where $\mathrm{EE}'=ds$, will give

$$\mathrm{U}'=\mathrm{A}\frac{d}{ds}(\mathrm{P}_0+\mathrm{P}_1)ds$$

$$=\frac{d}{ds}(\mathrm{P}_0+\mathrm{P}_1), \text{ if we take } \mathrm{A}ds=1.$$

$\frac{d}{ds}(\mathrm{P}_0+\mathrm{P}_1)$ is therefore due to a unit doublet of normal force at E, and from the term $-\frac{d\mathrm{P}}{ds}\tfrac{1}{2}\sigma[\mathrm{N}_2]$ we conclude that the discontinuity in N_2 at E has the same interior effect as a doublet of normal force at E of strength $-\tfrac{1}{2}\sigma[\mathrm{N}_2]$.

The other terms may be interpreted similarly. It does not seem possible to account on physical grounds for any except the principal terms of the solution given above. The principal terms are of course the same as those deduced in the ordinary theory from the 'Principle of the elastic equivalence of statically equipollent systems of load.'

With reference to the equivalence of mere discontinuities to line elements and doublets one or two remarks may be made. Discontinuity in the applied force will not produce infinite displacement at a line where it takes place, but a line element of load, and, *a fortiori*, a doublet, will do so. The permanent mode may therefore contain infinities at the edge which do not exist in the exact solution. There is really no difficulty in this, since the permanent mode does not purport to represent the strain, even approximately, in the immediate vicinity of the edge. The point may be illustrated by the permanent part of the infinite solid solution for a single force. This becomes infinite on the perpendicular through the source in a totally different way from the exact solution. A good deal of discussion took place at one time over a similar point in the flexural solution. This will be referred to again, but the considerations we have adduced seem to remove the chief part of the difficulty.

44. *Flexural strain.*

In this case N, S are odd functions, and Z an even function of z.

(i) *Permanent mode.*

This mode is defined in terms of one function F of (x, y) satisfying $\nabla^4 \mathrm{F} = 0$, and may be referred to simply as an F strain.

$$\phi = -(z\mathrm{F} - \tfrac{1}{6}z^3\nabla^2\mathrm{F}); \quad \theta = z\mathrm{F} - \tfrac{1}{6}z^3\nabla^2\mathrm{F} - 2h^2z\nabla^2\mathrm{F}$$

$$\left.\begin{aligned} p &= \frac{d}{dn}\left\{ -4(1-\sigma)(z\mathrm{F} - \tfrac{1}{6}z^3\nabla^2\mathrm{F}) + 2(\tfrac{1}{3}z^3 - h^2z)\nabla^2\mathrm{F} \right\} \\ q &= \frac{d}{ds}\left\{ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \right\} \\ w &= \qquad 4(1-\sigma)(\mathrm{F} - \tfrac{1}{2}z^2\nabla^2\mathrm{F}) + 2(z^2 - h^2)\nabla^2\mathrm{F} \end{aligned}\right]$$

For shortness in writing out the stresses, we shall work with symbols ϑ_1, ϑ_2, ϑ_3, denoting operations of differentiation applied to F, and defined by the equations

$$\left.\begin{aligned} \vartheta_1 &= 4\nabla^2 + 4(\sigma-1)\left(\frac{1}{\rho}\frac{d}{dn} + \frac{d^2}{ds^2}\right) \\ \vartheta_2 &= 4(1-\sigma)\left(\frac{d}{ds}\frac{d}{dn} - \frac{1}{\rho}\frac{d}{ds}\right) \\ \vartheta_3 &= 4\frac{d}{dn}\nabla^2 \end{aligned}\right\}$$

Then

$$\left.\begin{aligned} \frac{\widehat{nn}}{2\mu} &= -z\vartheta_1\mathrm{F} + \left\{ \frac{1}{6}z^3 + \frac{1}{2}\frac{1}{1-\sigma}\left(\frac{1}{3}z^3 - h^2z\right) \right\}\vartheta_1\nabla^2\mathrm{F} \\ \frac{ns}{2\mu} &= -z\vartheta_2\mathrm{F} + \left\{ \frac{1}{6}z^3 + \frac{1}{2}\frac{1}{1-\sigma}\left(\frac{1}{3}z^3 - h^2z\right) \right\}\vartheta_2\nabla^2\mathrm{F} \\ \frac{\widehat{nz}}{\mu 2} &= \frac{1}{2}(z^2 - h^2)\vartheta_3\mathrm{F} \end{aligned}\right]$$

(ii) *Rotational transitory modes.*

These are as in § 42 (ii), except that now

$$\psi = \sum_{n=0} \psi_n(x, y) \sin \frac{2n+1}{2} \frac{\pi z}{h}$$

where $$\nabla^2 \psi_n - (2n+1)^2 \frac{\pi^2}{4h^2} \psi_n = 0 .$$

(iii) *Dilatational transitory modes.*

$$\phi = \sinh \kappa z \, f(x, y) \; ; \; \theta = -\cosh 2\kappa h \cdot \phi$$

As in § 42 (iii),

$$\left.\begin{aligned} p &= \frac{df}{dn}\Big\{ (\alpha - \cosh 2\kappa h) \sinh \kappa z + 2\kappa z \cosh \kappa z \Big\} \\ q &= \frac{df}{ds}\Big\{ \\ w &= \kappa f \{ -(\alpha + \cosh 2\kappa h) \cosh \kappa z + 2\kappa z \sinh \kappa z \} \end{aligned}\right|$$

$$\left.\begin{aligned} \frac{\widehat{nn}}{2\mu} &= -\kappa^2 f \{ (3 - \cosh 2\kappa h) \sinh \kappa z + 2\kappa z \cosh \kappa z \} \\ &\quad -\Big(\frac{1}{\rho} \frac{df}{dn} + \frac{d^2 f}{ds^2} \Big) \Big\{ (\alpha - \cosh 2\kappa h) \sinh \kappa z + 2\kappa z \cosh \kappa z \Big\} \\ \frac{\widehat{ns}}{2\mu} &= \Big(\frac{d}{ds} \frac{df}{dn} - \frac{1}{\rho} \frac{df}{ds} \Big) \Big\{ (\alpha - \cosh 2\kappa h) \sinh \kappa z + 2\kappa z \cosh \kappa z \Big\} \\ \frac{\widehat{nz}}{2\mu} &= \kappa \frac{df}{dn} \Big\{ (1 - \cosh 2\kappa h) \cosh \kappa z + 2\kappa z \sinh \kappa z \Big\} \end{aligned}\right|$$

We have $\dfrac{df}{dn} = -i\kappa f$ to the lowest order,

or $f = \dfrac{i}{\kappa} \dfrac{df}{dn}$.

Hence if we put

$$\left.\begin{aligned} \kappa \frac{df}{dn} &= g_\kappa \\ i\{(\cosh 2\kappa h - 3) \sinh \kappa z - 2\kappa z \cosh \kappa z\} &= \mathrm{N}(\kappa z) \\ (1 - \cosh 2\kappa h) \cosh \kappa z + 2\kappa z \sinh \kappa z &= \mathrm{Z}(\kappa z) \end{aligned}\right\}$$

the above strain gives

$$\left.\begin{aligned} \frac{\widehat{nn}}{2\mu} &= g_\kappa \mathrm{N}(\kappa z) \\ \widehat{ns} &= 0 \end{aligned}\right\} \quad \text{with an error of relative order } h ,$$

$$\frac{\widehat{nz}}{2\mu} = g_\kappa \mathrm{Z}(\kappa z) \quad \text{exactly,}$$

and we may note that $\int_{-h}^{h} z \mathrm{N}(\kappa z) = 0$ and $\int_{-h}^{h} \mathrm{Z}(\kappa z) dz = 0$. (§ 41.)

The same remarks as in the extensional case might be made here about the *complementary* character of the types (ii), (iii) in regard to their contributions to edge displacement or traction, when h is small.

(iv) If we follow the lines of the discussion of the extensional case, we have now to consider the approximate allocation of a given system of edge traction among the three types of strain.

The investigation is this time more complex, chiefly in consequence of the presence of the stress $\widehat{nz}$ in the permanent mode. Since, moreover, flexure is much more important physically than extension, we shall give a fairly detailed discussion in the next article, but in the meantime we may examine what could be done with a solution in which, as in § 42 (iv), the principal part of the edge traction is of order zero for each type of strain, and the parts of higher order are neglected.

Such a solution would give

$$\left.\begin{aligned} \mathrm{N} &= \mathrm{N}_p &&+\mathrm{N}_a \\ \mathrm{S} &= \mathrm{S}_p + \mathrm{S}_r \\ \mathrm{Z} &= &&\mathrm{Z}_a \end{aligned}\right\}$$

But we see at once that we do not in this way get a perfectly general distribution of N, S, Z, since the last equation gives $\int_{-h}^{h} \mathrm{Z}dz = 0$. A closer examination is therefore necessary, and it will perhaps conduce to lucidity if we consider separately the three cases of normal, tangential, and perpendicular traction.

45. *Flexural forces.*

(i) *Normal traction.*

N being of order zero, we can satisfy the conditions by taking $z\mathrm{F}$, $\frac{d^2\psi}{dz^2}$, and g_κ all of this order, but besides the terms of order zero in the stresses, it will be necessary to take account of the terms of $\widehat{nz}$ which come from F and ψ, albeit these are of an order one higher. Then

$$\left.\begin{aligned} \frac{\mathrm{N}}{2\mu} &= -z\vartheta_1\mathrm{F} &&+ \Sigma g_\kappa \mathrm{N}(\kappa z) && . \quad . \quad . \quad (1) \\ 0 &= -z\vartheta_2\mathrm{F} &&+ \frac{d^2\psi}{dz^2} && . \quad . \quad . \quad (2) \\ 0 &= \tfrac{1}{2}(z^2 - h^2)\vartheta_3\mathrm{F} &&+ \frac{d^2\psi}{ds\,dz} + \Sigma g_\kappa \mathrm{Z}(\kappa z) && . \quad . \quad . \quad (3) \end{aligned}\right\}$$

Assuming these provisionally, multiply (1) by z and integrate from $-h$ to h.

Thus
$$\vartheta_1\mathrm{F} = -\frac{3}{4\mu h^3}\int_{-h}^{h} z\mathrm{N}dz \quad . \quad . \quad . \quad (4)$$

and
$$\Sigma g_\kappa \mathrm{N}(\kappa z) = \frac{\mathrm{N}}{2\mu} - \frac{3z}{4\mu h^3}\int_{-h}^{h} z\mathrm{N}dz \quad . \quad . \quad (5)$$

From (2), since ψ is odd in z, and $\frac{d\psi}{dz} = 0$ for $z = \pm h$, we get

$$\psi = (\tfrac{1}{6}z^3 - \tfrac{1}{2}h^2 z)\vartheta_2\mathrm{F} \quad . \quad . \quad . \quad (6)$$

In (3) the terms are of different orders; thus, with the help of (6),

$$\vartheta_3\mathrm{F} + \frac{d}{ds}\vartheta_2\mathrm{F} = 0 \quad . \quad . \quad . \quad (7)$$

$$\Sigma g_\kappa \mathrm{Z}(\kappa z) = 0 \quad . \quad . \quad . \quad (8)$$

(4) and (7) define F, (6) then gives the edge value of ψ, and (5), (8) determine g_κ, the functions to be expanded obviously satisfying the conditions of § 40 (ii).

Further, with the values of F, ψ, g_κ so determined, the equations (1), (2), (3) are all satisfied, the traction $\widehat{nz}$ vanishes exactly, and the residual tractions $\widehat{nn}$ $\widehat{ns}$ are of order 1.

(ii) *Tangential traction.*

S being of order 0, zF and $\frac{d^2\psi}{dz^2}$ will again be of this order, but it will be seen that g_κ is of order 1. For, making these suppositions, we have

$$\left.\begin{aligned} 0 &= -z\vartheta_1 \mathrm{F} && \text{(1)} \\ \frac{\mathrm{S}}{2\mu} &= -z\vartheta_2 \mathrm{F} + \frac{d^2\psi}{dz^2} && \text{(2)} \\ 0 &= \tfrac{1}{2}(z^2-h^2)\vartheta_3 \mathrm{F} + \frac{d^2\psi}{ds\,dz} + \Sigma g_\kappa \mathrm{Z}(\kappa z) && \text{(3)} \end{aligned}\right\}$$

$$\text{From (1)}, \qquad \vartheta_1 \mathrm{F} = 0 \qquad \text{(4)}$$

$$\text{From (2)}, \qquad \frac{1}{2\mu}\int_{-h}^{z} \mathrm{S}\,dz = -\tfrac{1}{2}(z^2-h^2)\vartheta\, \mathrm{F} + \frac{d\psi}{dz} \qquad \text{(5)}$$

Differentiate this with respect to s, and subtract from (3). Thus

$$-\frac{1}{2\mu}\frac{d}{ds}\int_{-h}^{z} \mathrm{S}\,dz = \tfrac{1}{2}(z^2-h^2)\left(\vartheta_3 \mathrm{F} + \frac{d}{ds}\vartheta_2 \mathrm{F}\right) + \Sigma g_\kappa \mathrm{Z}(\kappa z) \qquad \text{(6)}$$

and, integrating the last from $-h$ to h,

$$\vartheta_3 \mathrm{F} + \frac{d}{ds}\vartheta_2 \mathrm{F} = -\frac{3}{4\mu h^3}\frac{d}{ds}\int_{-h}^{h} z\mathrm{S}\,dz \qquad \text{(7)}$$

(4) and (7) define F, and (5) integrated from 0 to z gives ψ. With the values so determined, and with (6) satisfied by g_κ (and this is possible in virtue of (7)), the equations (1), (2), (3) are all satisfied, the traction $\widehat{nz}$ vanishes exactly, and the residual tractions $\widehat{nn}$, $\widehat{ns}$ are of order 1.

By combining this with the preceding case, we see that the results do in fact give a first approximation to the solution, since the residual stresses N, S are each of an order higher than their original given values.

The additional equation required to define g_κ might be found by carrying out the process of (i) with the residual normal traction $4\mu\frac{d}{ds}\frac{d\psi}{dn}$. The analysis would be practically the same as will be given in connection with the next case.

(iii) *Perpendicular traction.*

Z being of order zero, we shall have to take zF and $\frac{d^2\psi}{dz^2}$ of order -1, g_κ of order 0. On this hypothesis, we shall write down the exact expression for $\widehat{nz}$, and the terms of order -1 and 0 in $\widehat{nn}$ and $\widehat{ns}$. We are then to have

$$\left.\begin{aligned} 0 &= -z\vartheta_1 \mathrm{F} && +2\frac{d}{ds}\frac{d\psi}{dn} + \Sigma g_\kappa \mathrm{N}(\kappa z) && \text{(1)} \\ 0 &= -z\vartheta_2 \mathrm{F} + \frac{d^2\psi}{dz^2} && +\frac{2}{\rho}\frac{d\psi}{dn} && \text{(2)} \\ \frac{\mathrm{Z}}{2\mu} &= \tfrac{1}{2}(z^2-h^2)\vartheta_3 \mathrm{F} + \frac{d^2\psi}{ds\,dz} + \Sigma g_\kappa \mathrm{Z}(\kappa z) && && \text{(3)} \end{aligned}\right\}$$

In the first two stresses the terms of order -1 are annulled by taking

$$\vartheta_1 F = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\psi = (\tfrac{1}{6}z^3 - \tfrac{1}{2}h^2 z)\vartheta_2 F \quad . \quad . \quad . \quad . \quad (5)$$

Then integration of (3) from $-h$ to h gives

$$(\vartheta_3 + \frac{d}{ds}\vartheta_2)F = -\frac{3}{4\mu h^3}\int_{-h}^{h} Z dz \quad . \quad . \quad . \quad . \quad (6)$$

and from this with (3),

$$\Sigma g_\kappa Z(\kappa z) = \frac{Z}{2\mu} + \frac{3}{8\mu h^3}(z^2 - h^2)\int_{-h}^{h} Z dz \quad . \quad . \quad . \quad (7)$$

Another equation is required before g_κ is defined, but in view of the results of (i), (ii) we can already infer that the values of F, ψ determined by (4), (6) and (5) are the correct principal values, since the residual stresses $\widehat{nn}$, $\widehat{ns}$ are of order zero. As for g_κ, we note that we cannot complete its definition by annulling the residual stress $\widehat{nn}$, for the condition $\int_{-h}^{h} 2\frac{d}{ds}\frac{d\psi}{dn} z dz = 0$ is not satisfied. In order to get the remaining equation for g_κ we must therefore solve by the method of (i) for the residual normal traction $4\mu\frac{d}{ds}\frac{d\psi}{dn}$, so as to get the equation corresponding to (i) (5). The matter might be left at this stage, but it may be interesting actually to carry out the process of balancing the residual parts of $\widehat{nn}$ and $\widehat{ns}$. By so doing we shall not only define g_κ, but also obtain a second approximation to F and ψ.

We have to introduce into the solution F′, ψ', g_κ' with zF′ and $\frac{d^2\psi'}{dz^2}$ of order 0, g_κ' of order 1. The equations to be satisfied are

$$\left.\begin{aligned} -2\frac{d}{ds}\frac{d\psi}{dn} - \Sigma g_\kappa N(\kappa z) &= -z\vartheta_1 F' \quad . \quad . \quad . \quad (8)\\ -\frac{2}{\rho}\frac{d\psi}{dn} &= -z\vartheta_2 F' + \frac{d^2\psi'}{dz^2} \quad . \quad . \quad . \quad (9)\\ 0 &= \tfrac{1}{2}(z^2 - h^2)\vartheta_3 F' + \frac{d^2\psi'}{ds\,dz} + \Sigma g_\kappa' Z(\kappa z) \quad . \quad . \quad (10) \end{aligned}\right\}$$

We must first find the principal value of $\frac{d\psi}{dn}$ in terms of defined quantities. Now from (5),

$$\psi = (\tfrac{1}{6}z^3 - \tfrac{1}{2}h^2 z)\vartheta_2 F, \text{ at the edge.}$$

Hence, by Fourier's Theorem,

$$\psi = -\frac{32}{\pi^4}h^3\vartheta_2 F\left(\sin\frac{\pi z}{2h} - \frac{1}{3^4}\sin\frac{3\pi z}{2h} + \frac{1}{5^4}\sin\frac{5\pi z}{2h} - \ldots\right)$$

Within the plate, therefore,

$$\psi = -\frac{32}{\pi^4}h^3\left(\psi_0\sin\frac{\pi z}{2h} - \frac{1}{3^4}\psi_1\sin\frac{3\pi z}{2h} + \frac{1}{5^4}\psi_2\sin\frac{5\pi z}{2h} - \ldots\right)$$

where

$$\nabla^2\psi_m - \frac{(2m+1)^2\pi^2}{4h^2}\psi_m = 0,$$

and ψ_m has the edge value $\vartheta_2 F$.

Thus

$$\frac{d\psi}{dn} = -\frac{32}{\pi^4}h^3\left(\frac{d\psi_0}{dn}\sin\frac{\pi z}{2h} - \frac{1}{3^4}\frac{d\psi_1}{dn}\sin\frac{3\pi z}{2h} + \ldots\right)$$

the principal value of which is

$$-\frac{16}{\pi^3}h^2\vartheta_2 F\left(\sin\frac{\pi z}{2h} - \frac{1}{3^3}\sin\frac{3\pi z}{2h} + \frac{1}{5^3}\sin\frac{5\pi z}{2h} - \ldots\right)$$

Let

$$\beta(z/h) \equiv \frac{16}{\pi^3}\left(\sin\frac{\pi z}{2h} - \frac{1}{3^3}\sin\frac{3\pi z}{2h} + \frac{1}{5^3}\sin\frac{5\pi z}{2h} - \ldots\right) \quad . \quad (11)$$

then

$$\frac{d\psi}{dn} = -h^2\beta(z/h)\vartheta_2 F \quad . \quad . \quad . \quad (12)$$

and

$$\left.\begin{aligned}\int_{-h}^{h} z\frac{d\psi}{dn}\,dz &= -\frac{128}{\pi^5}h^4\left(1 + \frac{1}{3^5} + \frac{1}{5^5} + \ldots\right)\vartheta_2 F \\ &= -\frac{128}{\pi^5}\gamma_5 h^4\vartheta_2 F \\ \gamma_5 &\equiv 1 + \frac{1}{3^5} + \frac{1}{5^5} + \ldots.\end{aligned}\right\} \quad . \quad (13)$$

if

Now multiply (8) by z, and integrate from $-h$ to h.

Hence

$$\vartheta_1 F' = -\frac{384}{\pi^5}\gamma_5 h\frac{d}{ds}\vartheta_2 F \quad . \quad . \quad . \quad (14)$$

and

$$\sum g_\kappa N(\kappa z) = \left\{2h^2\beta(z/h) - 384\gamma_5 hz/\pi^5\right\}\frac{d}{ds}\vartheta_2 F \quad . \quad (15)$$

This, and equation (7), define g_κ.

As we do not require g_κ', we will eliminate it from (10) at once by integrating.

Thus

$$0 = -2\frac{h^3}{3}\vartheta_3 F' + \frac{d}{ds}\left[\psi'\right]_{-h}^{h}.$$

Multiply (9) by z and integrate. Then

$$-\frac{2}{\rho}\int_{-h}^{h} z\frac{d\psi}{dn}\,dz = -\frac{2h^3}{3}\vartheta_2 F' - \left[\psi'\right]_{-h}^{h}$$

and from this with the last

$$\frac{d}{ds}\left\{\frac{2}{\rho}\int_{-h}^{h} z\frac{d\psi}{dn}\,dz\right\} = 2\frac{h^3}{3}\left(\vartheta_2 F' + \frac{d}{ds}\vartheta_2 F'\right)$$

or, from (13),

$$\vartheta_3 F' + \frac{d}{ds}\vartheta_2 F' = -\frac{384}{\pi^5}\gamma_5 h\frac{d}{ds}\left(\frac{1}{\rho}\vartheta_2 F\right) \quad . \quad . \quad (16)$$

(14) and (16) give F′, and ψ' may be found from (9).

If we write $\overline{\mathrm{F}}$ for $\mathrm{F}+\mathrm{F}'$, we get from (4), (14)

$$\begin{aligned}\vartheta_1\overline{\mathrm{F}} &= -\frac{384}{\pi^5}\gamma_5 h\frac{d}{ds}\vartheta_2\mathrm{F}\\ &= -\frac{384}{\pi^5}\gamma_5 h\frac{d}{ds}\vartheta_2\overline{\mathrm{F}}\end{aligned}$$

neglecting the term in $h\mathrm{F}'$, which is of order h^2 relative to $\overline{\mathrm{F}}$.

This may be written

$$\left(\vartheta_1+\frac{384}{\pi^5}\gamma_5 h\frac{d}{ds}\vartheta_2\right)\overline{\mathrm{F}}=0 \qquad . \qquad . \qquad (17)$$

Similarly from (6), (16) we obtain

$$\left(\vartheta_3+\frac{d}{ds}\vartheta_2+\frac{384}{\pi^5}\gamma_5 h\frac{d}{ds}\frac{1}{\rho}\vartheta_2\right)\overline{\mathrm{F}}=-\frac{3}{4h^3}\int_{-h}^{h}\mathrm{Z}dz \qquad . \qquad . \qquad . \qquad (18)$$

We may regard (17) and (18) as the equations giving F to a second approximation. If we combine the results of the three cases of this article, we obtain

$$\left.\begin{aligned}\vartheta_1\mathrm{F} &= -\frac{3}{4\mu h^3}\int_{-h}^{h}z\mathrm{N}dz\\ \vartheta_3\mathrm{F}+\frac{d}{ds}\vartheta_2\mathrm{F} &= -\frac{3}{4\mu h^3}\left\{\int_{-h}^{h}\mathrm{Z}dz+\frac{d}{ds}\int_{-h}^{h}z\mathrm{S}dz\right\}\end{aligned}\right]. \qquad . \qquad (19)$$

These are the equations usually referred to as Kirchhoff's boundary conditions. The extension of the more approximate conditions (17), (18) to the general case will be given in the next article.

46. *Flexural strain under given edge tractions. The Green's function method for the permanent mode.*

The displacement at (x', y', z') due to tractions N, S, Z is defined in § 33, II. (i), in terms of the work difference from the system

$$\begin{aligned}\phi &= -(3/32\pi\mu h^3)(z\chi-\tfrac{1}{6}z^3\nabla^2\chi)\\ \epsilon &= (3/32\pi\mu h^3)(z\chi-\tfrac{1}{6}z^3\nabla^2\chi-2h^2z\nabla^2\chi).\end{aligned}$$

From this system let the terms conveying resultant stress be removed; the residue is still an F strain with $\mathrm{F}=\mathrm{F}_0$ say, and F_0 is of order h^{-3}.

We have to balance F_0 at the edge, and the edge displacements in the balanced solution being p', q', w', the work difference required (F_1 of § 33) is

$$\iint(\mathrm{N}p'+\mathrm{S}q'+\mathrm{Z}w')ds\,dz\,.$$

The problem is to determine p', q', w' as closely as is practicable.

The tractions to be balanced are

$$\left.\begin{aligned}\mathrm{N}/2\mu &= -z\vartheta_1\mathrm{F}_0\\ \mathrm{S}/2\mu &= -z\vartheta_2\mathrm{F}_0\end{aligned}\right\}\text{with terms in } z^3$$
$$\mathrm{Z}/2\mu = \tfrac{1}{2}(z^2-h^2)\vartheta_3\mathrm{F}_0\,.$$

With tractions of these orders, of quite general form in z, the analysis of last article would lead us to expect that in the balancing solution F would be of order -3, ψ of order 0, and g_κ of order -2. But in consequence of N being simply proportional to z, it will be noticed that § 45 (i) (5) gives $\Sigma g_\kappa \mathrm{N}(\kappa z) = 0$, and it follows that in the present case g_κ will be of order -1 at lowest. The displacements p', w' derived from the strain defined by the functions g_κ will therefore be of order 0, and q' of order 1. Thus we may anticipate that the first *two* terms of p', and the first *three* terms of q' and w' will be obtained in practicable forms, *i.e.* independently of series associated with the zeroes of $\sinh 2\kappa h - 2\kappa h$.

The tractions written down above may be balanced approximately by strains F′, ψ'. We require

$$\left.\begin{aligned} -z\vartheta_1(\mathrm{F}_0+\mathrm{F}') &= 0 \\ -z\vartheta_2(\mathrm{F}_0+\mathrm{F}') + \frac{d^2\psi'}{dz^2} &= 0 \\ \tfrac{1}{2}(z^2-h^2)\vartheta_3(\mathrm{F}_0+\mathrm{F}') + \frac{d^2\psi'}{ds\,dz} &= 0 \end{aligned}\right\}.$$

These are equivalent to

$$\left.\begin{aligned} \vartheta_1(\mathrm{F}_0+\mathrm{F}') &= 0 \\ \psi' &= (\tfrac{1}{6}z^3 - \tfrac{1}{2}h^2 z)\vartheta_2(\mathrm{F}_0+\mathrm{F}') \\ (\vartheta_3 + \frac{d}{ds}\vartheta_2)(\mathrm{F}_0+\mathrm{F}') &= 0 \end{aligned}\right\} \quad . \quad (1)$$

which determine F′, ψ'.

The principal terms of the residual tractions are

$$\left.\begin{aligned} \frac{\widehat{nn}}{2\mu} &= 2\frac{d}{ds}\frac{d\psi'}{dn} = -2h^2\beta(z/h)\frac{d}{ds}\vartheta_2(\mathrm{F}_0+\mathrm{F}') \\ \frac{\widehat{ns}}{2\mu} &= \frac{2}{\rho}\frac{d\psi'}{du} = -\frac{2h^2}{\rho}\beta(z/h)\vartheta_2(\mathrm{F}_0+\mathrm{F}') \end{aligned}\right\} \quad . \quad (2)$$

as in (iii) (12) of last article, and they may be dealt with in the manner illustrated there.

To balance them take strains defined by F″, ψ'', g_κ'' with $z\mathrm{F}''$, $\frac{d^2\psi''}{dz^2}$, g_κ'' of order -1.

We must therefore have

$$\left.\begin{aligned} 2h^2\beta(z/h)\frac{d}{ds}\vartheta_2(\mathrm{F}_0+\mathrm{F}') &= -z\vartheta_1\mathrm{F}'' && +\Sigma g_\kappa''\mathrm{N}(\kappa z) \\ (2h^2/\rho)\beta(z/h)\vartheta_2(\mathrm{F}_0+\mathrm{F}') &= -z\vartheta_2\mathrm{F}'' + \frac{d^2\psi''}{dz^2} && \\ 0 &= \frac{1}{2}(z^2-h^2)\vartheta_3\mathrm{F}'' + \frac{d^2\psi''}{ds\,dz} && +\Sigma g_\kappa''\mathrm{Z}(\kappa z) \end{aligned}\right\} \quad . \quad (3)$$

From these, as in last article,

$$\left.\begin{aligned} \vartheta_1\mathrm{F}'' &= -\frac{384}{\pi^5}\gamma_5 h\frac{d}{ds}\vartheta_2(\mathrm{F}_0+\mathrm{F}') \\ (\vartheta_3+\frac{d}{ds}\vartheta_2)\mathrm{F}'' &= -\frac{384}{\pi^5}\gamma_5 h\frac{d}{ds}\left\{\frac{1}{\rho}\vartheta_2(\mathrm{F}_0+\mathrm{F}')\right\} \\ \psi'' &= \left(\frac{1}{6}z^3 - \frac{1}{2}h^2 z\right)\vartheta_2\mathrm{F}'' - \frac{128}{\pi^5\rho}h^4\vartheta_2(\mathrm{F}_0+\mathrm{F}')\left(\sin\frac{\pi z}{2h} - \frac{1}{3^5}\sin\frac{3\pi z}{2h} + \cdots\right) \end{aligned}\right\} \quad (4)$$

whence

$$\frac{d\psi''}{dn} = -h^2\beta(z/h)\vartheta_2 F'' + \frac{2}{\rho}\left(\frac{1}{6}z^3 - \frac{1}{2}h^2 z\right)\vartheta_2(F_0 + F') \quad . \quad . \quad (5)$$

We are now left with tractions $\widehat{nn}$, $\widehat{ns}$ of order zero. In the solution balancing these, F''' will be of order -1, ψ''' of order 2, g_κ''' of order 0. We do not think it worth while to write down the equations defining F''', but it should be noticed that they can be found explicitly. In fact, although the residual tractions $\widehat{nn}$, $\widehat{ns}$ with which we are now dealing are partly defined by κ series, the integrals $\int_{-h}^{h} z\,\widehat{nn}\,dz$ and $\int_{-h}^{h} z\,\widehat{ns}\,dz$ will be found to vanish to the order concerned so far as they come from these series, in virtue of the relation $\sum g_\kappa'' Z(\kappa z) = 0$, which follows from (iii) (7) of last article.

The functions g_κ'' give terms of order zero in p', w'.

Hence, including in p' terms of orders $-2, -1$

. . . . . q' . . . . . $-2, -1, 0$

. . . . . w' . . . . . $3, -2, -1$

we have

$$\left.\begin{aligned} p' &= -4(1-\sigma)z\frac{d}{dn}(F_0 + F' + F'') \\ q' &= -4(1-\sigma)z\frac{d}{ds}(F_0 + F' + F'' + F''') + \left(\frac{4-2\sigma}{3}z^3 - 2h^2 z\right)\frac{d}{ds}\nabla^2(F_0 + F') \\ &\qquad - 2\frac{d}{dn}(\psi' + \psi'') \\ w' &= 4(1-\sigma)(F_0 + F' + F'' + F''') + 2(\sigma z^2 - h^2)\nabla^2(F_0 + F') . \end{aligned}\right\} \quad (6)$$

The value of $\frac{d\psi'}{dn}$ to a second approximation is

$$\frac{d\psi'}{dn} = -h^2\beta(z/h)\vartheta_2(F_0 + F') - \frac{1}{2\rho}\left(\frac{1}{6}z^3 - \frac{1}{2}h^2 z\right)\vartheta_2(F_0 + F') \quad . \quad (7)$$

and $\frac{d\psi''}{dn}$ is given by (5).

The function F_1 which (§ 33) defines the permanent solution is

$$\iint (Np' + Sq' + Zw')\,ds\,dz \quad . \quad . \quad . \quad (8)$$

For the case of a solid or hollow circular plate all the quantities in the right-hand members of (6) can actually be calculated, and we thus obtain the solution for normal traction to a *second* approximation, and for tangential or perpendicular traction to a *third* approximation, in a form, moreover, applicable without modification or addition even when the given edge stresses are discontinuous.

We conclude by deducing the equations corresponding to Kirchhoff's boundary conditions to a second approximation. (They might be found to one order higher in the case of vanishing normal traction.)

We suppose that the given tractions N, S are of the same order in h, and that Z is of an order one higher. Any case may be reduced to combinations of cases satisfying this condition.

If we write F for $F_0+F'+F''$, then it follows from (6), (7), (8) that the terms of two lowest orders in F_1 are given by

$$\frac{F_1}{4(1-\sigma)}=\int\left|\begin{array}{l}-\dfrac{dF}{dn}\displaystyle\int_{-h}^{h}zN\,dz-\frac{dF}{ds}\int_{-h}^{h}zS\,dz+F\int_{-h}^{h}Z\,dz\\ +\left(\dfrac{d}{ds}\dfrac{dF}{dn}-\dfrac{1}{\rho}\dfrac{dF}{ds}\right)\displaystyle\int_{-h}^{h}2h^2\beta(z/h)S\,dz\end{array}\right|ds \quad . \qquad (9)$$

As in the extensional case, § 43, the integral with respect to s may be modified by means of integration by parts so that only F and $\frac{dF}{dn}$ appear under the integral sign.

Write

$$\left.\begin{array}{l}\displaystyle\int_{-h}^{h}zN\,dz=N_1,\ \int_{-h}^{h}zS\,dz=S_1,\ \int_{-h}^{h}Z\,dz=Z_0\\ \displaystyle\int_{-h}^{h}2h^2\beta(z/h)S\,dz=S_2.\end{array}\right\} \quad . \quad . \qquad (10)$$

Then if N_1, S_1, Z_0, S_2 are continuous functions of s,

$$\frac{F_1}{4(1-\sigma)}=\int\left\{-\frac{dF}{dn}\left(N_1+\frac{dS_2}{ds}\right)+F\left(\frac{dS_1}{ds}+Z_0+\frac{d}{ds}\frac{S_2}{\rho}\right)\right\}ds \qquad (11)$$

Hence any two systems of traction will give the same permanent mode to a second approximation, provided the values of $N_1+\frac{dS_2}{ds}$ and $\frac{dS_1}{ds}+Z_0+\frac{d}{ds}\frac{S_2}{\rho}$ are the same for the two systems.

Now for the system F_1, § 44 (i),

$$\left.\begin{array}{l}N_1=-\dfrac{4}{3}\mu h^3\vartheta_1F_1,\ S_1=-\dfrac{4}{3}\mu h^3\vartheta_2F_1,\ Z_0=-\dfrac{4}{3}\mu h^3\vartheta_3F_1,\\ S_2=-\dfrac{4}{3}\mu h^3\dfrac{384}{\pi^5}\gamma_5h\vartheta_2F_1,\ \text{§ 45, (iii) (13).}\end{array}\right\} \quad . \qquad (12)$$

Thus, with $\vartheta_1, \vartheta_2, \vartheta_3$ defined as in § 44 (i), the boundary conditions are

$$\left.\begin{array}{l}-\dfrac{4\mu h^3}{3}\left(\vartheta_1F_1+\dfrac{384}{\pi^5}\gamma_5h\dfrac{d}{ds}\vartheta_2F_1\right)=N_1+\dfrac{dS_2}{ds}\\ -\dfrac{4\mu h^3}{3}\left(\dfrac{d}{ds}\vartheta_2F_1+\vartheta_3F_1+\dfrac{384}{\pi^5}\gamma_5h\dfrac{d}{ds}\dfrac{\vartheta_2F_1}{\rho}\right)=\dfrac{dS_1}{ds}+Z_0+\dfrac{d}{ds}\dfrac{S_2}{\rho}\end{array}\right\} \quad . \qquad (13)$$

When only the principal terms are retained, these reduce to Kirchhoff's conditions.

If S_1 or S_2 is discontinuous at any point of the edge, integrated terms will appear in equation (11), as in the extensional case, § 43. Thus, if the normal couple S_1 be discontinuous at a point P ($s=s'$), there will appear on the right of (11) a term

$$F(s')\mid S_1\mid_{s=s'-}^{s=s'+}$$

The method of dealing with such a discontinuity in any actual problem is obvious, for by (9) its effect is the same as that of an *element* $\mid S_1\mid$ of shearing traction applied at P, a result which on the 'elastic equivalence' theory may easily be obtained by a trifling modification of the process by which Thomson and Tait reconciled the conditions of Kirchhoff and Poisson.

Addition to Paper by J. Dougall on—

"An Analytical Theory of the Equilibrium of an Isotropic Elastic Plate."

(Note added May 20, 1904.)

The kindness of one of the referees enables me to supply the following references to recent work bearing on the subject of the paper.

(*a*) J. H. Michell, in a paper "On the direct determination of Stress in an Elastic Solid, with application to the Theory of Plates," *Proc. Lond. Math. Soc.*, vol. 31, 1899, shows how the stresses might be found without previous determination of the displacements. In the case of the stress $\widehat{zz}$ or R, he finds that $\nabla^4 R$ is a given function in the body of the plate, while R and dR/dz are given on the faces. If we neglect the conditions at the edge, which have practically no influence on the result, a value of R satisfying these conditions can be found, in terms of Fourier integrals for instance. Mr Michell does not determine R—this has been done in the present paper—but proceeds to deduce the forms of the remaining stresses, and the differential equation for the normal displacement of a point on the mid plane. One special case of normal force is worked out to a first approximation, and Lagrange's equation for this case deduced.

For the conditions at the edge, reference is made to the ordinary Thomson-Boussinesq theory, which uses the principle of equipollent loads.

(*b*) L. N. G. Filon, "On an approximate solution for the bending of a Beam of rectangular cross section under any system of Load, with special reference to points of concentrated or discontinuous Loading," *Phil. Trans. R. Soc. Lond.* (Sec. A), vol. 201 (1903).

Dr Filon's solution applies to a beam in which the ratios of breadth to depth, and of depth to length, are both small. The axis of z being taken in the direction of the breadth, the stress $\widehat{zz}$ is taken as negligible, and equations are deduced for the mean values, across the breadth, of the displacements u, v. These equations are the same as equations (90), page 182 of the present paper, with the body force null. In order to see the reason of this from our standpoint, we may notice that the assumption that $\widehat{zz}$ vanishes eliminates all the solutions of what we have called the dilatational transitory type, and that taking the mean of the displacements eliminates all the flexural solutions, as well as the rotational transitory solutions.

As regards the conditions at the ends, the beam is treated as a long rod.

It may be of interest to remark that the results of § 43 above furnish the data for a more approximate treatment of the problem on the lines followed by Dr Filon.

(*c*) A note appended to a paper by Professor Lamb in *Proc. Lond. Math. Soc.*, vol. 34 (1902), pp. 283, 284, contains a solution of a special case of the problem of face traction.

(*d*) In connection with existence theorems relating to the elastic equations, reference should be made to the work of Italian elasticians, as Somigliana, Lauricella, and Tedone.

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IX.—On the Measurement of Stress by Thermal Methods, with an Account of some Experiments on the Influence of Stress on the Thermal Expansion of Metals.

By **E. G. Coker**, M.A. (Cantab.), D.Sc. (Edin.), F.R.S.E.; Assistant Professor of Civil Engineering, M'Gill University, Montreal. (With Two Plates.)

(MS. received April 11, 1904. Read June 6, 1904. Issued separately September 2, 1904.)

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1. Introduction.

In the determination of the effects of stress upon different materials, the investigator has several methods of attack open to him, each of which has its own particular advantages. In the great majority of cases the material under investigation obeys the generalised Hooke's law, and the effects of a stress are therefore most easily followed and measured by observations of the strains produced. The strains being usually exceedingly minute, it is necessary to magnify them sufficiently to allow of accurate measurement. To this end many instruments have been devised for measuring the strains obtained by the action of different stresses, and in fact the great majority of our experimental knowledge has been obtained in this way. The application of polarised light to the determination of stress was first suggested by Brewster,* and he applied it to many problems, particularly the determination of the neutral axis of a glass beam. Neumann,† with a full knowledge of the work of Brewster, developed a theory of the analysis of strain by polarised light, and Maxwell‡ also independently developed a theory. A third method, which has assumed great prominence in recent years, is the microscopic examination of metals under stress, as developed by Ewing, and Rosenhain,‖ and others.

The present paper is mainly concerned with the measurement of stress by the temperature changes produced, a subject to which attention was first drawn by Weber,§ who found that when a wire was stretched suddenly a thermal effect was produced,

* *Trans. R.S.E.*, vol. iii.
† *Abhandlungen der k. Akademie der Wissenschaften zu Berlin*, 1841.
‡ "On the Equilibrium of Elastic Solids," *Trans. R.S.E.*, 1853.
‖ "On the Crystalline Structure of Metals," *Trans. R.S.*, 1900.
§ *Poggendorff's Annalen*, Bd. xx., 1830.

which he attributed to an internal cause. He deduced a theoretical formula in which the change of temperature t was shown to bear a linear relation to the stress p and to the coefficient of expansion k, which may be expressed in the form $t = c\ k\ p$, where c is a constant. He proved from his experiments that a wire when stretched within its elastic limit is lowered in temperature, and when compressed has its temperature raised. DUHAMEL * investigated the modifications which became necessary in POISSON'S elasticity equations, when allowance is made for change of temperature. The subject was placed on a sound basis by Lord KELVIN,† who deduced from the laws of thermo-dynamics the general equations of thermo-elasticity. He showed that the thermal effect H produced by stresses p_{xy} can be expressed in the form

$$H = \frac{t}{J} \sum \frac{dp_{xy}}{dt} e_{xy}$$

where t is the temperature, J is JOULE'S equivalent, and e_{xy} is the strain corresponding to the stress p_{xy}.

The general conclusions deduced were that "cold is produced whenever a solid is strained by opposing, and heat when it is strained by yielding to any elastic force of its own, the strength of which would diminish if the temperature were raised, but that, on the contrary, heat is produced when a solid is strained against, and cold when it is strained by yielding to, any elastic force of its own, the strength of which would increase if the temperature were raised."

These conclusions were experimentally verified by JOULE,‡ who showed that the thermal changes produced by stretching and compressing metals, timber, etc., and by the deflection of helical springs, was proportional to the stress applied, and obeyed the Thomson law. The change of temperature was measured by thermo-electric couples composed of iron and copper wires, either pressed against the specimen or inserted in holes drilled into them; a galvanometer was placed in circuit with the thermo-junction to indicate the change of temperature; to calibrate the galvanometer the test specimen was plunged into water, at a known temperature, to within a short distance of the junction.

EDLUND § applied the methods of JOULE to the determination of the effects of stress on wires, and instead of an iron-copper junction he used crystals of bismuth and antimony cut to a cylindrical form, and the cut ends were pressed against the wire, so that no variation in the thermo-electric power was possible, as might be the case if the natural cleavage plans were used. His results amply verify JOULE'S earlier work, and he, moreover, obtained an approximately correct value of the mechanical equivalent of

* "Mémoire sur le calcul des actions moleculaires développées par les changements de température dans le corps solides,"—Mémoires . . . par divers savans, vol. v., 1838.

† "On the Dynamical Theory of Heat," *Trans. R.S.E.*, 1851.

‡ "On some Thermo-Dynamical Properties of Solids," *Trans. R.S.*, 1853.

§ "Untersuchung über die bei voluniveränderung fester körper entstehenden wärmephänomene, sowie derein verhaltniss zu dabei geleisteten mechanischen arbeit," *Pogg. Annal.*, vol. cxiv., 1861, and "Quantitative bestimmung der bei volumveränderung der metalle enstehenden wärmephänomene und die mechanischen wärmeaequivalents, unabhängig von der inneren arbeit des metalls," *Pogg. Annal.*, vol. cxxvi., 1865.

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heat by a direct application of the Thomson formula. In a recent valuable paper by TURNER * the methods of JOULE and EDLUND are substantially followed, and a detailed account is given of experiments on tension stress for metal bars of a size needing a modern testing machine for the stressing agent.

2. THE THERMAL EFFECT OF TENSION AND COMPRESSION STRESS.

In the present paper the main object of inquiry is tension and compression, and for these stresses it is easy to deduce a simple form of the relation between the change of temperature and the stress from the equations of thermo-dynamics.

If l be the length of unit mass of a rod subjected to a compression stress of intensity p, which shortens the bar by an amount dl, and E be the initial intrinsic energy of the bar and dH the amount of heat developed by the compression, we have

$$\begin{aligned} d\mathrm{E} + p \cdot dl &= \mathrm{J}dh \cdot \\ &= \mathrm{J}\left\{ \left(\frac{\partial \mathrm{H}}{\partial p}\right)_t \cdot dp + \left(\frac{\partial \mathrm{H}}{\partial t}\right)_p \cdot dt \right\} \\ &= \mathrm{J}\left\{ p_t \cdot dp + t_p \cdot dt \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1) \end{aligned}$$

where p_t is written for the more cumbersome symbol $\left(\frac{\partial \mathrm{H}}{\partial p}\right)_t$.

Now the alteration of length is a function of the pressure and temperature, and hence we have

$$dl = \frac{\partial l}{\partial p} dp + \frac{\partial l}{\partial t} dt .$$

Therefore

$$d\mathrm{E} = \left(\mathrm{J}p_t - p\frac{\partial l}{\partial p}\right) dp + \left(\mathrm{J}t_p - p\frac{\partial l}{\partial t}\right) dt$$

a perfect differential, whence

$$\frac{\partial}{\partial t}\left(\mathrm{J}p_t - p\frac{\partial l}{\partial p}\right) = \frac{\partial}{\partial p}\left(\mathrm{J}t_p - p\frac{\partial l}{\partial t}\right),$$

giving

$$\mathrm{J}\left(\frac{\partial t_p}{\partial p} - \frac{\partial p_t}{\partial t}\right) = \frac{\partial l}{\partial t} .$$

Equation (1) can be written in the form

$$\frac{d\mathrm{H}}{t} = \frac{p_t}{t} dp + \frac{t_p}{t} dt$$

a perfect differential also, from which we obtain

$$-\frac{p_t}{t} = \left(\frac{\partial t_p}{\partial p} - \frac{\partial p_t}{\partial t}\right) = \frac{1}{\mathrm{J}} \frac{\partial l}{\partial t} .$$

Now if a body be compressed adiabatically

$$t_p dt + p_t \cdot dp = 0$$

or

$$\frac{dt}{dp} = -\frac{p_t}{t_p} = \frac{t}{\mathrm{J}} \frac{\partial l}{\partial t} \frac{1}{t_p} .$$

* "Thermo-Electric Measurement of Stress," *Trans. Amer. Soc. C.E.*, 1902.

If a be the coefficient of expansion of the bar for unit increase of temperature, we can write this in the form

$$\frac{dt}{dp} = \frac{tal}{Jt_p},$$

and for small changes of pressure and temperature we can write the equation in the form

$$\Delta p = \Delta t \frac{Jt_p}{tal}$$

where the sign of Δt depends upon the signs of a and Δp, since all the other quantities are essentially positive.

For metals a is in general positive, and hence a compression stress will raise the temperature, while a tensional stress will lower it. Strictly speaking, the equation only holds for infinitesimal changes of p and t, and it is therefore essential to show what limitations, if any, are to be imposed in its application to bodies under great ranges of stress. The effect of a varying load upon the specific heat of a body has not been determined, so far as I am aware, but it is unlikely to differ by an appreciable amount from the specific heat at atmospheric pressure. The coefficient of expansion when the stress is varied may change to a small extent, and experiments were made by JOULE * to examine the effect of stress upon the expansion of various timbers, and he found an increase of expansibility with tension. As far as I am aware, there are no experiments showing what effect stress has upon the thermal expansion of metals; and as the coefficient of expansion enters into the fundamental equation, a special investigation was made of the thermal expansion of brass and steel under different tension loads.

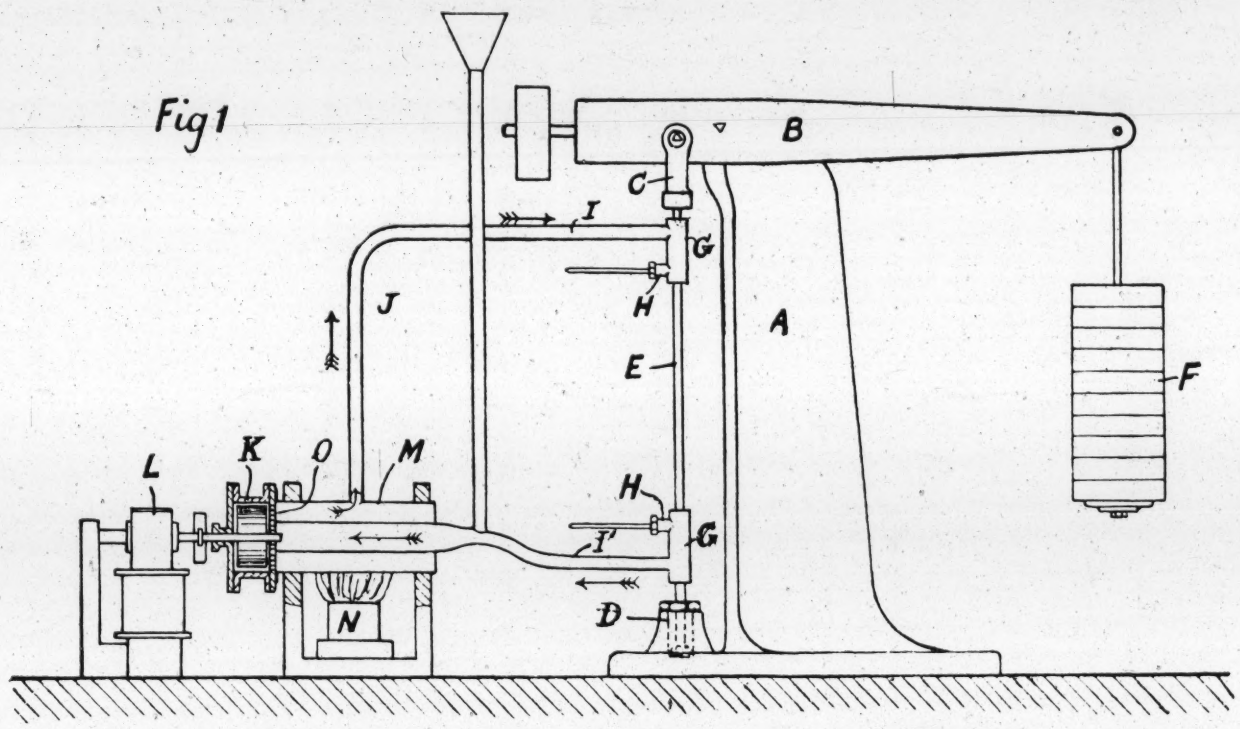
3. THE THERMAL EXPANSION OF BRASS AND STEEL UNDER TENSION STRESS.

The general arrangement of apparatus adopted is shown in fig. 1, in which A is the standard of a small single-lever testing machine provided with a weigh-beam B and shackles C D for securing the test-piece E in position. The loading of the specimen is effected by suspending dead weights F from the end of the beam, and the maximum load which could be safely applied was 175 pounds. The ratio of the arms of the lever was 20 to 1, and hence the maximum stress obtainable was 3500 pounds.

In order to carry the experiments past the elastic limit, it was necessary to have a specimen of very small section; and on account of the difficulty of maintaining a solid specimen at a uniform temperature, and at the same time observing the change of length, the specimens were chosen of seamless drawn tube, very uniform in diameter, thereby permitting the outside being turned in a lathe to a manageable section. The ends of the tube were soldered into thick ferrules G, having side tubes H for the insertion of thermometers, and inlet and outlet tubes I I′ were provided, connecting to a pipe system J, in which water could be circulated at any desired temperature. The circulation was effected by a small centrifugal pump K driven by an electric

* "On some Thermo-Dynamic Properties of Solids," *Phil. Trans.*, 1859.

motor L, and provided with an extension barrel M, heated by a gas flame N. The rotation of the pump caused water to be drawn in at the eye of the pump from the pipe I′, and to be discharged through the perforated partition O into the encircling heater M. From thence it flowed through the rising pipe J and the specimen, as indicated by arrows. A very vigorous circulation was maintained, and the pipes were thickly lagged, so that there was practically no difference of temperature between the end points of the tube under measurement. The readings of the thermometers were found to be practically coincident at all temperatures, and therefore in the tabulated results one set of temperature readings is omitted. The alteration in the length of



the tube with increase of temperature was determined by aid of a Ewing extensometer of the original pattern. The instrument (fig. 2) consists of a pair of clips A B, secured to the tube by set screws C; the upper clip carries a frame D, provided with a calibrating screw P and a reading microscope F focussed on the edge of a thick wire W carried by the lower clip. Any alteration in the length of the tube causes a movement of the wire relative to the microscope, and by aid of a glass scale inside the eye-piece the alteration in length can be determined. The dimensions of the instrument were such that a movement of 0·00002 inch corresponded to one unit of the scale, and the micrometer screw enabled a calibration of the scale to be effected while the instrument was attached to the specimen — a great advantage with an extensometer. The construction of the apparatus permitted a thick layer of lagging K to be applied. It was proposed at first to surround the specimen by a water-jacket, fed from the circulating system, to ensure the temperature of the tube being absolutely uniform throughout, but this was not done, as the lagging was found to be very efficient, and the extra complication did not seem warranted, particularly as only

comparative readings were required. The thermometer bulbs I were inserted in the main tubes, and secured by flexible rubber joints J,* as indicated.

In carrying out a test the centrifugal pump was first set in operation, and when the readings of the extensometer became steady, the gas flame was applied to the extension barrel of the pump, and the readings were taken, at intervals, of the

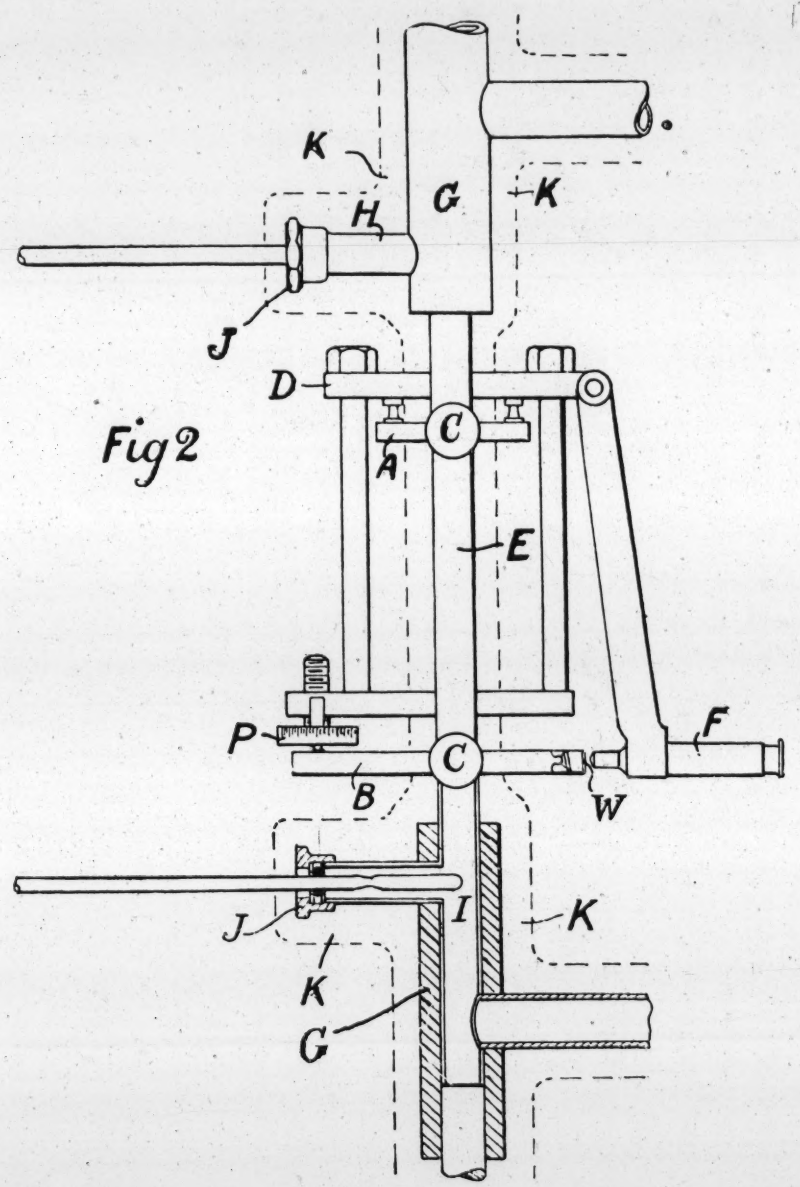


Fig 2

temperature and the extension, until a temperature of about 180° Fahr. was reached A new experiment was then begun with a different loading. The testing machine used was not sufficiently powerful to overstrain the tubes used, as these latter were exceptionally hard, owing to their manner of production; and for the experiments with a permanent overstrain the tubes were taken out and stressed in a Riehle testing machine of 60,000 pounds capacity.

The results of the tests on the brass tube are shown in Table I., from which it will

* "On a flexible Joint for securing Thermometer and like Stems and Tubes," E. G. COKER, *Physical Review*, 1903.

Table I.

Brass specimen.—Internal diameter = 0·378 inches.
External diameter = 0·525 ,,
Area = 0·1042 square inches.

Stress in lbs. per sq. inch	960		9,597		19,194		28,791		33,590		$\frac{1}{16}$-inch overstrain		$\frac{1}{8}$-inch overstrain		$\frac{1}{4}$-inch overstrain	
Load, lbs.	100		1,000		2,000		3,000		3,500							
Temp. Fahr.	Reads.	Δ	Reads.	Δ	Reads.	Δ	Reads.	Δ	Reads.	Δ	Reads.	Δ	Reads.	Δ	Reads.	Δ
70	0		0		0		0		0		0		0		0	
		− 20		− 23		− 22		− 21		− 20		− 22		− 22		− 22
75	20		23		22		21		20		22		22		22	
		− 21		− 21		− 21		− 21		− 21		− 21		− 21		− 24
80	41		44		43		42		41		45		43		46	
		− 23		− 22		− 21		− 24		− 21		− 22		− 23		− 21
85	64		66		64		66		62		67		66		67	
		− 23		− 21		− 22		− 19		− 20		− 19		− 21		− 25
90	87		87		86		85		82		86		87		92	
		− 22		− 21		− 20		− 21		− 20		− 23		− 22		− 21
95	109		108		106		106		102		109		109		113	
		− 22		− 21		− 21		− 21		− 19		− 21		− 22		− 21
100	131		129		127		127		121		130		131		134	
		− 22		− 21		− 23		− 21		− 19		− 21		− 22		− 22
105	153		150		150		148		140		151		153		156	
		− 22		− 21		− 22		− 22		− 20		− 21		− 22		− 23
110	175		171		172		170		160		172		175		179	
		− 21		− 21		− 21		− 23		− 20		− 21		− 21		− 24
115	196		192		193		193		180		193		196		203	
		− 23		− 21		− 21		− 21		− 21		− 22		− 23		− 21
120	219		213		214		214		201		215		219		224	
		− 22		− 21		− 21		− 23		− 20		− 21		− 21		− 24
125	241		234		235		237		221		236		240		248	
		− 22		− 21		− 21		− 24		− 21		− 22		− 21		− 23
130	263		255		256		261		242		258		261		271	
		− 22		− 21		− 21		− 24		− 22		− 22		− 20		− 25
135	285		276		277		285		264		280		281		296	
		− 22		− 21		− 21		− 21		− 23		− 22		− 22		− 24
140	307		297		298		306		287		302		303		320	
		− 21		− 21		− 22		− 18		− 24		− 22		− 22		− 23
145	328		318		320		324		311		324		325		343	
		− 22		− 21		− 22		− 22		− 25		− 21		− 26		− 26
150	350		339		342		346		336		345		351		369	
		− 22		− 21		− 22		− 22		− 23		− 22		− 20		− 22
155	372		360		364		368		359		367		371		391	
		− 22		− 21		− 21		− 23		− 23		− 24		− 24		− 26
160	394		381		385		391		382		391		395		417	
		− 22		− 22		− 22		− 24		− 23		− 23		− 24		− 26
165	416		403		407		415		405		414		419		443	
		− 22		− 23		− 21		− 20		− 24		− 25		− 22		− 28
170	438		426		428		435		429		439		441		471	
		− 21		− 24		− 25		− 24		− 23		− 25		− 23		− 24
175	459		450		453		459		452		464		464		495	
		− 21		− 24		− 21		− 23		− 25		− 24		− 25		
180	480		479		474		482		477		480		489		—	

be seen that the extension for all the different loads below the yield point of the material are practically constant. This is shown graphically by fig. 3, in which the readings are plotted as ordinates, with the temperatures as abscissæ. The curves for stresses below the yield point are very nearly straight lines, with the exception of No. IV., when a load of 3000 pounds was applied, corresponding to a stress of 28,790 pounds per square inch. We may neglect the small deviation there shown, since at an increased stress of 33,570 pounds per square inch it disappears, and we may assume that the expansion is practically linear. The mean value of the coefficient of expansion for these five different experiments corresponds to a linear expansion of ·00001953 for 1° centigrade, the maximum deviation therefrom being slightly less than 1 per cent.

For the overstrained tube the coefficient of expansion was greater, the values obtained being as follows:—

TABLE II.

Total Permanent Extension. Inches.	Coefficients of Expansion per 1° Centigrade.
$\frac{1}{16}$	·00001963
$\frac{1}{8}$	·00002004
$\frac{1}{4}$	·00002121

A similar experiment upon a steel tube having very thin walls was made, and the results are given in Table III.

TABLE III.

Steel specimen.—Internal diameter = 0·406 inches.
External diameter = 0·482 inches.
Area = 0·067 square inches.

Stress in lbs. per sq. inch	1,493		14,926		29,852		35,824		37,157	
Load, lbs.	100		1,000		2,000		2,400 $\frac{1}{8}$-in. overstrain		2,500 $\frac{1}{4}$-in. overstrain	
Temp. Fahr.	Readg.	Δ	Readg.	Δ	Readg.	Δ	Readg.	Δ	Readg.	Δ
70	0		0		0		0		0	
		−12		−13		−11		−12		−12
75	12		13		11		12		12	
		−14		−13		−12		−12		−12
80	26		26		23		24		24	
		−12		−12		−13		−11		−10
85	38		38		36		35		34	
		−12		−12		−13		−13		−12
90	50		50		49		48		46	
		−12		−13		−12		−13		−11
95	62		63		61		61		57	
		−13		−12		−11		−12		−12
100	75		75		72		73		69	
		−11		−11		−12		−12		−13
105	86		86		84		85		82	
		−13		−12		−13		−12		−11
110	99		98		97		97		93	
		−14		−13		−13		−12		−13
115	113		111		110		109		106	
		−13		−11		−12		−13		−13
120	126		122		122		122		119	
		−12		−12		−13		−11		−13
125	138		134		135		133		132	
		−13		−12		−13		−12		−11
130	151		146		148		145		143	
		−13		−11		−12		−13		−12
135	164		157		160		158		155	
		−12		−13		−12		−14		−16
140	176		170		172		172		171	
		−13		−12		−13		−11		−11
145	189		182		185		183		182	
		−13		−13		−14		−13		−12
150	202		195		199		196		194	
		−14		−11		−12		−12		−14
155	216		206		211		208		208	
		−12		−12		−13		−12		−13
160	228		218		224		220		221	
		−13		−12		−14		−15		−13
165	241		230		238		235		234	
		−13		−13		−12		−15		−13
170	254		243		250		250		247	
				−16		−13		−15		−14
175	—		259		263		265		260	
						−13		−15		
180	—		—		276		280		—	

This differed somewhat from the brass tube previously experimented upon, in showing a practically uniform coefficient of expansion under every load below and above the yield point (fig. 4). The mean value of the coefficient was found to be ·00001121, the maximum deviation being for the two lowest loads, and amounting to nearly 2·4 per cent., while for the remaining loads the deviation was less than 1 per cent. The general accuracy of the results was checked by comparison with the known values of the coefficient of expansion at atmospheric pressure, and the agreement is sufficiently close to make it clear that the observations were accurately taken, having regard to the fact that the seamless tubes experimented upon had been subjected to exceptional treatment in the process of manufacture, and were probably in a very different physical condition from solid bars of the same material rolled or cast in the ordinary way.

We may conclude from these experiments that there is practically no difference in the linear expansion of brass and steel within the range of stress up to the yield point of the material, and that for brass there is probably an increase in the coefficient beyond the yield point, but that there is no increase for steel. In this part of the work I was greatly assisted by Mr Charles M'Kergow, Demonstrator in the Civil Engineering Department, M'Gill University, who kindly undertook the major part of the work of observation, and who also rendered me very able assistance in the experimental work detailed below.

4. The Behaviour of Iron and Steel under Tensile Stress.

In applying the thermal method of measuring stress, the most convenient arrangement of apparatus consists of a thermal junction or pile of the necessary delicacy connected to a galvanometer giving a sufficiently wide range of readings for the small difference of temperature produced. In nearly all testing laboratories the presence of iron in large quantities makes it necessary to choose a galvanometer which is not influenced by the proximity of iron, and this was especially necessary in the present case, since in the M'Gill University testing laboratory the main testing machines have, in the course of time, become magnetised, owing to the subsidiary mechanism being operated by electric motors. These difficulties are easily overcome by the use of a D'Arsonval galvanometer, the field of which is very uniform, even in the neighbourhood of large masses of feebly magnetised iron. The galvanometer coil was specially wound for me by Dr Tory, and was of approximately the same resistance as the thermopile used in the majority of the experiments, so that the arrangement was as sensitive as possible. To avoid short-circuiting of the pile when in contact with the metal specimen under test, it is convenient to insulate it therefrom, and I have found a thin sheet of paper, as suggested and used by Joule, the most convenient. The connections of the galvanometer to the thermopile were made by soldered joints, which were afterwards wrapped in paper to insulate them from one another, and then tied together and lagged with cotton-wool.

In making observations with a galvanometer provided with a moving coil of considerable weight, such as that of the D'Arsonval type, the indications may not be a faithful record unless certain precautions are observed, for the galvanometer does not take up its position of equilibrium at once, and therefore any error due to this lag will make a considerable difference in the results. It was found experimentally that the time rate of loading could be so determined by trial that the reading of the galvanometer was very approximately a maximum for the stress at any given instant, and by making special experiments for each bar, the rate at which the loading ought to be applied could be easily determined. As an example of the accuracy with which the loading could be applied to keep step with the galvanometer, reference may be made to the results obtained from a steel bar of rectangular section, having a breadth of 0·86 inches and a thickness of 0·315 inches, which was loaded at a uniform rate until a maximum of 4000 pounds was reached, corresponding to a stress of 14,760 pounds per square inch.

The readings obtained were as follows :—

TABLE IV.

Load, pounds.	Time in seconds.	Observed Reading.	Corrected Reading.
0	0	0	0
2000	25	4·0	4·23
3000	45	7·0	7·71
4000	60	9·8	11·12
4000	90	9·0	10·82
4000	110	8·0	9·98
4000	135	7·0	9·13
4000	165	6·0	8·23
4000	200	5·0	7·25
4000	260	4·0	6·34
4000	340	3·0	5·60
4000	410	2·5	4·81

Notes.—Scale distance, 83½ inches: 1 division = 0·5 inches on scale.
Resistance of thermopile and leads = 6·07 ohms.
Resistance of moving coil of galvanometer = 5·2 ohms.
Temperature 68° Fahr.

From which it will be seen that as soon as the loading reached a maximum, the readings also attained a maximum, and then began to decrease. The ascending portion of the

curve showing the relation of the thermal change to the stress is, however, influenced by losses due to conduction and radiation, and hence it is necessary to correct for these in order to obtain a correct relation. If it be allowed that the lag of the galvanometer is not a factor in this correction, the required result may be obtained as follows. Let θ_0 be the diminution of temperature per second due to the application of a stress increasing uniformly with the time, and let θ be the actual difference of temperature at any time t from the commencement of the application of stress, then $\theta < \theta_0 t$, since there is a loss due to conduction and radiation, depending on the difference of temperature between the specimen and surrounding bodies. The loss due to this cause can be very approximately determined by observation of the subsequent readings when the application of stress has ceased, and it was found in all cases that the loss was very accurately proportional to the first power of the difference of temperature. In an interval of time dt, therefore, the diminution of temperature for a tension specimen under uniformly applied stress will be $\theta_0 dt - k\theta \cdot dt$, where k is a constant to be determined. The actual decrease of temperature in the time dt is $\frac{d\theta}{dt}dt$.
Hence

$$\frac{d\theta}{dt} + k\theta = \theta_0 \qquad (1),$$

an integrating factor of which is obviously ϵ^{kt}.
Hence

$$\epsilon^{kt}\theta = \theta_0 \int \epsilon^{kt} \cdot dt + c$$

$$= \frac{\theta_0}{k} \cdot \epsilon^{kt} + c$$

$$\text{or} \qquad \theta = \theta_0/k + c\epsilon^{-kt}.$$

To determine the constant c we have the condition that θ is zero at the commencement of the application of the load; hence $c = -\theta_0/k$, and we have

$$\theta = \theta_0/k \cdot (1 - \epsilon^{-kt})$$

$$\text{or} \qquad \theta_0 = k\theta/(1 - \epsilon^{-kt})$$

$$\theta_0 t = k\theta \Big/ \left(1 - \frac{kt}{2!} + \frac{k^2t^2}{3!} - \text{etc.}\right).$$

Now the denominator can be expanded provided the value of the variable t is such that the expression in the bracket remains convergent, and it is evident if $kt/2 < 1$ i.e. $t < 2/k$ this condition will be satisfied. Hence we obtain as a sufficiently near approximation

$$\theta_0 t = \theta\left(1 + \frac{kt}{2!} - \frac{k^2t^2}{3!}\right);$$

and since k is a very small quantity, this reduces to

$$\theta_0 t = \theta\left(1 + \frac{kt}{2}\right).$$

Now $\theta_0 t$ is the actual decrement of temperature D_t due to the stress up to the time t,

and θ is the observed value D_A. Hence the observations for the ascending part of the curve must be corrected by the formula

$$D_t = D_a\left(1 + \frac{kt}{2}\right).$$

The value of k is determined in each case by the second part of the curve; and in the example shown and in all others described in this paper, it is of the exponential type. In the present example the value of the deflection D at any time after the loading ceased was found to be

$$D = 9{\cdot}8\epsilon^{-{\cdot}0045t}$$

where t is the time in seconds from the cessation of the load. The curve showing the readings corrected for the radiation loss during the loading is shown dotted in fig. 5. There is also a correction for the change in resistance of the galvanometer coil and leads, owing to the change in the temperature of the room. The testing laboratory was very favourably situated in this respect, as its temperature rarely varied more than two or three degrees, and hence this correction was unnecessary. A further correction might be made since the current strength i in the thermopile circuit, and therefore the deflection of the galvanometer, $= k\frac{\phi}{\cos\phi}$, and where ϕ is the angle turned through by the moving coil, but in all cases the deflection was so small in comparison with the distance of the scale from the moving coil and mirror attached thereto that the correction was negligible.

5. The Relation of Stress to Strain and Thermal Change in Tension Members.

The variation of strain with regard to tension stress follows a linear law very approximately over a certain range in the case of most metals, and in the case of iron and steel this linear relation holds for a considerable part of the whole range of stress up to rupture. This is easily shown by delicate extensometers, such as those devised by Unwin, Ewing, Martens and others. It becomes of importance to determine what is the relation of the thermal change to stress and to strain. The only previous experiments of which I am aware are those of Turner,* who has experimented upon the relation of thermal change to stress; and from the known properties of iron and steel as regards strain, he has deduced from his results that "the thermal limit of proportionality is lower than what is considered the true primitive elastic limit of the metal." He suggests that there exists from the thermal point of view a well-defined range of almost perfect elasticity, beyond which "there is a considerable, in fact nearly equal, range of imperfect elasticity, before reaching the limits of apparent elasticity of shape." This is a matter of considerable importance in regard to the question of repeated stress, since if this is so, it may have an important bearing on the results of Wöhler and others.

* *Loc. cit., ante.*

In order to test the truth of this, several experiments were made on bars in tension, using a thermopile for measuring the cooling effect caused by the stress applied, and an extensometer to determine the strain. This latter was of the usual Unwin pattern, except that the metal distance-pieces were replaced by mahogany rods, previously soaked in paraffin wax, and suitably capped. This precaution renders negligible any error due to any slight changes of temperature, as the coefficient of expansion of mahogany is extremely small. The steel bar quoted above, for which the correction factor for radiation and conduction had already been determined, may be quoted as an illustration. The load was applied as uniformly as possible in a Buckton testing machine at a uniform rate of 4000 pounds per minute, and the galvanometer and extensometer readings were taken at each interval of 1000 pounds. The following readings were obtained :—

TABLE V.

Time, seconds.	Load, pounds.	Extensometer Reading.	Observed Galvanometer Reading.	Corrected Galvanometer Reading. $k = \cdot0045$.
0	0	0	0	0
16	1,000	...	2·5	2·59
30	2,000	200	5·0	5·34
		− 105		
44	3,000	305	8·0	8·79
		− 105		
60	4,000	410	10·5	11·92
		− 105		
75	5,000	520	13·3	15·54
		− 105		
91	6,000	625	16·3	19·60
		− 100		
...	7,000	725	18·3	...
		− 100		
120	8,000	825	20·5	26·04
		− 102		
...	9,000	927	22·8	...
		− 103		
150	10,000	1,030	25·0	33·44
		− 100		
...	11,000	1,130	26·8	...
		− 120		
180	12,000	1,250	28·7	40·32
195	13,000	...	30·0	43·16
	14,000	Went off scale at 13,600 lbs.	Galvanometer reading went off scale	

Note.—Distance of scale from mirror of galvanometer = 6′ 11½″.

The galvanometer used in all the experiments mentioned in this paper was provided with a coil of resistance 5·2 ohms, and having about 300 turns; the thermopile was

approximately square in section, with 31 couples, and had a resistance of 5·55 ohms. The short connecting wires or leads had a resistance of 0·52 ohms. In a few experiments, which are specially noted, a linear pile was used of 10 couples and of 0·18 ohms resistance, and also long connecting wires or leads of 1·81 ohms resistance were used in some cases. A plot of these readings is shown on fig. 6, in which curve I shows the relation of the stress to the galvanometer readings. In order to obtain the true reading, correction must be made for the losses due to radiation and change of resistance. The first is the only important one. The readings corrected by the formula $D_t = D_a\left(1+\frac{kt}{2}\right)$ are shown in the table above, and the plot of these, with the load as abscissæ, is shown by curve II, giving almost exactly a straight line to near the yield point (fig. 6). The stress-strain relation obtained from the extensometer readings is plotted for comparison upon the same diagram, the unit of extension being 0·00001 inch, and this also exhibits a nearly linear relation up to the yield point. The result of the experiment appears to show that the thermal changes do not indicate a range of imperfect elasticity within the apparent limits of elasticity of shape.

A second experiment upon a wrought-iron specimen having a section 2 inches by 0·25 inches was next subjected to stress in the testing machine in a similar manner, and the observations are recorded in Table VI., and a plot of the readings is shown in fig. 7.

The observations made to determine the radiation loss are omitted, as they are of a similar character to the example quoted above. The value of k obtained was 0·0031, the time being measured in seconds.

The general character of the diagram is the same as in the last case; there is a gradual bending over of the galvanometer readings towards the time axis, the deviation from a straight line being nearly in a geometrical progression with regard to time. The apparent coincidence of the lower readings with the dotted straight line is probably not exact. It should be noted that the stress-strain curve would practically coincide with the corrected thermal stress curve if sheared over, except near the upper end, where the heating effect begins to play a part. In both cases the thermal readings begin to show deviations from a linear relation to the stress at about the same value of the stress. In other experiments upon different bars of iron and steel, results were obtained confirming those quoted above. It therefore appears probable that the thermal change is very nearly proportional to the stress, in the same manner as the strain; and that, for the material experimented upon, there appears to be no range of imperfect elasticity as measured by thermal change, coinciding with a part of the range of perfect elasticity as determined by the strain.

6. The Relation of Stress to Strain and Thermal Change in Short Compression Members.

It is well known that the relation of stress to strain in short compression members of wrought-iron and steel follows a linear relation for a considerable range of stress, and

TABLE VI.

Time, seconds.	Load, pounds.	Extensometer Readings.		Observed Galvanometer Readings.	Corrected Galvanometer Readings. $k = \cdot0031$.
0	1,000	0		0	0
			− 55		
10	2,000	65		1·70	1·73
			− 60		
...	3,000	115		3·30	...
			− 52		
35	4,000	167		5·20	5·48
			− 52		
45	5,000	219		6·90	7·38
			− 53		
55	6,000	272		8·60	9·38
			− 58		
63	7,000	330		10·30	11·31
			− 60		
75	8,000	390		12·40	13·84
			− 48		
83	9,000	438		13·90	15·69
			− 59		
93	10,000	497		15·50	17·57
			− 60		
103	11,000	557		17·10	19·83
			− 53		
115	12,000	610		18·60	21·91
			− 60		
123	13,000	670		19·10	22·74
			− 57		
132	14,000	727		21·50	25·92
			− 58		
145	15,000	785		23·00	28·17
			− 63		
150	16,000	848		23·90	29·46
155	17,000	...		24·90	30·88
...	17,500	...		...	...

Note.—Distance of scale from galvometer mirror = 6′ 0½″.

that generally there is no very definite yield point—the strain gradually increasing beyond a certain load, so that the curve showing the relation of stress to strain is well rounded, and therefore the yield point is not so well defined. In order to obtain pure compression stress without bending, it is necessary to keep the specimen as short as possible, and experiments were first made upon compression specimens only long enough to accommodate the thermopile, the strain-measuring apparatus being secured to the compression plates of the testing machine. This arrangement did not give satisfactory results, and after several trials new specimens were prepared, sufficiently long to allow of a strain-measuring instrument being applied to them in addition to the thermopile. The shortest specimen which could be used under these conditions was 4·5 inches long, and, as might be expected, the specimen usually failed by buckling, so that it was not

possible generally to trace the relation of stress to strain and thermal change for pure compression stress up to the point where the departure from Hooke's law was very definitely marked; but sufficient work was accomplished to show that the strain and thermal change are both proportional to the stress throughout the greater part of the elastic range, and it seems highly probable, from the evidence obtained in the tension experiments, that this will hold throughout the whole elastic range of stress, as determined by strain measurements.

In order to indicate the nature of the results the following experiment may be quoted. The specimen was of wrought-iron, 0·9 × 0·39 inches in section and 4·5 inches long. The strain-measuring instrument used was one specially designed by Professor Ewing for compression, and similar in principle to the extensometer used in a previous experiment, except that the distance between the grips was 1·25 inches, and there was no calibrating screw. The instrument was first calibrated on a Whitworth measuring machine, and the position of the micrometer eye-piece determined, so that one division of the scale corresponded exactly to $\frac{1}{250{,}000}$ of an inch. The instrument was then set up on the specimen, and the thermopile applied to the broad face. The specimen was, for convenience, stressed in a small press, provided with an hydraulic diaphragm, accurately calibrated up to 21,000 pounds, and the load was applied as uniformly as possible. Preliminary experiments were made to obtain the correction factor for radiation and conduction, and the value of k was found to be ·0096. A load was applied at the rate of 2000 pounds in ten seconds, until the specimen failed by buckling. The following readings were obtained in this way :—

Table VII.

Time.	Load.	Compressometer Reading.	Galvanometer Reading.	Corrected Reading. $k = ·0096$.
0	0	0	0	0
		− 58		
10	2,000	58	3·2	3·45
		− 60		
20	4,000	118	6·6	7 23
		− 59		
30	6,000	177	9·8	11·20
		− 57		
40	8,000	234	12·2	14·54
		− 60		
50	10,000	294	15·0	18·60
		− 58		
60	12,000	352	17·6	22·66
		− 58		
70	14,000	410	19·6	26·18
	Failed by bending			

Notes.—Long connecting leads.

Distance of scale from galvanometer mirror = 10′ 2″.

The specimen failed by bending at the lower end, due no doubt to the compression plate being provided with a hemispherical seat.

The galvanometer readings uncorrected (fig. 8) show a marked deviation from a straight line, but the corrected readings follow a linear law almost exactly, and show no trace of the bending stress, as the thermopile was set against the top of the specimen farthest from the place of failure.

The strain readings are plotted on the figure for comparison, and it is evident that the linear correspondence is quite definite. As a further example, the following test may be quoted of a steel specimen $1{\cdot}01 \times {\cdot}38$ inches in section and $4\frac{3}{4}$ inches in length. The value of k found by a preliminary test was ·0089, and the test gave the following readings :—

TABLE VIII.

Load, pounds.	Time, seconds.	Extensometer Reading.		Galvanometer Readings. Uncorrected.	Galvanometer Readings. Corrected. $k = {\cdot}0089$.
1,000	0	0		0	0
			−57		
3,000	10	57		2·8	2·93
			−55		
5,000	20	112		6·0	6·52
			−56		
7,000	29·5	168		9·3	10·50
			−53		
9,000	40	221		11·5	13·50
			−60		
11,000	51	281		13·9	17·00
			−62		
13,000	60	343		16·6	21·00
			−60		
15,000	70	403		18·5	24·30
			−58		
17,000	81	461		20·1	27·30
			−65		
19,000	90	526		22·2	31·20
	100	Went off scale		23·5	34·00
	110				

Notes.—Long connecting leads.
Distance of scale from galvanometer mirror = 10′ 2″.

The specimen showed a very slight lateral set near the centre after the application of the stress, showing that the stress had been carried nearly to the point of failure, but this is not apparent in the plot of fig. 9.

Other compression tests showed the same general characteristics, and we may therefore conclude that the thermal change accompanying compression stress for iron and steel is linearly related to the stress applied through approximately the same range as the strain.

7. The Variation of Compression Stress in a Long Compression Member.

The stresses in any but a short bar are always influenced by bending, and if we assume, in pillars of rectangular section, that the lateral deflection is proportional to the square of the length l, and inversely as the thickness t, it is easy to show, with the usual assumptions of technical elasticity, that the maximum stress in a long column is greater than the stress in a short specimen of the same section, by an amount cl^2/t^2, where c is a factor depending upon the fixing of the ends. The value of c for the case of a pillar with fixed or squared ends is often taken as $\frac{1}{3000}$, where l and t are expressed in the same units. For technical applications this formula is widely used in the slightly modified form proposed by Rankine, viz. :—

$$p = f \Big/ 1 + c\frac{l^2}{r^2},$$

where p is the allowable working stress, f is the safe working stress in direct compression, and r the least radius of gyration of the section, the constant c being adjusted to agree as closely as possible with experimental values. As an example of the ease with which the stresses at different parts of the same column may be compared by thermal methods, the following result on a specimen with squared ends 15 inches long and 1·375 by ·625 inches in section may be quoted. The load was applied at the rate of 4000 pounds in 25 seconds, and the galvanometer showed no lag with the rate of loading. The thermopile was a linear one, and it was pressed against the broad face of the specimen in the direction of the breadth. The maximum deflections for different positions of the thermopile were as follows :—

Table IX.

Expt.	Distance of Pile from the top end of the specimen in inches.	Time of application of Load, seconds.	Load, pounds.	Maximum Deflection.	Deflection corrected to 70 – 4000 lbs.
1	1·38	25	70–4000	2·65	2·65
2	3·88	25	70–4000	3·00	3·00
3	7·50	25	70–4000	3·05	3·05
4	11·25	25	70–3900	2·95	3·03
5	13·38	25	100–4000	2·90	2·90

Notes.—Linear pile, long connecting leads.
Distance of scale from galvanometer mirror = 10′ 5½″.

As the maximum deflections differed very little from one another, and there was no perceptible lag of the galvanometer, no correction for radiation was necessary. The actual readings obtained are plotted as ordinates on fig. 10, the distances along the bar

being used as abscissæ, and the corrected maximum readings of column are plotted in the same way to show the variation of thermal effect.

It will be seen that the corrected readings are not symmetrical with regard to the centre of the length of the pillar, so that we may infer that the ends were not in exactly the same condition as regards fixture, and therefore it would be difficult to draw any definite conclusions; but the experiments serve to demonstrate the value of the method, and it appears probable that further experiments in this direction will be fruitful of results. As a further example of the application of the method, we may quote some experiments on the variation of thermal effect in beams.

8. The Variation of Stress in the Cross Section of a Beam.

The assumptions of the Bernouilli-Eulerian hypothesis for beams lead to the simple result that there is a neutral plane perpendicular to the plane of symmetry, and that the stress at any point of the section varies as the bending moment and as the distance from the neutral plane. The assumptions of the above theory have been shown to be false by Pearson,* who has proved that for a beam of circular cross section, subject to a surface load perpendicular to the axis of the beam, the stress does not vary according to the distance from a neutral axis, nor according to the bending moment. The variation of stress at the surface of a beam has been determined by more than one experimenter, chiefly by observations of the strains; and in order to establish the value of the thermal method for determinations of this kind, a steel I beam was chosen of the section shown in fig. 11, and this was subjected to a uniform bending moment by applying equal loads at two points, each distant 4 inches from the central section of a span of 5 feet. In this way the bending moment at the central section was made as uniform as possible. The thermopile was pressed against the beam at five different places in succession, and the deflections of the galvanometer were noted for approximately the same loading applied at a uniform rate. The value of the correction factor for each experiment was determined in the usual manner, and its value was found to be very constant, except in the last set of readings. The observed and corrected readings are plotted in place upon fig. 11, and from them a curve has been drawn, the ordinates of which represent to a reduced scale the maximum readings for a total load of 5000 pounds. The variation of thermal change is seen to be proportional to the distance from a point slightly above the centre line, and (fig. 11) to obey a linear law almost exactly. These results agree in general with those obtained by Professor Bovey, F.R.S.,† who used a very delicate roller extensometer. He found an approximately linear relation for the strains, and in most cases the neutral axis was somewhat above the centre of gravity of the section towards the compression side, a result which may be expected, having regard to the probable distortion of the section by the bending moment. The

* "On the Flexure of Heavy Beams subjected to Continuous Systems of Load," *Quart. Jour. Math.*, 1889.
† "A New Extensometer," *Trans. Roy. Soc. Canada*, 1901.

TABLE X.

Time, seconds.	Thermopile on top of Beam. 4″ above centre. k = ·0044, 5000 lbs. in 60 seconds.		Thermopile 1·72 above centre. k = ·0046, 4950 lbs. in 60 seconds.		Thermopile $\frac{1}{16}$″ below centre.		Thermopile 1·81″ below centre. k = ·0044, 5000 lbs. in 60 seconds.		Thermopile 4″ below centre. k = ·0054, 5080 lbs. in 60 seconds.	
	Galvr. Readg.	Corr. do.	Galvr. Readg.	Corr. do.	Galvr. Readg.	Corr. do.	Galvr. Readg.	Corr. do.	Galvr. Readg.	Corr. do.
0	0	0	0	0	0	0	0	0	0	0
10	1·30	1·53	·65	·66	0		– ·85	– ·87	– 1·5	– 1·54
20	2·50	2·61	1·75	1·83	– ·03		...	...	– 3·5	– 3·70
30	3·55	3·78	2·38	2·54	– ·07		– 2·15	– 2·3	– 5·8	– 6·27
40	4·90	5·33	2·75	3·0	– ·10		– 2·50	– 2·72	– 6·5	– 7·20
50	5·80	6·43	3·00	3·35	– ·12		– 2·75	– 3·05	– 7·1	– 8·06
60	7·00	7·92	3·15	3·59	– ·14		– 3·10	– 3·51	– 7·7	– 8·95
75	7·60	8·85	3·08	3·61	– ·14		– 3·30	– 3·84	– 7·7	– 9·10
90	7·40	8·87	2·90	3·50	– ·14		– 3·25	– 3·89	– 7·1	– 8·8
105	...	...	...	...	...		– 3·00	– 3·69	...	...

TABLE XI.

Distance of pile from centre line of beam in inches.	Load, pounds.	Corrected Reading for Radiation.	Corrected to 5000 lbs.
4·00	5000	8·87	8·87
1·72	4950	3·61	3·63
– ·063	5000	– ·14	– ·14
– 1·81	5000	– 3·89	– 3·89
– 4·00	5080	– 9·10	– 9·09

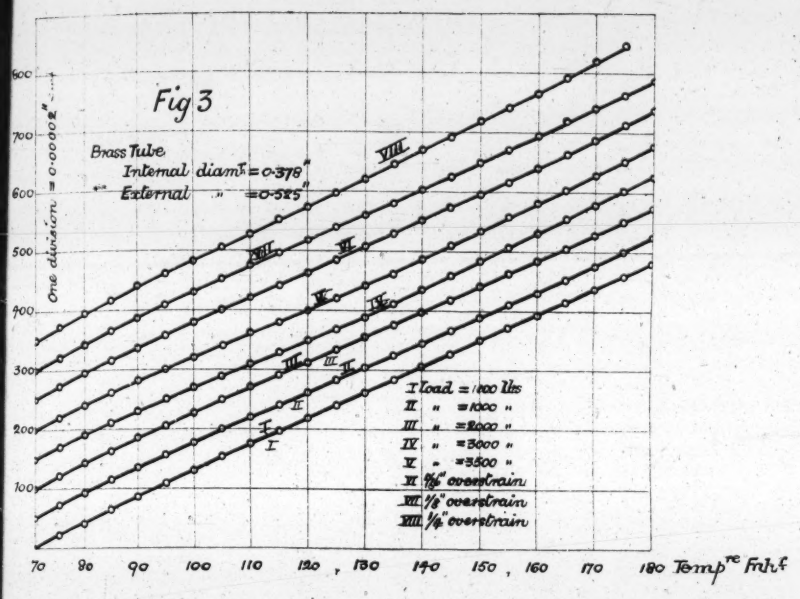
results obtained were confirmed for the same loading by other tests. In some experiments on the variation of stress in cement beams two months old with steel reinforcing, the thermal method did not give satisfactory results, and apparently the combination does not appear to behave like a true elastic solid. On the other hand, specimens of cement of considerable age behave exactly like iron and steel.

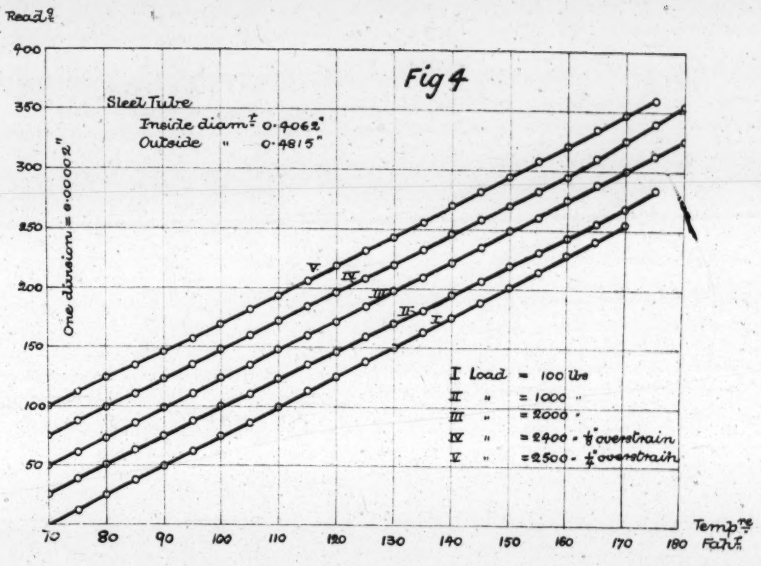
Conclusion.

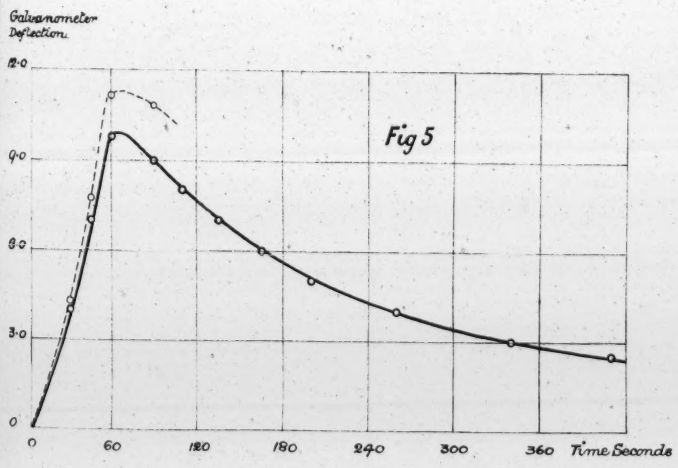
The application of the thermopile to determine the thermal change in a body, and hence the intensity of the stress, has an extremely wide application, since the thermo-elastic equations deduced by Lord Kelvin are generally applicable to elastic bodies subjected to every type of stress. Only a limited number of cases have been treated in the preceding pages, chiefly with a view of showing the range of application of the method. The thermopile, while probably forming the most convenient method of determining the thermal state of a body under stress, is not the only one which could be applied. During the summer of 1903, the author, by the kindness of Professor Cox, Director of the Physics Building, M'Gill University, was enabled to experiment with a Boys radio-micrometer, which was set up in close proximity to specimens in tension and compression, and the readings were found to be proportional to the load; but, with the limited experience of the author, it was found to be much less convenient than a thermopile, mainly because of the extreme delicacy of the apparatus, and the difficulty of setting it horizontally upon a steady base near the testing machine.

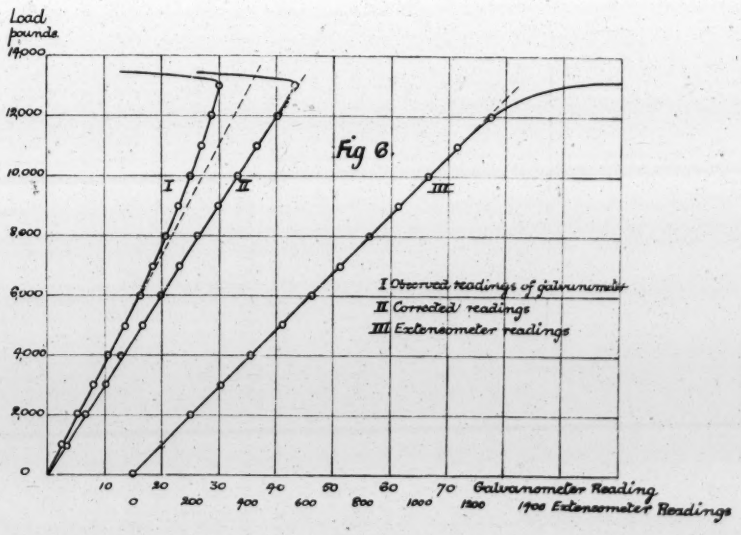
In conclusion, the author desires to express his warmest thanks to Professor Bovey, F.R.S., for the use of his well-equipped testing laboratory, and to Professor Rutherford, F.R.S., for valuable suggestions during the progress of the work; also to Mr M'Kergow, Demonstrator in Civil Engineering, for his untiring assistance in making observations.

COKER: ON THE MEASUREMENT OF STRESS BY THERMAL METHODS. — PLATE I.



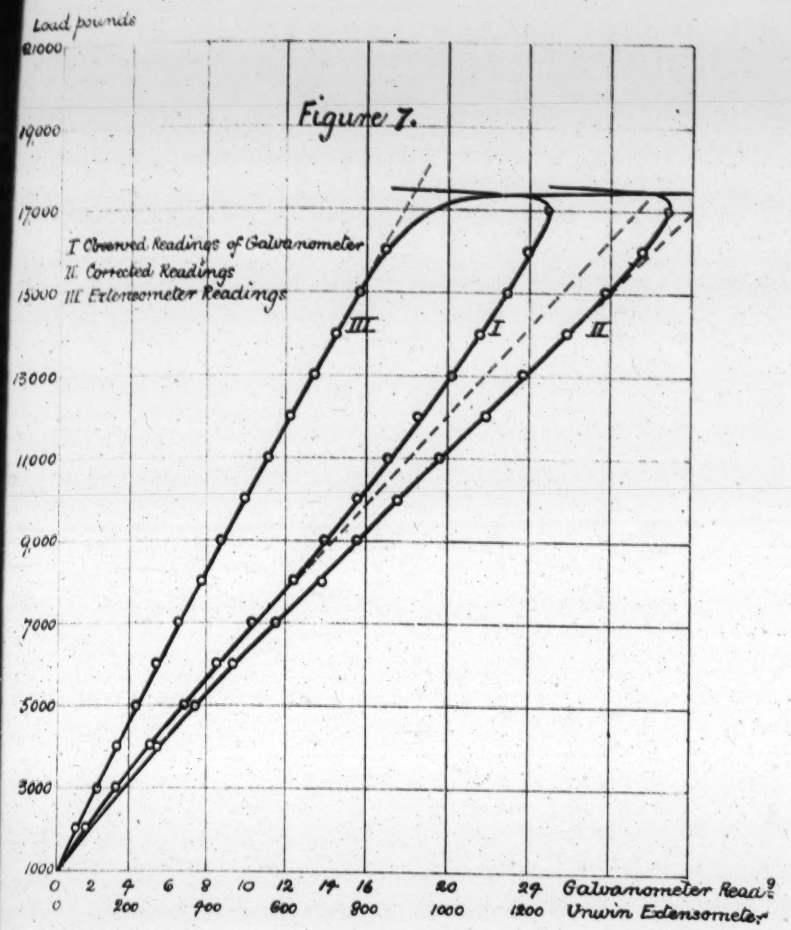


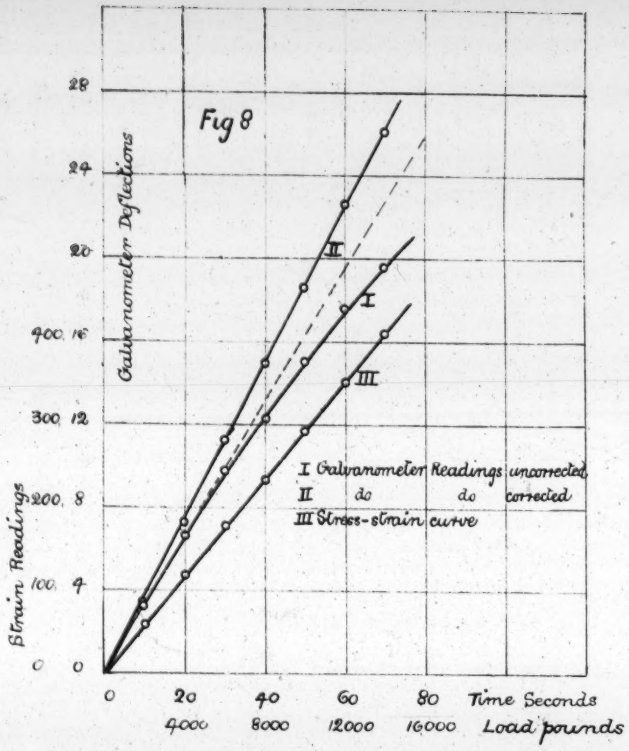


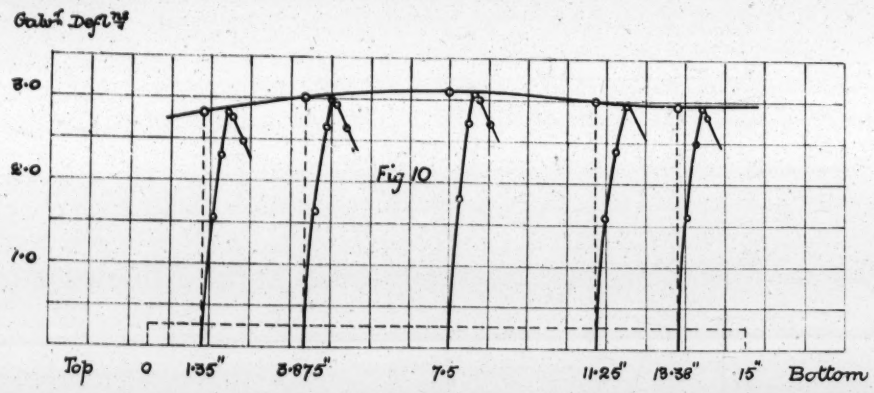


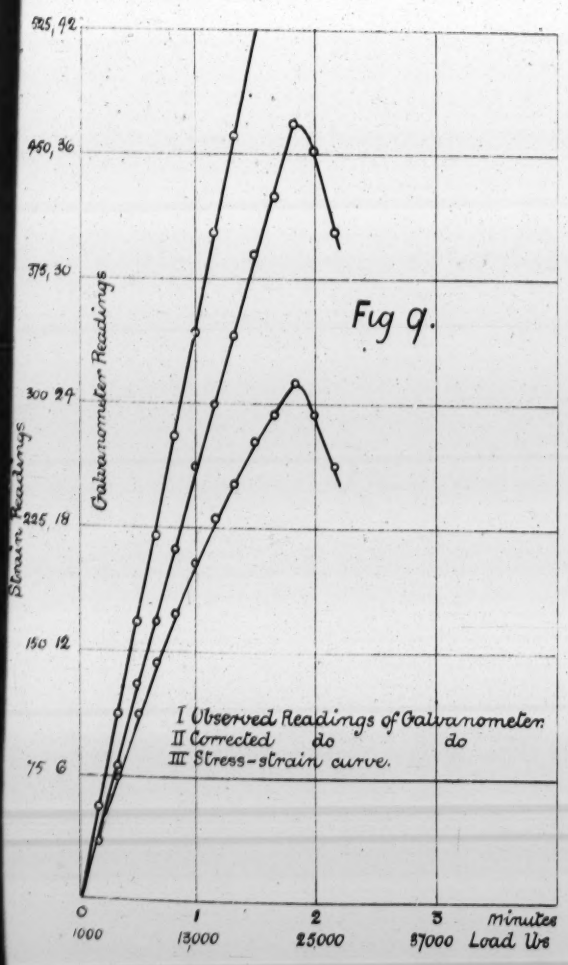
A. RITCHIE & SON, EDIN.

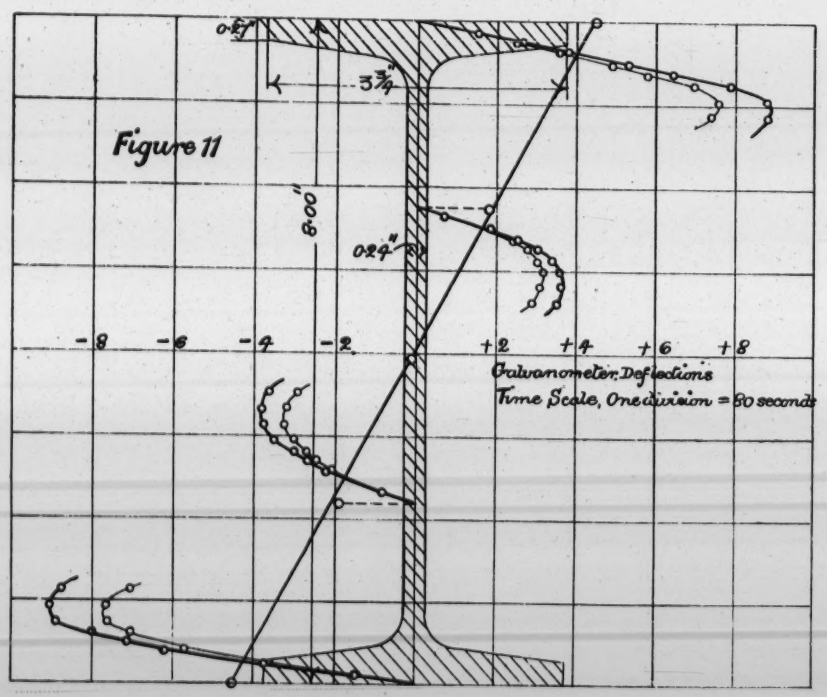
COKER: ON THE MEASUREMENT OF STRESS BY THERMAL METHODS. — PLATE II.











X.—On the Spectrum of Nova Persei and the Structure of its Bands, as photographed at Glasgow.

By L. Becker, Ph.D., Professor of Astronomy in the University of Glasgow. (With Three Plates.)

(MS. received May 2, 1904. Read June 6, 1904. Issued separately September 9, 1904.)

The spectrum of the new star in Perseus, which Dr ANDERSON, of Edinburgh, discovered 1901 February 21, was photographed at the Glasgow Observatory from 1901 March 3 till 1903 January. From the early photographs one gains the impression that the spectrum consists of a number of bright bands of different lengths, fading towards the ends, and overlapping each other, thus producing a series of maxima and minima of brightness. Near wave-length 5000 the intensity rapidly falls off towards the less refrangible side, and the bands appear detached. The middle of each of the three most intense maxima approximately coincide with the hydrogen lines H_β, H_γ, H_δ, and on two photo-plates, March 18 to 20 and March 25, each of the bands is crossed by a sharp Fraunhofer line. On the photo-plates taken after 1901 August 1 the bands are all detached; some, including the two bands whose middles approximately coincide with the principal nebular lines, have almost the same lengths, and suggest a line spectrum in which the lines have been broadened into bands, others are considerably longer and have pronounced maxima.

While it was probable that the three hydrogen lines and the two principal nebular lines were represented in the spectrum by bands, it remained to be proved that the wave-length of a definite point of the band bore a definite relation to the wave-length of the line to which it belonged. As a result of my investigations, founded on micrometric measurements and estimates of intensity, I shall show that the bands which contain a series of reversals are similar in type, and that the ratio of the distance between any two points in a band to that between corresponding points in another band is the ratio between the wave-lengths. The spectrum of Nova Aurigæ resembled that of Nova Persei very closely; its changes followed the same course, and it showed the considerable broadening of the lines into bands, the structure of which has, however, never been investigated. The systematic broadening of the spectral lines into bands, which for Nova Persei amounted to a 35th of the wave-length in March and April, and a 100th after August, seems to be a feature of new stars, and ought to be accounted for in an explanation of these objects.

2. *The Spectrograph.*—The spectrograph of 8 cm. aperture is connected to the Breadalbane reflector of 51 cm. aperture and 446 cm. focal length. The equatorial mounting of this instrument, probably made by the late Thomas Grubb some fifty years

ago, is remarkable in so far as its inclined stand anticipates the chief structural feature of the Potsdam astrographic refractor. The reflector, which had proved useless in its old condition, obtained in 1895 a new silver-on-glass mirror, driving clock with electric control, driving sector and declination clamp with slow motion, all from the works of Sir Howard Grubb. The mounting of the spectrograph was supplied by a local smith. A platform forming a right-angled triangle extends from the upper end of the tube to the free end of the declination axis, and its plane is inclined 35 degrees to the optical axis. Parallel to it, the central ray of the reflector is reflected by a plane mirror. The platform is a stiff structure for its weight. It consists of two layers of corrugated iron, with the corrugations crossed and bolted at every point of contact, and it is strengthened by thin sheet steel ribs. To it is clamped a quarter-inch steel sole-plate, with adjustable bearings for the two tubes of the spectrograph, and on this sole-plate a small cast-iron table carrying the prism-box can be adjusted and clamped. The platform rests at its upper end, a corner of the triangle, on a casting which is bolted to the tube of the reflector; at its lower end, the shortest side of the triangle, it is screwed to a strong cast-iron arm, which is fixed to the declination axis, in place of the balancing weights, at right angles to this axis and the axis of the tube. As I had the declination axis lengthened, and the telescope tube shortened and placed more favourably in its cradle, the movable part of the instrument weighs now less than in its old condition.

The object-glass of the collimator has an aperture of 8·2 cm. and focal length of 74 cm.; that of the camera, a Cooke triplet, 8·9 cm. by 149 cm. The focal length of both combined has a large temperature coefficient, 0·13 mm. for a degree centigrade. The prism made by Hilger of white Jena flint glass measures 16·5 cm. on a side, and is 9·5 cm. in height. Since it was re-annealed its separating power is most satisfactory. The central portion of the spectrograph is enclosed in a box, and by means of a small heating apparatus the temperature of the prism and the object-glasses can, at least to some extent, be kept under control. Unfortunately, the instrument cannot be used in summer after a sunny day, because in the iron dome the large prism is heated in such a way that the definition becomes too bad for accurate work.

The jaws of the slit are formed by the two halves of a circular mirror 2·5 cm. in diameter, and they open symmetrically 0·15 mm. for a revolution of the screw. The width here employed was usually 0·018 mm. The plane of the mirrors which form the jaws of the slit is inclined 7 degrees to the plane normal to the central ray. If the image of the star does not fall on the slit, the rays are reflected towards a small mirror which is fixed to the telescope tube, and thence towards a viewing "telescope" (which is focussed on the slit) of 7 cm. aperture and 30 cm. focal length (two object-glasses mounted close together). It lies almost parallel to the collimator. Owing to the large size of the jaws of the slit, the effective field is half a degree, which is a great convenience in finding a star and setting it on the slit. The spark apparatus is hinged to the platform in front of the slit. When turned into position, the optical axis of its lens coincides with that of the collimator.

The collimator lens, though a fine object-glass, F/9, did not prove to be achromatised for photographic rays; the focal curve of the spectrograph runs along zero from D to H_β, then gradually rises, and at wave-length 3700 the ordinate is 2·8 cm. I had therefore to incline the photographic plate up to 30°.

When the telescope is turned through twenty-four hours of hour-angle, the image in the focal plane of the spectrograph oscillates in simple harmonic motion, with an amplitude of 0·8 mm., along a line which is slightly inclined to the spectral lines. I compensated this deficiency of the mounting by making the plate-holder adjustable in the direction of the spectral lines, and at right angles to them, the motion being effected by two micrometer screws of 0·5 mm. pitch. The plate-holder takes the plate, 11 by 4 cm., in its upper half, which lies central in the camera, and in the space below it carries an eye-piece with stout cross wires. As fiducial line for keeping the plate stationary with regard to the spectrum, I employed the magnesium line 4481, which is almost covered by the stout wire in the eye-piece. A split spring, which presses against the jaws of the slit, cuts out a short line from the upper portion of the slit 1·5 cm. above the optical axis, and the magnesium terminals are placed close to it, inside a short glass tube, to guard the slit against tarnishing. By these means I am able to keep a line of the spectrum always on the same place of the photographic plate, and to replace the plate after days in its old position. The differential change of dispersion due to changes of temperature is, of course, not taken into account. During an exposure of the plate I moved the plate-holder every time 0·01 revolution of the micrometer screws, at intervals given by a table, and checked the position once an hour direct on the magnesium line. To illustrate the efficiency of this method, I mention that on photo-plate No. 23, comparison lines at a distance of 0·04 mm. appear separated, though they were exposed on twelve different occasions, five seconds each time, on two days, and at different hour-angles.

At the time the new star was announced, wave-length 5200 t.m. was in the centre of the field of the camera, and 4000 at the end of the plate. No change was made in the position of the camera until the beginning of October 1901, when wave-length 4170 was placed in the centre.

The distance of the hydrogen lines H_β and H_γ is 20 mm. on the plate, and one tenth-metre or Angstrom unit is represented on the plate by 0·1 mm. at $\lambda = 3500$, 0·05 mm. at $\lambda = 4300$, and 0·025 mm. at $\lambda = 5200$ t.m.

3. *The Measurements and their Reduction to Wave-Lengths.*—The plates taken in March and April 1901, and again those after January 1902, were difficult to measure,—the former, owing to the gradual change of intensity of the spectrum, which presented few definite points to set on; the latter, owing to the faintness of the spectrum, some parts of which could barely be distinguished from the accidental markings on the film. I finally adopted the rule to measure every point to which the eye was drawn, except those which I thought to be defects in the film. With respect to these, I became more careful as the work advanced; and it is possible that the earlier plates may contain

more than were actually measured of the minima, or reversals, which were present in the spectrum during the whole period. On the plates taken between August 1901 and January 1902 the structure of the bands, including the minima, is easily seen, and in some bands it is visible to the unaided eye. The intensity of the spectrum between every two points measured was estimated on an arbitrary scale, the estimated "degrees" of intensity increasing with the intensity. In this paper the "intensity of the spectrum" stands for the intensity of blackness on the negative, while "intensity of radiation" is used for the intensity of light in the focal plane of the spectrograph. I measured each plate about four times, alternately in opposite directions. Each series includes a number of settings on the lines of the comparison spectrum, iron-calcium until September, and iron-titanium afterwards. The points measured on the same plate were then identified by a graphical process, and all those were discarded which had not been repeatedly observed. Only on plate No. 5 I made an exception, where, after the discussion was finished, I included two minima which had only once been measured. Since the measuring occupied about half a year, and no measurements were taken after the reductions were begun, the results of the different plates may be considered independent. For each plate I reduced each series of measurements separately to wave-lengths, and then combined them to mean values. The tables used in the reductions give the position of the micrometer screw of the measuring apparatus, referred to an arbitrary zero, for each wave-length at an interval of 1 tenth-metre; they are based on Ketteler's formula of dispersion,* and were prepared for the angles of inclination at which the plates were exposed.

The comparison spectrum determines the correction curve of the zero of the table.

The Spectrum in March and April 1901.

4. *Results of Measurements.*—The results derived from the measurements made on the photo-plates Nos. 1 to 7 are given in Table I. The first column contains the mean wave-length, the second the average difference of one measurement from the adopted mean value, and the third, under the heading "Intensity," the estimated degrees of intensity. The notation $^{9}_{7}$ indicates that the intensity gradually changes from degree 9 to 7.

* *Annalen der Physik und Chemie*, 1881, 12.

TABLE I.

Photo-Plate No. 1, 1901 *March* 3.

Angle of inclination of plate = 0°; width of slit 0·018 mm.; orthochromatic plate; exposure 2·4 hours; the spectrum was measured 6 times and referred to 50 standard lines; good definition of spectrum; the more refrangible end beginning at λ = 4500 is increasingly out of focus.

λ	*v.*	Intensity.	λ	*v.*	Intensity.	λ	*v.*	Intensity.	λ	*v.*	Intensity.
3967		5	4317·7	0·5	4	4844·4	0·7	9	5007	1·6	3
3972	1·3	6	4320·2	0·6	3	4847·6	0·8	=15	5024	6	3
4010	2·4	5	4321·8	0·4	6	4850·3	1·5	=15	5046	2	1
4049	1·2	4	4325·5	0·3	6	4851·6	1·0	14=15	5121	2	0·3
4077·8	0·8	2	4329·3	0·6	8	4852·9	0·5	=15	5143	2	1
4086·2	1·1	5	4334·1	0·8	12	4854·8	1·2	12	5160	1	0
4090·5	0·8	10	4359·7	0·7	11	4869	1·4	8	5210	1·6	3
4097·7	0·4	9	4367	1·5	10	4873·2	0·5	14	5241	3	1·5
4117·9	0·4	9	4412·8	0·8	9	4883·2	0·7	8	5271	1·6	0·5
4123·7	0·8	7	4435	2	7	4890·0	0·8	6	5306	6	0
4145·6	0·6	6	4454·2	1·0	6	4896·9	0·5	3	5347	2	1
4161·3	1·2	5	4476	1·7	7	4902	1	2·5	5362	1	3
4186·5	0·7	7	4584 *	3	9	4907	1·3	2·5	5381	2	1·5
4200·8	0·9	6	4619	4	10	4925·0	0·7	5	5520	2	0
4209	1·6	5	4640	7	6	4934·2	0	5	5544	9	2
4221·6	0·5	4	4686	3	7·5	4943·1	0·8	4	5586	1	4
4253·6	1·0	5	4801	2	8	4959	4	3	5678	3	4·5
4281	5	6	4835·2	0·5	7	4981	2	2	5711	4	4
4312·5	0·5	4	4839·3	0·9	6	4985	8	1	5747	3	2
4317·7	0·5	5	4844·4	0·7	5	5007	1·6	0·3			1
		4			4			2			0·3
					8						

* The enlargement shows a maximum within this space.

Photo-Plate No. 2, 1901 *March* 6 *and* 8.

Angle of inclination of plate = 0°; width of slit 0·018 mm.; orthochromatic plate; exposure 1·8 hours, 1 h. on March 6 and 0·8 h. on March 8; the plate was intensified and is slightly fogged; the spectrum is not so dark as on No. 1; it was measured 4 times, 3 times on the intensified plate, and referred to 22 standard lines; good definition; focus same as on No. 1.

λ	*v.*	Intensity.	λ	*v.*	Intensity.	λ	*v.*	Intensity.	λ	*v.*	Intensity.
4089	2	2	4359·2	0·4	4	4545	3	4	4852·3	0	10
4118·2	0·3	0	4365·7	0·7	3	4600	1·8	3	4855·4	0·9	7
4296	2	1	4375·6	1·3	2	4616·7	0·4	4	4857·7	0·3	10
4319	1·5	0	4392·0	0·2	1	4643·6	0·6	3	4861·8	0·4	8
4324·6	0·4	3	4413·1	0·4	2	4691	2	2	4870	2	10
4328·6	0·4	8	4466·0	0·7	0	4720·2	1·1	1	4879·7	0·9	7
4337·8		8	4477	5	2	4840	3	0	4881·8	1·1	4
4347	2	9	4499	4	3	4846·4	0·7	2	4883·3	0·1	2
4355·5	0·6	6	4528·0	0	3·5	4849·9	0·7	7	4892	2	1
4359·2	0·4		4545	3	3	4852·3	0	10	4928	3	0·5

Table I. is continued at the top of the next four pages.

TABLE I.—*continued.*

Photo-Plate No. 3, 1901 *March* 18 *and* 20.

Angle of inclination of plate = 13°; width of slit 0·018 mm.; Imperial plate; exposure 4·4 hours, 1·2 h. on March 18 and 3·2 h. on March 20; clear negative; the spectrum was measured 6 times, the H-bands 12 times, and referred to 50 standard lines; good definition.

λ	*v.*	Intensity.	λ	*v.*	Intensity.	λ	*v.*	Intensity.	λ	*v.*	Intensity.
4028	1·5	1	4251	4	7	4512	4	10·5	4871·0		13 = 9
4034·8	0·4	1·5	4260	4	6	4535	2	10	4875·1		13 = 9
4037·3	1	2	4284·7	0·8	6	4551	2	10	4879·4		11
4044·0	0	2	4315·31		7 = 0	4562	4	11	4883·9		8
4055	2	3	4318·4		7	4590	2	11 10	4895	2	7
4061	3	3·5	4319·03		0	4602	3	9	4900		6 3
4080·76		4	4319·6		0	4607·2	0·1	8	4911·4		3 4
4081·51		0	4326·9		9	4611	1·3	9	4922		3·5
4082·2		6	4335·6		11 12	4620	3	10	4931·2	0·2	3
4085·7		7	4340·9		13	4643	6	10 11	4941·0	0·5	2
4090·4		8	4343		14	4679	8	10	4956	2	1 1
4097·8		9	4348·7		14 = 10	4700	3	8	4978	1	1·5
4102·7		10	4358·0		14 12	4735	7	6	5003	4	2
4117·9		10	4368·9		10	4794	7	5	5025	2	1·5
4121·1		9	4382	5	9	4833·46		5 = 0	5042	3	1
4132	5	6	4396	1·4	9 10	4836·5		5	507–	6	0·5
4148		6	4423	3	10	4837·43		0	509–	10	0·2
4154·5	1·1	5	4427·8	0·6	9	4838·4		8	5133		0·3
4165	2	5	4442	2	9 9	4844·1		10	5154		0
4187	3	6	4459	3	10	4849·4		13	5175		0·2 0·2
4203·9	0·7	5·5	4473	3	10·5	4855·8		9	5188		0·1 0·5
4219	1·3	5	4492	3	11	4858·0		13	5207		0·1
4233	1·6	6	4502	6	10·5	4864·1		13 = 9	5265		0·1 0
4251	4	7	4512	4	10	4871·0		= 9	5295		

5. *The three H-bands.*—On photo-plates Nos. 3 and 4 three well-defined Fraunhofer lines are a prominent feature, appearing respectively in the neighbourhood of the hydrogen lines H_β H_γ H_δ, and the difference between the wave-length of a Fraunhofer line and that of a corresponding H-line is proportional to the wave-length. Towards the less refrangible side of the Fraunhofer lines there are bright bands, which also occur on the other five photo-plates. If the wave-lengths of corresponding points of the three bands be compared with the wave-lengths of the three hydrogen lines, the differences are found also to be proportional to the wave-lengths. I conclude that the

TABLE I.—*continued.*

Photo-Plate No. 4, 1901 *March* 25.

Angle of inclination of plate = 13°; width of slit 0·020 mm.; Imperial plate; exposure 4 hours; the plate is fogged and the spectrum is much fainter than on plate No. 3; the spectrum was measured 4 times and referred to 30 standard lines; the definition is not so good as on plates Nos. 1, 2, 3, 5.

λ	v.	Intensity.	λ	v.	Intensity.	λ	v.	Intensity.	λ	v.	Intensity.
4041·5	0·1	0·2	4318·7	0·2	0	4540	4	4	4851·8	0·9	8
4049	1·4	0·5	4320·0	0·6	3·5	4559·7	0·3	3	4858·0		3
4056·4	0·2	1	4325	1·3	4·5	4589	1·3	5·5	4859·6		8
4065·0	0·5	2	4328·9	0·4	5·5	4613	2	3	4863·4	0·7	2
4072·6	0·7	3	4335·2		3	4617	1·1	3·5	4865·0		
4081·1	0·4	0	4337·8		7	4626·6	0·8	4	4866·1	0·4	8
4083·4	1·1	4	4344·9	0·4	8	4648·4	0·7	6	4871·0		0
4092·5	1·0	5·5	4354·0		0	4660	2	8	4872·8		7·5
4100·8	0·9	6	4354·6		7	4669·1	0·6	7	4879·9	0·7	4
4109	1·6	5	4360·0	0·3	6	4682	4	6	4882		1
4115·6	1·0	4	4365·2	0·3	5	4691·0	0·5	8	4884		3
4119	2	3	4372	1·2	4	4701	3	7	4885·5	0·7	2
4122·0	1·0	2	4384	1·7	3·5	4722·0	0·8	6	4894	1	1·5
4135	1·8	1	4398	5	3	4729	1·8	4	4905·7	0·6	1
4157	3	0	4428·7	0	2	4739·9	0·4	3	4912		0
4165	1·6	1	4434	1·3	2·5	4750	1·1	2·5	4929		1·2
4207	5	2	4454·6	1·0	3	4770	4	1·5	4947	1·1	1
4261	1·6	1	4459·3	0·7	4	4826·3	0·9	1	4968	1·5	0·3
4277	1·7	1·5	4480·5	0·4	5	4834·4	1·1	0·3	4986·2	0·2	0·1
4282·8	0·8	2·5	4493	4	6	4836·1		2	5006	6	0·3
4307	1·7	3	4503·9	0·8	5	4837·9	0·1	0	5020	4	0·5
4317·7	0·2	0	4527	3	3	4846·2	0·6	3	5045·8	0·8	0·4
4318·7	0·2		4540	4	4	4851·8	0·9	5			
					4·5						

bands are due to hydrogen radiations, which, under ordinary conditions, would produce the three hydrogen lines. Tables II., III., IV. prove the statement.

Let λ_0 be the wave-length of a hydrogen line, λ the observed wave-length of a point in the band, s the correction for the orbital motion of the earth, then

(1) $$\lambda + s = \lambda_0 + a_0 \frac{\lambda}{4500}$$

determines a_0, belonging to a certain point of a band, and it must be shown that a_0 has the same value for corresponding points of all three bands. In Table II. the values of a_0 are compiled. The positions of the first, third, fourth and fifth minima agree as

TABLE I.—*continued.*

Photo-Plate No. 5, 1901 *March* 25.

Angle of inclination of plate = 13°; width of slit 0·020 mm.; Imperial plate; exposure 3·2 hours; the plate is badly fogged; the spectrum was measured 5 times and referred to 35 standard lines; good definition.

λ	*v.*	Intensity.	λ	*v.*	Intensity.	λ	*v.*	Intensity.	λ	*v.*	Intensity.
4032·9	0·3		4235·0	0·8		4377	5		4825·5	0·7	
		0·5			5			7			6
4038·2	0·5		4248	1·1		4405·5	0·9		4839·9	0·2	
		1			4·5			6			8
4054·4	0·8		4279	3		4427·2	0·6		4846·0		
		1·5			4			7			0
4062·7	1·0		4298·4	0·4		4454	1·2		4848·8	0·5	
		2·5			5			8			11
4082	1·8	4	4313·8	0·5	6	4471·3	0·8		4855·8	0·8	max.
		|			|			8·5			11
4089	1·2	6	4325·2 *		7	4477·3	0·8		4867·9	0 8	max.
		7			3			9			11
4099·1	0·4		4326·9	0·5		4486·3	0·9	8	4874·5	0·8	
					8·5			|			9
4106	1·3	min.	4336·5	0·4		4507	2	7·5	4878	2	7
		}			6						|
4113	1·3	8 min.	4337·3			4556	4	7	4887	1·4	4
		}			9			8			2
4117·3	0·4		4343·5	1·0		4582	4		4898·9	1·0	
		7			7			6			0·5
4124·6	0·2	6	4345·9			4605·7	0·5		4910·1	1·0	
		|			10		7				2
4129	1·2	5	4353·3	0·3		4615·4	0·6		4932	1·7	
		4·5			9			9			1·5
4145	1·7		4358·4	0·5		4660	3		4941·6	0·7	
		4			8·5			8			0·5
4163·5	0·5		4362·0		=4	4697·8	0·5	7	4950·6	0·9	
		5			8·5			6·5			1
4179	1·3		4364·5	0·6		4724·7	0·8	5·5	4963	2	
		3·5			4			5			0·5
4220·9	0·1		4366·5		8	4753·8	1·0	4	5037	1·6	
		4			|			3·5			0·2
4235·0	0·8		4377	5	7	4806	2		5090·7	1·0	
								3			
						4825·5	0·7				

* One measurement.

closely as may be expected from the accuracy of the measurements. The sixth minimum appears as such only in two bands on photo-plates Nos. 5 and 6, but each being measured repeatedly and independently, there can be no doubt that it really exists; besides, the other five settings made at this point appear to me to suggest that it existed also at those places, though it was not appreciated. The second minimum is questionable, because both positions rest on only one measurement.

The estimated degrees of intensity at corresponding points of the same H-band do not agree on the seven plates. The observations are too few to determine the reductions to an average scale of the estimated degrees of intensity as a function of the degrees; but since the differences of the estimates made on two plates at corresponding places are almost the same for all degrees, I reduce the estimates to the scale employed on photo-plate No. 3 by adding to the degrees on plates Nos. 1, 2, 4, 5, 6 respectively 0, 4, 5, 2, 5. The result is given in Table III.

TABLE I.—*continued.*

Photo-Plate No. 6, 1901 *April* 1 *and* 5.

Angle of inclination of plate = 0°; width of slit 0·020 mm.; orthochromatic plate; exposure 6 hours, 3·2 h. on April 1 and 2·8 h. on April 5; plate was intensified and the spectrum is faint; the spectrum was measured 4 times and referred to 30 standard lines; fair definition; focus same as on plate No. 1.

λ	*v.*	Intensity.	λ	*v.*	Intensity.	λ	*v.*	Intensity.	λ	*v.*	Intensity.
4087	?	0·5	4442	4	3	4731	1·6	0·5	4933	1·3	0·5
4101	?	0·8	4462·2	0·7	3·5	476–	8	0·3 1	4943	3	0
4114	?	0	4483	1·3	3	4800	3	2	5037	6	1 ?
4310·7	0·1	0·3	4498	3	2·5	4846·5	0·6	5	5063	3	0
4328·4	0·8	3	4512	?	*	4853	1·8	7	5160	2	0·2 ?
4337·5	1·6	=0 4	4563	?	3	4863·8	0·6	0	5177·0	0·1	0
4344	1·8	5	4568	3	3·5	4865·1	0·6	8	5299	6	0·2
4350	1·6	7 =0	4593	3	3	4870·7	0·1	9 =0	5383	4	0·5
4355·9	0·5	6	4614	2	3·5	4876·1	1·0	5	5447	7	1
4358·8	0·9	4 2	4638	4	4	4878·9	0·9	4	5498	7	0·8
4370	2	1	4663·1	0·8	3·5	4882	2	3	5578	2	0
4385·1	0·2	0·2	4678·1	0·8	3	4895·4	0·7	1	5604	3	0·5
4402·4	0·5	0·5	4690·0	0·6	2·5	4911	4	0·2	5677·7	0·1	0·8
4430·5	0·4	2	4709·4	0·7	1	4925	1·7	0·5	5713	1·7	0·2
4442	4	2·5	4731	1·6		4933	1·3	0	5739	4	

* Defect in film?

Photo-Plate No. 7, 1901 *April* 10 *to May* 3.

Angle of inclination of plate = 0°; width of slit 0·020 mm.; orthochromatic plate; exposure, near horizon, 9 hours on 7 days; the plate is fogged; the spectrum was measured twice and is very faint; bad definition.

λ	Intensity.	λ	Intensity.	λ	Intensity.	λ	Intensity.	λ	Intensity.
4325	3	4455	1	4614	3	4845	3	5005	1
4360		4504	3	4718		4882		5021	
		4536							

In Table IV. I have combined the results obtained from three plates to mean values, in order to exhibit the agreement in position of corresponding points of the three H-bands as reduced to λ = 4500, and also to show in what manner the observed degrees differ. Towards the less refrangible end H_γ declines in intensity 4 degrees on 25 tenth-metres and passes into a bright spectrum, while H_β decreases 8 degrees and fades into

Table II.

Table of a_0 in Tenth-Metres of Corresponding Points of the H-bands, showing that the Bands are similar in March and April 1901, *and the Broadening proportional to the Wave-Length.*

	No. of Point.	H_δ. Index Number of Plate.						H_γ. Index Number of Plate.							H_β. Index Number of Plate.						
		1	2	3	4	5	6	1	2	3	4	5	6	7	1	2	3	4	5	6	7
Intensity	1			− 45	− 40·7	− 43·2			− 47			− 44·2									
increases	2				− 32·4			− 29·6			− 36	− 28·3	− 31·4					− 33·0	− 33·7		
1. Minimum	3	− 26·8		− 23·4	− 23·0	− 22		− 24·3	− 23·0	− 23·4	− 24·1				− 24·7		− 23·5	− 25·5			
				− 22·59						− 22·8	− 23·1						− 22·66				
	4			− 21·8	− 20·5			− 21·7		− 22·2	− 21·8				− 20·9	− 20·0	− 21·8	− 22·2	− 20·4		
	5	− 17·6		− 18·0			− 17														
2. Minimum	6					− 14·5		− 16·2	− 17·1	− 14·6	− 16·2	− 16·4	− 13·1	− 16·0	− 16·2	− 14·4	− 16·5	− 14·5	− 14·7	− 14·2	− 15·6
	7											− 14·6							− 12·1		
	8	− 12·8	− 15·	− 12·8	− 10·5			− 12·2	− 13·0		− 12·5				− 13·2	− 11·1	− 11·6	− 9·4		− 8	
3. Minimum	9										− 6·0	− 4·7				− 6·0	− 5·6	− 3·6			
	10	− 4·9		− 4·7		− 3·3	− 1	− 7·3	− 3·4	− 5·6	− 3·3	− 3·8	− 3·6		− 6·6	− 3·9	− 3·6	− 2·1			
4. Minimum	11											+ 2·6						+ 1·4		+ 1·9	
	12			+ 0·7	− 1·4	+ 3·8				+ 2	+ 4·0	+ 5·1	+ 3			− 0·1	+ 2·0	+ 3·9		+ 3·1	
5. Minimum	13														+ 6·7			+ 8·4			
	14				+ 7				+ 6	+ 8·0			+ 9		+ 10·5	+ 7·8	+ 8·4	+ 10·1		+ 8·2	
6. Minimum	15										+ 13·5										
	16				+ 14·8	+ 11·5	+ 13				+ 14·1	+ 12·8					+ 12·2		+ 11·7	+ 13·2	
Intensity	17	+ 17·2	+ 17·7	+ 17·3	+ 18·5	+ 16·7			+ 14·9	+ 17·6		+ 18.0	+ 15			+ 16·5	+ 16·2	+ 16·7	+ 15·3	+ 15·8	
	18			+ 20·8	+ 21·8			+ 19·3	+ 18·8		+ 19·7		+ 18	+ 19·9	+ 19·7	+ 19·8	+ 20·4	+ 21·9		+ 19	+ 18·5
	19	+ 23·6				+ 24·7		+ 27	+ 25·5		+ 25·1	+ 24·3			+ 26·0	+ 28		+ 28	+ 23·9		
	20			+ 32·8		+ 29·8				+ 29	+ 31·7		+ 29·9		+ 32·4		+ 31·1		+ 34·3	+ 31	
decreases	21				+ 36·5				+ 36			+ 37·8			+ 37·0						

Table III.

Table giving the Reduced Degrees of Intensity at Corresponding Points of the H-bands.

	No. of Point.	H_δ. Index Number of Plate.						H_γ. Index Number of Plate.							H_β. Index Number of Plate.						
		1	2	3	4	5	6	1	2	3	4	5	6	7	1	2	3	4	5	6	7
	1	4	...	3·5	6	3·5		5	?	7	7·5	6	?		5	?	5	?	5	6	
	2	4	...	4	7	4·5		5	5	7	7·5	7	?		5	?	5	?	5		
	3	4	...	4	8	4·5		4	5	7	8	8	?		5	?	5	7	8		
1. Minimum	4	2	...	0	0	6		4 3	?	0	0		?		4 3	?	0	0	8		
	5	2	...	6	9			6	?	9	8·5		?		3	6	8	8	10		
	6	5	...	7	9	8	5	6	?	9	8·5	9	?		8	6	8	8	10	7	
2. Minimum	7	...	...	...	...	...	...	...	...	...	...	5	...		...	...	...	...	2	...	
	8	5	...	7	9	8	5	8	7	11	9·5	10·5	8		9	11	10	10	13	10	
	9	10	6	8	10·5	9	5	12	12	11	10·5	10·5	9		14	14	13	13	13	12	
3. Minimum	10	...	...	...	...	...	...	...	10	...	8	8	<5		...	...	9	8	...	...	
	11	9	6	9 10	10·5	10	6	11	12	12 14	12	11	10		12	14	13	13	13	12	
4. Minimum	12	...	...	...	...	min.	...	...	...	...	...	9	...		...	...	9	7	...	<5	
	13	9	6	10	11	10	6	11	12	14	13	12	12		12	12	13	13	13	13	
5. Minimum	14	...	...	...	...	...	...	...	...	10	...	...	<5		8	...	9	5	...	<5	
	15	9	6	10	10	10	6	11	13	14	13	12	11		14	14	13	13	13	14	
6. Minimum	16	...	...	...	...	min.		...	...	...	5	...	...		...	...	9	...	...	...	
	17	9	6	10	9	10		11	13	14	12	11	11		14	14	13	13	11	10	
	18	9		10 9	8	9		11	10	12	12	10·5	9		14	11	11 9	9 8	9	9 8	
	19	7		9	7	9		10	8		11	10·5	7		8	6 5	9	7	6	6	
	20	6		6	7	8 7		9	7	10	10	10	6		6 3		7	6·5 6	4	6	
	21	6		6	7	6·5		9		10	9		5		2·5		6	6	2·5	5	
		6		6	7	6·5		9	6	9	9	9	5		2·5			6	2·5	5	

a faint band. The difference cannot be due to a change in sensitiveness of the photographic film, as its effect is inappreciable within the extent of a band. On the other hand, it will be proved from the later plates that these bands must be similar also in intensity, and therefore other radiations must be superposed on the hydrogen radiations at the ends of the bands.

TABLE IV.

Mean of the Results from three Plates, Nos. 3, 4, and 5, showing the Similarity of Structure of the three H-bands and the more rapid Falling-off in Intensity of the Hβ-band.

No. of Point, Tables II., III.	Hδ		Hγ		Hβ	
	a_0	Int^y.	a_0	Int^y.	a_0	Int^y.
1	−43	4	−44	7		5
2	−32	5	−32	7	−33	5
3	−23	5	−24	8	−24	7
4*	−21	0	−22	0	−21	0
5	−18	7		9		9
6	−14	8	−16	9	−16	9
7		8	−14	5	−14	2
8	−12		−12	10	−11	11
9	−4	9	−6	11	−5	13
10			−4	8	−3	9
11	+1	10	+2	12	+1	13
12		min.	+4	9	+3	8
13	+7	10	+8	13	+8	13
14				=10	+10	7
15	+13	10	+13	13	+12	13
16		min.	+14	5		= 9
17	+17	10	+18	12	+16	12
18	+21	9	+20	11·5	+21	10
19	+25	8	+25	11	+26	9
20	+31	7	+30	10	+33	8
21	+36	6	+38	9·5		7
		6		9·5		6
		6		9		5
						5

TABLE V.

Mean Structure of Bands.			Number of Bands showing Minimum.	Index Nos. of Plates.
No.	a_0	Int^y.		
1	−73	0		
2	−55	5		
3	−43	5		
4	−31·5	6		
5	−23·9	7		
6*	−21·8	0	6	3,4
7	−17·5	8		
8	−16·6	9		
9	−14·4	3	2	5
10	−12·0	11		
11	−5·8	13		
12	−3·8	7	7	2,3,4,5,6
13	+1·2	13		
14	+3·2	7	5	3,4,5,6
15	+6·8	12·5		
16	+9·5	7	6	1,3,4,6
17	+12·4	12·5		
18	+13·0	7	3	3,4,5
19	+16·8	11·5		
20	+20·1	10		
21	+25·1	8		
22	+31·7	7		
23	+34	5·5		
24	+36·4	3·5		
25	+40·5	2·5		
26	+56	1		
		0·5		

* The Fraunhofer lines occur only on Plates Nos. 3 and 4, March 18 to 20 and March 25.

6. *The Continuous Spectrum.*—The question whether the continuous spectrum has a share in these radiations is settled by photo-plate No. 1, which was orthochromatic. There are five minima recorded between wave-length 496 and 538, whose intensities range between 0 and 0·3 degrees. I obtain the reduction for relative sensitiveness for different wave-lengths from a series of exposures on the sky spectrum. On the supposition that the intensity curves of the spectrum of the sky and of the star are identical, the intensity of the continuous spectrum of the star does not reach 1 degree within the area photographed, and probably is less, considering the observed colour of the star. Not to complicate the problem unnecessarily, I have disregarded the continuous spectrum, and it will be seen from what follows that the only effect this omission can have on the result is to increase slightly the calculated degrees of intensity of the bands.

7. *Mean Structure of the Bands.*—The continuous spectrum being very faint, the difference under discussion must be caused by superposed bands. From the plates taken after August I shall prove that not only the H-bands, but all the bands, are similar. I assume that the same holds good for the earlier plates, and that therefore the superposed bands are similar to the H-bands. The unknown structure of the bands thus enters not only the H-bands, but also the bands overlapping their ends, and that structure must be so determined as to give the observed degree of intensity. In the solution of this problem two other questions are involved: First, in what way does the intensity curve of the band vary if the intensity of its maximum alters? Secondly, what is the resultant intensity on my degree scale if two radiations which singly produce certain degrees of intensity on the photograph be superposed?

The two bands, $\lambda = 4902$ to 4959 and $\lambda = 4981$ to 5046, agree in extent and in position of their maxima with the bands calculated from the bands of Table IV. by formula (1), and the wave-lengths of their zero, λ_0, are found respectively to be 4922 and 5016. I compared the observed degrees of the H_β-band, which in first approximation served as standard, with those at corresponding points of the faint bands $\lambda_0 = 4922$ and 5016, discarding the points where the bands overlap, and deduced by interpolation the relations given in Table VI. For instance, a band which has at different points the

TABLE VI.—*Corresponding Degrees of Intensity.*

13	11	9	7	5	3	1
11	9·3	7·5	5·8	4·1	2·3	0·6
9	7·5	6·1	4·6	3·1	1·7	0·2
7	5·8	4·6	3·4	2·2	1·0	
5	4·1	3·1	2·2	1·3	0·4	
3	2·3	1·7	1·0	0·4		
1	0·6	0·2				

degrees of intensity 13, 11, 9, 7, etc., has at corresponding points the degrees, say, 5, 4·1, 3·1, 2·2, etc. (Also see § 9.)

The bands contained on the first seven photographs here under discussion give no evidence as to the second question. On the later plates there are two bands which can with certainty be identified as consisting each of two bands, while the structure of the standard band is independently determined from detached bands. From Table XII., where the later observations are compiled along with the calculated bands, it will be seen that the sum of the calculated degrees of intensity due to radiations of the same wave-length nearly agrees with the observed intensity. I adopt this additive rule here as a working hypothesis, the accuracy of which will be investigated in § 9.

I deduce the common structure of the bands by successive approximation. Choosing first the H_β-band of Table IV., I calculate the band for $\lambda_0 = 4922$, and employing Table VI., reduce the degrees of intensity so that the degree of intensity of the maximum of this band agrees with the observed intensity of the maximum. I then subtract the calculated degrees from their observed values, and find the degrees of intensity at the different points of the H_β-band freed at its less refrangible end from the superposed band 4922. From the corrected H_β-band I calculate the band for $\lambda_0 = 4265$, which a preliminary discussion had shown to overlap the more refrangible end of the H_γ-band, and proceeding as before, I obtain the intensities at different points of the more refrangible end of the H_γ-band. In second approximation I combine these results, and repeating the calculation, find the mean structure as contained in Table V., the values a_0 being the means, with regard to weights, of the measurements given in Table II.

8. *Resolving of the Spectrum into Bands.*—I set myself the problem to find the wave-lengths λ_0 of the zero of each band, and the degree of intensity of its maximum, which I shall call the intensity of the band, so that the superposed bands represent the observed intensity curve. I found this research on the following basis:—1. The continuous spectrum is faint and may be neglected. 2. The bands are similar to the band given in Table V., and determined by formula (1), λ_0 being unknown. 3. The intensity curve of each band is defined by the unknown maximum intensity and the data contained in Tables V. and VI. 4. At places where bands are superposed, the resultant degree of intensity is the sum of the degrees of intensity which the radiations would singly produce on the photographic plate.

The last assumption is merely a convenient rule, which, though not strictly correct, is sufficient for our purpose, as will be proved in § 9. I may mention here the considerations which induced me to undertake a research which at first sight appears to be hopeless. I suppose that two bands have been identified in the spectrum, and draw their intensity curves as calculated from Tables V. and VI., together with the observed intensity curve of the spectrum. The length, in the direction of the axis of wave-lengths, of the area bordered by the three curves is independent of the manner in which the ordinates of the two bands are deducted from those of the observed intensity curve.

If this length agrees with the extent of one band, which is almost constant for the brighter bands, the position of a band can be adjusted in this area, *i.e.* the wave-length, λ_0, of the zero of the band can be found. Fainter bands are shorter in length, and two or more fainter bands may be combined so that their total extent is that of a bright band. On the understanding that a band selected in the manner described may be replaced by two or more fainter bands whose values of λ_0 differ little, the wave-length of the zero of the band is independent of the rule according to which the degrees of intensity are combined, and depends only on the wave-lengths λ_0 of the zeros of the two adjoining bands. The error of the rule will appear only in the residuals. To a lesser degree, this holds also good for a space belonging to two or more bands, and the greater the space to be covered, the more important it will be that the intensities be correctly compounded. For this reason I have divided the spectrum between H_β and H_γ into four parts, introducing three bands, whose maxima I make to coincide with pronounced maxima of the observed intensity curve. The intermediate spaces were explained by as few bands as possible, with the intention that each could afterwards be replaced by two or more fainter bands, should this improve the agreement of the spectrum with one which otherwise resembled it.

A. For the discussion of the spectrum between H_β and H_γ I employ the intensity curves of photo-plates Nos. 3 and 4. As before, I add 5 to the degrees of intensity of No. 4 to refer them to the same scale as was chosen for No. 3. The spectrum on No. 4 differs somewhat from those on the other plates, of which No. 3 is the best representative.

On plate No. 4 there are four prominent maxima between H_β and H_γ, viz., 4459 to 4480, intensity 6 ; 4560 to 4589, intensity 5·5 ; 4627 to 4648, intensity 8 ; 4682 to 4691, intensity 8 ; all of them fading off several degrees towards both sides. I assume that they are due to bands whose λ_0 is respectively 4470, 4574, 4637, 4687, and I calculate the intensity curves of these bands from the data given in Tables V. and VI. The calculated degrees of intensity were written out at intervals of two tenth-metres, and subtracted arithmetically from the observed intensities. To explain the residuals, I chose, in accordance with the above, as few bands as possible, and introduced further the condition that the same bands be selected for both photo-plates. Between 4341 and 4470 at least three bands were required for plate 3 and two for plate 4 ; between 4470 and 4574 three for plate 3 and two for plate 4 ; between 4574 and 4637 one each for plates 3 and 4 ; between 4637 and 4687 one for plate 3 and none for plate 4 ; between 4687 and 4861 two bands for both plates. By a lengthy process of trials in which the wave-lengths and the intensities of the bands were altered, including those of the above bands, I found the wave-lengths λ_0 of the zeros of the bands and the degrees of the intensity of their maxima, as given in Table VII. under the heading A. The three intensity curves calculated from these data at intervals of two tenth-metres are represented on Plates I. and II. under A, together with the observed curves, which are dotted. The straight lines drawn at the top of the plates show the extent of each band and the number of superpositions at each point.

Table VII. further contains the hydrogen spectrum observed by WILSING and the spectrum of Nova Aurigæ by VOGEL, both copied from the table in WILSING'S memoir *Untersuchungen ueber das Spectrum der Nova Aurigæ*. I ought to mention here that during the progress of my work I did not consult any previous researches or observations on new stars, and that I arrived at the result A without bias. Between 4067 and 4341 the wave-length λ_0 of the zeros of the bands agree well with the wave-lengths measured in the spectrum of Nova Aurigæ, and the two lines 4922 and 5016 are also present in both. On the other hand, there are marked differences between H_β and H_γ, the region which, owing to the large interval between two identifiable bands, presented the greatest difficulty to division into bands. The same remarks apply to the hydrogen spectrum. It is of no moment that some of the lines, as 4388, 4472, etc., are found intensely bright, because, in accordance with the above, each might be split into two or more fainter lines. The fact that the hydrogen line 4581 is the only bright line of intensity above 6 which is not represented in the spectrum of Nova Persei, while it occurs in the spectrum of Nova Aurigæ, appears to suggest that the observed maximum 4560 to 4589 is not due to the band $\lambda_0 = 4570$, which is one of those used in subdividing the spectrum.

B. I therefore repeated the work between H_β and H_γ, and subdivided the spectrum as before, but chose 4581 instead of 4570; 4634 instead of 4637; and 4684 instead of 4687. I further introduced the condition that as few lines as possible should be chosen, and that where the wave-length of the zero of a band fell near that of a hydrogen line, the wave-length of the H-line should be taken. The introduction of the line 4581 instead of 4570 as zero of a band changed the position of all the bands as far as the H_γ-band; 4570 being the mean of 4559 and 4581, the two bands belonging to them share almost equally in producing the maximum formerly ascribed to 4570. Each band entails the introduction of a series of bands fitting into one another, and there are thus 14 bands required to represent the intensity curve compared with 7 bands before. In other regions I altered some of the wave-lengths slightly to make them agree with those of hydrogen. The band $\lambda_0 = 4768$, which does not occur in the hydrogen spectrum, is perhaps due to a series of faint lines. The result of this new analysis is given in Table VII., B, and the intensity curve calculated at intervals of 2 tenth-metres is drawn on Plates I. and II. under B. It is remarkable that the lines of the hydrogen spectrum, which I have been forced to take from WILSING'S table as being within 3 t.m. of the zeros of bands actually obtained, include all the hydrogen lines whose intensities exceed 2 between 4341 and 4861, though no heed was taken of their intensity. The spectrum as defined by λ_0 agrees well with that of Nova Aurigæ. In the same table I have further entered all the lines of helium except those of the two second subordinate series. All of them have corresponding lines in the Nova spectrum.

On the assumption of the additive rule, the brightest lines of the hydrogen and helium spectrum represent the intensity curve, each line being broadened according to Table V. and formula 1; and I consider the conjecture that hydrogen actually pro-

duced the spectrum not only possible, but probable, for these reasons:—(1) Towards the more refrangible side of H_γ, lines coinciding within admissible limits with prominent H-lines, were found without any bias, and they belong to that region of the spectrum which from the outset offered the least difficulty to the splitting up of the intensity curve into bands. (2) The intensity curve of the band belonging to 4637 or 4634, also detected without bias, follows the intensity curve of the March 25 plate in that part of the spectrum without requiring the introduction of another bright band, and this line is, together with H_β, H_γ, H_δ, the most prominent line of the hydrogen spectrum. (3) If, instead of the line 4570 found under A, which does not occur in the hydrogen spectrum, the prominent H-line 4581 is introduced, the lines then required to represent the intensity curve agree in position more closely than before with those of Nova Aurigæ, *i.e.*, a star whose spectrum is, as I shall show, identical with that of Nova Persei after August. (4) The lines of the hydrogen spectrum, which I have been forced to take from WILSING'S table as being within 3 t.m. of the zeros of bands actually obtained, include all the hydrogen lines whose intensities exceed 2 between 4341 and 4861, though no heed was taken of their intensities.

9. *Proof that the Error of the Additive Rule does not affect the Result of* § 8.—The results just derived rest on the assumption that the radiations of intensity $i_1 \ldots i_n$ which individually give on the photo-plate a blackness of degree $m_1 \ldots m_n$ for the same exposure t produce if acting together during the same time t a blackness equal to Σm. It is, however, well known that this cannot be correct for all degrees of blackness. With the view of determining the error introduced by the use of the additive rule, I exposed several plates on a continuous spectrum, each plate containing five spectra, the exposures of which were proportional to 1, 2, 4, 8, 16. I estimated the degrees of blackness of the spectra in the same way as done on the star photographs. Any two degrees of blackness could then be superposed, and compared with those estimated for another exposure. I find that for the degrees of blackness occurring on the star photographs, the blackness of the film is about proportional to the time of exposure for a constant intensity of radiation, and that the degrees of my scale are about proportional to the blackness. Since this relation cannot hold good for the highest degrees of blackness, I take it to be only approximately true for the lowest degrees, and put

$$\text{(2)} \qquad \frac{t_1}{t_2} = \frac{f(m_1)}{f(m_2)} \quad \text{for a radiation of constant intensity } i,$$

and choose

$$\text{(3)} \qquad f(m) = 10^{am} - 1,$$

where m is the degree of blackness, t the time, and a a constant.

SCHEINER'S *Die Photographie der Gestirne* contains on p. 246 a table, the results of experiments by MICHALKE, which gives the times of exposure for intensities of radiation varying from 1 to 36, to produce the same degree of blackness on the photographic film. I find this table is sufficiently well represented by

$$\text{(4)} \qquad ti^b = \text{constant} \quad b = 1{\cdot}08, \text{ for a constant blackness } m.$$

TABLE VII.—*Spectrum of Nova Persei in March and April* 1901.

Nova Persei.								Hydrogen * (Wilsing).		Helium.			Nova Aurigæ * (Vogel).	
	A. Intensity.				B. Intensity.									
λ_0.	March 3.	March 18 and 20.	March 25.	λ_0.	March 3.	March 18 and 20.	March 25.	λ.	I.	λ.	I.	Series.	λ.	I.
4026	5	1	5	4026	5	1	5	4027	4 3	4026	5	II, 1		
								4044	3 0					
								4055	3 2					
								4063	6 6					
4067	...	...	2·5	4067	...	...	2·5	4067	5 5				4067	b 3
								4070	5 5					
								4088	3 1					
								4097	2 1					
4102	9	10	10	4102	9	10	10	4101	5 1				4102	b 3
4122	2	2	2	4132	3	3	3	4132	4 2				4125	2
4145	2	2	2	4145	...	1	1	4145	3 1	4144	2	II, 1		
								4157	5 2					
				4160	1·5	1·5	1·5	4163	5 2				4158	2
								4171	4 3					
4174	6	6	6	4177	5·5	4·5	4·5	4177	8 8				4176	vb 3
								4182	3 0					
								4189	2 1					
								4196	3 3					
4210	2	2	3	4208	1·5	1·5	1·5	4205	6 4					
								4213	7 5					
								4222	4 3					
4234	4·5	4·5	4·5	4233	4·5	5	5	4233	3 1				4230	b 3
								4243	1 1					
								4253	2 1					
4265	3·5	4·5	4·5	4265	3·5	4·5	4·5	4265	1 0				4262	vb 3
								4293	1 0				4288	b 3
4304	1	3	3	4305	1	3	3	4305	3 3					
								4312	2 0				4315	b 3
								4330	2 0					
4341	9	10·5	10·5	4341	9	10·5	10·5	4341	13 3				4341	b 4
				4364	2	3	2							
				4377	1	...	2	4377	0 1					
								4382	2 0				4383	3
4388	8·5	8·5	8·5	4388	5·5	4·5	4	4389	2 1	4388	3	II, 1		
								4402	2 1					
4410	...	2	...	4410	4	5	3	4410	4 3					
								4413	2 0					
								4419	2 0				4417	b 3
4436	5	5	3·5	4425	1	2·5	2	4425	4 4				4435	b 3
				4448	3·5	3	3	4448	2 4				4445	2
				4459	...	3	3·5	4459	3 4					
4472	4	8·5	9	4472	3	4	3			4472	6	I, 1	4473	b 2

I. = intensity.
I = helium, II = parhelium, p = principal series, 1 = first subordinate series.
In the last column b = broad, 3 = bright, 4 = very bright.
* Copied from *Publicationen des Astrophysikalischen Observatoriums zu Potsdam*, xii. p. 96.

TABLE VII.—*continued.*

Nova Persei.								Hydrogen (Wilsing).			Helium.			Nova Aurigæ (Vogel).	
	A. Intensity.				B Intensity.										
λ_0.	March 3.	March 18 and 20.	March 25.	λ_0.	March 3.	March 18 and 20.	March 25.	λ.	I.		λ.	I.	Series.	λ.	I.
								4482	1	1				4480	b 3
				4488	3	3	4	4488	2	4					
4502	5	3	...	4500	3	4	1	4498	2	0				4495	
								4501	2	4				4507	2
4520	5	6	6	4522	2·5	3	5	4521	2	2				4520	b 3
								4523	1	3					
				4535	4	3	2	4535	2	2				4530	
								4551	2	2					
								4555	2	2					
4558	4	2·5	2	4559	6	5	4	4559	5	2				4557	b 2
4570	7	8	8												
				4581	3	5·5	5	4581	6	10				4583	b 3
								4597	2	1					
4612	3·5	5	3	4608	2	4	1	4608	2	1					
								4619	2	1					
4637	3·5	3·5	8	4634	5	6	9	4634	9	15				4628	b 3
4660	3	6	...	4662	2·5	2·5	...	4662	1	2					
				4672	...	3	...	4672	1	3					
4687	5	5·5	10	4684	5	6	10	4684	3	3					
								4710	2	2					
								4719	2	2					
4725	4·5	4·5	5	4724	4	4	6	4724	2	3					
4768	5·5	5·5	5·5	4768	5·5	5·5	5·5								
								4797	0	1					
4861	13	13	13	4861	13	13	13	4862	7	2				4862	b 3
4922	4	4	4	4922	4	4	4				4922	4	II, 1	4924	2
								4932	0	2					
								4973							
5016	3·5	0·5	2	5016	3	0·5	2	5014			5016	6	II, p	5016	2
								5055							
5132	1			5132	1										
5178	2			5178	2									5167	vb 3
5200	2			5200	2									523	2
														528	2
5327	2			5327	3									5317	4
								5388							
				5405	2										
5456	3			5451	3										
								5481							
5500	2·5			5495	2·5			5499							
								5505							
5544	4			5537	4			5537							
5589	3			5584	3										
5635	2·5			5640	3										
5680	2			5689	2			5689							

The hydrogen lines 4973 to 5689 were measured by Hasselberg.

On p. 247 Prof. SCHEINER further gives the results of his experiments on artificial stars produced by means of a Zoellner photometer. He finds that if the time of exposure be increased 2·5 times, the faintest stars recorded on the plate are only 0·7 mg. fainter than before, therefore $b = 1{\cdot}4$.

Let a radiation of intensity i produce a degree of blackness m, and a radiation of intensity i' a blackness m', both in time t. Let t' be the time required for i to produce m', then by (2) and (4)

$$t\,i''^{b} = t'\,i'^{b} \qquad \text{and} \qquad \frac{t}{t'} = \frac{f(m)}{f(m')}, \qquad \text{therefore}$$

$$(5) \qquad \frac{f(m)}{f(m')} = \left(\frac{i}{i'}\right)^{b} \quad \text{for the same exposure on radiations of intensities } i \text{ and } i'.$$

I assume that the broadening of the lines into bands is due to the same physical cause, and that the ratio of the intensities of the radiations at any two corresponding points of two bands is a constant for these two bands; therefore by (5)

$$(6) \qquad \frac{fm}{f(m')} = \frac{f\mu}{f(\mu')} = \text{constant}, \qquad \text{and} \qquad \frac{f(m)}{f(\mu)} = \frac{f(m')}{f(\mu')} = \text{constant},$$

where μ and m are the degrees of blackness at two points of a band, and μ' and m' those at corresponding points of a second band. By means of (6) I determine the constant a in $f(m)$ from the observed corresponding degrees of blackness contained in Table VI. The result is $a = 0{\cdot}04$, with which I have calculated the following table.

TABLE VIII.

Calculated corresponding degrees of blackness for $a = 0{\cdot}04$.

μ	13	11	9	7	5	3	1
m	11	9·2	7·4	5·7	4·0	2·3	0·7
	9	7·4	5·9	4·5	3·1	1·8	0·5
	7	5·7	4·5	3·4	2·2	1·2	0·4
	5	4·0	3·1	2·2	1·5	0·8	0·2
	3	2·3	1·8	1·2	0·8	0·4	0·1
	1	0·7	0·5	0·4	0·2	0·1	

For instance, if the maximum of a band of degree 13 is reduced to degree 5 in another band, blacknesses 7 and 5 at other points of the band become respectively 2·2 and 1·5, while Table VI. gives 2·2 and 1·3. The quantities in this table differ from those in Table VI. for all degrees of blackness greater than 0·8 by less than 0·2 degrees, and the average error is 0·1, but the differences increase to 0·4 for the degrees lower than 0·8. The function (3) therefore represents the observations satisfactorily. For $a = 0{\cdot}03$ and 0·05 the residuals are respectively 30 and 50 per cent. higher than for $a = 0{\cdot}04$, and for

$a = 0{\cdot}07$ they increase to $2\frac{1}{2}$ times the amount for $a = 0{\cdot}04$. For double the exposure the degrees of blackness 1, 3, 5 and 10 become respectively 1·9, 5·3, 8·4 and 15.

Let the intensities of radiation $i_1 \ldots i_n$ give respectively the blacknesses of degree $m_1 \ldots m_n$ when exposed time t. The question is, what is the degree m_0 if a radiation of intensity Σi be exposed the same time t? Formula (5) gives

$$i_n \mathrm{F}(m_0) = \mathrm{F}(m_n)\Sigma i \quad \text{where} \quad \mathrm{F}(m) = (f(m))^{\frac{1}{b}}, \quad \text{therefore}$$

$$\mathrm{F}(m_0) = \Sigma \mathrm{F}(m_n). \tag{7}$$

An extract of the F-Table is given below.

TABLE IX.

m.	F(m).		m.	F(m).	
	$b = 1{\cdot}08$.	$b = 1{\cdot}4$.		$b = 1{\cdot}08$.	$b = 1{\cdot}4$.
0	0·00	0·00	8	1·08	1·06
1	0·11	0·19	9	1·27	1·20
2	0·23	0·32	10	1·47	1·34
3	0·35	0·44	11	1·68	1·49
4	0·47	0·56	12	1·92	1·65
5	0·61	0·68	13	2·17	1·82
6	0·75	0·81	14	2·45	2·00
7	0·91	0·93			

For instance, radiations which would singly give the degrees 1, 2, 3, 4 produce when superposed $m_0 = 8{\cdot}4$ for $b = 1{\cdot}08$ (by Table IX., $\mathrm{F}(m_0) = 0{\cdot}11 + 0{\cdot}23 + 0{\cdot}35 + 0{\cdot}47 = 1{\cdot}16$) and 11·1 for $b = 1{\cdot}4$, as compared with $\Sigma m = 10$.

The difference $m_0 - \Sigma m$ depends on Σm and the number of radiations which are superposed. It varies most for the intensity curve which shows the greatest range between maxima and minima, and which has a different number of superposed bands at different places. On this account the calculated intensity curve, Σm, belonging to photo-plate No. 4, which on hypothesis B has from 3 to 6 superposed radiations in the region between H_β and H_γ is the most likely to differ from an intensity curve m_0 calculated by formula (7). I assume the same bands, and in first approximation the same degrees m which contribute to the Σm-curve of Plate I., but I compound them according to formula (7). The resulting m_0-curve is then brought to agreement with the observed intensity curve by suitable changes of the intensities of the bands, and the question is whether this curve satisfies the observed curve as well as the Σm-curve does. Table X. contains the calculation in detail at an interval of 10 t.m., while it was made for every 5 t.m. The first columns show, under the heading m, the degrees of blackness at each point which the radiations would singly produce on the film. Σm is the ordinate of the calculated intensity curve on Plate I., which was made to agree as near as possible with the observed intensity curve (see § 8); m_0 is the resultant if the degrees m be compounded

according to formula (7). $m_0 - \Sigma m$ ranges between $-0{\cdot}6$ and $-2{\cdot}2$ for $b = 1{\cdot}08$, and between $+0{\cdot}4$ and $+2{\cdot}4$ for $b = 1{\cdot}4$.

Let $\delta m_1 \ldots \delta m_n$ be the corrections of the degrees $m_1 \ldots m_n$ which change the degree of blackness m_0 by δm_0 therefore,

$$\Sigma \frac{d\mathrm{F}(m_n)}{dm_n}\delta m_n = \frac{d\mathrm{F}(m_0)}{dm_0}\delta m_0 . \tag{8}$$

A change δm_1 at wave-length λ of band 1 can be brought about only by all the degrees of blackness being changed at every place of this band. I express δm_1 by the change $\delta \mu_1$ of the degree of blackness μ_1 of the maximum of band 1; μ_1 being the quantity which was shortly called the intensity of the band, and I do the same for all the indices 1 to n. Employing (6) I replace in (8) δm_n by $\delta \mu_n$ and eliminate function f by F.

$$\Sigma \mathrm{F}(m_n)\frac{d \log \mathrm{F}(\mu_n)}{d\mu_n}\delta \mu_n = \frac{d\mathrm{F}(m_0)}{dm_0}\delta m_0 . \tag{9}$$

To reduce the work of calculating, I change the degrees of the maximum of every band by an amount $\delta \mu_n$ determined by

$$x = \frac{d \log \mathrm{F}(\mu_n)}{d\mu_n}\delta \mu_n , \tag{10}$$

and obtain from (9) and (7)

$$x = \frac{d \log \mathrm{F}(m_0)}{dm_0}\delta m_0 \tag{11}$$

$$(m_0) = m_0 + \delta m_0 .$$

I determine x by (11) with $\delta m_0 = \Sigma m - m_0$, and calculate δm_0 at each point from the mean value of x. For $b = 1{\cdot}4$, μ of band $\lambda_0 = 4425$ was reduced by 1 in addition to $\delta \mu$, and the intensities of the last four bands were not changed at all. $(m_0) = m_0 + \delta m_0$ which appears in Table X. is then the calculated intensity curve if the degrees of blackness of the maxima of the bands given in Table VII. be changed by certain amounts to be calculated from (10), and the degrees at each point be compounded according to (7). The last three columns give the residuals left in the observed intensity curve. Σm differs from the observed intensity curve on an average $0{\cdot}58$ degree, (m_0) differs $0{\cdot}57$ for $b = 1{\cdot}08$, and $0{\cdot}63$ for $b = 1{\cdot}4$, while (m_0) differs on an average $0{\cdot}26$ from Σm. The observed intensity curve is therefore equally well represented by Σm and by (m_0), and therefore the hydrogen lines of Table VII. represent the spectrum, no matter whether the degrees of intensity of their bands be compounded by mere addition, or according to formula (7). Combinations of bands which, if compounded according to the additive rule, leave inadmissible residuals in the observed intensity curve, must give errors of the same order if formula (7) be employed; and I conclude that if this formula had been used at the outset in analysing the spectrum into bands, the result would have been identical with that contained in Table VII.

A similar calculation for the intensity curve of photo-plate No. 3, assumption A, gave an average error, observed $m - \Sigma m = 0{\cdot}63$, observed $m - (m_0) = 0{\cdot}59$ for $b = 1{\cdot}08$, and $0{\cdot}58$ for $b = 1{\cdot}4$, while (m_0) differs on an average $0{\cdot}25$ from Σm for both values of b. This agreement proves again that the use of the additive rule cannot have influenced the analysing of the spectrum into bands.

TABLE X.

Calculated Intensity Curves, Σm and (m_0), for Photographic Plate No. 4.

λ	0 = Observed Intensity + 5.	Intensity Curves of the different Superposed Bands.						Σ_m.	(m_0) by Formulæ (7) and (11).						Residuals.		
									$b=1\cdot08$.			$b=1\cdot4$.				$0-(m_0)$.	
		m_1.	m_2.	m_3.	m_4.	m_5.	m_6.		m_0.	$\Sigma m - m_0$.	(m_0).	m_0.	$\Sigma m - m_0$	(m_0).	$0-\Sigma m$.	$b=1\cdot08$.	$b=1\cdot4$.
4360	12	6·5	2·0	0·7	1·6	0·4		11·2	9·4	+1·8	10·8	12·2	−1·0	10·7	+0·8	+1·2	+1·3
70	9·5	4·2	1·9	2·1	2·0	0·6		10·8	8·9	+1·9	10·2	12·2	−1·4	10·7	−1·3	−0·7	−1·2
80	9		1·2	1·9	4·0	1 0		8·1	7·2	+0·9	8·3	9·6	−1·5	8·3	+0·9	+0·7	+0·7
90	8·2	0·2		1·6	3·8	1·4		7·0	6·4	+0·6	7·4	8·4	−1·4	7·2	+1·2	+0·8	+1·0
4400	7	0·4	0·4	0·5	3·8	3·0		8·1	7·2	+0·9	8·3	9·6	−1·5	8·0	−1·1	−1·3	−1·0
10	7	0·6	0·9		1·9	3·0	0·6	7·0	6·2	+0·8	7·2	8·8	−1·8	7·0	0·0	−0·2	0·0
20	7	1·0	2·0	0·4		2·8	0·8	7·0	6·3	+0·7	7·3	9·0	−2·0	6·7	0·0	−0·3	+0·3
30	7·5	1·4	1·9	0·5	0·4	1 5	1·3	7·0	6·2	+0·8	7·2	9·4	−2·4	7·1	+0·5	+0·3	+0·4
40	8	3·0	1·6	0·6	0·8	0·5	1·7	8·2	7·0	+1·2	8·1	10·4	−2·2	8·2	−0·2	−0·1	−0·2
50	8	3·0	0·4	1·1	1·2		3·5	9·2	7·8	+1·4	9·0	10·8	−1·6	9·1	−1·2	−1·0	−1·1
60	10	2·8		1·6	2·3	0·4	3·5	10·6	8·7	+1·9	10·0	12·0	−1·4	10·5	−0·6	0·0	−0·5
70	11	1·1		2·2	3·0	1·3	3·3	10·9	8·9	+2·0	10·2	12·2	−1·3	10·7	+0·1	+0·8	+0·3
80	10		0·1	4·0	2·8	1·7	1·5	10·1	8·4	+1·7	9·7	11·4	−1·3	9·9	−0·1	+0·3	+0·1
90	10		1·0	3·8	1·5	2·2		8·5	7·4	+1·1	8·6	10·0	−1·5	8·6	+1·5	+1·4	+1·4
4500	8	0·6	1·0	3·6	0·5	2·7	0·1	8·5	7·3	+1·2	8·3	10·4	−1·9	9·0	−0·5	−0·3	−1·0
10	9	0·8	0·9	2·1		4·1	0·4	8·3	7·2	+1·1	8·3	10·0	−1·7	8·6	+0·7	+0·7	+0·4
20	9	1·1	0·2	1·0	0·4	5·0	0·9	8·6	7·6	+1·0	8·8	10·4	−1·8	9·0	+0·4	+0·2	0·0
30	9·5	1·6			1·3	4·8	2·0	9·7	8·4	+1·3	9·7	11·0	−1·3	9·6	−0·2	−0·2	−0·1
40	9·5	2·1			1·7	3·8	1·9	9·5	8·0	+1·5	9·2	10·8	−1·3	9·4	0·0	+0·3	+0·1
50	8·5	4·0			2·2	1·5	1·6	9·3	7·9	+1·4	9·1	10·6	−1·3	9·2	−0·8	−0·6	−0·7
60	8	4·0			2·7		0·4	7·1	6·5	+0·6	7·5	8·1	−1·0	6·9	+0·9	+0·5	+1·1
70	10·5	3·8	1·9		5·0			10·7	9·0	+1·7	10·3	11·2	−0·5	10·0	−0·2	+0·2	+0·5
80	10·5	2·1	3·2		5·0			10·3	8·8	+1·5	10·1	11·0	−0·7	10·0	+0·2	+0·4	+0·5
90	8	1·0	3·9	0·2	4·8			9·9	8·5	+1·4	9·8	10·8	−0·9	10·3	−1·9	−1·8	−2·3
4600	8		3·9	1·0	2·7			7·6	6·8	+0·8	7·9	8·6	−1·0	8·1	+0·4	+0·1	−0·1
10	8	0·4	5·3	0·9	1·5			8·1	7·2	+0·9	8·3	9·4	−1·3	9·1	−0·1	−0·3	−1·1
20	11	2·4	7·5	0·8				10·7	9·5	+1·2	10·8	11·2	−0·5	11·2	+0·3	+0·2	−0·2
30	13	3·6	9·0					12·6	11·0	+1·6	12·5	12·4	+0·2	12·4	+0·4	+0·5	+0·6
40	13	4·4	8·6					13·0	11·1	+1·9	12·6	12·6	+0·4	12·6	0·0	+0·4	+0·4
50	12	4·8	7·3					12·1	10·4	+1·7	11·9	11·8	+0·3	11·8	−0·1	+0·1	+0·2
60	10·5	6·0	3·5	0·8				10·3	9·2	+1·1	10·5	11·0	−0·7	11·0	+0·2	0·0	−0·5
70	11	8·4	1·3	1·7				11·4	10·1	+1·3	11·5	11·8	−0·4	11·8	−0·4	−0·5	−0·8
80	11	10·0		2·2				12·2	11·2	+1·0	12·7	12·2	0·0	12·2	−1·2	−1·7	−1·2
90	13	9 6		2·5				12·1	11·0	+1·1	12·3	12·0	+0·1	12·0	+0·9	+0·7	+1·0
4700	12	8·8	0·2	3·4				12·4	10·9	+1·5	12·4	12·4	0·0	12·4	−0·4	−0·4	−0·4

The Spectrum from 1901 August 1 to 1902 November.

10. *The Mean Spectrum.*—The results derived from the photo-plates Nos. 8 to 21, 1901 August 1 to 1902 January, agree closely with each other, and it is unnecessary that the results be given separately. Those derived from the later plates, Nos. 22 to 27, are considerably less accurate, owing to the faintness of the spectrum, but they suffice to show that the bands did not change in position. Their results are also not printed separately. The changes which the spectrum underwent belong to the intensity, and they appear for the whole period in Table XVII.

I combined the wave-lengths and the estimates of intensity to mean values, which are given in the first columns of Table XII. For most bands seven to eight plates contributed to the mean, and for the band near H_γ thirteen plates, all belonging to the period August to November. The two bands at wave-lengths 386 and 397 were outside the range of the plate until the beginning of October, and they rest on the results of the plates Nos. 18, 19, 21. In Table XII. the intensities of the bands, therefore, do not belong to the same epoch. The average error of a tabulated wave-length is 0·3 t.m.

With reference to the faint bands, whose intensities do not exceed 1·5, and which were difficult to measure, most of the detail had to be discarded, because it was seen only on one plate. The neglected measurements are about five per cent. of the total number. The wave-lengths of these faint bands may be several tenth-metres wrong.

11. *The Common Structure of the Bands.*—Of the detached bands, the first two have the most pronounced intensity curve (see Plate III.). I shall show (see § 12) that their structure is similar, and further, that the similarity extends to all the other bands. The wave-lengths of the lines to which the bands belong being unknown, I introduce λ_m, the mean of the wave-lengths of the three minima, and determine a_m from

$$\lambda = \lambda_m + a_m \frac{\lambda}{4500}. \tag{12}$$

a_m is given in Table XIII. for the first two bands, and two other prominent bands, whose zeros λ_m equal 4364 and 4726. The latter merge into fainter bands, which overlap their more refrangible ends; the intensities at these places are bracketed in the table, and I do not take them into account here. The adopted values of Table XIV. are the means of the figures contained in Table XIII., with the exception of a few which were corrected so as to represent other bands better. I have also calculated a_m for all the other bands except one, and drawn on Plate III. their intensity curves with a_m as abscissa.

12. *The Calculated Spectrum.*—I decide whether all the brighter bands are of the same type by calculating the different points of the bands from a_m of Table XIV. by means of formula (12). About six well-defined points contributed to the final λ_m, and its calculated error is on an average 0·2 t.m. The degrees of intensity have been obtained from Tables VIII. and XIV. I have given the calculated bands in Table XII. Including all points, I find that the calculated wave-lengths differ from the observed wave-lengths on an average 0·5 t.m., as compared with a calculated average error of 0·3 t.m. for λ, of

TABLE XI.—*Photographic Plates*, 1901 *August to* 1903 *February*.

Index Number of Plate.	Date.	Exposure.	Range of Temperature.	Definition.		Remarks.	Number of Points Measured.		Number of Reversals.	Measurements of Plate.
		Hours.	Degrees Centigrade.	Star.	Fe.		Star.	Fe, Ti.		
8	1901. Aug. 1	1·7	0·5	4	3	Intensified	35	25	4	6
9	Aug. 15	4·0	2·0	2	2	Intensified	93	24	15	8
10	Aug. 21	6·5	2·9	2	2		98	32	14	6
11	Aug. 26	5·8	1·8	2	2		109	40	18	4
12	Aug. 27	5·9	1·5	3	3	Intensified	77	24	10	3
13	Aug. 30	4·3	1·8	2	2	Intensified	94	25	14	4
14	Sept. 4	7·7	1·9	2	2	Very clear	114	33	20	4
15	Sept. 20	4·7	0·2	4 F	3	Intensified	49	17	5	2
16	Oct. 2	3·7	3·0	4 F	3	Intensified	45	25	2	2
	Oct. 4	1·9								
17	Oct. 6	7·0	1·3	2	2	Intensified	108	40	15	4
18	Oct. 31	8·5	1·2	2	2		64	30	12	4
19	Nov. 1	6·0	0·2	2	2		77	36	12	4
20	1901. Nov. 13	5·5	0·5	2	2	Intensified	133	45	17	4
21	1902. Jan. 12	5·8	1·1	2	1	Intensified	87	50	5	4
22	Jan. 26	1·6	0·7	F	2	Intensified	77	45	2	4
	Jan. 28	6·3								
23	Jan. 29	7·6	2·2	F	1	Intensified	80	35	1	4
	Jan. 31	7·6								
24	Feb. 9	4·8	2·0	F	2	Intensified	81	42	4	4
	Feb. 11	4·8								
	Feb. 12	4·3								
	Mar. 21	2·0								
25	April 1	2·6	5·0	v F	1	Intensified	78	52	2	4
	April 2	1·8								
	April 17	1·5								
	April 26	1·5								
	April 30	0·8								
	May 1	0·9								
	May 2	1·4								
26	Oct. 20	2·2	3·3	F	2	Intensified	98	48	3	4
	Oct. 26	9·0								
	Oct. 30	3·3								
	Nov. 1	8·5								
27	Nov. 17	6·7	1·7	v F	1	Intensified	46	32	2	4
	Nov. 18	6·7								
	Nov. 20	6·9								
	1902. Nov. 21	2·2								
	1903. Jan. 7	0·5								
	Jan. 24	0·5								
	Feb. 1	0·5								

Width of slit:—0·018 mm. for plates Nos. 8 to 25; 0·020 mm. for plate No. 26; and 0·022 mm. for No. 27.

Angle of inclination of plate:— 13° for Nos. 8 to 17; 16° for Nos. 20, 22, 23, 25; 30° for Nos. 18, 19, 21, 24, 26; 8° for No. 27.

Definition:—1 excellent, 4 inferior, F faint.

Table XII.

The Observed Spectrum, 1901 *August* 1 *to* 1902 *January, and the Calculated Bands.*

Observed.		Calculated.			Observed.		Calculated.		
λ	Intensity.	No. of point Table XIV.	λ	Intensity.	λ	Intensity.	No. of point Table XIV.	λ	Intensity.
3813	0·5	$\lambda_m = 3869{\cdot}5$					$\lambda_m = 3968{\cdot}0$		
3835	0				3978·5	6	14 }	78·0	6·2
3853·5	3	1	53·6	2·1	3980·7	2	15 }		
3856·0	4	2	56·3	3·2	3982·3	1	16	80·3	2·6
3858·1	7	3	58·1	9·5	3986	0	17	81·9	0·9
3859·4	11	4	59·6	11·5	4027	1	18	85·8	
3861·1	8	5	61·4	9·5	4045	0			
3862·6	1	6	62·7	1·1	4063	0·5			
3863·8	3	7	63·9	2·1	4071·5	1			
3865·2	11	8	65·2	10·5	4079·7	0	$\lambda_m = 4103{\cdot}4$		
3867·9	6	9	68·3	5·3	4082·6	2	1	86·1	0·4
3870·8	10	10	70·5	10·5	4088·3	1·5	2	89·4	0·7
3872·9	9	11	72·8	9·5	4091·1		3	91·3	2·6
3874·1	5	12	74·1	3·2			4	92·9	3·4
3876·5	10	13	76·4	8·4		3	5	94·8	2·6
3878·9	8	14 } 15 }	79·2	7·4	4095·8		6	96·2	0·2
3881·4	3	16	81·5	3·2		1	7	97·5	0·4
3883	1	17	83·1	1·1	4098·1		8	98·8	3·0
3889	0·3	18	86·9		4102·8	2·5	9	02·1	1·2
3900	0	$\lambda_m = 3968{\cdot}0$			4105·3	0·7	10	04·5	3·0
3936·6	0·5					2·5	11	06·9	2·6
3943·2	1				4109·2	0·5	12	08·3	0·7
3951·8 ?	0 ?	1	51·3	1·7	4110·4		13	10·7	2·2
3954·3	2	2	54·4	2·6		2·5	14 } 15 }	13·7	1·9
3955·7	4	3	56·3	8·0	4115·3		16	16·2	0·7
3957·5	10	4	57·9	9·9		1·5	17	17·8	0·2
3959·4	7	5	59·7	8·0	4121·3	0	18	21·8	
3961·4	2	6	61·1	0·9	4140 ±	0·5			
3963·9	10	7	62·2	1·7	4165 ±	0			
3966·5	4	8	63·6	9·0	4220 ±	0·5			
3969·3	9	9	66·8	4·4	4265 ±	0			
3972·9	3	10	69·0	9·0	4300 ±	0·3			
3974·4	7	11	71·3	8·0	4306	0·7			
		12	72·8	2·6	4314·2	1			
		13	75·0	7·1					

TABLE XII.—*continued.*

Observed.		Calculated.								
λ	Intensity.	No. of Point Table XIV.	λ	Intensity.	No. of Point Table XIV.	λ	Intensity.	No. of Point Table XIV.	λ	Intensity.
		$\lambda_m = 4341{\cdot}0$								
4323·2	1·7	1	22·8	0·6						
4326·6	2	2	26·1	1·0						
4328·8		3	28·2	3·6						
	4	4	29·9	4·5						
4333·2		5	31·9	3·6						
	1·5	6	33·5	0·3						
4335·9	4	7	34·7	0·6						
4340·1	0	8	36·2	4·0						
4342·0		9	39·6	1·7						
	4·5	10	42·2	4·0		$\lambda_m = 4364{\cdot}5$				
4346·3	1	11	44·7	3·6	1	46·2	2·0			
4347·4	6	12	46·2	1·0	2	49·6	3·0			
4349·6	9	13	48·7	3·0	3	51·6	9·0			
4351·7		14, 15	51·9		4	53·3	11·0			
	12	16	54·5	2·6	5	55·4	9·0			
4355·4	7	17	56·2	1·0	6	56·9	1·0			
4357·4	1			0·3	7	58·2	2·0			
4359·0	10	18	60·5		8	59·6	10·0			
4361·4	9				9	63·1	5·0			
4363·3	5				10	65·7	10·0			
4365·3	10				11	68·2	9·0			
4367·9	8				12	69·7	3·0			
4369·8	3				13	72·3	8			
4371·8	8				14	74·9	5			
4374·5	5				15	76·1	7			
4375·7	7				16	78·1	3			
4378·4	3				17	79·8	1			
4382·0	1				18	84·1				

TABLE XII.—*continued.*

Observed.		Calculated.								
λ	Intensity.	No. of Point Table XIV.	λ	Intensity.	No. of Point Table XIV.	λ	Intensity.	No. of Point Table XIV.	λ	Intensity.
4388·7	0·8									
4393·3	0·2									
4398	0									
4405	0·3									
4446·7	1									
4457·4	1·5									
4488	0									
4503	1·5									
4554	1									
4570	1·5									
4578 3	1	$\lambda_m = 4612{\cdot}6$								
4590										
	1·5	1	93·2	0·2						
		2	96·8	0·3						
4598·8		3	99·0	1·3						
	2	4	00·8	1·7						
		5	03·0	1·3						
4604·4		6	04·6	0·1						
	1·5	7	05·9	0·2						
		8	07·5	1·5						
4611·0	1	9	11·1	0·6					$(\lambda_m = 4635{\cdot}3)$	
4614·5		10	13·9	1·5						
	1·5	11	16·5	1·3				1	(15·8)	0·8
4619·3		12	18·1	0·3		$\lambda_m = 4642{\cdot}0$				
		13	20·8					2	(19·4)	1·2
	2							3	(21·6)	4·3
4623·3		14	24·2	1·1	1	22·5	0·8	4	(23·4)	5·6
		15		0·9	2	26·1	1·2	5	(25·6)	4·3
	2·5	16	26·9	0·3	3	28·3	4·3	6	(27·3)	0·4
4627·7		17	28·8					7	(28·6)	0·8
	5			0·1	4	30·1	5·6	8	(30·1)	
		18	33·3		5	32·3	4·3			5·0
4634·0					6	34·0	0·4	9	(33·9)	2·2
	2									

TABLE XII.—*continued.*

Observed.		Calculated.								
λ	Intensity.	No. of Point Table XIV.	λ	Intensity.	No. of Point Table XIV.	λ	Intensity.	No. of Point Table XIV.	λ	Intensity.
					7	35·3	0·8			
4636·4	5				8	36·8	5·0	10	(36·5)	5·0
								11	(39·2)	4·3
4641·1	3				9	40·6	2·2	12	(40·9)	1·2
4643·5	5				10	43·2	5·0	13	(43·6)	3·8
4645·3					11	45·9	4·3	14, 15	(47·0)	3·3
	4·5				12	47·6	1·2			
4648·9	4				13	50·3	3·8	16	(49·7)	1·2
4652·0	3·5				14, 15	53·7	3·3	17	(51·6)	0·4
4655·0	3				16	56·4	1·2	18	(56·1)	
4658·4	1	$\lambda_m = 4687{\cdot}8$			17	58·3	0·4			
					18	62·8				
4669·1	3	1	68·1	0·9						
		2	71·8	1·4						
4673·6	5	3	73·9	4·8						
		4	75·8	6·2						
		5	78·0	4·8						
4678·8	1	6	79·7	0·5						
		7	81·0	0·9						
4681·1	6	8	82·6	5·5						
4687		9	86·3	2·5						
		10	89·1	5·5						
4691·4	5	11	91·8	4·8						
4694·2	4	12	93·4	1·4						
	2									
					1	95·8	0·9			
4697·2	5	13	96·1	4·2						
4699·7		14, 15	99·6	3·6	2	99·5	1·4			
	7	16	02·4	1·4	3	01·7	4·8			
					4	03·5	6·2			
4704·0	6	17	04·3		5	05·7	4·8	$\lambda_m = 4726{\cdot}1$		
				0·5				1	06·3	1·4
4707·4	2·5				6	07·4	0·5			
		18	08·8		7	08·8	0·9			

TABLE XII.—*continued.*

Observed.		Calculated.									
λ	Intensity.	No. of Point Table XIV.	λ	Intensity.	No. of Point Table XIV.	λ	Intensity.	No. of Point Table XIV.	λ	Intensity.
					$\lambda_m = 4715{\cdot}6$			$\lambda_m = 4726{\cdot}1$		
4710·2	9				8	10·4	5·5	2	09·9	2·2
4712·5								3	12·1	7·1
	11				9	14·1	2·5	4	14·0	8·8
					10	16·9	5·5	5	16·2	7·1
4717·2	7							6	17·9	0·7
					11	19·6	4·8	7	19·3	1·4
4721·0	10				12	21·3	1·4	8	20·9	8·0
4725·0	7				13	24·0	4·2	9	24·6	3·8
4727·0	9				14, 15	27·4	3·6	10	27·4	8·0
					16	30·3	1·4	11	30·1	7·1
4731·7	2				17	32·2	0·5	12	31·8	2·2
4735·4								13	34·5	6·2
	5·5				18	36·8		14, 15	38·0	5·4
4740·3	3							16	40·8	2·2
4742·6	1							17	42·7	0·7
4747·3	0·5							18	47·3	
4757	0·3									
4768	0									
4776	0·5									
4786	0									
4799	0·5									
4810	0									
4824	0·5									
4834·2	1		$\lambda_m = 4862{\cdot}8$							
4840·3	0									
4842·5	1	1	42·4	0·7						
4845·4	2·7	2	46·2	1·1						
4848·3	5	3	48·4	4·0						
		4	50·4	5·0						
4851·9	4·5	5	52·6	4·0						
4853·8	1	6	54·4	0·4						
		7	55·8	0·7						
4856·9	4	8	57·4	4·5						
4861·7	1·5	9	61·3	2·0						

TABLE XII.—*continued.*

Observed.		Calculated.			Observed.		Calculated.		
λ	Intensity.	No. of Point Table XIV.	λ	Intensity.	λ	Intensity.	No. of Point Table XIV.	λ	Intensity.
4864·2	4·0	10	64·1	4·5			$\lambda_m = 4959{\cdot}4$		
4866·6	3·5	11	66·9	4·0			14, 15	71·9	1·2
4870·0	2·8	12, 13	68·6, 71·4	1·1, 3·4	4973	0·8	16	74·8	0·4
4875·4	1·5	14, 15	75·0	2·9	4978	0·2	17	76·8	0·1
4878·7	1	16	77·9	1·1			18	81·7	
4883·1	0	17	79·9	0·4			$\lambda_m = 5007{\cdot}2$		
4886	0·5	18	84·6		4986·2	0·5	1	86·2	0·7
4900	0				4992·1	5	2	90·1	1·1
4917	0·3	$\lambda_m = 4959{\cdot}4$					3	92·4	4·0
4924	0·5				4998·3	0·5	4	94·4	5·0
4938	1	1	38·6	0·3	5001·0	4	5	96·7	4·0
4944·8	2·5	2	42·4	0·4	5006·5	0	6	98·6	0·4
4950·5	0·4	3	44·7	1·7	5009·9	3	7	00·0	0·7
4954·2	2	4	46·7	2·3	5017·5	2·8	8	01·6	4·5
4964·0	1	5	49·0	1·7	5022·0	1·7	9	05·6	2·0
		6	50·8	0·1	5026·1	0·7	10	08·5	4·5
		7	52·3	0·3	5031	0·3	11	11·4	4·0
		8	53·9	2·0	5053		12, 13	13·2, 16·1	1·1, 3·4
		9	57·9	0·8			14, 15	19·8	2·7
		10	60·7	2·0			16	22·8	1·1
		11	63·6	1·7			17	24·8	0·4
		12	65·4	0·4			18	29·7	
		13	68·2	1·5					

TABLE XIII.—*Table of a_m of Corresponding Points of Four Bands and of the Intensities, showing that the Bands are similar in regard to their Wave-Lengths after August* 1, 1901.

No.	a in Tenth-metres.				Degrees of Intensity.			
	λ_m				λ_m			
	3869·4	3968·0	4364·4	4726·3	3869·4	3968·0	4364·4	4726·3
1	−18·5							
2	−15·6	−15·5	−15·3	−15·3	3			
3	−13·1	−13·9	−13·1	−13·1	4	2	(9)	(9)
4	−11·6	−11·9			7	4	(12)	(11)
5	− 9·6	− 9·8	− 9·3		11	10	(12)	(11)
6	− 7·9	− 7·5	− 7·2	− 8·7	8	7	7	(11)
7	− 6·5				1	2	1	(7)
8	− 4·9	− 4·6	− 5·6	− 5·1	3	2	1	(7)
9	− 1·7	− 1·7	− 1·1	− 1·2	11	10	10	10
10	+ 1·6	+ 1·5	+ 0·9	+ 0·7	6	4	5	7
11	+ 4·1		+ 3·6		10	9	10	9
12	+ 5·5	+ 5·6	+ 5·6	+ 5·1	9	9	8	9
13	+ 8·3	+ 7·3	+ 7·6	+ 8·7	5	3	3	2
14	+11·0	+11·9	+10·4		10	7	8	5·5
15			+11·6		...	...	5	...
16	+14·0	+14·4	+14·4	+13·3	8	6	7	5·5
17	+15·8	+16·2	...	+15·5	3	2	3	3
18	+22·8	+20	+18·1	+20·0	1	1	(3)	1

TABLE XIV.—*Structure of the Bands after August* 1901.

Mean a_m T. M.	Adopted Degrees of Intensity.
−18·9	2
−15·4	3
−13·3	9
−11·5	11
− 9·4	9
− 7·8	1
− 6·5	2
− 5·0	10
− 1·4	5
+ 1·2	10
+ 3·8	9
+ 5·4	3
+ 8·0	8
+10·7	5
+11·9	7
+14·0	3
+15·8	1
+20·2	

0·2 t.m. for a_m, and 0·2 t.m. for λ_m. The observed degrees of intensity also agree satisfactorily with the calculated ones, or their sums at those places where two or three bands are superposed. The average difference, apart from signs, is 0·7 degrees of intensity; 91 of the discrepancies lie between 0 and 0·5, 60 between 0·5 and 1, 26 between 1, and 2 and 6 are greater. I consider it therefore proved that the bands are in every way defined by λ_m, the degree of intensity of their maximum, and the quantities given in Table XIV.

13. *Permanency of Structure.*—Table XV. gives the number of observations of a minimum, and the period during which it was observed. The bands $\lambda_m = 3869$ and 3968 were outside the range of the photo-plates Nos. 8 to 17. Their position is nevertheless well determined, since the plates Nos. 18 and 19 contain all the six minima, and Nos. 20 and 21 each four. The first minimum appears to have been the most pronounced. In all, it was recognised 90 times, against 47 for the second minimum and 38 for the third minimum. Of the total of 177 minima, 158 belong to the 1901 plates. The number of the minima that have been found seems to depend on the brightness of the bands, and still more on the linear width of the minimum, which at wave-length 3870 was 0·23 mm., and at 5006 only 0·08 mm. In conformity with this, the

majority of the minima detected on the plates of 1902 belongs to the first two bands, which at first were the brightest of the spectrum, and then second in brightness only to the 5006-band. They are present on each of the three photo-plates on which they were in focus, plates Nos. 21, 24, 26, 1902 January to November. During this time there is also no change in the extent and the position of the maximum of the bands. The evidence therefore points to the conclusion that from August 1 1901 to the end of 1902 the structure of the bands remained unchanged.

TABLE XV.—*Table showing the Number of Photo-Plates on which the Minima have been measured, and the Periods to which they belong.*

λm of Bands.	Minimum 1.		Minimum 2.		Minimum 3.	
	Number.	Period.	Number.	Period.	Number.	Period.
3869·5	6	1901 Oct. to 1902 Mar.	4	1901 Oct. to 1902, Mar.	4	1901 Oct. to 1902 Nov.
3968·0	4	Oct. to Jan.	6	Oct. to Nov.	4	Oct. to Jan.
4103·4	7	Aug. to Nov.	6	Aug. to 1903 Jan.	5	Aug. to May
4341·0	8	Aug. to 1901 Nov.	7	Aug. to 1902 Nov.	5	Aug. to 1901 Nov.
4364·5	13	Aug. to 1902 Jan.	9	Aug. to 1901 Nov.	7	Aug. to Nov.
4612·6	1	Sept. 4.	3	Aug. to Nov.	1	Aug. 15.
4642·0	4	Aug. to 1901 Sept.	1	Aug. 26.	...	
4687·8	9	Aug. to Nov.	...		8	Aug. to Nov.
4715·6	11	Aug. to Nov.	...		...	
4726·1	8	Aug. to Nov.	5	Aug. to Nov.	3	Oct. to 1902 Mar.
4862·8	7	Aug. to Oct.	2	Aug. 21, Sept. 4.	...	
4959·4	4	Aug.	1	Nov. 13.	...	
5007·2	8	Aug. to 1903 Jan.	3	Aug. to 1901 Sept.	1	Sept. 4.

A fourth minimum was measured twice in band $\lambda_m = 4364.5$ on 1901, August 21 and 26.

14. *Identification of the Bands.*—The wave-lengths λ_m of the zeros of the principal bands are compiled in the first column of Table XVI. Five of these can with certainty be identified as the hydrogen lines H_β, H_ν, H_δ, and the two principal nebular lines. λ_m is the wave-length of an arbitrary zero of the band, viz., approximately the mean of the wave-lengths of the three minima. I change the zero and make it coincide with the wave-length λ_0 of the line to which the band would be reduced under ordinary conditions. Let $[s]$ be the mean of the corrections for the orbital motion of the earth on the days on which the photographs were taken, then

$$\lambda = \lambda_0 + a_0 \frac{\lambda}{4500} = \lambda_m + [s] + a_m \frac{\lambda}{4500}$$

$$\delta a_m = a_0 - a_m = \frac{4500}{\lambda}(\lambda_m + [s] - \lambda_0) \tag{13}$$

$$\lambda_0 = \lambda_m - \frac{\lambda}{4500}\delta a_m + [s]$$

TABLE XVI.

λ_m	[s]	δa_m	λ Hydrogen and Nebulæ.	δa_m
3869·5	+0·1			
3968·0	0·0			
4103·4	+0·3		4101·8	+2·1
4341·0	+0·3		4340·7	+0·6
4364·5	+0·3			
4612·6	+0·4			
4642·0	+0·4			
or (4635·3)	+0·4		(4633·8)	
4687·8	+0·4			
4715·6	+0·4			
4726·1	+0·4			
4862·8	+0·4		4861·5	+1·6
4959·4	+0·4		4959·0	+0·7
5007·2	+0·4		5007·0	+0·6

The five lines mentioned above give on an average $\delta a_m = +1{\cdot}1$ t.m. λ_0 calculated by the third formula is comparable with the wave-lengths of elements, and also with the λ_0 derived from the March-April plates (Table VII.). The motion of the new star in the line of sight is here eliminated. If one should succeed in deriving from experiments, or theoretical considerations, the wave-length λ' of a certain point a_0 of the band belonging to a line λ'_0, a' can be calculated from $a' = (\lambda' - \lambda'_0)4500/\lambda'$, and the ratio $(a_0 - a')/4500$ is the ratio of the velocity in the line of sight and the velocity of light.

The residuals entered in the column headed "Difference" of Table XVII. exceed the quantity that might be expected from the average error of λ_m, and perhaps the fact that they rest on fewer minima than those of the brighter bands made them less accurate than the latter. Some of the bands call for special remarks. The zero of the second band lies 1·6 t.m. from the calcium line 3968·6, and 3·2 t.m. from the hydrogen line 3970·2. Owing to the good definition of its minima and its isolated position, its wave-length is one of the best determined of the spectrum, and its error is not likely to be greater than the calculated average error. As δa_m cannot be so much in error, I take the band to belong neither to calcium nor to hydrogen. The bright band whose zero has the wave-length 4641·3 cannot be identified. This band overlaps the band, $\lambda_0 = 4611{\cdot}9$, and only two of the three minima were measured. The zero would almost agree with the bright hydrogen line 4633·8 if the observed minima were not the first and second of the standard band, but the second and third; an assumption which changes the wave-length of the zero by the distance of two minima. This identification is bracketed as an alternative, though it is questionable, because the first calculated maximum of the band has not been observed (see Table XII.). The next band, $\lambda_0 = 4687{\cdot}1$, is certainly

not due to the hydrogen radiation of wave-length 4684. Near $\lambda_0 = 4725{\cdot}3$ is the hydrogen line 4723·6, which in the March-April spectrum was identified with a band

TABLE XVII.—*Spectrum of Nova Persei from* 1901 *August* 1 *to* 1903 *January.*

Nova Persei.										Planetary Nebulæ. (Campbell).		Nova Aurigæ. (Campbell).	
		Degree of Intensity of Maximum of Band on Photo-Plate.				$\frac{I}{I_0}$ Relative Intensities of the Radiations.							
λ_0	Difference.	1901, August 1 to October 6.	1901, October 31 to November 13.	1902, January.	1902, October to November.	1901, August 1 to October 6.	1901, October 31 to November 13.	1902, January.	1902, October to November.	λ	I	λ	I
3868·7			11·0	7·0	0·4		30	24	5	3868	4,5		
3967·0			9·5	5·0	0·8		25	18	8	3969	4,5	396	0·5
4102·7	+0·9	3·1	1·3	0·7	0·8	7	5	4	8	4101·8	5,6	4098	2
4340·2	−0·5	5·2	3·8	3·0	0·5	11	11	11	6	4340·7	5,6	4336	1
4363·7		10·1	7·7	2·3		20	20	9		4363·8	2,4	4358	8
4611·9		1·4	1·7	0·2	0·3	4	5	2	4	4610	0,1	460	1
4641·3		5·5	3·1	1·0	0·1	11	9	5	2				
or (4634·6)										4637	0,2	4630	7
4687·1		5·2	4·0	1·2	0·3	11	11	6	4	4687	2,5	4681	4
4714·8		6·5	2·6	0·5	0·1	13	8	3	2	4715	2,4	471	1
4725·3		5·3	6·0	1·2		11	16	6					
4862·0	+0·5	4·8	1·5	0·3	0·1	10	5	2	2	4861·5	v.b	4857	10
4958·6	−0·4	2·6	2·0	0·5	0·4	6	6	3	5	4959·0	v.b	4953	30
5006·4	−0·6	4·9	3·5	2·5	1·0	10	10	10	10	5007·0	v.b	5002	100
Faint Bands.													
3813 – 3835					0·5				6				
3889 – 3952			0·2		0·2		1		3	3889	0,4		
4027 – 4045				0·8				4		4026	0,4		
4063 – 4080		0·6	0·1	0·5	0·1	2	1	3	2				
4140 – 4165		0·5		0·2		2		2					
4220 – 4265		0·5	0·5	0·5	0·3	2	2	3	4	424	0,1	423	1
										4265	0,1	426	1
4300 – 4323		0·8	0·8	0·2	0·1	2	3	2	2				
4382 – 4398		0·8	0·8	0·3	0·3	2	3	2	4	4390	0,4	438	1
4405 – 4488		1·3	1·5	0·7	0·4	3	5	4	5	4472·6	0,5	4466	1
4503 – 4590		0·9	1·0	0·7	0·3	3	4	4	4	4574	0,2	451	1
										4597	0,1		
										4662	1,4		
4747 – 4768		0·4	1·5	0·6	0·1	2	5	3	2	4744	2,4		
4776 – 4786			1·0	0·5	1·0?		4	3	10?				
4799 – 4810				0·6	0·5			3	6				
4824 – 4840		0·3		0·6		1		3					
4886 – 4900		0·1		0·5	0·3	0		3	4				
4917 – 4938		0·1	0·7	0·3	0·3	0	3	2	4				
5031 – 5053		0·1	0·5	0·3		0	2	2					

of medium intensity. As neither the wave-length nor the intensity of the maximum agree with those of the hydrogen line, it probably is not due to hydrogen, though the

possibility is not excluded that the March-April identification is wrong, and that both the earlier and the later bands belong to the same radiations.

In the second half of the table appear the twelve corresponding lines which Campbell photographed in the spectra of five planetary nebulæ, and the range of their intensities in these five spectra, 1 standing for "faint," and 6 for "very bright." Besides these, there are only two lines, $\lambda = 4662$, intensity 1 to 4, and $\lambda = 4744$, intensity 2 to 4, which Campbell found present in each of the five nebulæ. The first falls within the range of the two bright bands $\lambda_0 = 4642$ and 4688 of the Nova spectrum, and if faint, would be masked by them; while the second is probably not represented by the faint band 4747 to 4768 of the Nova spectrum lower down in the table. All the prominent lines of the nebulæ spectrum are present in that of the Nova, 3868 and 4364, in addition to the principal nebular lines and the hydrogen lines, and their wave-lengths agree within their probable errors. I have already said that the second line of Nova Persei could not be the hydrogen line 3970·2. The planetary spectrum is not decisive on this point. All the hydrogen lines are bright, and the intensity of 3969 fits into their series, while its wave-length may be a t.m. in error. If it were the hydrogen line, the Nova spectrum after January 1902 could be reconciled with it. It is possible that a faint hydrogen band was superposed on the bright band $\lambda_0 = 3967{\cdot}0$, and that after January 1902, when the band faded and the measurements became difficult and less accurate, its principal constituent was the band $\lambda_0 = 3970$. The only prominent band of the Nova spectrum which has no counterpart in the nebular spectrum belongs to wave-length 4725·3.

15. *Variation of the Bands and of the Radiations in Intensity.*—The degree of intensity μ of the maximum of a band determines the intensity curve of the band. It alone requires to be discussed. From 1901 August 1 to October 6 the observed values of μ agree with each other within their probable errors, and I have combined them to mean values. In this period the photo-plates Nos. 8 to 17 were all taken at the same angle of inclination. I have also combined the estimates made in three other periods, using in each period photo-plates taken at angles of 30° and 16°, and discarding those estimates which belong to bands out of focus. The results, which are corrected for the superposed bands, are tabulated in Table XVII.

By means of the formulæ given in § 9 the relative changes in intensity of the radiations which produced the bands can be calculated. Let a radiation of intensity i in the focal plane of the spectrograph produce in time t a degree μ of blackness on a photographic plate whose sensitiveness is s for the wave-length of this radiation, and I designate these conditions by (i, s, μ, t), and let a radiation i_0 produce in the same time t on the same plate, for sensitiveness s_0 a degree μ_0 of blackness (i_0, s_0, μ_0, t), and for sensitiveness s a degree μ'_0, (i_0, s, μ'_0, t).

I define sensitiveness by $st = constant$ for the same intensity of radiation and the same degree of blackness. i/i_0 is the quantity wanted. I apply formula (5) to (i, s, μ, t) and (i_0, s, μ'_0, t).

$$\left(\frac{i}{i_0}\right)^s = \frac{f(\mu)}{f(\mu'_0)}$$

$f(\mu'_0)$ can be eliminated by $f(\mu_0)$. I apply formula (2) to (i_0, s_0, μ_0, t) and (i_0, s, μ'_0, t), which can be replaced by $(i_0, s_0, \mu'_0, \frac{ts}{s_0})$, then

$$\frac{s}{s_0} = \frac{f(\mu'_0)}{f(\mu_0)} \tag{14}$$

$$\frac{I}{I_0} = \frac{is^{\frac{1}{b}}}{i_0 s_0^{\frac{1}{b}}} = \frac{F(\mu)}{F(\mu_0)} \tag{15}$$

Since the sensitiveness is constant for the same wave-length, I/I_0 suffices for our purpose. It appears in Table XVII., where it is referred to the intensity of the radiation at $\lambda_0 = 5007$.

Compared with the March-April spectrum all the bright bands except the three H-bands, and perhaps the bands $\lambda_0 = 4634{\cdot}6$ and $4611{\cdot}9$, have come into existence, or have grown much in intensity. The radiation at $\lambda_0 = 4363{\cdot}7$ is twice as intense as that of the three H-bands, which in the spring were the most prominent bands of the spectrum. The other bands observed in the spring have almost entirely disappeared. None of the radiations faded at a slower rate than the nebular radiation $\lambda = 5007$. Several radiations began to decrease already in October, and others started in November. The decline of the bands $\lambda = 4363{\cdot}7$ and 4725 in the two months 1901 November 13 to 1902 January 12 is further remarkable. Four radiations faded at the same rate, H_δ, 4612, 4959, 5007, and perhaps H_γ, while H_β certainly decreased at a much greater rate in October. I may mention that I calculated s_0/s by (14) from estimates of degrees of blackness made on a photograph of the sky spectrum. The values i/i_0 calculated by (15) show that, with the only exception of the radiation at $\lambda_0 = 3869$, that at $\lambda = 5007$ was the most intense already in August 1901.

In October and November 1902 the relative intensities agree better with those of the nebular spectrum than before. It must of course be borne in mind that the spectrum on the last plates was extremely difficult to see, the intensity of the maximum of the band $\lambda_0 = 5007$ being only of degree 1, and that the figures belonging to that period are only a rough approximation. The trend of the table is certainly to show that the intensities are approaching those of the average nebular state.

16. *The Faint Bands.*—One may conclude by analogy that the faint bands would be reduced to lines under ordinary conditions. In that case, on account of their breadth, several must be due to multiple lines. On a whole they agree fairly well with the maxima of the intensity curve observed in March and April. Considering the uncertainty of the wave-lengths of these faint bands, about 5 t.m., a convincing proof as to their origin cannot be brought, though it is probable that they are the remnants of the bright spectrum in the first months. It may be mentioned that the hydrogen lines given under B in Table VII. also explain them, provided seven of them be excluded.

17. *Last Visual Observation of the Spectrum.*—On March 3 1903 I inspected the spectrum of Nova Persei in the focal plane of the spectrograph without using an eye-piece, a method which I usually employed prior to the exposure, to make sure that the proper star had been set on the slit. I saw only one bright spot in the whole range of the spectrum which coincided with the air band at 5004. Several times I gained the impression that there was a faint spot near the place of the magnesium line 4481. The comparison was made in this way, that when the eye had been fixed on the spot the spark was switched on for an instant.

18. *Curious relation between* λ^{-1} *of Four Prominent Lines.*—The wave-lengths of the zeros of the brightest bands are 3869, 3967, 4364 and 5007. The oscillation frequencies of the first, third, and fourth zeros almost form an arithmetic series, which, continued to the less refrangible side, gives the wave-length of the helium line D_3, a line which was measured by others in the spectrum of the new star, and also belongs to the nebular spectrum. In the following table I give the wave-length of the helium line, Keeler's determination of the nebular line, and my determination of the other two lines, reduced to the two nebular lines as standards.

The formula

$$\lambda^{-1} = 17014{\cdot}2 + 2957{\cdot}6n - 5{\cdot}5n^2, \quad n = 0, 1, 2, 3$$

determines λ as entered in the last column. The agreement is perfect. Should this be merely a casual coincidence?

λ	$\frac{1}{\lambda}$ Vacuum.	Difference.	Calculated λ.
3869·2	25837·8		3869·2
		2931·0	
4364·3	22906·8		4364·3
		2940·5	
5007·05	19966·3		5007·05
		2952·1	
5875·87	17014·2		5875·87

19. *Similarity of the Structure of the Bands in March-April* 1901 *and after August* 1901.—I add $\delta a_m = +1.1$ t.m. to a_m of Table XIV., which reduces them to the same zero as was employed for the March-April bands. Both bands are given in Table XVIII., and also on Plate II. I include the second minimum of the March-April band, though it rests on only two single measurements in two bands, because it seems to fill up a gap in the order of the minima. The extent of the maximum and the position of the minima agree with each other. There is only this difference, that while the March-April band declines to nothing from $a_0 = -12$ to -73 t.m., and from $+13$ to $+56$ t.m., the later bands fade abruptly on 6 t.m. Between April and August the ends of the bands have therefore decreased at a greater rate than the central maximum portion. I repeat again that from August 1901 to January 1902 no change took place in the structure, and that the extent of the maximum remained unaltered during 1902. It appears that the spectrum converges towards a nebular spectrum, in which each line is broadened 27 t.m.

TABLE XVIII.—*The Structure of the Bands in March–April* 1901, *and after August* 1901, *referred to* λ_0 *as zero, and reduced to* $\lambda_0 = 4500$.

Structure of the Bands.				Reversals.					
March–April.		After August.		March–April.			After August.		
a_0 T.M.	Intensity.	a_0 T.M.	Intensity.	a_0 T.M.	Number of Bands containing Reversals.	Number of Plates.	a_0 T.M.	Number of Bands containing Reversals.	Index Number of Plates.
− 73	0								
− 55	5								
− 43	5								
− 31·5	6								
− 23·9	7								
− 21·8	0*			− 22·8 *	8	3			
− 17·5	8								
− 16·6	9	− 17·8	2						
− 14·4	3	− 14·3		− 15·5 ±	2	1			
− 12 0	11	− 12·2	3						
		− 10·4	9						
	13	− 8·3	11						
− 5·8		− 6·7	9						
	7	− 5·4	1						
− 3·8		− 3·9	2	− 4·8	7	5	− 5·3	89	8 to 24
+ 1·2	13	− 0·3	10						
+ 3·2	7	+ 2·3	5	+ 2·2	5	4	+ 1·0	48	9 to 27
	12·5	+ 4·9	10						
+ 6·8		+ 6·5	9						
+ 9·5	7	+ 9·1	3	+ 8·1	6	4	+ 7·8	38	9 to 26
+ 12·4	12·5	+ 11·8	8						
+ 13·0	7	+ 13·0	5	+ 12·7	3	3	+ 12·4	2	10 and 11
	11·5	+ 15·1	7						
+ 16·8	10	+ 16·9	3						
+ 20·1	8	+ 21·3	1						
+ 25·1	7								
+ 31·7	5·5								
+ 34	3·5								
+ 36·4	2 5								
+ 40·5	1								
+ 56	0·5								

* Sharp Fraunhofer line, 1901, March 18–20 and March 25.

20. *Results.*—1. The spectrum consists of a line spectrum in which each line is broadened into a band, the broadening being proportional to the wave-length of the line and independent of the element. Tables V. and XIV. give the common structure of the bands.

The position of the maxima and of the minima or reversals remains unchanged during the whole period 1901 March to 1902 November. (See Table XVIII. and Plate II.)

2. The intensity curve of the spectrum in March and April is satisfied by the hydrogen and helium lines, some of which vary in intensity during this period. (See Table VII. B, and Plates I. and II.) It is probable that the spectrum is due to hydrogen and helium.

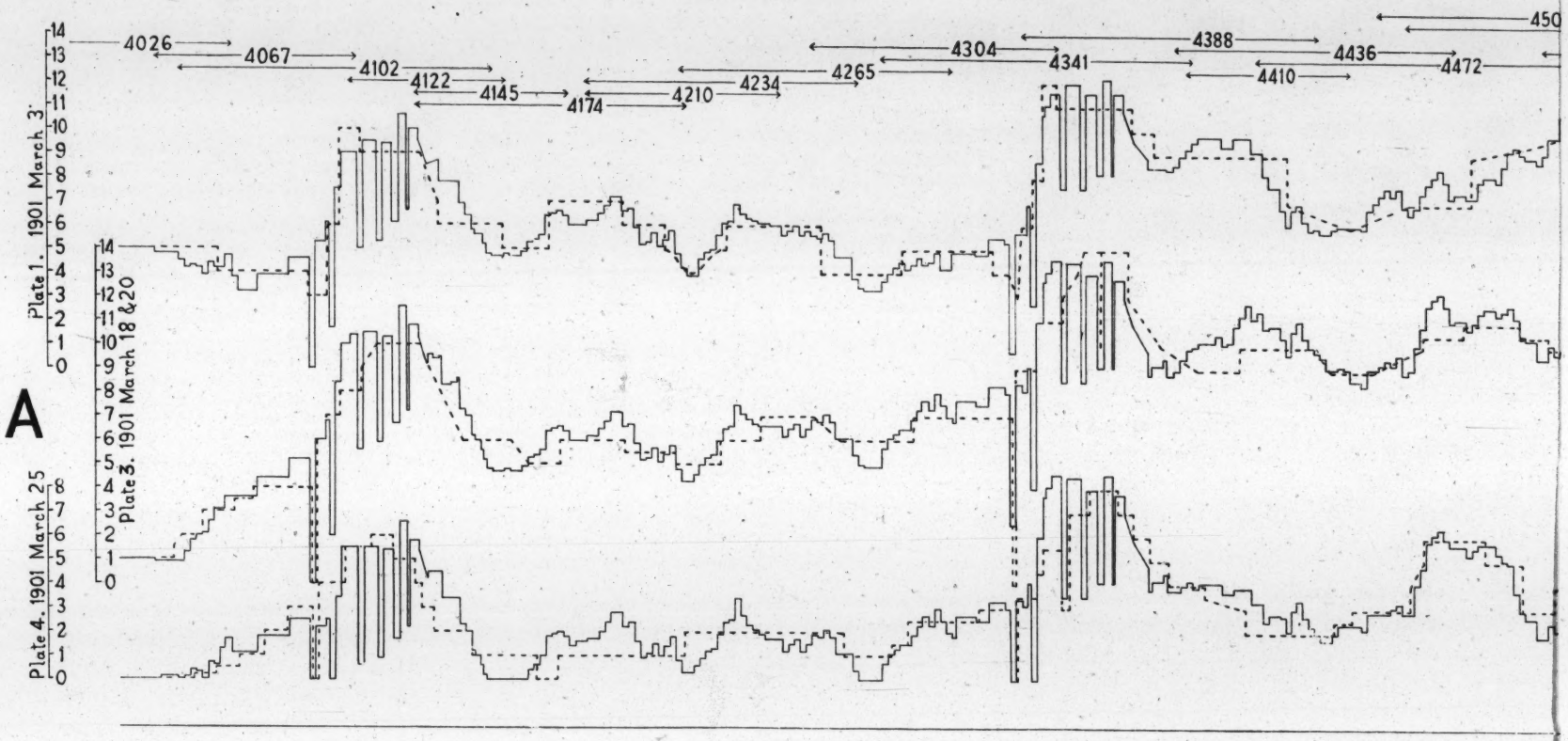
3. From August 1 1901 to the end of 1902 the bands belong to the lines of the spectrum of planetary nebulæ, and their relative intensities converge towards those of the average nebular spectrum. (See Table XVII. and Plate II.) Probably the March-April spectrum is also faintly present during the whole period.

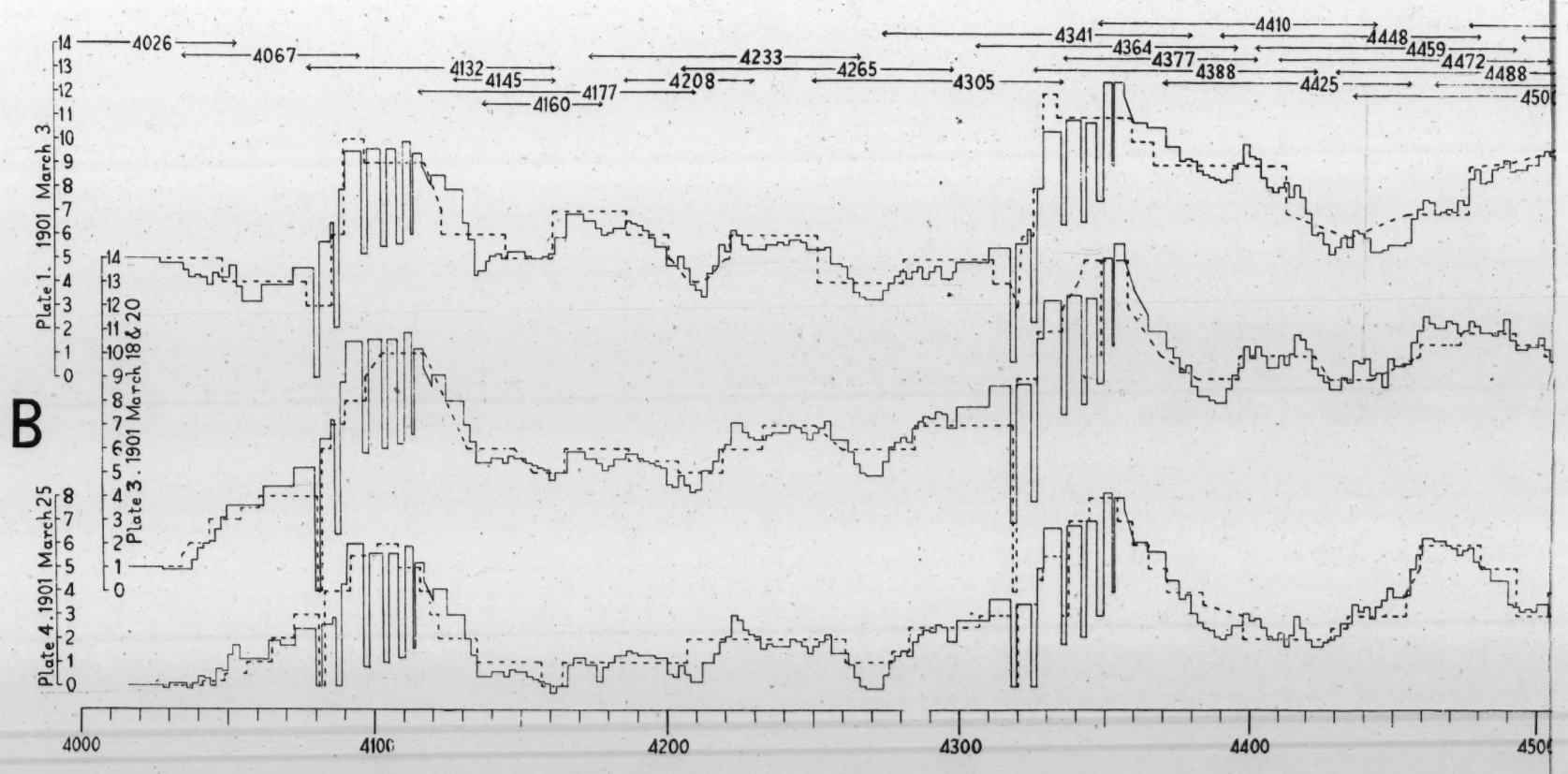
I wish to acknowledge the help I have received from my Assistant, Mr JAMES CONNELL, who attended to the guiding of the telescope and plotted the curves given in the plates accompanying this paper.

BECKER: ON THE SPECTRUM

Intensity curves of the spectra of Nova Persei,

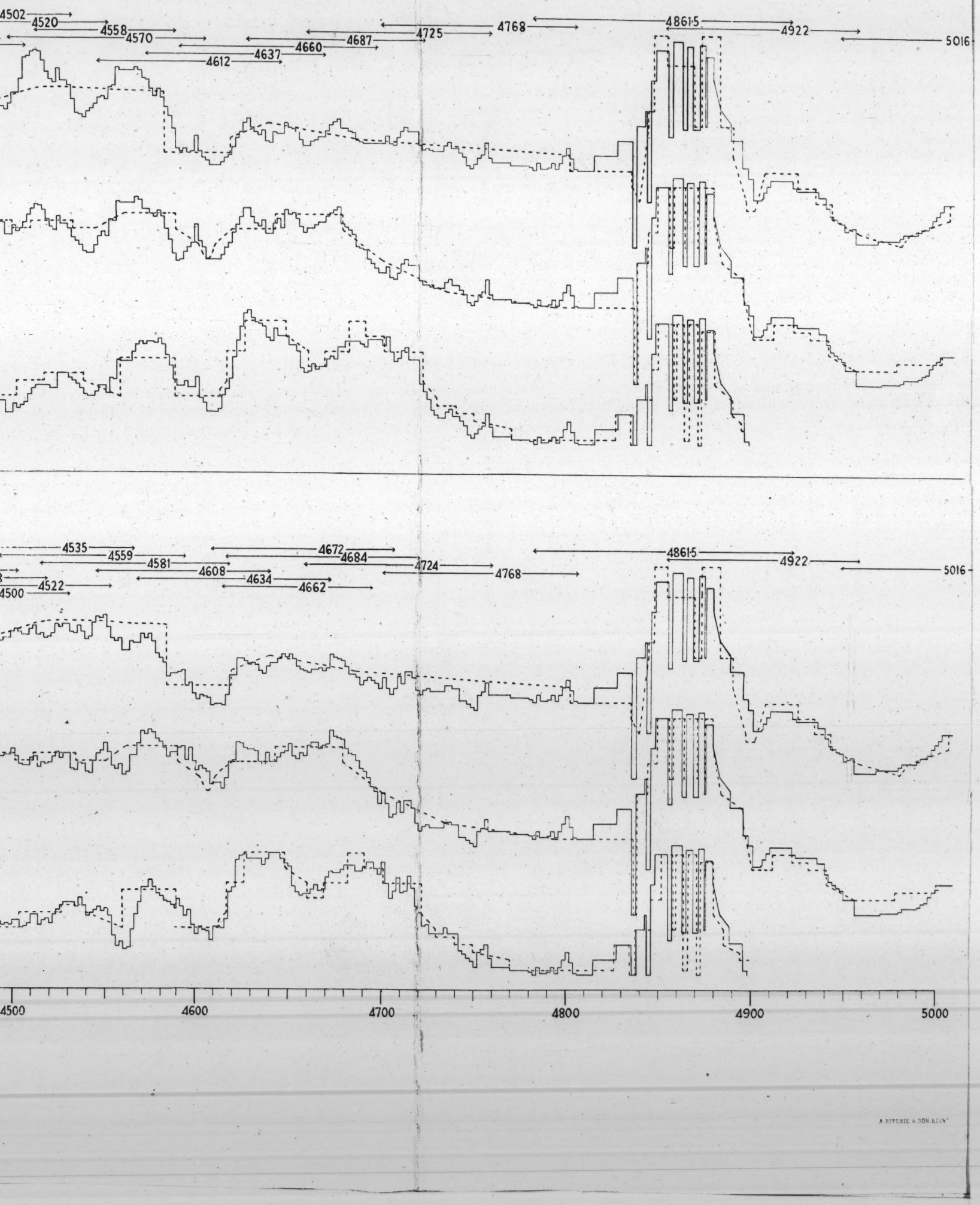
calculated ———. observed ·····





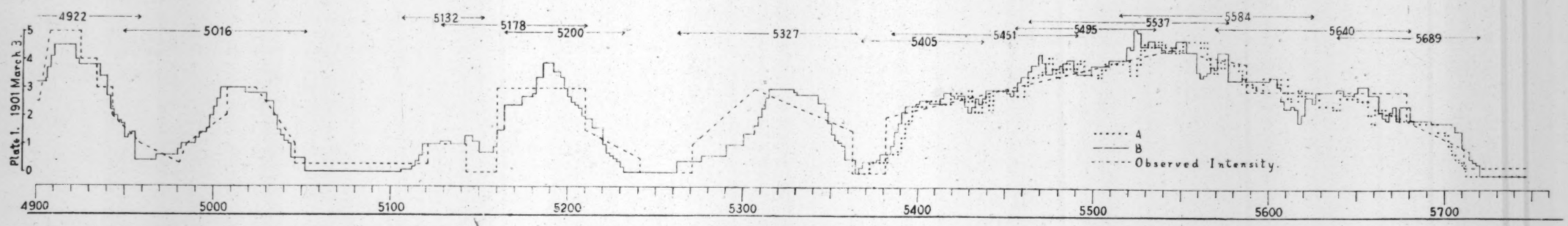
OF NOVA PERSEI. — PLATE I.

i, 1901, March 3, March 18-20, and March 25.

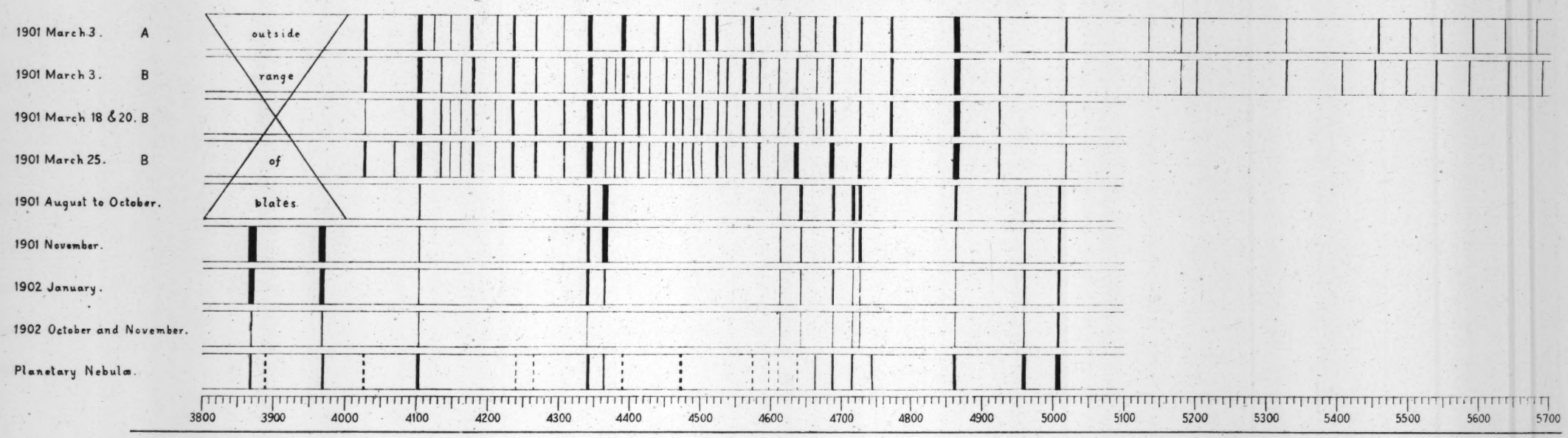


A. RITCHIE & SON, EDIN.

Intensity Curve, 1901, March 3 — continued.

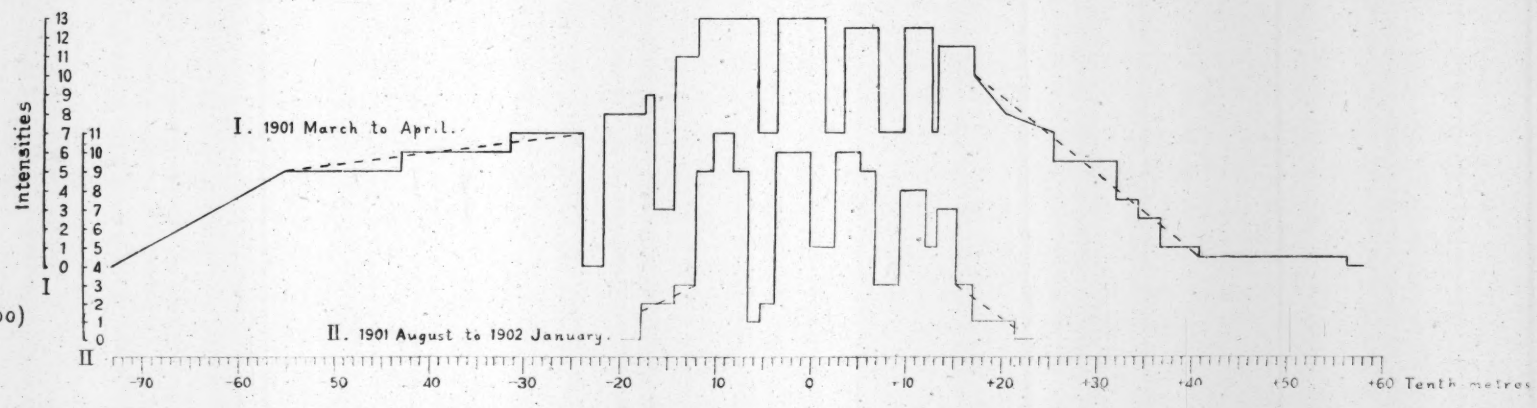


Spectrum of Nova Persei reduced to lines.



Structure of Bands.

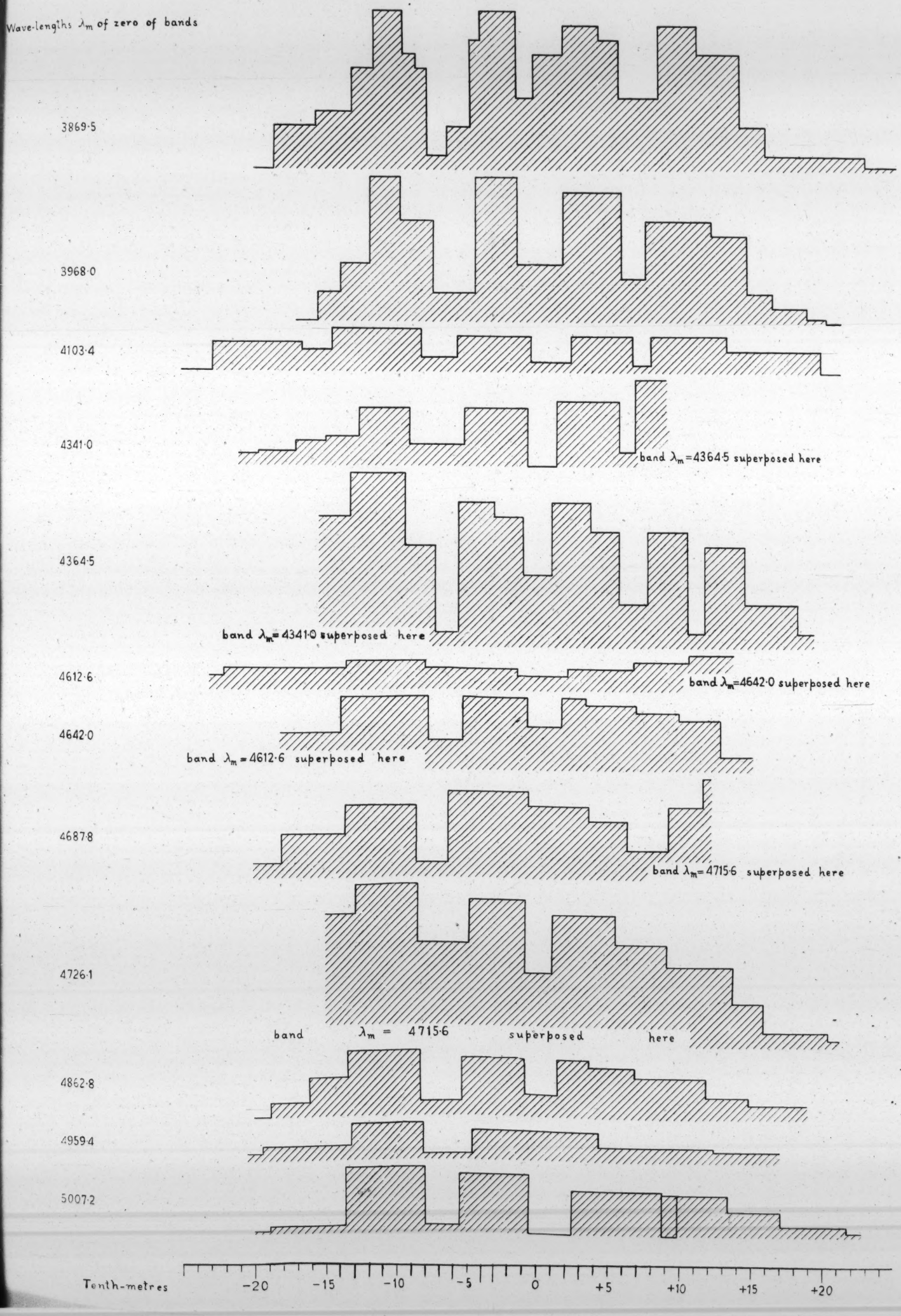
(reduced to wave-length 4500)



BECKER: ON THE SPECTRUM OF NOVA PERSEI. — PLATE III.

Curves of observed intensity of 12 bands in the spectrum of Nova Persei.

1901 August to November. abscissæ a_m (see § 11)



XI.—The Histology of the Blood of the Larva of *Lepidosiren paradoxa*. Part I. Structure of the Resting and Dividing Corpuscles.

By Thomas H. Bryce, M.A., M.D. (With Five Plates.)

(Read January 18, 1904; MS. received March 19, 1904. Issued separately November 19, 1904.)

The material for the observations recorded in this paper has been kindly lent to me by Mr Graham Kerr, Professor of Zoology in the University of Glasgow. It consisted of some of his beautiful series of cut embryos, and of some freshly-sectioned material which I stained specially for the purposes of the research.

The blood corpuscles of the embryo Lepidosiren are exceptionally favourable objects for the study not only of the morphology of the blood, but also of cell structure. The karyokinesis in the red corpuscles presents features of considerable interest—and the phenomena are presented to the observer on such a scale as to render them almost diagrammatic.

In the present paper I shall deal with the structure of the corpuscles and the mitotic phases in the erythrocytes, reserving for a future communication the results of studies on the origin and histogenesis of the elements.

Methods.

For the study of the dividing red corpuscles I selected a stage in which the embryo was small enough to have permitted perfect penetration of the fixative fluids, and yet sufficiently advanced to have its cells free of yolk.

The stage selected was that represented in pl. x. fig. 32 of Mr Graham Kerr's memoir* on "The External Features in the Development of *Lepidosiren paradoxa* (Fitz.)," a larva twenty-four days after hatching.

The embryos chosen had been fixed in sublimo-acetic fluid, and the fixation leaves nothing to be desired.

The sections were cut at 10 μ, which was rather thick for some points, but the nature of the material, owing to the mass of yolk, did not permit of thin sections.

The stain employed was in the first instance iron hæmatoxylin, with a counter stain of eosin. It was, however, discovered that even at this early stage several varieties of leucocytes were present in the blood, and for the study of these a stain of methylene blue and eosin, and the mixture of Ehrlich known as Triacid were employed.

The best results in some respects were obtained with the first named, especially for the centrosome of the erythrocytes, but for the resting red corpuscles and the leucocytes the methylene blue and eosin gave a finer differential colorisation.

* *Phil. Trans.*, vol. cxcii. B. 182, 1899.

The two dyes were applied successively, and not in a mixture, according to a method described by Dr GULLAND, and communicated to me by Dr GOODALL. The sections were stained first for about five minutes with a saturated watery solution of eosin, and then after washing, with a saturated watery solution of methylene blue for two or three minutes. They were then washed and differentiated if necessary in 90° alcohol, dehydrated and cleared in pure xylol.

The sections stained by this method are as bright after nine months as they were at first.

I. STRUCTURE OF THE ERYTHROCYTES.

(*a*) *Resting Corpuscles.*

The red blood corpuscles are oval biconvex discs, varying in size from 42 to 50 μ in length, 30 to 36 μ in breadth, and 12 to 15 μ in thickness. The nucleus occupies the centre of the disc (Pl. I. fig. 1, Pl. IV. fig. 32). It is also oval in shape, measuring 20 to 27 μ in length, 12 to 15 μ in breadth, and 9 to 12 μ in thickness.

(1) *Cytoplasm.*

The corpuscle is surrounded by a delicate membrane. The cell body shows a peripheral ring or band, within which there is a coarse meshwork structure. The meshwork is not very regular, but the thickness of the sections intensifies the appearance of irregularity. The meshes are from 3 to 4 μ in diameter. The whole reticulum centres on the nucleus, having a general radial direction from nucleus to periphery. At the nodal points there are strongly refractile granules of considerable size.

In some corpuscles the fibrillæ of the reticulum in the central nuclear portion of the corpuscle are arranged as parallel running threads between the nucleus and the periphery, but it is not quite clear how far this is a normal appearance.

The staining reactions of the meshwork are as follows:—With iron hæmatoxylin it is grey, while the microsomes are black (Pl. I. figs. 1 and 2); with methylene blue and eosin, the meshwork stains bright red and the microsomes are dark red spots; with triacid it is yellow, and the microsomes stand out as darker yellowish-brown points. In some favourable stainings with the last-named mixture the alveoli had a faint pink tinge.

In all the larger corpuscles there is a large vacuole, with structureless contents, showing no differential reaction to any of the stains used.

Round the equator of the cell there is a remarkable band about 3 μ in diameter. It forms a complete peripheral ring, when the corpuscle is seen on the flat (Pl. I. fig. 1, Pl. IV. fig. 32). In the fixed cell its appearance is distinctly fibrillar. The fibrillæ run

concentrically; and though they seem for the most part parallel, there is a considerable amount of apparent crossing and recrossing.

In profile views (Pl. I. fig. 2 and Pl. IV. fig. 33) there is to be observed at each end of the corpuscle an area free of reticular formation, but occupied by a number of fine points arranged generally in a straight line. These I take to be the cross sections of what appear to be fibrillæ seen on the flat.

Such a peripheral ring has been described in the red blood corpuscles of the chick embryo by DEHLER,* and in amphibian corpuscles by NICOLAS† and MEVES.‡ The latter has demonstrated that in Salamandra the ring is fibrillar, consisting of very fine threads running parallel to one another, or a single unbroken thread developed into a skein in the wall of the corpuscle. This is displaced inwards at the beginning of mitosis, undergoes a loosening, and then disappears as such, its substance being apparently employed for the formation of the achromatic figure.

The structure thus described by MEVES is evidently of exactly the same nature as the band in the Lepidosiren corpuscles, but he finds no network such as I have described, and the question here arises whether that structure is not a precipitation product.

A reticular or meshwork structure has been described in amphibian erythrocytes by a number of authors (LEYDIG, FROMMANN, AUERBACH, FOÁ, and others), but it has been variously interpreted. GIGLIO-TOS§ figures a reticulum identically like that I have described, in the erythrocytes of the lamprey. What I have named the microsomes he calls hæmoglobigenic granules. Recently RUŻIĆKA‖ has represented the corpuscles of Rana as having a reticular structure closely resembling that seen in the Lepidosiren cells. BÜTSCHLI,¶ on the other hand, attributes to the outer portion of the corpuscles in Rana an alveolar structure bounded by a distinct membrane. Within this outer zone is an inner girdle-like zone of finely meshed internal protoplasm, while the central nuclear portion is occupied by a space containing stuctureless enchylema, in which there are radiating tracts of protoplasm.

I do not propose to discuss the history of opinion on the structure of the red discs, but I may mention that ROLLETT** in a recent paper concludes for an alveolar stroma, while WEIDENREICH'S†† recent observations support SCHÄFER'S conclusions (published in *Quain's Anatomy*), that the contents are fluid and structureless, enclosed by a membrane. In this case I feel no doubt of the existence of a membrane, but reserve is necessary as to the reticulum. It must be noted, however, that I am dealing with young corpuscles.‡‡

* *Archiv f. mikr. Anat.*, Bd. 46, 1895.
† *Bibliographie anatomique*, 1896.
‡ *Anat. Anzeiger*, Bd. 23, 1903.
§ GIGLIO-TOS, *Mem. Accad. delle Sc. Torino*, T. xlvi., 1896.
‖ *Anat. Anzeiger*, July 1903, Bd. 23.
¶ *Protoplasm*, etc., English trans., 1894, p. 125.
** Pflüger's *Archiv f. Physiologie*, Bd. 82, 1900.
†† *Arch. f. mikr. Anat.*, Bd. 61, 1902, p. 459.

‡‡ MEVES, in a paper published since this paper was written (*Anat. Anzeiger*, vol. xxiv. No. 18), holds that there is no membrane in the amphibian corpuscles. The peripheral ring of fibrillæ is the only structural arrangement in Salamandra, but he states that in *Rana* there is, in addition, a 'Fadenwerk,' which is collected further round the nucleus, especially at its poles, and he quotes HENSEN (*Zeitschr. f. wiss. Zool.*, Bd. 11, 1862) as having described in the corpuscles of the Frog a granular material round the nucleus, from which threads pass to the periphery.

The Lepidosiren corpuscles thus resemble those of Salamandra in the possession of a very distinct equatorial band, but in their reticular structure they seem to correspond more to the description given of the corpuscles of the Frog. Taking all the possibilities into account, I adopt the view that the reticulum is not an artifact, but that it represents a protoplasmic framework. This is possibly alveolar in arrangement, but it is clear that the meshes of the reticulum exceed considerably the limit laid down by BÜTSCHLI for the true protoplasmic alveoli, and greatly exceed those of the optical reticulum seen in the protoplasm of the leucocytes. The erythrocyte is a much differentiated cell, and the structure described is evidently a secondary one. The whole protoplasm is fibrillar, but the framework is not necessarily fibrous or fixed. I believe rather that it is colloidal. I derive it from a vacuolated condition, in which the active protoplasm (Hyaloplasm) is greatly reduced, and it may well be that an original alveolar arrangement has been lost by the breaking through of mesh walls.

The peripheral band must be either the cause or the consequence of the shape of the corpuscle. It disappears when the disc begins to round up for division. This suggests the possibility that the appearance is due to a massing of the mesh walls. Further, in the angular interval between the upper and under layers of the membrane round the equator, there is a space (fig. 2, Pl. I.) occupied by the fibrillæ of the ring cut across. When the corpuscle rounds up, this space disappears, and the band is replaced by a reticular formation.

These considerations, combined with observations on young corpuscles, incline me to the view that the ring may rather be the consequence than the mechanical cause of the shape of the corpuscle, but the matter will come up for discussion again in the second part of these studies, when I am in the position to deal with the histogenesis of the cells.*

The question here arises whether the corpuscles which have assumed the biconvex disc shape are capable of division. Besides the corpuscles with oval nuclei, there are others with round nuclei, and a smaller cell body showing a finer reticular structure. These do not assume the disc shape, though they are oval in form. They are found in active division. In the second part of this memoir I shall discuss the relationship between these two forms. Meantime it has to be determined whether both classes of cells are dividing elements. In the later stages of mitosis there is little to distinguish the one class from the other, for all dividing corpuscles are spherical. Variation in the size of the chromosomes would indicate a derivation from a coarser or finer chromatin network, and the round-nucleated corpuscles have distinctly a finer network of chromatin than those with oval nuclei. Direct observation, however, shows that by far the greater number of nuclei showing prophase stages are oval in shape, and between

* MEVES, in a recent paper cited in the note to page 293, concludes that the band is the cause of the biconvex shape of the corpuscle. His explanation of the mechanism does not seem to apply very satisfactorily to the Lepidosiren corpuscles, but I must postpone a discussion of the question until all the stages in their histogenesis have been worked out. It seems to me that it is only by a study of the developmental stages that the significance of the band or ring can be determined.

the rounded daughter corpuscles, and the biconvex disc-shaped corpuscles, are to be seen all varieties of intermediate stages. These must necessarily be corpuscles either assuming the disc form or rounding up again for division. These intermediate forms sometimes possess two centrosomes; sometimes no centrosome can be demonstrated. The corpuscles showing early prophase stages of the nucleus always have two very distinct centrosomes, and they are either quite spherical (fig. 8, Pl. I.) or are oval and often irregular, showing, according to the plane of the section, one or two lateral projections (Pl. I. figs. 3 and 4).

Corpuscles such as that drawn in fig. 4, Pl. I. are frequently met with, in which two very distinct centrosomes are present, although the oval nucleus shows still a coarse network.

I believe I am justified in stating that, while it is possible that the corpuscles with vesicular nuclei may not divide, at any rate the smaller disc-shaped ones represent the resting phase of the dividing cells.

In the resting stage, if this be so, no centrosome is present in any form in which it can be distinguished from the microsomes.

(2) *Nucleus.*

As mentioned above, the nucleus is an oval body. It has a very coarse chromatin network (Pl. I. figs. 1 to 5), with large karyosomes close packed. In a considerable number of corpuscles the nucleus is to all appearance a solid mass of chromatin.

The reactions of the nucleus to the various dyes is interesting. In iron hæmatoxylin material the chromatin holds the stain with great persistency, so that the erythrocyte nuclei are still intensely black after all the other nuclei have completely surrendered it. With methylene blue and eosin, the network has a blackish violet colour, quite different from the lighter violet of the nuclei of the leucocytes, and again from the pure blue of the nuclei of the cells of the mesenchyme.

The chromatin network again selects the orange from Ehrlich's mixture, and has a golden colour. The alveoli are occupied by a delicate green staining, but no linin threads can be made out. In a successfully stained specimen the chromatin of the mesenchyme nuclei selects the basic dye, and their green colour contrasts with the golden yellow of the nuclei of the red corpuscles.

Notwithstanding this behaviour to the dyes, the rounded masses in the nuclei are not true nucleoli, but karyosomes,* or at any rate they are local accumulations of the same substance as forms the intervening bars, and, as later, is uniformly distributed along the spireme thread.

(*b*) *Mitosis.*

As I have already stated, no centrosome is to be seen in any recognisable form in any of the resting corpuscles, large or small.

* *Cf.* Pappenheim, *Virchow's Archiv*, vol. 145.

The first evidence of the onset of mitosis is the formation of a bulging of the central nuclear portion of the corpuscle on one side. In this projection are seen in the vast majority of cases two centrosomes lying side by side, and close to the surface of the corpuscle, and remote from the nucleus (Pl. I. fig. 4). Each centrosome is the focal point of far-reaching radiations, which are clearly directly continuous with the reticulum of the corpuscle. They have every appearance of being simply a radially disposed portion of the general network. The centrosomes are not connected directly by intervening fibres. In fig. 4, Pl. I. an appearance seen in that, as well as other corpuscles, is suggested. On the left of the nucleus the meshes of the network appear drawn out towards the site of the centrosomes, and the radial fibrillæ can be traced far out forming the walls of the meshes of the network. This appearance is transitory. In the next stage (Pl. I. fig. 8, Pl. IV. fig. 34) the lateral wings have been drawn in, and the corpuscle has become spherical. The radiations are confined to one pole of the cell, the centrosomes remaining near together and close to the surface.

(1) *Structure of Centrosome.*

The structure of the centrosome varies according to the character of fixation and the manner of staining. In iron hæmatoxylin sections the appearances depend on the degree of abstraction of the stain. When much of the stain is left, the body is a very large one, and the black colour is even continued out along the radial fibrillæ. When the decoloration is carried far, there is a much smaller dark point in the centre of a halo staining red in preparations counterstained with eosin. The fibrillæ spring from the circumference of this halo (Pl. II. fig. 13). This is clearly an instance of concentric decoloration, and the black spot is not a true centriole in Boveri's* sense. I have not been able to demonstrate a single such centriole or pair of centrioles at any stage of mitosis, but frequently the centrosome has the appearance of a grey spot, containing a group of centrioles. A slightly lobed appearance of a solid centrosome points to the same structure, even though no separate granules are to be made out. The question arises whether this is a 'fragmentation' of the centrosome (Boveri)* or the true structure.

With the other dyes used the centrosome is not so vividly differentiated as with iron hæmatoxylin, but in view of the tendency of that stain to mask a finer structure by remaining lodged between the smaller elements, a truer picture is perhaps obtained by their use. In methylene blue and eosin sections the centrosome is a red area occupied by fine granules of the same size as the microsomes, but darker in colour, having a neutral tint—an appearance very possibly due to their being massed together. In the same way in triacid preparations the centrosome is yellow, with brownish yellow granules.

* *Zellen-Studien*, Heft iv., "Über die Natur der Centrosomen," 1900.

(2) *Origin of Centrosome.*

The origin of the two centrosomes is very difficult to arrive at. I have seen only a very few corpuscles with a localised radial disposition of the reticulum in which there were not two centrosomes, either together in the same, or one in each of the adjoining sections of the series, and these few cases are difficult of interpretation. There is not a single clearly defined centrosome, but an area (fig. 3, Pl. I.) on which the radii converge. This is occupied by granules in every way similar to the microsomes. The extreme rarity of this stage, if it be a stage, shows that it must be a short-lived one, and that almost immediately the centre of activity is duplicated.

It is clear that the centrosome or centrosomes described have no direct relationship to that of the previous division. The resting cells have no centrosome, and no granules distinguishable from the microsomes which can be recognised as centrioles. Further, the new centrosomes do not appear where the old disappear, and therefore, unless on the purely theoretical assumption that the centrioles are scattered in the protoplasm, and though indistinguishable retain their identity,* to become the new focus or foci, the centrosome must be considered to appear *de novo*. That this is actually the case is strongly supported by instances such as these figured on Pl. I. figs. 5 and 6, in which two centrosomes are seen so far removed from one another that it is hardly possible to believe that they have not appeared quite independently of one another. It is remarkable, however, that in later stages, when the centrosomes are very far apart, presumably successors of a stage such as figured (Pl. I. fig. 6), they are still single. I have seen no multipolar figures, and in such cases there is a suggestion that the chromosomes are forming themselves into two groups round the two asters. I have seen only a small number of such figures, but even the one or two I have seen seem to prove that the two centrosomes *may* appear independently; and the fact that the independent centrosomes do not divide and form multipolar figures further suggests the possibility, in the absence of any stage in which a single definite centrosome can be confidently asserted, that the two adjacent centrosomes are independent from the first—that is, as definite stainable and recognisable foci.

(3) *History of the Nucleus during Mitosis.*

It will not be necessary to deal in detail with the history of the chromatin as it presents only the well-known evolutions; a few points only require to be mentioned.

The spireme thread is not beaded; that is, there is no distinction between a linin basis and chromatin granules imbedded in it. The whole thread stains uniformly. In this respect it differs from the thread seen in the prophases in the nuclei of the

* Meves, *Verhand. anat. Gesellschaft*, 1902.

leucocytes. The longitudinal splitting takes place early. The Vs are unequal, with one short and one long leg. The latter in the metaphase is of such length that when all seen in one section it extends round a third of the circumference of the cell. This makes the metaphase figures so complicated that I cannot be certain of the number of chromosomes.

In the late anaphases the chromosomes are merged again into a seemingly solid mass of chromatin, which no amount of extraction will resolve into separate elements. The long tails are gradually drawn into the common mass and an oval solid nucleus is formed. In many resting cells, as mentioned above, the nucleus has the same character, and the appearances point to the coarse reticular stage being reached by a sort of vacuolation. Throughout all the phases the chromatin retains the staining reactions described for the resting nucleus.

(4) *History of the Achromatic Figure.*

At the stage at which we left the centrosomes when they lay close together, and the corpuscle has rounded up for division, we noticed that there were no direct connecting threads between them. On their outer sides the radiations are strong and join the general reticulum. As the centrosomes draw apart (Plate I. fig. 9) it becomes clear that there are still no fibres directly joining the centrosomes, and that the radiations are stronger on the side of the nucleus. Both at the equator of the spindle figure and where the radiations of the asters meet, the fibres seem to branch and anastomose. I think the appearances are in favour of an anastomosis rather than of a mere crossing of the fibres; one never sees a loose end at any stage of the process.

When both centrosomes are sharply in focus at the same time, the axis of the spindle system is seen to be occupied only by a faint system of branching and anastomosing fibrillæ.

There is, strictly speaking, no 'central spindle' spun out between the centrosomes, but only two systems, mainly of mantle fibres, which join one another round the equator (Pl. I. figs. 11, 12). In a cross section of the metaphase figure there is no core of fibres representing a cross-cut central spindle in the heart of the equatorial crown; only a few fine fibrillæ are to be made out.

The appearances point, not to any new formation of radiating fibres, but to a conversion, step by step, of the general network into radiating tracts, until it has all been drawn into the opposing systems, and the achromatic figure comes to be placed symmetrically in the corpuscle.

As the daughter chromosomes move apart, the axis of the spindle system is seen to be occupied by loosely-arranged, irregularly-disposed fibres; and as the anaphase progresses, the 'subequatorial fibres' (Meves) come out more and more clearly, while the axial system becomes more loosely arranged (Pl. II. figs. 13, 14, 15; Pl. V. fig. 39), until we have a central space traversed by coarse much-branched fibres, and bounded

laterally by very distinct fibrillæ. These branch, and the branches join at the equator those of the fibres of the opposite astral system, while the outermost threads abut against the cell membrane, and those from opposite poles are seen to meet at the point where it is becoming infolded (Pl. II. fig. 15 ; Pl. V. fig. 39). There is no thickening of the membrane at the point of infolding.

I have shown that at all stages the axial system of fibres is very feebly developed. The contortion of these fibres in the anaphase cannot be due to any 'pushing' force exerted along them, but rather I believe to the accumulation at, or determination of fluid to the equator of the corpuscle. In some preparations I have seen an actual vacuole occupying the spindle axis, as if the protoplasmic threads had been wholly withdrawn towards the poles.

The subequatorial fibres become more strongly marked at this stage, and it is certainly suggested that the lines of force are now directed on the cell periphery, and the picture gives the idea that the force that is exerted by or along the lines of these threads is rather a tractive than a pushing one. The determination of fluid to the equator seems coincident with the passage of the chromosomes to the spindle poles. It is to be noticed that the distance between the spindle poles is distinctly increased at this stage.

Stages intermediate between that represented in Pl. II. fig. 15 and that shown in fig. 17 are rare, suggesting that once the infolding is produced, the cell division is quickly completed.

The subequatorial threads, still attached to the cell membrane at the bottom of the furrow, come to be stretched in a straight line between the spindle poles (Pl. V. fig. 40), and at a later stage (Pl. II. fig. 16 ; Pl. V. fig. 41) form, with the loose fibres in the axis of the spindle, an hourglass-shaped system of fibrillæ. These are grouped apparently in bundles, which contract into the 'mid-body' when division is complete. This has not the ring form seen in some cells, but is a large single body, probably formed from the smaller single granules on the bundles of threads of the previous stage (Pl. II. fig. 17). It becomes drawn out into a longish thread when the daughter corpuscles separate from one another (Pl. II. fig. 18 ; Pl. V. fig. 43).

The centrosome undergoes little increase in size during mitosis. There are no phenomena comparable to the enlargement of the sphere which occurs in dividing ova. In the late anaphases it is drawn out somewhat tangentially, and in the telophases it begins to dwindle. It lies in the hollow of the reconstructing nucleus and is difficult to detect, but in oblique sections it is seen standing out clear of the nucleus ; and in such sections, although I have given much attention to the point, I have not been able to convince myself that it was in any case duplicated (*cf.* fig. 17, Pl. II.).

In fig. 18, Pl. II. an appearance suggestive of a division is drawn, but careful examination proved that the radiations were all focussed on one point, and that the appearance was an accidental one, due probably to defective fixation. Similar deformations of the reticulum are met with in other cells removed from the centrosomal area.

I am obliged, therefore, to conclude that, as a recognisable structure, the centrosome disappears completely when mitosis is over, and that, in the absence of any proof that a contained centriole divides in the telophase to survive to the next generation, the centrioles also disappear as such, whether they be twin centrioles or a group of centrioles that could be supposed to persist.

Since the demonstration by WILSON,* following the earlier observations of MORGAN, that centrosomes arise *de novo* in the echinoderm egg during artificial parthenogenesis, a conclusion such as I have come to will seem less improbable than it would have some years ago.

(*c*) *Interpretation.*

As stated above, the conclusion was arrived at that the meshwork seen in the fixed corpuscles represented a protoplasmic framework in the living cells. Certain features of the resting cells, and certain appearances observed during mitosis, suggest that the protoplasm is of a specially viscous or ductile nature. The early history of the achromatic figure and of the centrosome preclude the application to this particular case *in sensu stricto* of either the fibrillar hypothesis (VAN BENEDEN), or of the doctrine of the organic radii (HEIDENHAIN). Both involve a structure of the resting cell which does not exist in the erythrocytes. In the conceptions of RHUMBLER,† however, I find room for a free formation of the centrosome; and the interesting feature of this case is, that the theoretical conditions of his model of cell division are fulfilled more closely perhaps than in any hitherto described.

The general reticulum is in the resting cell centred on the nucleus. It is under some degree of elastic tension, but the focus of that tension is not a centrosome, and therefore the conditions are not such as represented in HEIDENHAIN'S‡ model. On the appearance of the centrosomes, the reticulum begins to show a new disposition. It is now centred on these bodies, and round them is converted into radially directed threads. This radial arrangement of the reticulum is probably brought about by the withdrawal of the mesh walls circumferentially disposed into those radially disposed to the centrosomes.

Apart altogether from the why and wherefore, the centrosomes and their radiations are a manifestation of a tendency of the protoplasm to retract or concentrate itself at two focal points. The first effect of the retraction is the rounding up of the corpuscle; the second effect is the separation of the centrosomes.

When there are two centrosomes at some distance apart (Pl. I. figs. 5 and 6), the reticulum becomes converted into a symmetrical aster round each. When they lie close together, the asters are not symmetrical, for between them the protoplasmic material is limited, and is in large measure retracted on to the opposing centres. The progressive condensation or retraction of the threads on the outer sides being thus in excess of that

* *Arch. f. Entwickelungsmek.*, Bd. xii., 1901.

† *Ibid.*, Bd. iii., iv., xvii.

‡ *Ver. anat. Gesell.*, Berlin, 1896. *Arch. f. Entwickelungsmek.*, Bd. i.

between the centres, the result necessarily is to cause the centres to separate from one another.

Thus the separation of the centrosomes cannot here be due to the growth of a central spindle between them, but more probably to the contraction or retraction of the astral rays on their outer sides.

When the centrosomes have drawn apart to such a distance that the forces of which each is the expression can dominate half the cell, the whole protoplasm arranges itself symmetrically round them, and a position of equipoise is reached. The cytoplasm is already divided into two exactly equal portions, with a neutral zone between.

When the nuclear membrane disappears on the side next the spindle, it is noticed that the spindle system is disproportioned. The greater development on the nuclear side is possibly due to the taking up by the two centres of nuclear substance, rather than to an increase in the growth of the astral rays on that side, as has been suggested. It has been a point much discussed, how far the nuclear substance shares in the formation of the spindle system. In this case, while the system is formed apparently wholly in the cytoplasm, it seems almost certain that the achromatic substance of the nucleus is also drawn into and divided in it.

The arrangement of the spindle fibres in the anaphases is much like that of the outer polar fibres in BOVERI's figures of the dividing eggs of *Ascaris*,* but there is no plate at the equator. That the chromosomes are separated by a pushing force on the central spindle is excluded here by the absence of a developed central mass of fibrils.

The subequatorial fibres become exposed on the separation of the daughter chromosomes, and I believe they are related here to the division of the cell body, not in virtue of a pushing or expansive force, as MEVES described,† but of a contractive force. In fig. 15, Pl. II. the condition of things is pretty clear. In the axis of the spindle system there is a very loosely arranged mass of fibres, with large spaces between the threads, while peripherally, from under the reconstructing nuclei the subequatorial fibres extend towards the equator, and are there continuous with the cell membrane. The threads from the opposite poles meet exactly at the equator on the surface of the cell. At this stage the spindle poles are separated from one another, whether by a determination of the enchylema, or substance from the contracting daughter nuclei, to the equator or otherwise, and the consequence must be to put the longest subequatorial threads, *i.e.* those reaching the surface at the equator, on the stretch, and if they be of a colloid nature, they will, by their elastic tension, tend to retract on to their centres.

Thus we have produced a disposition of the protoplasmic threads, which is roughly indicated in a rude model which I have constructed (text-fig. 1). It is an indiarubber balloon, with a band applied round the equator, to which threads are attached. The threads are brought out through tubes, the inner orifices of which are carried some distance into the interior. When the balloon is inflated through one of the tubes with the threads loose, the result is such as represented in fig. 1 ; when they are drawn tight

* *Zellen-Studien*, Heft 2, 1888.

† *Arch. f. Entwickelungsmek.*, Bd. v.

up, on the other hand, the balloon is divided into two (fig. 2). This simple model is not required, of course, to prove that such a system of threads, if contractile, or under elastic tension, and attached to a cell membrane at the equator, will produce, or at any rate initiate, cell division. HEIDENHAIN's or RHUMBLER's models show this quite well, but the device described imitates in this one respect, I think, even better what I believe actually occurs in this special case.

There is no apparent sign of growth of the cell membrane at the equator, which is one of RHUMBLER's secondary factors. When once the furrow is produced it quickly completes itself, because the external pressure is now related to the two centres, and division takes place in the neutral zone between them. That the subequatorial threads

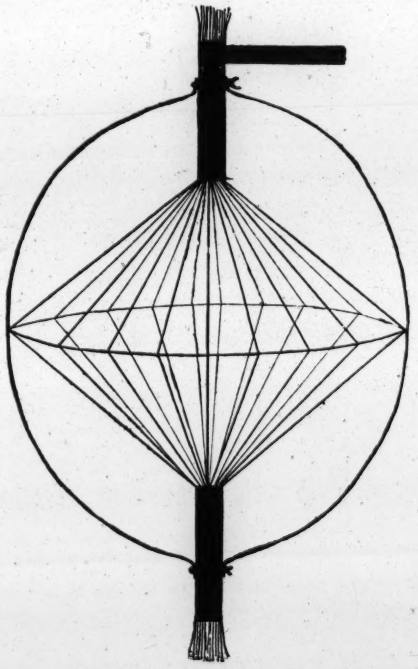


FIG. 1.

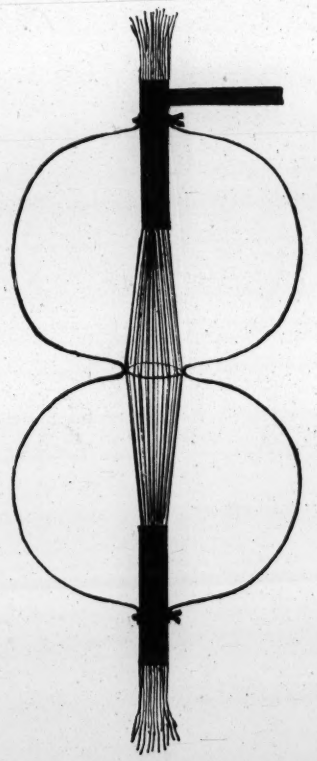


FIG. 2.

become stretched out in the axial line between the centres is seen in the photograph Pl. V. fig. 40, which closely resembles text, fig. 2 representing the model.

That the protoplasm has considerable ductility seems to be indicated by the tardy return to the reticular or alveolar condition, and also by the drawing out of the spindle remnant between the daughter cells into a thread of some length.

Turning for a moment to alternative hypotheses as to the structure of the corpuscles, I think the idea that the phenomena are to be attributed to lines of strain in a homogeneous and continuous colloid substance may be put aside. Although the alveolar theory of BÜTSCHLI is excluded in the strict sense of the term by the size of the alveoli, the protoplasmic framework behaves much as the hyaloplasmic framework does in RHUMBLER's theory and the elastic framework in his model.

It would be beyond the scope of this paper to enter on the possible theories as to the changes underlying the retractive phenomena in the protoplasm, but it may be noted that as the centrosome does not enlarge during mitosis, there can be no actual centripetal movement of the protoplasm on to that body. The same difficulty presents itself if we supposed, with RHUMBLER, that the centrosome acts by the abstraction of water from the hyaloplasmic framework, causing it to thicken and condense, unless it were further supposed that the water entered into new combinations in the centrosome, which it is not very easy to accept.

The account I have given above is an attempt to explain merely the phenomena as they are presented in the individual case, and does not involve a general theory of the mechanism of mitosis. It seems at first sight radically different from that given by various observers (WILSON,* TEICHMANN,† and others) of the appearances in dividing ova, in which the radiations are conceived as manifestations of an actual centripetal movement of the hyaloplasm. It may, however, be that the contradiction is one of appearance only. The essential factor is the same in both cases—a centripetal condensation of the hyaloplasm. In very fluid protoplasm like that of the ovum, there may well be an actual centripetal movement; but in very viscous protoplasm like that of the red corpuscles, which are undoubtedly firm and elastic bodies, the condensation may involve only retraction without a flowing movement. If the framework is fixed peripherally the retraction would involve increased tension and the rays would become contractile fibrils. Thus no one explanation will apply to all cases; for if the centrosome and its radiations are the expression of a condensation of the active protoplasm, due to chemical or physical causes, the mechanical results will vary with the consistency of the medium in which such condensation occurs.

II. STRUCTURE OF THE LEUCOCYTES.

Though it is now well known that in all classes of vertebrates the blood of the adult contains leucocytes of several different varieties, showing very different reactions to various dyes, little is known about the first appearance of the white elements in the blood of the embryo. The stage of embryonic life at which they appear seems to vary. In Lepidosiren the blood is already at a very early stage provided with several different kinds of leucocytes, but in the present writing I shall describe merely the morphology of the different kinds of free cells I have found in the blood and tissues of the embryo, reserving for a future communication the questions regarding the origin of the different varieties, and the interrelation between them.

(1) *Small Mononuclear Hyaline Corpuscles.*

This form occurs sparsely in the blood, but more abundantly in the spaces adjoining the posterior cardinal sinus.

* *Arch. f. Entwickelungsmek.*, Bd. xiii. † *Ibid.*, Bd. xvi.

It measures 14 to 16 microms in diameter, and possesses a small halo of very delicate protoplasm, which varies in amount from a zone hardly to be made out except under a high power, to a well-defined envelope to the nucleus (fig. 23, Pl. III.; fig. 44, Pl. V.). The protoplasm is nongranular, is hyaline in appearance, and even under a magnification of 1500 diameters it is not possible to make out more than the vaguest suggestion of reticular formation. In methylene blue and eosin preparations it is very delicately stained by the basic dye, while in those tinted with Ehrlich's 'triacid' mixture it has a faint grey tinge.

I cannot with certainty demonstrate a centrosome. The nucleus is round, with a coarsish chromatin reticulum, loosely arranged. It colours violet with methylene blue and eosin, and no part of the nucleus is oxyphil, the linin taking a cold blue tint, while the karyosomes are deep violet.

There is some doubt whether in all instances the nucleus is round, or whether there is a notching at one pole. I have observed some such nuclei, and it is obvious that the notching could only be seen if the section passed through a plane at right angles to it, and through the centre of the body.

(2) *Large Mononuclear Hyaline Corpuscles.*

This variety occurs more frequently than the last, and is the commonest form seen in the blood stream. It measures 24 to 26 μ in diameter. The protoplasm varies in amount, but is always merely a narrow zone surrounding the nucleus.

In methylene blue and eosin preparations it has a very delicate blue tint, and high magnification reveals a very delicate meshwork, with microsomes at the nodal points, which stain brightly with the blue dye (Pl. III. fig. 24; Pl. V. fig. 45). The nucleus is spherical; the chromatin network is very loosely arranged, and therefore in a section (fig. 24) one sees only rounded bodies with delicate threads radiating from them. These are not true nucleoli or plasmosomes, but karyosomes. So far as I can discover, plasmosomes do not occur in these embryonic nuclei. The staining reactions are interesting. In iron hæmatoxylin and eosin preparations the karyosomes are black and the general network red, but the chromatin parts more readily with the black stain than the chromatin of the red corpuscles, so that in sections which are suitable for a study of the latter the white cells are almost purely red. In methylene blue and eosin preparations the karyosomes are deep violet and the network takes a blue shade, but, as in the small corpuscle, there are no purely oxyphil granules. The deep violet blue stands out in strong contrast to the delicate pure methylene blue staining of the protoplasm.

In triacid material (fig. 25, Pl. III.) the network is green, and the karyosomes almost invariably retain some of the acid dye. The colour is sharply distinguished from the golden yellow of the chromatin of the red corpuscles, but also from that of the general mesenchyme nuclei, which stain pure green in preparations which show a yellow tinge in the leucocytes. In cells which show this staining, the chromosomes during division

have the same yellow tint as the karyosomes, showing that the material for the chromosomes is at any rate chiefly drawn from them.

Though in the great majority of sections the nucleus appears spherical, it is more than probable that corpuscles showing characters identical in other respects, but with a notch at one pole, as represented in fig. 25, are merely cells cut in a plane at right angles to that in which the rounded nucleated cells are cut. I have observed, however, all degrees of notching, from a slight bay to an angular depression, such as seen in fig. 25, or even to a linear fissure reaching to the centre of the nucleus, the two walls of which are in close contact. In sections such as that drawn in fig. 25, Pl. III. there is found lying opposite the notch a very imperfectly developed centrosome. I have drawn it as a nearly circular darker portion of the protoplasm, which is the 'attraction sphere,' staining like the protoplasm a neutral tint with triacid. At the centre is a slightly darker circular spot, which I take for the centrosome, but it is impossible to make out either a radial structure in the 'sphere' or rays extending from it into the surrounding protoplasm. In iron hæmatoxylin and eosin preparations the same spot in the cell comes out as a homogeneous area, staining of a darker red tint than the rest of the cytoplasm, but I have never seen a darker spot in its centre.

There is no doubt that we have here to do with a protoplasmic area, which corresponds to the area to be described in the next variety of leucocyte, showing an active and operative centrosome, with its attraction sphere and rays.

In triacid preparations the protoplasm stains of a somewhat indefinite neutral tint, and no granules are ever to be made out.

The cell represented in fig. 25, Pl. III. is certainly a leucoblast, but there is some reason to believe that certain of the cells like that figured in fig. 24, Pl. III. bear a relation to the ERYTHROBLASTS. These are cells of the same dimensions, but with larger karyosomes and a coarser intervening network, and showing a concentric fibrillation of the basiphil protoplasm. They will be dealt with in Part II.

(3) *Polymorphonuclear Corpuscles.*

This variety I have named in keeping with the general terminology of blood histology, on account of the lobed form of the nucleus. This body may, however, have many forms (as seen in figs. 19, 20, Pl. II.; 27–29, Pl. III.; and photographs 46, 47, 48, Pl. V.). Sometimes, as in fig. 27, Pl. III., the superficial appearance is that of a multinuclear corpuscle, but in reality the nucleus of that cell was single, but much lobulated.

This group of corpuscles is characterised by the possession of a well-marked centrosome in active operation. They are frequently seen in active diapedesis. Further, they always show, or almost always show, granules in their protoplasm. I shall first describe the centrosome. In fig. 19, which is the same cell photographed in fig. 49, I have drawn the body without filling in the granular cytoplasm.

The centrosome is a large body, which stains a delicate grey in iron hæmatoxylin preparations, many degrees lighter in tint than the intensely black centrosome of the dividing erythrocytes. In sections counterstained with eosin the body is red. It does not stain, therefore, like the chromatin. In methylene blue and eosin preparations it is very faint, coming only very indistinctly out as a slightly darker area on the faintly bluish-red protoplasm. In triacid preparations it is very distinct (fig. 31, Pl. III., and fig. 48, Pl. V.), and has a neutral tint. I have never seen in my material any finer structure within the centrosome, nor can I make out any granules, single, double, or multiple. I have therefore not adopted the term microcentum (Heidenhain),* but have used the word centrosome in Boveri's sense.

Round the centrosome there is a sphere distinctly radiate (as Van Beneden, Heidenhain, and others have described), bounded by a circle of microsomes. This separates the central from an astral zone, into which the fibres of the central sphere pass. There are no outer circles of microsomes concentrically arranged, as described for some cells (Drüner). The radii are at first straight, then becoming wavy, they seem to branch and join the general meshwork (fig. 20, Pl. II.).†

In fig. 21, Pl. II. is represented an amœboid leucocyte. The extended pseudopodium is not straight, but wavy, and its axis is occupied by a core of seeming fibrillæ passing from the centrosome. The cytoplasm around this central core shows an exceedingly delicate meshwork structure, but it is difficult to be quite sure of this towards the end of the pseudopodium, which is not well defined, and composed of very delicate substance. From the centrosome to the sides of the nucleus there seems to pass a core of seeming fibrillæ, represented in the section by the two lateral strands figured.

Another wandering cell is drawn in fig. 26, Pl. IV. It is passing through a space between a number of other mesenchyme cells. The manner in which the polymorphic nucleus is doubled up is interesting. The meshwork and radiations as well as the centrosome are very obscurely revealed in these granular cells by methylene blue and eosin, but it could be made out that at the extremity of the body the meshwork was drawn out, and that delicate radiations from the centrosome passed into it.

Fig. 22, Pl. II. represents a corpuscle in which the centrosome has divided. The 'attraction sphere' has apparently enlarged, and is not now bounded by a circle of microsomes. Round the two centrosomes new radiations are developing within the old sphere, which has now the appearance of an extremely fine feltwork, so fine that it is difficult to convince oneself that there is any structure at all.

The protoplasm of these leucocytes is basophil when free from granules; when only a few granules are present, it stains a delicate warm blue with methylene blue and eosin, and shows a very delicate faint meshwork. The granules colour intensely with eosin, are copper red after treatment with triacid, and blacken after iron hæmatoxylin. They vary in size in the same cell and in

* *Archiv f. mikr. Anat.*, Bd. xliii.

† *Cf.* Gulland, *Jour. of Phys.*, vol. xix.

number in different cells. In many they are scattered very sparingly through the protoplasm (fig. 27, Pl. III.), in others they are closely packed (fig. 29, Pl. III.), while in a smaller number the whole cell body is a uniform mass of closely apposed large granules of nearly uniform size (fig. 30, Pl. III.). These last-described corpuscles may be classified as

(4) *Eosinophil Leucocytes.*

Though they have all the appearances of eosinophils described in adult animals, it is not contended that they necessarily belong to the same category.

In leucocytes in which the granules are sparingly distributed, their dimensions vary from the minutest particle to bodies 2 to 4 μ in diameter. They are highly refractile, and this gives rise to some difficulty in determining their relations to the meshwork. In the drawings I have represented them as seen surrounded by their halo of refraction.

Apart from their actual relationships to the alveoli of the meshwork, it is quite certain they are not the nodal points of a reticulum, but are clearly metaplastic, probably minute drops in the protoplasm, which run together to form larger granules.

Exactly similar granules are found in the yolk cells, so that it is probable that these in the leucocytes are derived from that source, in which case they could be neither secretory nor excretory products; but to this question I shall return in the second part of this memoir.

There is always a space clear of granules round the centrosome (figs. 29 and 31, Pl. III.).

The nucleus varies greatly in shape, but in its other characters it agrees with that of the mononuclear leucocytes. The simplest form in which it is found is the horse-shoe shape (Pl. II. fig. 19; Pl. V. fig. 49). Between this and the complicated lobed condition there are all varieties. Sometimes it is ring-shaped (Pl. V. fig. 47), while sometimes it is formed of quite a number of lobes (Pl. II. fig. 20; Pl. III. fig. 27).

In no case are the lobes detached from one another to produce a multinucleated cell. They are always joined by attenuated portions of the nucleus. The leucocyte figured in fig. 27, Pl. III. has all the appearance of a polynuclear cell, but careful scrutiny proved that the several lobes were connected together like the two lobes of the nucleus seen in fig. 29 on the same plate.

I have found all the varieties of the leucocytes in mitotic phases, but these are few in number at the stages examined compared with the dividing erythrocytes. The polymorphonuclear may divide with the nucleus in the horseshoe form. The general character of the karyokinetic phases is the same as in the erythrocytes, but the chromosomes arising from a relatively loose and scantier chromatin reticulum are much finer and smaller, and the achromatic structures are of great delicacy. It is interesting to note that, as has been observed in other cases, the centrosome which is so large in the resting phases is reduced during mitosis to a very fine granule, hardly to be demonstrated.

As the object of the present writing is more descriptive than theoretical, I do not

propose to enter into a detailed discussion of all the theoretical points on which my observations bear. The problems relating to the histogenesis of the erythrocytes, and the origin and interrelations of the leucocytes, will be dealt with in the second part of the memoir. Here I shall only deal briefly with certain more strictly cytological considerations, and refer to certain conclusions which the contrasting characters of the erythrocytes and leucocytes seem to warrant.

It is to be noted that as the close juxtaposition of the blood elements has exposed them to identical conditions both of fixation and staining, the different pictures presented afford a secure ground for comparative study.

In the matter of the protoplasm, the material affords examples of all grades between a purely structureless condition and a coarse reticular formation. I have already sufficiently discussed the reticular structure of the erythrocytes, and expressed my belief that it represents an actual disposition of the protoplasm in the living cell. During mitosis it has been demonstrated that this framework is converted into the achromatic figure. In the leucocytes which show a definite but extremely delicate reticulum, this plays the same part in mitosis as the large network of the erythrocytes. The appearances differ in degree only, not in kind, and therefore there is no sharp line to be drawn between the two, whether one accepts the reticular or the alveolar hypothesis. It would seem to be more or less a matter of the relative proportion of active and passive elements, and of variations in consistency.

The differential staining of the chromatin of the erythrocytes is a point of suggestive interest, but I am not competent to deal with the questions of cell chemistry involved. While the behaviour to iron hæmatoxylin might be due merely to differing physical properties, the reaction to the other dyes indicates a chemical differentiation of the chromatin which must be in some way connected with the functions of the corpuscles.

The most suggestive of the contrasts observed is that in the characters of the centrosomes.

In the erythrocytes, which are passive bodies, in the resting stages there is no centrososome discernible. The body is related only to the mitotic phenomena, and when the kinetic phase is past it disappears as such. Every fact in its history points to its being merely the central point of a cytoplasmic condensation, whatever may be the physical or chemical changes involved.

In the leucocytes the centrosome always stains with the cytoplasm. In its full panoply of sphere and aster it is only seen in the leucocytes which undergo amœboid movements. This fact supports the view that it is related to these movements, and this is actually demonstrated by leucocytes caught in amœboid movement (fig. 21, Pl. II.). That some cytoplasmic activity exists, centred on the centrosome, is clear, but what the nature of the activity may be is another matter.

My observations are too few to warrant my going into this question. One would require to see many more amœboid leucocytes at all stages than I have done to form any opinion on the general question. I put forward the facts I have observed merely

in support of the supposition, that whatever may be the explanation of the outputting of the pseudopodium in the case of the leucocytes, the withdrawal and forward movement must be referred to the centrosome, with its sphere and radiations; and so far as the appearances go, they are in favour of the idea that the centrosome in this case also is the centre of a contractive force.

In the conversion of the leucoblast into the leucocyte, the attraction sphere and aster are gradually unfolded as the cell body increases in size. The nucleus is at first central, but later assumes an eccentric position, while the sphere moves towards the centre of the cell. I would explain this in the same terms as I explained the rounding up of the erythrocyte, and the separation of its centrosomes. The centrosome being here single, however, it comes to a position of equipoise in the centre of the protoplasm.

My reading of the structure of the leucocyte is different from that of HEIDENHAIN,* in so far as the radii seem to me to branch and join the general reticulum, which I believe (with the necessary reservations) probably represents an alveolar disposition of the protoplasm, but they act quite like his organic radii, in respect of the movement of the sphere. With regard to the form of the nucleus in relation to the movement of the sphere, my observations, so far as they go, seem to agree with his in matter of fact, but I have not followed out the point in such detail as to follow him into the domain of theory.

EXPLANATION OF PLATES.

The drawings were done, in everything but the very minute detail, by aid of the camera lucida (Abbé). The lenses employed were the 3 mm. and the 2 mm., both 1·4 numerical aperture, apochromatic objectives of Zeiss, combined with the compensating oculars 8 or 12. The magnification indicated was ascertained by the stage micrometer. It is rather greater than the magnification given by the combinations used, the excess depending on the depression of inclined drawing-table beyond the visual distance.

The coloured drawings were tinted with the same stains as used for staining the sections. The watery solutions of aniline dyes colour smooth Bristol board very delicately, and permit of a degree of verisimilitude difficult to attain with ordinary water-colours.

The photographs were all taken with the 3 mm. 1·4 numerical aperture achromatic objective and No. 4 projection eyepiece, at a distance which gave a magnification of 800, with the exception of fig. 37, in which the magnification is 1500.

PLATE I.

Fig. 1. Section red blood corpuscle in plane parallel to surface of disc. × 1200 d. Compare photograph, Pl. IV. fig. 32.

Fig. 2. Section of same in place at right angles to surface of disc. × 1200 d. Compare photograph, Pl. IV. fig. 33.

Fig. 3. First stage of mitosis. Section passes through corpuscle in a plane vertical to its flat face. It is rounding up for division. Possible phase of single centrosome placed in a projection which has risen from centre of disc. × 1200 d.

Fig. 4. Similar stage in larger corpuscle. Two centrosomes. × 1200 d.

Fig. 5. Corpuscle with two independent centrosomes. × 1200 d.

Fig. 6. Corpuscle with two centrosomes which have appeared separately as in last. × 800 d.

* *Loc. cit.*

Fig. 7. Case of want of synchronism between centrosomal and nuclear cycles. Polar view of corpuscle with nucleus in stage of spireme, while centrosomes still close together × 800 d.

Fig. 8. Corpuscle which has rounded up for division. Nucleus in tight spireme stage. × 1200 d.

Fig. 9. Drawing apart of the centrosomes. Chromatin thread divided into V-shaped loops, each of which has a long tail-like process from one of its limbs. × 800 d.

Fig. 10. Same, a little later. × 2000 d. Compare photograph, Pl. IV. fig. 36.

Fig. 11. Stage in rotation of spindle system. × 2000 d. Compare photograph, Pl. IV. fig. 37.

Fig. 12. Fully developed metaphase. × 2000 d.

Plate II.

Figs. 13, 14, 15. Three stages in anaphase, illustrating especially the emergence of the 'subequatorial fibres.' × 2000 d. With fig. 15 compare fig. 39, Pl. IV.

Figs. 16, 17, 18. Telophases illustrating division of cell body; formation of intercellular spindle remnant; reduction of same to form mid-body, which disappears by being drawn out into a thread of some length; disappearance of centrosome *in situ* without previous division; condensation of chromosomes to form a continuous and homogeneous mass. Fig. 16 × 2000 d.; figs. 18, 19, 1200 d.

Fig. 19. One variety of polymorphonuclear leucocyte, showing large centrosome, radiate sphere, and single circle of microsomes. × 1500 d.

Fig. 20. Same, with ring-shaped and lobed nucleus. The cytoplasmic meshwork indicated. There are a few minute granules. × 1500 d.

Fig. 21. Amœboid leucocyte. × 2000 d.

Fig. 22. Polymorphonuclear leucocyte, showing division of centrosome within old sphere. × 1800 d.

Plate III.

Fig. 23. Small mononuclear hyaline leucocyte, methylene blue and eosin. × 2000 d.

Fig. 24. Large mononuclear cell, methylene blue and eosin. × 2000 d.

Fig. 25. Large mononuclear leucocyte, with notched nucleus and centrosome; Triacid. × 2000 d.

Fig. 26. Polymorphonuclear leucocyte; amœboid. × 1500 d.

Fig. 27. Another form of same, with much-lobed nucleus and scattered small granules. Centrosome not in section. Methylene blue and eosin. × 1500 d.

Fig. 28. Another form of same, showing nature of granulation in protoplasm. Centrosome not in section. × 2000 d.

Fig. 29. Same, with closely packed granules. × 2000 d.

Fig. 30. A further stage of same, showing all the characters of an eosinophil leucocyte. × 2000 d.

Fig. 31. Same, showing reaction to triacid. × 2000 d.

Plates IV. and V.

[All the photographs were taken at a magnification of 800 d.]

Figs. 32 and 33. Resting red blood corpuscles.

Figs. 34 to 43. Sequence of mitotic phases. Fig. 37 × 1500 d.

Fig. 44. Small mononuclear leucocyte.

Fig. 45. Large mononuclear leucocyte.

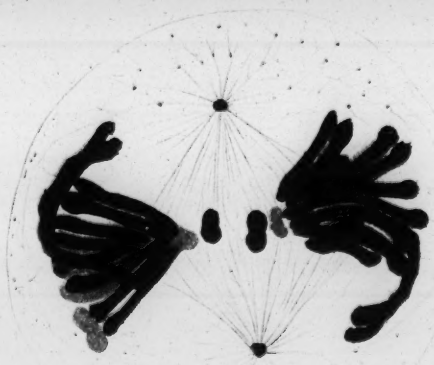
Fig. 46. Polymorphonuclear leucocyte, with centrosome and sphere.

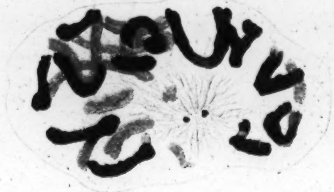
Fig. 47. Same, with ring-shaped nucleus.

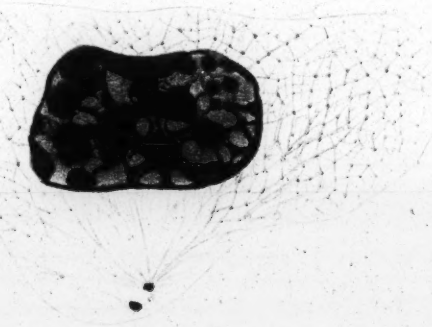
Fig. 48. Same, with fine granules.

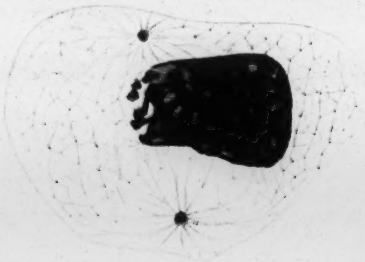
Fig. 49. Same, with large closely-packed granules, *i.e.* eosinophil leucocyte.

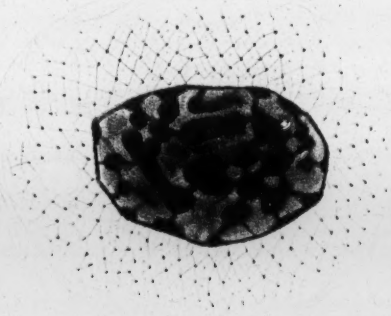
The photographs in reproduction have lost much of the delicacy of detail seen in the gelatino-chloride prints.

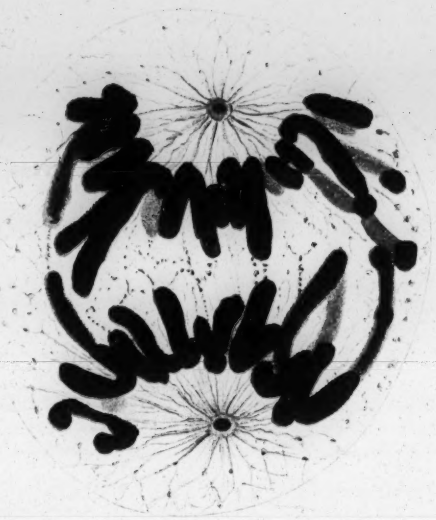












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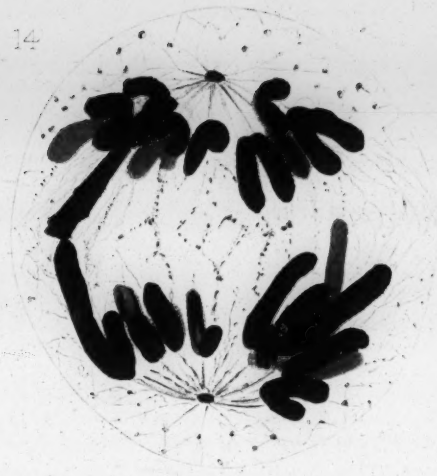

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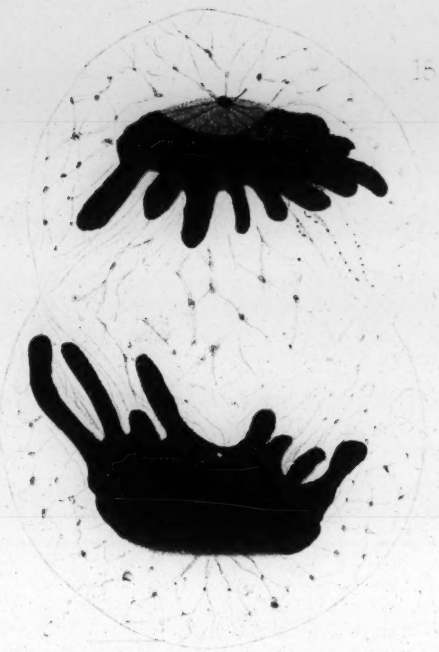

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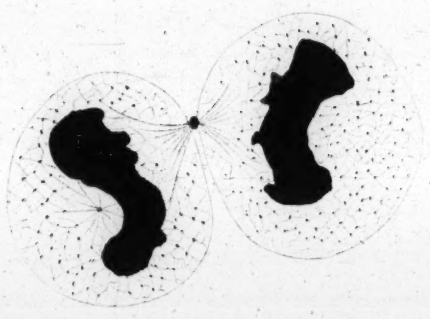

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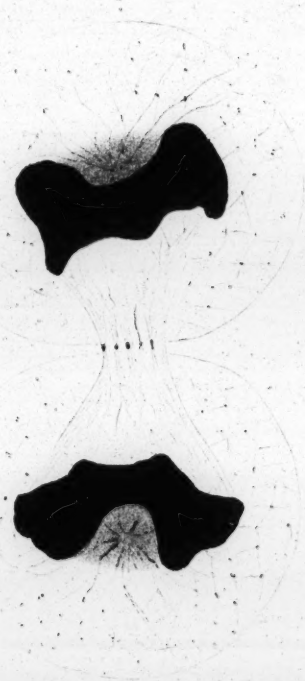

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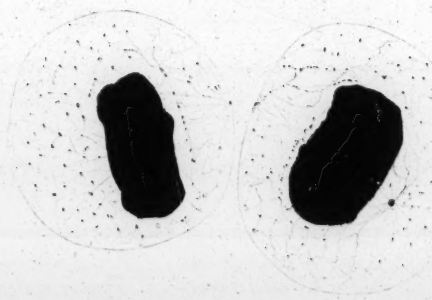

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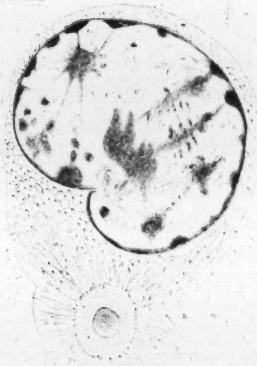

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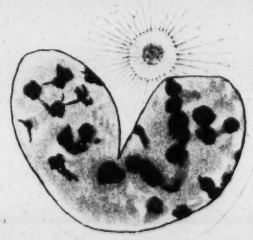

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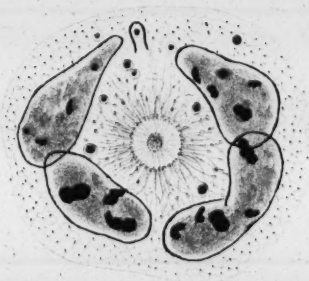

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Larvae. Histology of the Blood of the Larva of Lepidosiren Paradoxa. Part I. Plate III.

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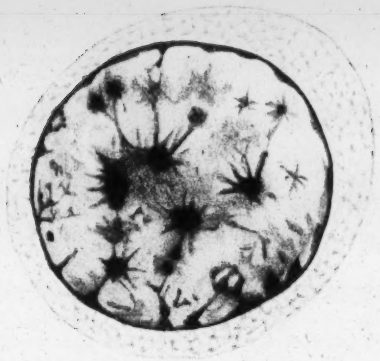

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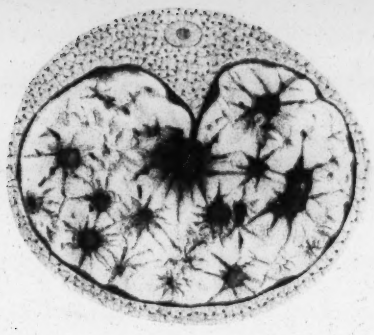

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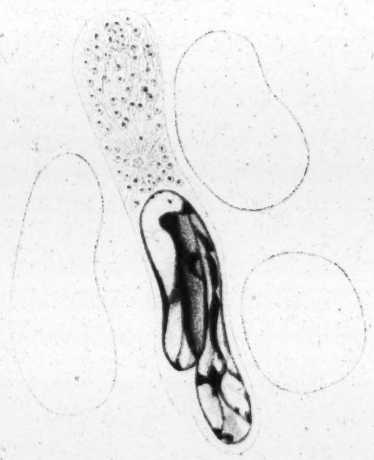

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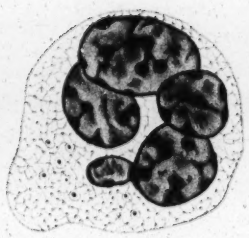

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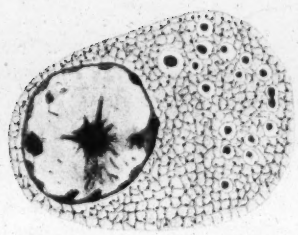

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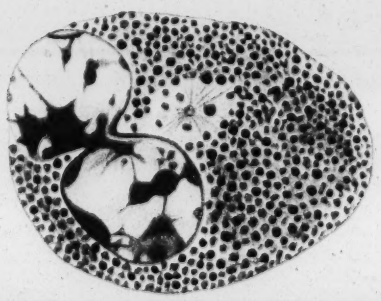

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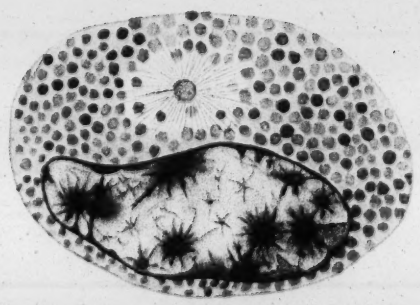

BRYCE: HISTOLOGY OF THE BLOOD OF THE LARVA OF LEPIDOSIREN PARADOXA.
PART I.

FIG. 32.

FIG. 33.

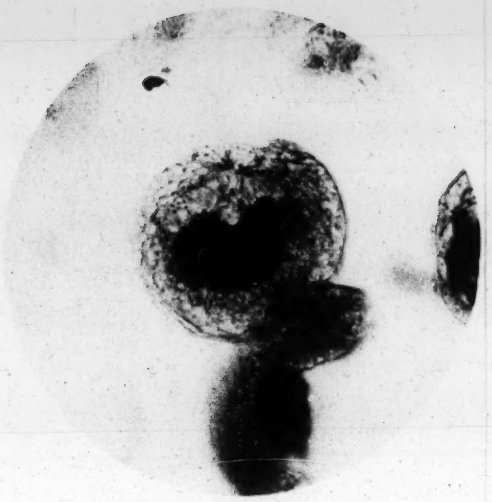

FIG. 34.

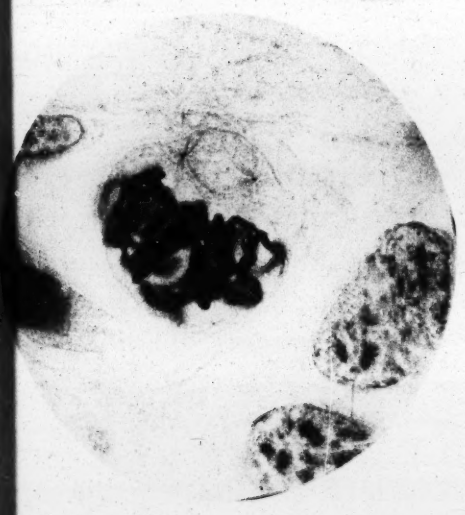

FIG. 35.

FIG. 36.

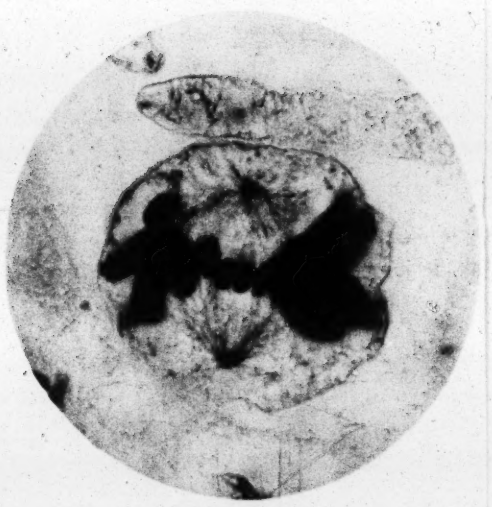

FIG. 37.

FIG. 38.

FIG. 39.

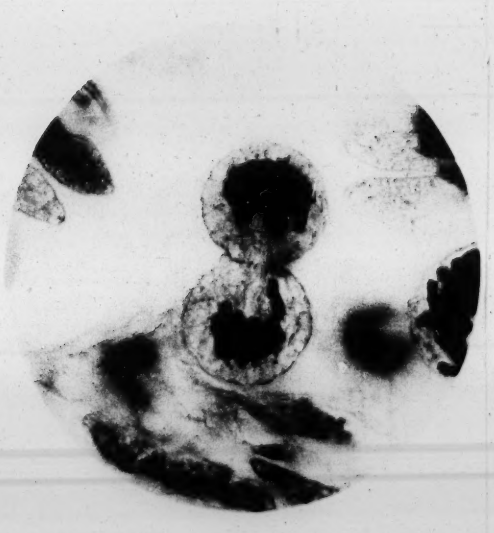

FIG. 40.

BRYCE: HISTOLOGY OF THE BLOOD OF THE LARVA OF LEPIDOSIREN PARADOXA.
PART I.

FIG. 41.

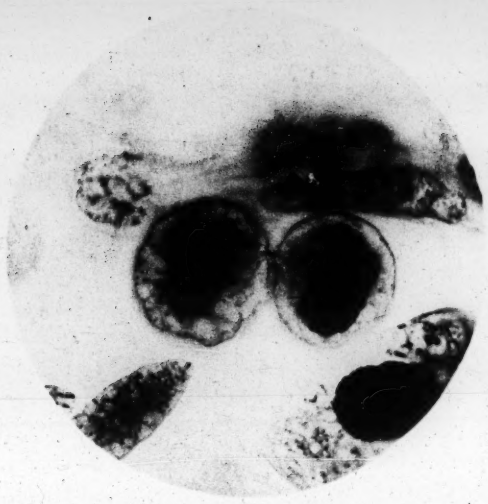

FIG. 42.

FIG. 43.

FIG. 44.

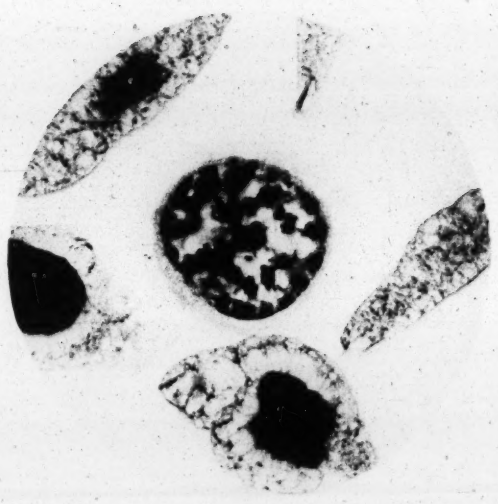

FIG. 45.

FIG. 46.

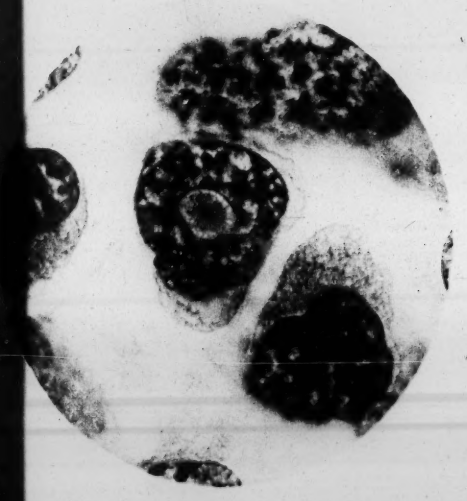

FIG. 47.

FIG. 48.

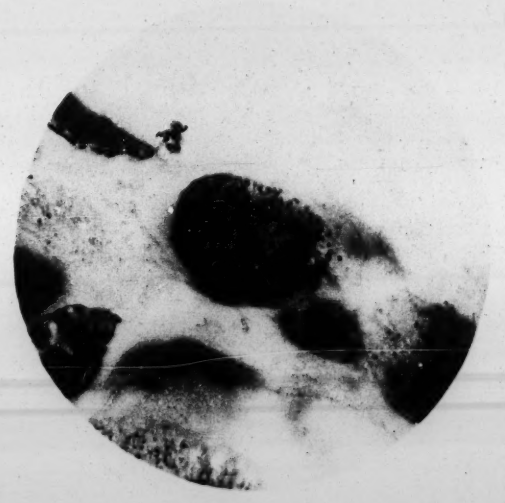

FIG. 49.

XII.—The Action of Chloroform upon the Heart and Arteries.

By **E. A. Schäfer**, F.R.S., and **H. J. Scharlieb**, M.D., C.M.G. (*From the Physiological Laboratory of the University of Edinburgh.*)

(Communicated March 21, 1904. MS. received August 17, 1904. Issued separately December 14, 1904.)

The original design of this research was to determine whether the extract of suprarenal medulla (or its active principle) has the power of antagonising the effects of an overdose of chloroform upon the heart and arterial system. Incidentally the research became extended so as to cover the action of certain other antagonising agents. It further appeared necessary, as the investigation proceeded, to subject the action of chloroform upon the vascular system to renewed study. For although, as the result of numerous recent researches, physiologists are in agreement regarding the general effect of the drug upon the heart, there yet remain various points requiring elucidation both as regards its effect on the heart and on the arteries.

Effect of Chloroform upon the Arteries.

Singularly little is precisely known as to the effect of the drug upon the arterial system. The most generally received opinion has been that adopted by Bowditch and Minot* to the effect that chloroform exerts, besides a specific action on the heart, a paralysing influence upon the whole vasomotor system, and that the fall of blood-pressure which accompanies its administration is due as well to the dilatation of vessels as to the effect which it produces upon the cardiac musculature. On the other hand, Arloing,† as the result of observations on the rate of flow through the carotid, made by means of the hæmadromograph, inferred that a constriction of arterioles is produced by the drug. Dastre‡ came to the same conclusion, and referred to it the pallor of the face which is seen in chloroform administration. But it is obvious that a diminution of rate in the carotid might be caused by dilatation of vessels in the splanchnic area, so that these observations cannot be regarded as conclusive. Gaskell and Shore,§ in their cross-circulation experiments, obtained distinct evidence of stimulation of the vasomotor centre; constriction of arterioles and rise of blood-pressure occurring as the result of the action of the drug upon the medulla oblongata. Roy and Sherrington‖ inferred that constriction of cerebral vessels is produced by chloro-

* *Boston Med. and Surg. Jour.*, 1874. *Cf.* Leonard Hill, *Brit. Med. Jour.*, April 1897.
† Thèse, Paris, 1879.
‡ *Les Anesthétiques*, 1890.
§ *Brit. Med. Jour.*, 1893, vol. i.
‖ *Jour. Physiol.*, vol. xi. p. 97, 1890.

form, while HÜRTHLE* found evidence of dilatation of these vessels, followed by constriction. NEWMAN† observed constriction of pulmonary capillaries in the frog as the result of chloroform inhalation. SHERRINGTON and SOWTON,‡ working with the isolated mammalian heart perfused with Ringer's solution by Langendorff's method, observed a diminished flow of the perfusion fluid when chloroform was added to it; this they were inclined to ascribe to a contraction of coronary vessels under its influence. C. J. MARTIN§ has suggested that this diminution of flow through the coronary vessels may be accounted for, without the necessity of assuming constriction of those vessels, by the fact that a diminished action of the cardiac musculature, such as chloroform produces, may tend by itself to diminish the rate of circulation through its vessels. To this we may add that in a heart which is separated from its surroundings and fed by the perfusion of fluid under pressure into the root of the aorta, in which therefore the mechanical conditions are very different from those which obtain normally, the aortic valves do not necessarily act efficiently, but often permit of some passage of fluid into the cavity of the left ventricle, and through this into the left auricle, and so out by the cut pulmonary veins; and the extent of this valvular defect with the consequent leakage will vary with the condition of tone of the heart and the force of its contractions.

Opinions on this subject being thus divided, it appeared important in the first instance to determine what is precisely the action of chloroform upon the arterial system. The method which we have used for this purpose is the classical one of perfusing the vessels with blood or saline fluid containing the drug in solution. The chloroform used for this purpose and in most of our experiments has been Duncan & Flockhart's, sp.gr. 1·49. The result of our preliminary experiments‖ showed that a solution containing from 1 gramme to 5 grammes of chloroform to the litre of circulating fluid produces a marked constriction of the frog's arterioles, and that this constriction is apparent whether the medulla oblongata and spinal cord are left intact or destroyed. These observations established the fact that for high percentages of chloroform (5 grammes per litre is approximately a saturated solution, and 1 gramme per litre is therefore one-fifth saturated) there is a pronounced excitation by the drug of the musculature of the arterioles—whether operating directly or through the vasomotor nerve-endings, our preliminary experiments did not decide—which may contract under its influence to such an extent as almost to arrest the flow of circulating fluid.

Since the publication of these preliminary results, C. J. MARTIN,§ in confirmation of earlier experiments in conjunction with EMBLEY,¶ has made observations upon the mammalian kidney by the plethysmographic method which appear to indicate that in dilute solution—the actual dosage was not determined, but the perfusing fluid (blood) was first passed through the lungs, into which a mixture of air and chloroform vapour was pumped—chloroform has the effect upon the blood-vessels ascribed

* *Pflüger's Arch.*, vol. xliv. p. 596, 1889.
† *Jour. Anat. and Phys.* vol. xiv., 1879, p. 495.
‡ *Thompson-Yates Laboratories Reports*, 1903, vol. v.
§ Private communication.
‖ Communication to the Physiol. Soc.; *Jour. Phys.*, vol. xxix., 1903.
¶ *Brit. Med. Jour.*, April 1902. *Lancet*, 1902.

to it by Bowditch and Minot, viz., that of producing dilatation. In repeating and extending our experiments we have therefore included the effect of the perfusion of mammalian organs with various strengths of chloroform dissolved in Ringer's solution.

Perfusion of Frog Vessels with Chloroform dissolved in Ringer's Solution.

The Ringer's fluid used consisted of NaCl, 6 grammes; $CaCl_2$, 0·1 gramme; KCl, 0·075 gramme; $NaHCO_3$, 0·1 gramme per litre. The chloroform was either dissolved in this solution in proportion determined by weight, or some of the fluid was saturated by being shaken up with and kept over an excess of chloroform, and was assumed to contain 1 part chloroform to 200 Ringer, this being the amount water will take up at the ordinary temperature of the air (15° C.). This saturated solution was mixed with varying proportions of normal Ringer. A fine cannula having been tied into the bulbus aortæ of the frog (*R. esculenta* or *R. temporaria*), the fluid was allowed to pass by gravity, at a pressure varying in different experiments from 50 mm. to 150 mm. of water, through the vascular system, and to drip from the extremities of the toes. In our earlier experiments the mode of determining the rate of flow was to count the number of drops per minute; but this method, although serving to indicate any differences of vascular calibre which are marked, is not sufficiently accurate for slight variations, since the size of the drops is liable to vary somewhat with differences of surface tension of the fluid, and the amount of dissolved chloroform or of intermixed blood and lymph may affect its surface tension. In all later experiments, therefore, the amount of fluid perfusing in a given time was accurately measured. Only the results thus obtained are included in this communication.

The result of these perfusion experiments with Ringer's fluid containing dissolved chloroform may be shortly stated as follows:—

With the strongest solutions, *i.e.*, from saturated (1 in 200) down to solutions containing 1 in 500, a very marked constriction of the arterioles is the result of perfusing with chloroform-Ringer, so that the flow of the perfusing fluid becomes very slow, and may almost cease. With increase of dilution the amount of constriction, as registered by rate of flow, becomes less; but although very slight when the dilution is considerable, we have been able to substantiate constriction with solutions as weak as 1 in 20,000. On the other hand, no solution of any strength when perfused through the frog's vessels has given evidence of dilatation of arterioles, the weaker solutions having simply shown themselves inert. If for the chloroform-Ringer which has been passed for some minutes through the vessels, and has produced the diminutions of flow above indicated, normal Ringer be now substituted, the flow again becomes more rapid, but the original rate is rarely again obtained; in fact, after the chloroform solution has been in action for some minutes, even if the strength of the solution be such as to be insufficient to cause actual constriction of arterioles, there is a tendency towards a gradual diminution in the rate of flow, which appears to be caused by œdema of

the tissues; the effect of the chloroform solution on the endothelium of the vessels being such as to render the capillaries more permeable to saline solution.

The following experiment will serve as an example of the results obtained with weak solutions of chloroform in Ringer's solution. The numbers represent the amount flowing through the vessels in equal periods of time.

Immediately before chloroform perfusion		28·5 c.c.
During perfusion of fluid containing 1 part chloroform in 20,000		22·5 „
Immediately after perfusion of chloroform-Ringer	1st period	17·5 „
	2nd „	21 „
	3rd „	25 „

Perfusion of Mammalian Vessels with Chloroform dissolved in Ringer's Fluid.

The kidney, leg, and heart of the cat, rabbit, or dog were employed, and the method of perfusion was the same as for the frog, except that the head of pressure was higher (80 to 100 mm. Hg.). The Ringer solution had the composition: NaCl, 9 grammes; $CaCl_2$, 0·24 gramme; KCl, 0·42 gramme; $NaHCO_3$, 0·1 gramme; distilled water, 1 litre, and was warmed to 38° C. by being passed through a glass spiral contained in a water-bath before being conducted to the organ to be perfused, which was itself also kept in a warm chamber at the same temperature. The perfused fluid was either collected in a graduated measure and the amount flowing in a given time recorded, or it was caused to work an automatic "tilter," so arranged that every 7 c.c. of fluid produced a see-saw of the tilter, and this was recorded by a magnetic signal. In some experiments Ringer's solution, containing a known percentage of chloroform, was, after the normal record of flow had been obtained, allowed to pass for a certain time through the vessels in place of the ordinary Ringer, and was then again replaced by ordinary Ringer, the rate being recorded before, during, and after the passage. In other experiments a chloroform-Ringer of known strength was injected by a fine hypodermic needle through the indiarubber supply-tube of the perfusion apparatus, so as to mix with the inflowing normal Ringer. The amount of dilution of the chloroform-Ringer so perfused was calculated from the amount of fluid flowing through the kidney during the actual time occupied by the injection. This method has the advantages (1) that the chloroform solution only acts for a short time upon the kidney vessels, and is less liable to cause a permanently deleterious effect; and (2) that the conditions of flow are maintained the same throughout, for if the injection is performed very gradually, no perceptible increase of pressure is caused by it. (It is scarcely possible to change the flow from one vessel to another, as in the ordinary method of testing perfusion, without causing a temporary effect of some kind upon the pressure of the perfusing fluid.)

The results yielded by these methods show that in mammalian as in frog's vessels the effect of chloroform solutions of a certain strength is to cause marked constriction of the arterioles and consequent diminution in the rate of flow of the perfusing liquid. If the flow lasts for a short time only, the rate is soon recovered, but prolonged perfusion

—even with (so-called) normal Ringer, and much more with Ringer containing chloroform—is accompanied by œdema and a consequent permanent diminution in rate of flow.

Kidney.—The strengths of solution which produce constriction in the vessels of the kidney are from saturated (1 in 200) to 1 in 1000 (perhaps somewhat more dilute). But whereas, in the entire frog, solutions weaker than those which cause constriction do not produce dilatation, and have no apparent effect apart from the gradual œdema which results from prolonged action, in the mammalian kidney the effect of weaker solutions is, as C. J. MARTIN has stated, to cause dilatation. This result has been obtained with solutions of from 1 to 1500 to 1 in 20,000 (in one instance); weaker solutions gave no result.

Coronary Vessels.—In employing the heart we have always taken the precaution of tying the pulmonary veins, to prevent loss of circulating fluid by regurgitation past the aortic and mitral valves. The effect of chloroform upon the coronary arteries is to produce constriction in all strengths from saturated to 1 in 10,000. The stronger solutions of chloroform cause so marked a diminution of the rate through the coronary vessels as to almost arrest the flow of fluid; and this is not due to arrested cardiac action, for on substituting normal Ringer for chloroform-Ringer the rate of flow returns to normal long before the action of the heart recommences. With weaker solutions the effect is also to produce diminished flow, and at no condition of dilution have we obtained evidence of dilatation of vessels.

Limbs.—For this purpose the hind limbs of the rabbit and cat have been used. The results are precisely the same as in the case of the coronary arteries of the mammal and the systemic arteries of the frog. Evidence of constriction has been obtained with all strengths from 1 in 200 (which arrests the flow altogether) to 1 in 10,000 (which causes a slight diminution). More dilute solutions are inactive; we have obtained no evidence of dilatation in these vessels.

The following may serve as examples of the results:—

Kidney of Rabbit:

Amount flowing	before	chloroform	perfusion		42 c.c.
,,	,,	during	,,	(1 in 1500 Ringer)	49 ,,
,,	,,	after	,,		35 ,,

The same Kidney, later:

Amount flowing	before	chloroform	perfusion		39 c.c.
,,	,,	during	,,	(1 in 700)	17·5 ,,

Kidneys of Kitten:

Amount flowing	before	chloroform	perfusion		57 c.c.
,,	,,	during	,,	(1 in 20,000)	60 ,,
,,	,,	after	,,		62 ,,

In this and the next experiment, as the increase was progressive and there was no return towards normal, it is possible that the increase of rate may not have been due to the chloroform. But it is clear that the drug has not caused constriction of the kidney vessels.

The same Kidneys, later:

Amount flowing	before	chloroform	perfusion	. . . . .	. .	64 c.c.
,,	,,	during	,,	,, (1 in 5000)	. .	67 ,,
,,	,,	during	,,	,, (1 in 2000)	. .	70 ,,

Subsequent perfusion with suprarenal produced strong constriction, showing that the arterioles were still active.

The effects of stronger and weaker solutions upon the kidney vessels is also well illustrated in the accompanying tracings, which show (in fig. 1) the effect (marked constriction of vessels with rise of pressure

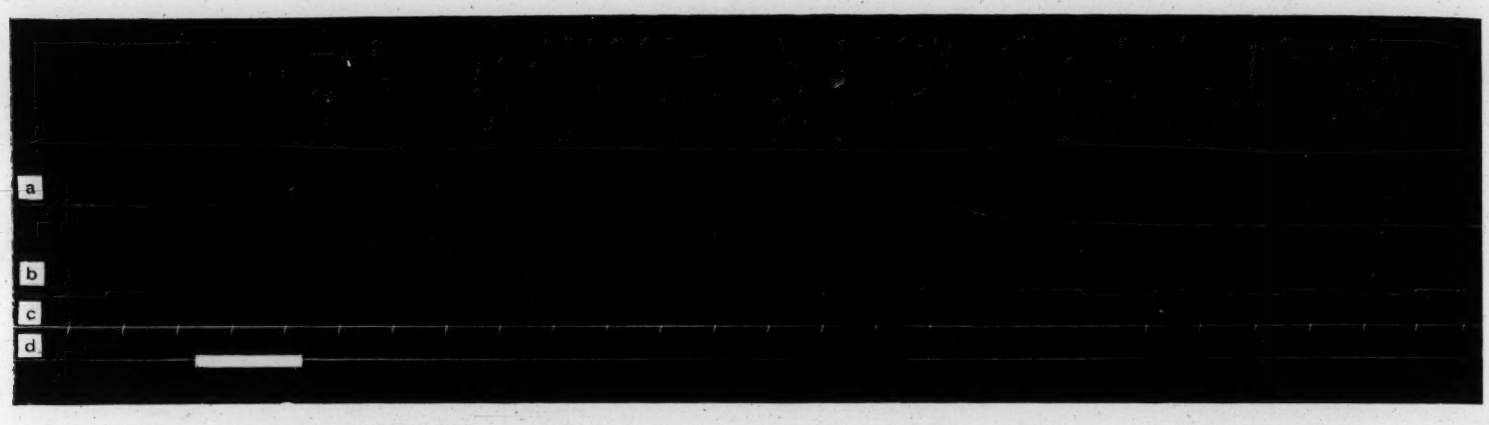

FIG. 1.—Perfusion of rabbit's kidney with Ringer's solution. At the time marked by the signal 4 c.c. of the same solution, containing 0·5 per cent. of chloroform, and at the same temperature, was slowly injected into the supply tube. The mixture of this solution with that passing through the tube at the time gave a fluid containing 0·2 per cent. (1 in 500) actually perfused. Notice the rise of pressure due to constriction of the bloodvessels, followed after the passage of the fluid by a dilatation; also the great diminution in outflow, followed by a slight increase.

a, Register of mercury manometer; *b*, movements of "see-saw," registered by air transmission: each up or down movement represents 7 c.c. of fluid discharged; *c*, time in 10 secs.; *d*, signal line and pressure abscissa.

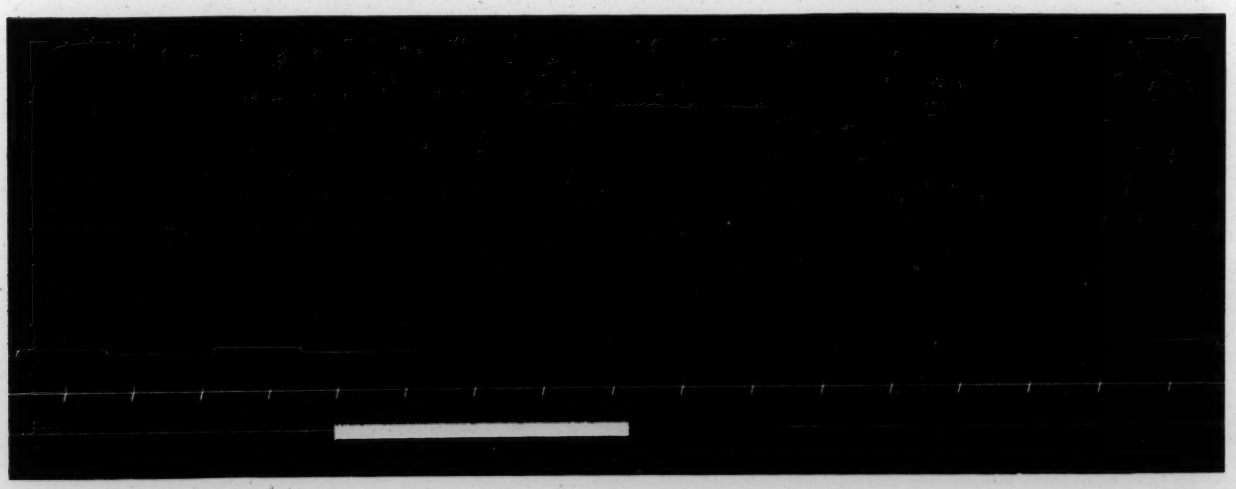

FIG. 2.—Tracing similar to that shown in fig. 1, but with injection of 4 c.c. of 1 in 1000 chloroform, and actual perfusion of 1 in 6000. Notice the fall of pressure and the increase in rate of discharge, indicating dilatation of arteries.

and great diminution of flow) of perfusing a solution of Ringer containing 1 in 500 chloroform; and (in fig. 2) the effect (dilatation of vessels with fall of manometric pressure, and increase of rate of flow) of passing a solution containing 1 in 6000 through the kidney.

Heart of Cat:

Amount flowing	through	coronary	system	before	chloroform	. . . . .	44	c.c.
,,	,,	,,	,,	,,	during ,,	(1 in 10,000) . . .	41·5	,,
,,	,,	,,	,,	,,	after ,,	. . . . .	44	,,

Another Heart:

Amount flowing through coronary system					before	chloroform	.	.	.	.	45 c.c.
,,	,,	,,	,,	,,	during	,,	(1 in 4000)	.	.	.	32 ,,
,,	,,	,,	,,	,,	after	,,	.	.	.	.	42 ,,

Heart of Rabbit:

Amount flowing through coronary system					before	chloroform	.	.	.	.	58 c.c.
,,	,,	,,	,,	,,	during	,,	(1 in 1500)	.	.	.	40 ,,
,,	,,	,,	,,	,,	after	,,	.	.	.	.	48 ,,

The same Heart, later:

Amount flowing through coronary system					before	chloroform	.	.	.	.	48 c.c.
,,	,,	,,	,,	,,	during	,,	(1 in 1000)	.	.	.	34 ,,
,,	,,	,,	,,	,,	after	,,	.	.	.	.	42 ,,

The tracing shown in fig. 3 gives an illustration of the effect produced upon the heart by a still stronger chloroform solution, the rate of perfusion falling during the passage of the chloroform from 21 c.c. per minute to 5 c.c. per minute, and gradually recovering as the chloroform was washed away. It will be noticed that the recovery of the vessels appears before the contractions of the heart reappear. It can also be seen that the latency of the arterial contraction is longer than that of the heart paralysis which the chloroform produces.

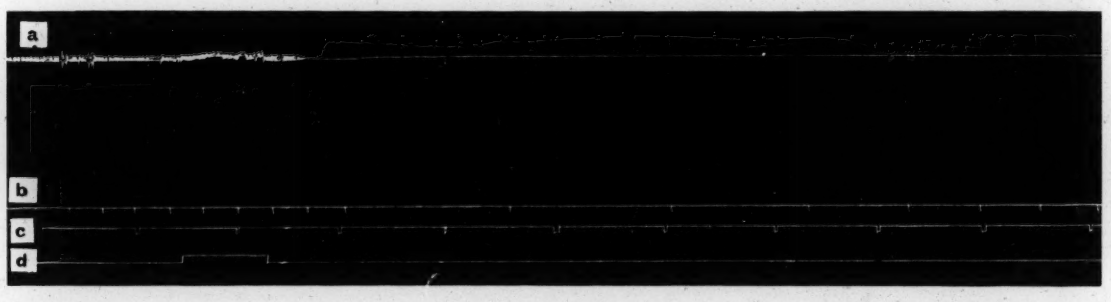

FIG. 3.—Effect of perfusing 20 c.c. of chloroform-Ringer (=1 in 500) through the coronary vessels of the rabbit.

a, tracing of manometer connected with supply cannula; *b*, register of flow from coronary veins: each interval represents 7 c.c.; *c*, time in minutes; *d*, signal marking period of injection into supply tube.

Note the diminution in rate of flow, and subsequent commencing recovery although the heart remains in a condition of arrest. The rate of movement of the paper is too slow for the individual heart-beats to be seen on the manometer tracing.

Hind Limbs of Rabbit:

Amount flowing through limbs					before	chloroform	.	.	.	.	86 c.c.
,,	,,	,,	,,	during	,,	(1 in 10,000)	.	.	.	.	77 ,,
,,	,,	,,	,,	after	,,	.	.	.	.	.	84 ,,

Hind Limb of Rabbit:

Amount flowing through limb					before	chloroform	.	.	.	.	44 c.c.
,,	,,	,,	,,	after passage of 10 c.c. of 1 in 2000 chloroform-Ringer							39 ,,
,,	,,	,,	,,	in subsequent period			.	.	.	.	42 ,,

These observations show that the kidney differs from the other organs investigated in the fact that the more dilute solutions of chloroform produce an increased flow through the kidney vessels, whereas in the other organs (heart, limbs) the effect of the drug is always in the direction of vasoconstriction. The difference is a remarkable one; but without discussing it at greater length, we may point out that dilatation of the renal vessels is the normal response of the organ to all but a very few excitants, whereas the normal response of most vessels to an excitation is contraction, and it is possible therefore that the explanation is connected with this difference of "habit" of the kidney vessels as compared with the systemic vessels generally.

The action of chloroform upon the vessels when perfused through the isolated organs is a direct action upon the muscular tissue, and not, as in the case of suprarenal extract, upon the terminal apparatus of the vasomotor nerves. This is shown by the fact that apocodeine, which in sufficient dose abolishes the effect of adrenalin,* does not abolish the effect of chloroform in producing vasoconstriction.

The following experiment may be quoted in illustration of this statement :—

Hind Limb of Rabbit.—Perfusion with 1 in 2000 chloroform reduced the rate of flow during each period of time from 44 c.c. to 39 c.c. After recovery, perfusion with 0·0001 gramme hemisine (adrenalin) added to the normal Ringer brought it down to 10 c.c. After recovery, perfusion with Ringer solution, containing 0·0001 gramme hemisine and 0·0075 gramme apocodeine, caused only a slight reduction, soon disappearing. After recovery, admixture with the perfusing fluid of 10 c.c. of chloroform-Ringer (= 1 in 500), containing 0·02 gramme apocodeine, caused an almost complete arrest of flow during several minutes. Fig. 4 is a graphic record of this experiment.

In all cases the drugs were injected into the tube which supplied the normal Ringer, and the solution became mixed with a certain proportion of this, and warmed to the same temperature by passing through the glass spiral before reaching the organ which was perfused.

As a further proof that chloroform acts upon the muscular tissue of the arterioles in

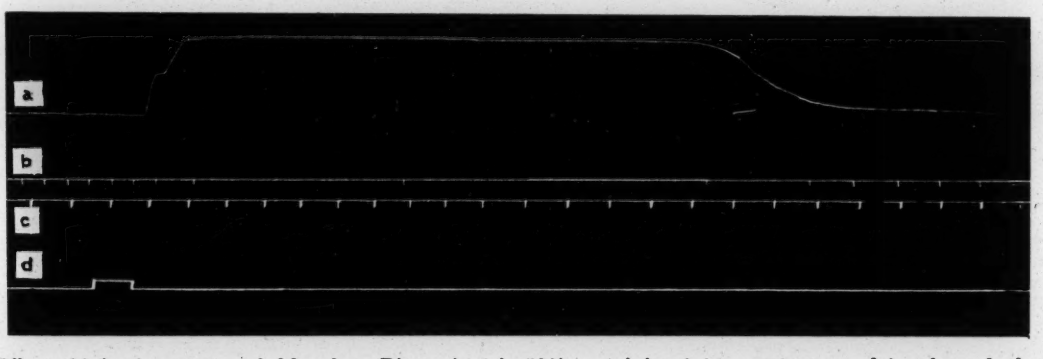

FIG. 4.—Effect of injecting 10 c.c. of chloroform-Ringer (=1 in 500) containing 0·02 gramme apocodeine through the vessels of the hind limb of the rabbit.

a, *b*, *c*, *d* as in fig. 3. The tracing is taken on a more slowly moving surface than that in fig. 3. The initial pressure of the perfusion fluid was lower in this experiment than in the experiment shown in fig. 3, and the supply less free: this, as well as more complete constriction of the arterioles, accounts for the fact that the manometer tracing is much affected in the one case and scarcely at all in the other.

producing contraction may be adduced the observation that its action can be got after the neuro-muscular end-apparatus has lost its irritability. Thus in the kidney of a rabbit, which had been killed three hours previously and in which the injection of 0·0003 gramme hemisine (adrenalin) produced no effect whatever upon the rate of flow, injection of 20 c.c. of 1 in 200 chloroform-Ringer into the supply-tube reduced the rate from 56 c.c. to 28 c.c.

All recent observers are agreed that the fall of blood-pressure which is caused by chloroform is essentially due to its effect upon the heart muscle, the action of which is weakened and eventually paralysed by the drug. MARTIN and EMBLEY are, as we have seen, inclined to ascribe the fall—in a minor degree—partly to the dilatation which may be produced in the peripheral vessels by small doses of the drug. But since GASKELL and SHORE have shown that the effect of chloroform is to excite the vasomotor centre in the medulla oblongata, and thus to cause contraction of the

* DIXON, *Jour. Phys.*, vol. XXX., p. 97, 1904. Also BRODIE and DIXON, *ibid.*, p. 476.

arterioles—an observation which MARTIN and EMBLEY themselves confirm—it is extremely improbable that, while this centre is in action, the dilating effect, if really existent, upon the periphery would be at all apparent. On the other hand, with stronger dosage of chloroform, the direct effect upon the arterioles is always one of constriction. It follows, therefore, that at the beginning of a chloroform inhalation there will be a tendency to counteract the fall of blood-pressure due to heart weakening, by

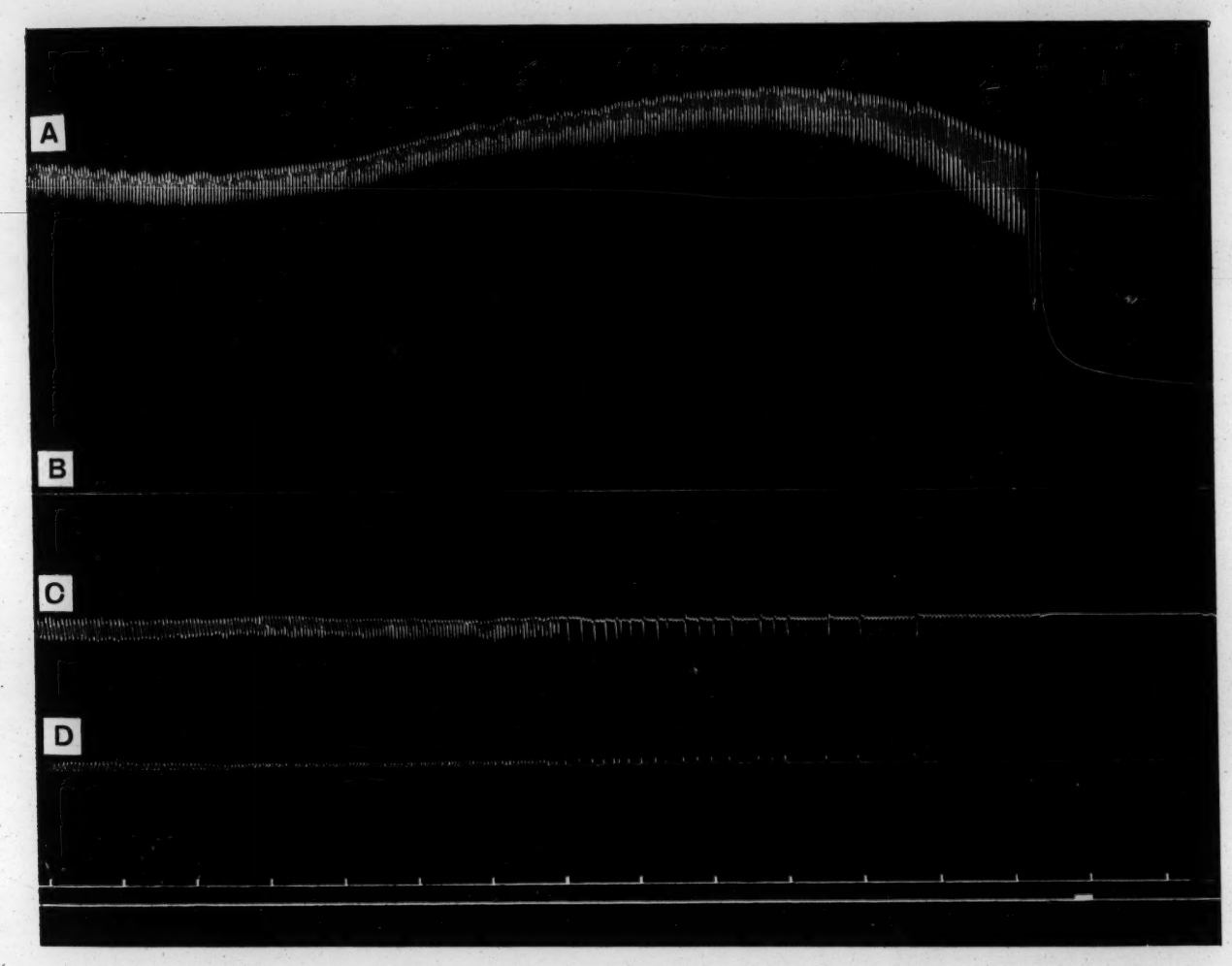

FIG. 5.—End of a fatal chloroform inhalation. Dog, 6570 g. Inhalation through trachea tube of air strongly charged with chloroform vapour.

A, blood-pressure curve; B, line 1 centimetre below zero of blood-pressure; C, costal respiration (the small waves upon this are heart movements); D, diaphragmatic respiration. Time in 10 secs. The signal marks the removal of chloroform. Respirations ceased 20 secs. before the heart. The subterminal rise in blood-pressure which sometimes occurs is shown in this tracing. The increase in size of the manometer excursions is due to a gradual slowing of the heart rhythm, and does not represent an increase of force of the contractions.

excitation of the vasomotor centre, and later on, while this may still be active,* a similar tendency to counteract the fall by direct excitation of the peripheral arterioles. As a general rule, the action of the drug upon the heart is so marked as to more than counter-balance the arterial constriction, however produced. But in certain cases a

* Reflex constriction of bloodvessels can be obtained, on stimulating an afferent nerve, even if chloroform anæsthesia is very pronounced, showing that even in deep anæsthesia the vasomotor centre is still active, although its activity is no doubt lessened. *Cf.* BOWDITCH and MINOT, *op. cit.* Further, chloroform does not diminish the excitability of the peripheral vasomotor nerves (SCHEINENSSON, *Centralbl. f. d. med. Wiss.*, 1869, p. 105).

brief rise of pressure at the beginning of an experiment is sufficiently manifest, and in others again there is apparent, even when the respirations are maintained, a rise towards the end of the experiment. Since in these cases asphyxia is eliminated, and the heart is probably not beating more but less strongly, such a subterminal rise can only be ascribed to the excitatory action of the drug upon the vasomotor centre, or directly upon the arterioles. It is exemplified in fig. 5, which shows the tracing from the latter half of an acute chloroform poisoning, terminating by a slowing and arrest of the respirations, followed after a few seconds, suddenly, by complete arrest of the heart. In this tracing it will be observed that long before the failure of the respiration begins to show itself there is a decided tendency to rise on the part of the arterial pressure, although the heart at this time is not beating more but rather less strongly (the increase in size of

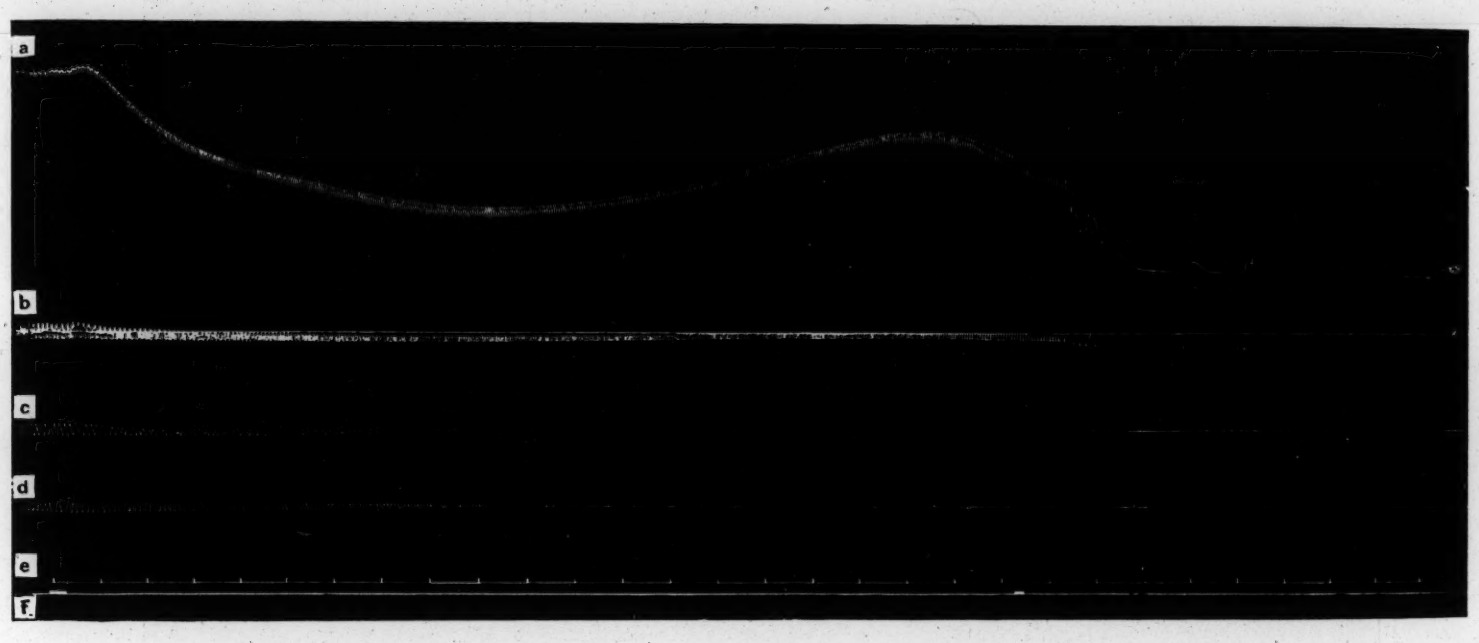

FIG. 6.—Tracing (dog) showing marked secondary rise of blood-pressure during chloroform inhalation, probably due to early failure of respiration. The chloroform was administered between the two marks on the signal line. Notice the cardiac inhibition, which in this case is more gradually developed than usual, and the subsequent escape of the ventricle, which continued to beat feebly for a minute or two, but with hardly any rise of blood-pressure. Artificial respiration by pump commenced 10 minutes after natural respiration had ceased, failed to effect recovery.

a, blood-pressure; *b*, tracing from needle passed through chest wall into ventricle; *c*, thoracic movements; *d*, diaphragmatic respirations; *e*, time in 10 seconds; *f*, signal. The horizontal line at *b* is 10 mm. below the abscissa of blood-pressure.

the arterial pulsations seen near the end is a result of slowing of the cardiac rhythm). This rise of pressure therefore must be due to arterial constriction caused by the drug. The chloroform was given as concentrated vapour, producing abolition of corneal reflex in one minute and death in about four minutes; but how far the constriction was due to direct action upon the arterioles, and how far to an action upon the vasomotor centre, the experiment does not determine. The continuation of the rise in the tracing may perhaps be ascribed to a condition produced by the commencing failure of respiration, the vasomotor centre being stimulated by the venous blood; especially as it is accompanied by a certain amount of cardiac inhibition. Such asphyxial rise may be very marked

when the respirations become shallow early in the administration, as is shown strikingly in the tracing given in fig. 6.

While cases showing such a marked subterminal rise are uncommon, it is not unusual to find a subterminal arrest of the fall of blood-pressure, so that the curve remains for a minute or two at the same level, or shows a more gradual fall than immediately before and immediately after. Such arrest of fall, when unaccompanied by failure of respiration, may also be explained by the constricting action of the drug on the arterioles, acting either through the vasomotor centre or directly. This con-

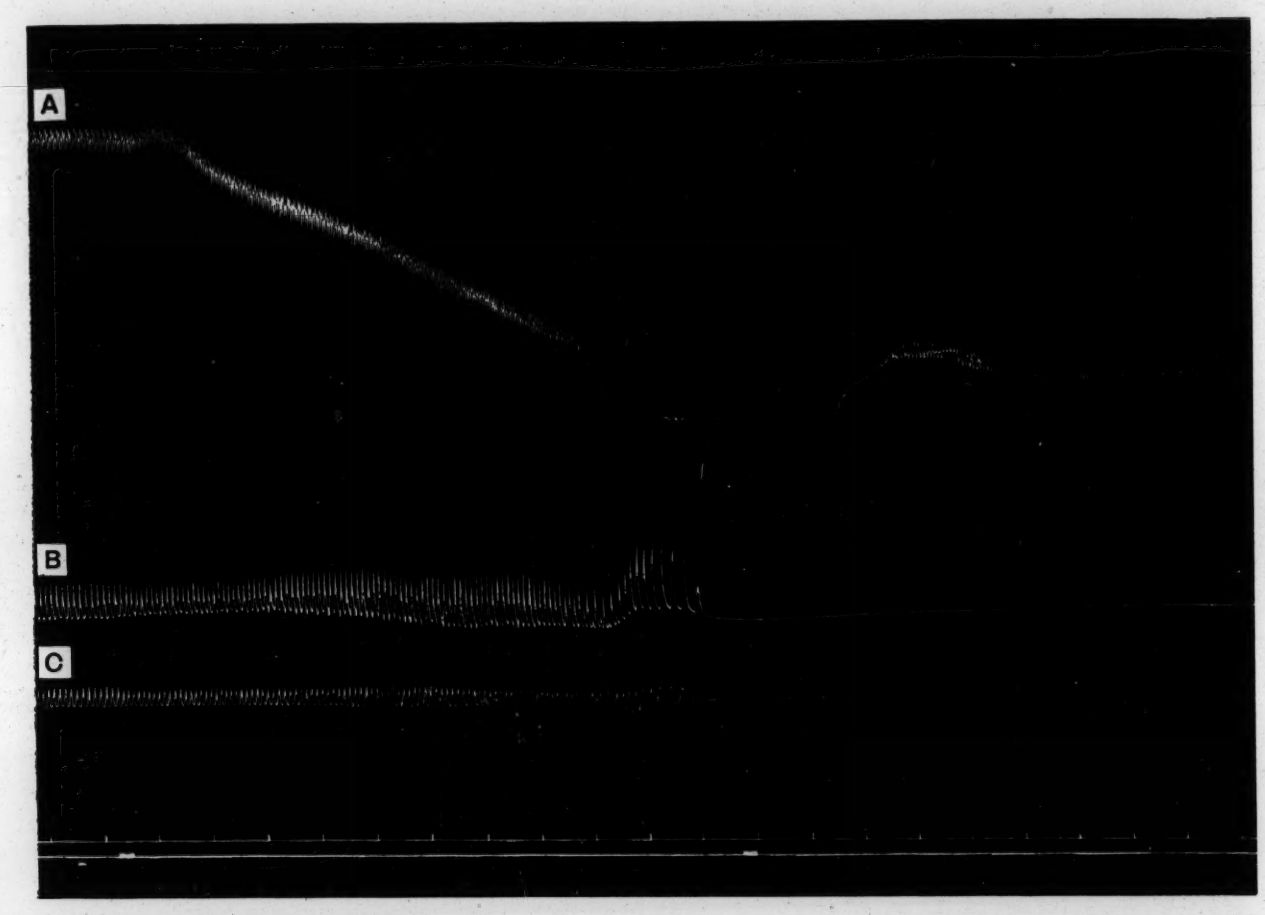

FIG. 7.—Dog, weight 7000 g. Effect of inhalation through trachea tube of air nearly saturated with chloroform vapour.

The uppermost tracing (A) is that of the blood-pressure; the second tracing (B) is costal respiration; the third (C) abdominal respiration; the fourth, time in 10 seconds; and the fifth, the signal marking when chloroform was admitted and stopped.

In this experiment the heart failed before the respiration, and about 30 seconds later showed spontaneous recovery, which was, however, only temporary. There was no recovery of respiration.

striction, although insufficient entirely to compensate for the continual and gradual weakening and slowing of the heart which is going on the whole time, interferes with the continuous and uniform fall of pressure, which would otherwise show itself. At a much later stage the ventricular contractions, although greatly weakened, produce large fluctuations of pressure in the arterial system, which is then comparatively empty, owing to the accumulation of blood in the great veins and in the dilated heart cavities.

Effect of Chloroform upon the Heart.

Our experiments abundantly illustrate the fact, which is now unreservedly conceded, that of all ordinary anæsthetics chloroform is the one which produces the most deleterious effect upon both heart and respirations. Nothing is more striking than the comparison of tracings from chloroform experiments, such as those shown in figs. 5, 6, and 7, with one in which ether is the anæsthetic agent (fig. 8). We have further

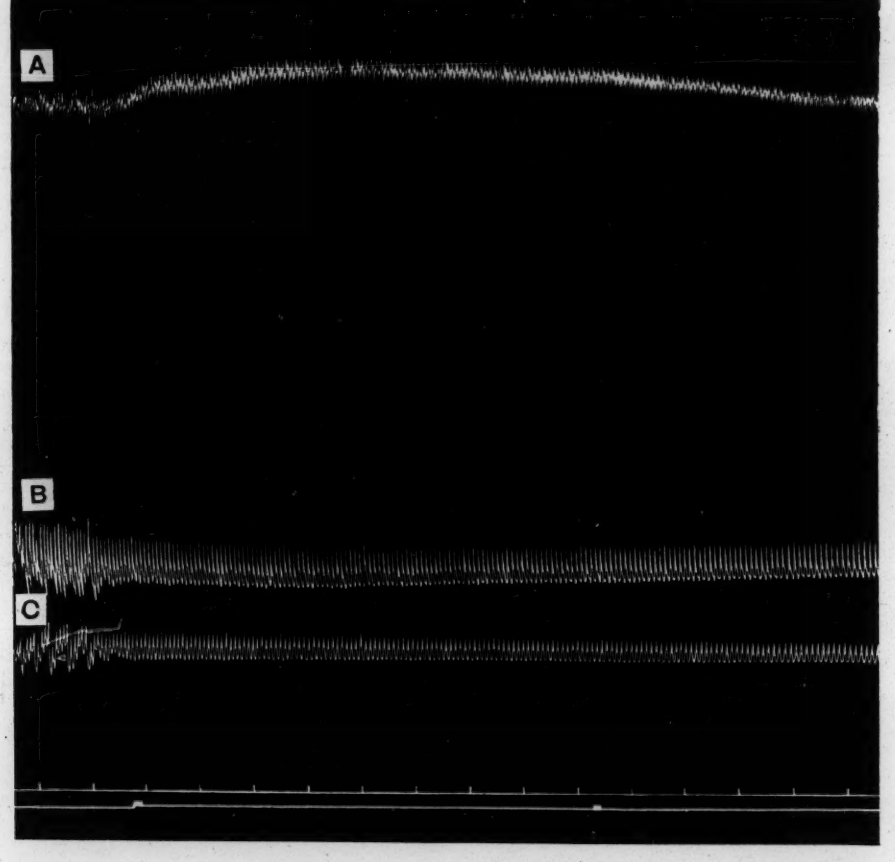

Fig. 8.—This tracing was taken from the same animal as fig. 7, and immediately before it. Air saturated or nearly saturated with ether was inhaled through the trachea tube, between the two points marked by the signal. Previously to this the dog was very lightly anæsthetised with chloroform, corneal reflex being present. Compare the effect of ether upon the blood-pressure and respiration with chloroform, in these tracings.

investigated the action of chloroform upon the heart *in situ* with the chest opened, after the method used by M'William;* and also after removal of the organ from the body, with the coronary vessels perfused by Langendorff's method.†

In confirmation of previous observers, we find that the effect upon the organ when its nervous connections are severed, or when the activity of the vagus is abolished by atropine (fig. 9), is to produce a gradual weakening of the contractions (without any marked slowing, although this may appear towards the end of a fatal experiment)

* *Jour. Phys.*, vol. xiii. p. 860, 1892.

† *Pflüger's Arch*, vol. lxi. p. 291, 1895.

affecting both auricles and ventricles; but the former are affected much more than the latter. This is also the case when the cardio-inhibitory centre is reflexly stimulated, even in presence of atropine (fig. 10). On the other hand, when the nervous connections are intact, the administration of a strong dose of chloroform not only greatly weakens the force of the cardiac contractions, and thereby causes a marked and progressive fall of blood-pressure, but often, after a primary acceleration, produces gradual slowing of the rhythm, and this is frequently followed by abrupt and complete cardiac arrest (figs. 6 and 7). The abrupt cessation of contraction may affect both auricles and ventricles simultaneously, or the auricles may first stop; the ventricles, either at once or after a

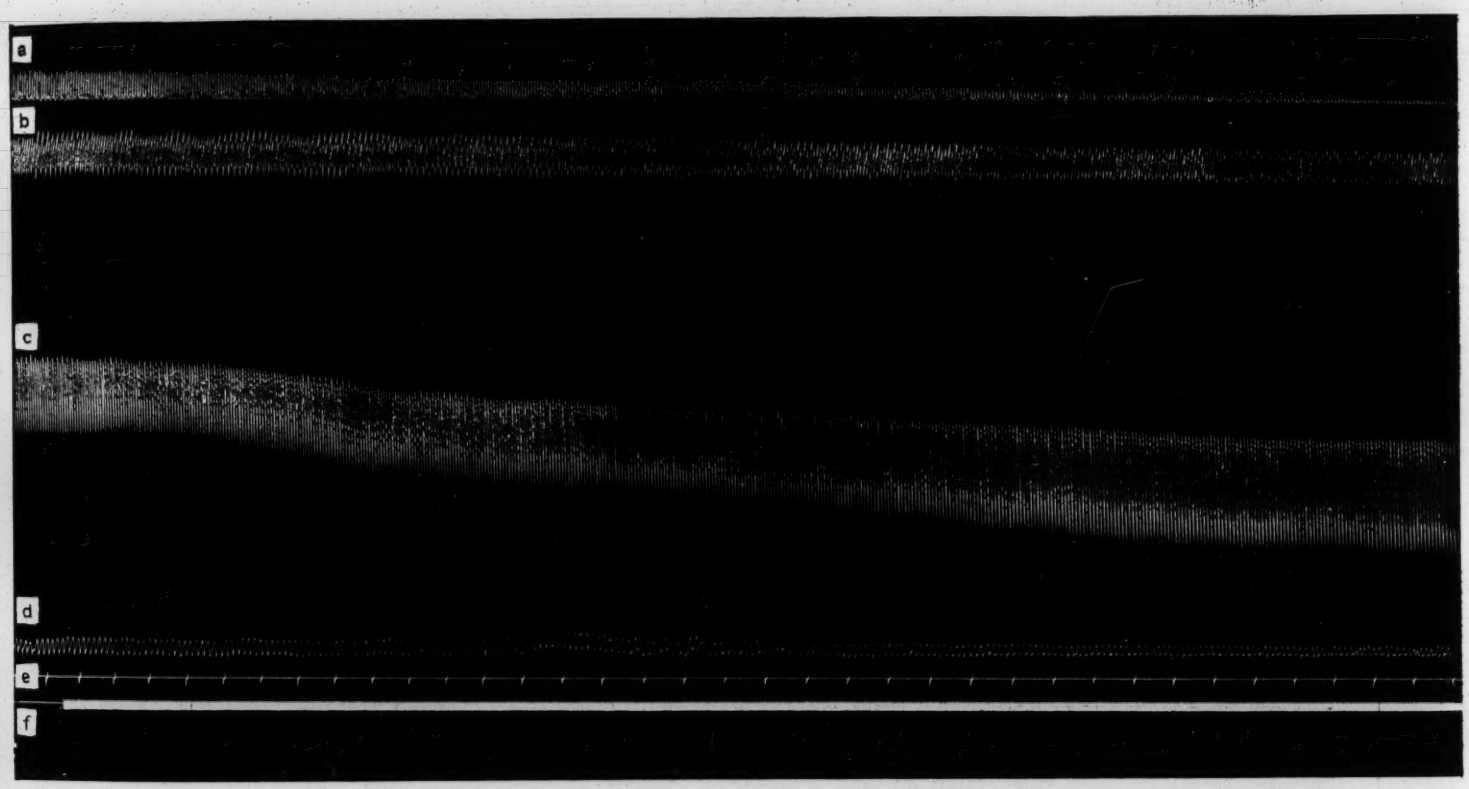

FIG. 9.—Dog, weighing about 10 kilos. The animal had received some 3 hours previously ·00054 gramme ($\frac{1}{120}$ gr.) atropine sulphate administered hypodermically. The effect of this was to abolish arrest of the heart on stimulation of the cardio-inhibitory centre (see fig. 10), whilst permitting a diminution in force of the beats, especially of the auricle. It will be seen from these tracings that exactly the same effect is produced in an animal under the influence of atropine by chloroform alone in strong dose as is caused by reflex excitation of the cardio-inhibitory centre, except that the result is attained more gradually.

a, auricular tracing; *b*, ventricular tracing; *c*, blood-pressure (femoral); *d*, respiratory movements of the thorax, which are continued in spite of the fact that artificial respiration is carried on by perflation; *e*, time in 10 seconds; *f*, signal.

short period of arrest, resuming their action with a rhythm of their own (figs. 6, 7, and 11).

The effect entirely resembles that produced by vagal excitation, with the exception that vagal excitation does not, as a rule, by itself produce permanent arrest of cardiac action. But at a certain stage of chloroform anæsthesia the arrest produced by artificial excitation of the vagus may be permanent, or so prolonged as to lead to death. The cessation of the heart's action brings the blood-pressure to zero, and by arresting the

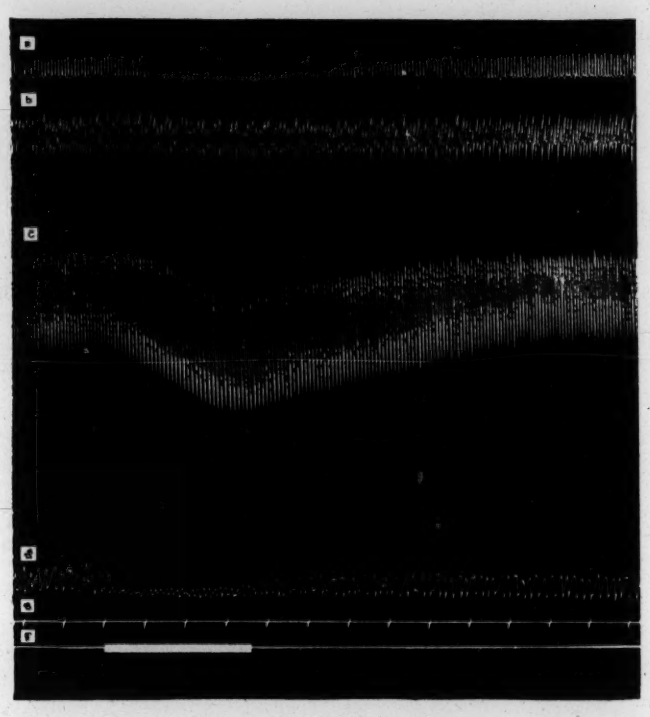

FIG. 10.—Testing of cardiac centre by stimulation of central end of one vagus—the other being intact—in the dog from which the tracing given in fig. 9 was taken, and immediately previous to that tracing.

a, *b*, *c*, *d*, *e*, *f* as before. Notice, as with chloroform, the great diminution in force of the auricular beats, but without arrest or slowing, the diminution in force of the ventricular beats being hardly perceptible. The fall in blood-pressure may be in part due to depressor action, but a similar fall was obtained by stimulation of the peripheral cut vagus.

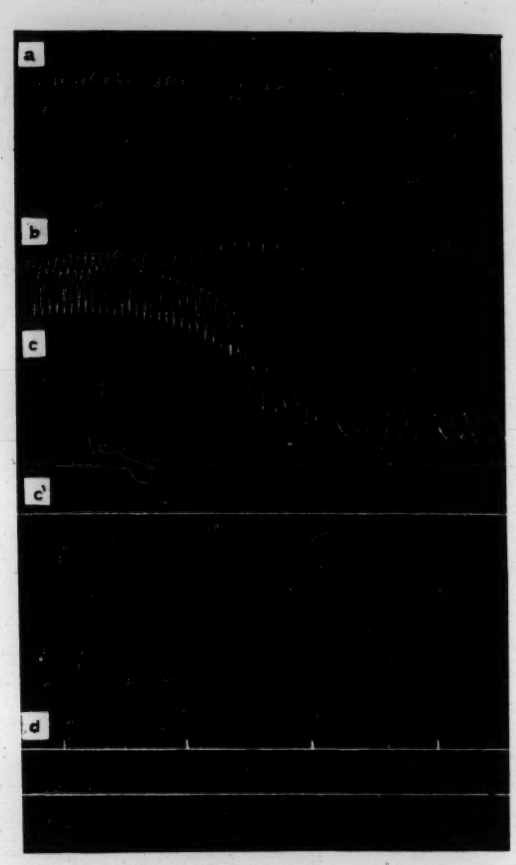

FIG. 11.—Chloroform inhalation. Showing cessation of auricle before ventricle, the latter then assuming its independent rhythm, with larger excursions.

a, auricle; *b*, ventricle; *c*, blood-pressure (the alignment of this pen is a little in advance of the others); c^1, zero of blood-pressure; *d*, time in 10 seconds. Artificial respiration by perflation.

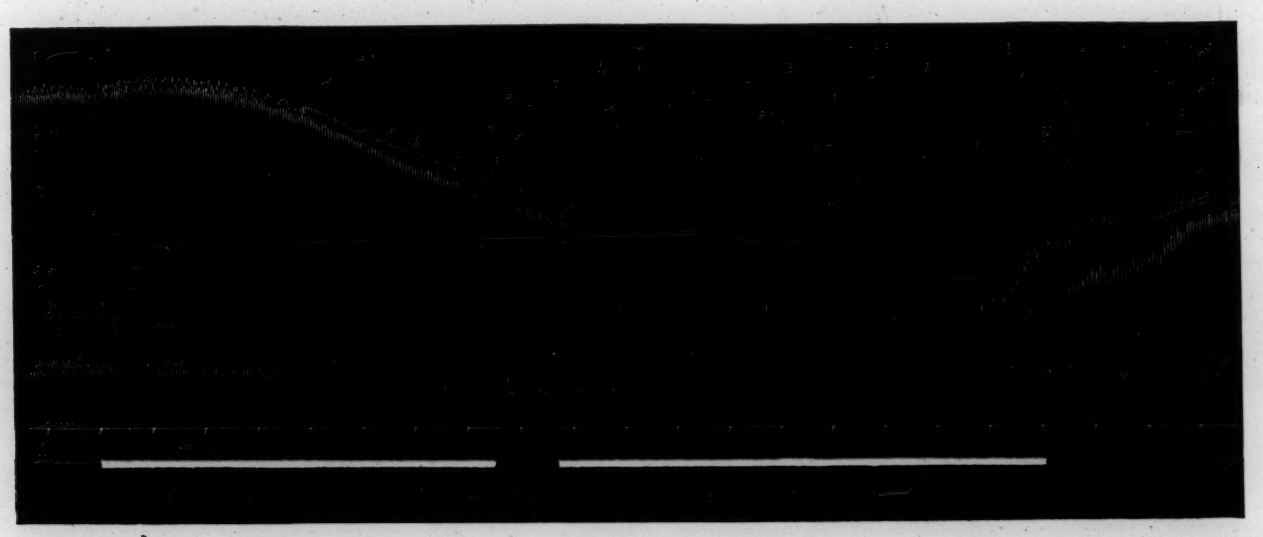

FIG. 12.—Effect of excitation (with coil 100) of the peripheral vagus (second signal) during moderately deep chloroform anæsthesia, strong chloroform vapour having just previously been administered during 1½ minutes (first signal). The result is seen to be an immediate heart arrest, with the blood-pressure falling to zero; the respirations cease 30 seconds after the heart has stopped, but are only gradually arrested. In this case the heart (ventricle ?) begins to escape from the arrest after 40 seconds, beating at first very slowly, but after a minute faster: as the heart recovers, the respirations are also renewed.

circulation causes also a failure of the respiratory centre. This cause of respiratory failure has been clearly demonstrated by EMBLEY.*

The complete similarity between the tracings obtained under the influence of chloroform alone on the one hand (when a concentrated vapour is inhaled), and of

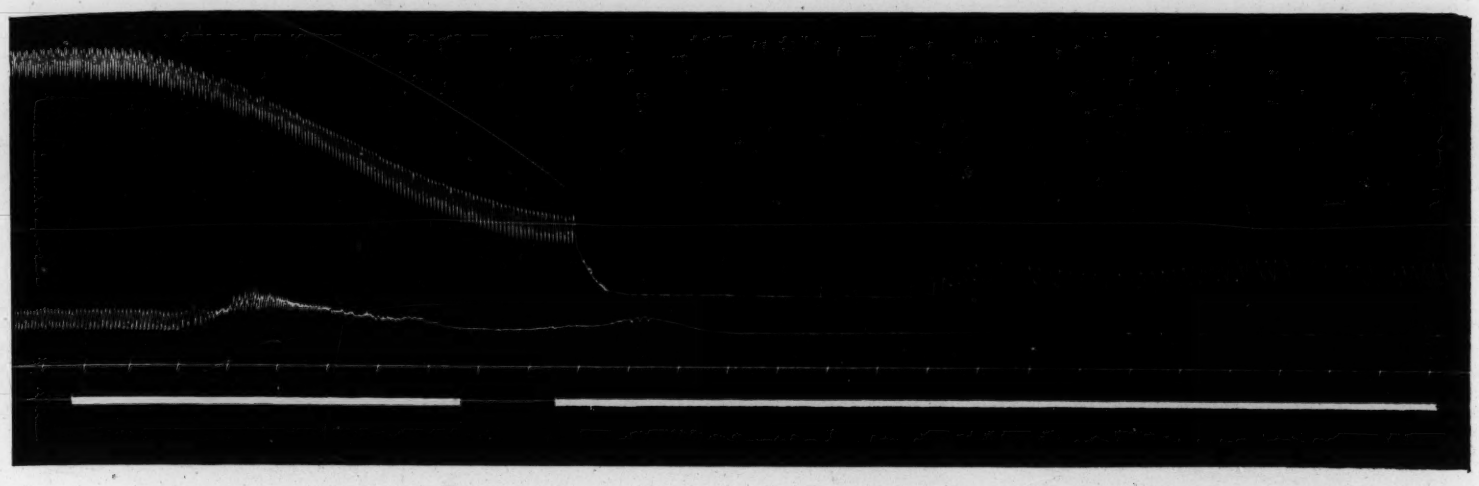

FIG. 13.—Tracing showing the effect of moderate vagus stimulation (coil 100) during deeper chloroform anæsthesia. In this case the excitation was not applied until the respiratory movements had nearly ceased. The effect is to produce complete arrest of the heart, which, however, shows a beginning of escape from the arrest after the lapse of about a minute, and 20 seconds later resumes beating slowly and feebly, and with but little effect upon the blood-pressure. Respirations are not resumed spontaneously, but the animal was recovered 5 minutes after the respirations had ceased, by artificial respiration (compression of thorax) continued during about 2 minutes. The first signal mark shows the period of strong chloroform administration: the second that of vagus stimulation.

chloroform *plus* artificial vagus excitation on the other, shows conclusively that in the former case, as in the latter, the actual cause of the arrest of the heart (and of the

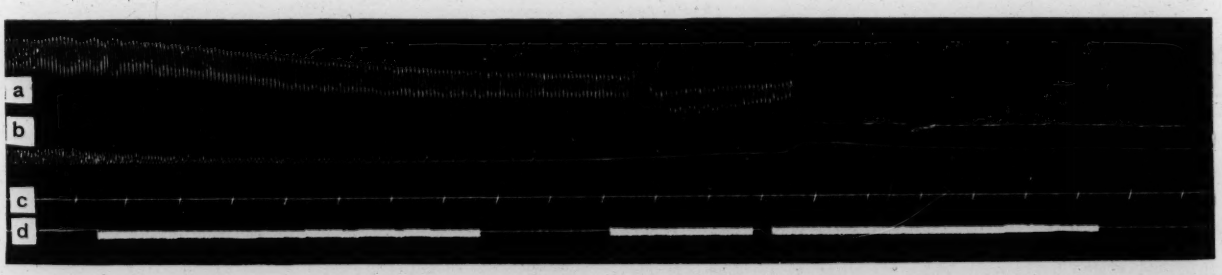

FIG. 14.—Tracing showing the effect upon the heart (1) of weak and (2) of stronger vagus excitation during extreme chloroform anæsthesia. The first signal mark shows the period of strong chloroform administration: the second that of weak vagus stimulation: the third that of stronger vagus stimulation. The chloroform was given until the respiration had ceased and the blood-pressure had fallen to 20 mm. Hg. Excitation of the vagus by induction shocks of very moderate strength produced only a momentary arrest of the heart, but stronger excitation caused instant and permanent arrest.

respiration as a secondary effect) is inhibition excited through the vagus. Thus figs. 5, 6, 7, and 15 show such a cardiac and respiratory arrest produced by strong chloroform alone, and figs. 12, 13, and 14 the same phenomenon produced under varying degrees of chloroform anæsthesia by vagal stimulation. In figs. 12 and 13 it is seen that the ventricle has escaped from the inhibition and has resumed contraction

* *Op. cit.*

with an independent (slower) rhythm. In all such cases as these recovery may be effected if artificial respiration be started soon (fig. 15), before the respiratory centre has been allowed to remain too long under the influence of the chloroform *plus* anæmia; or if the anæsthetisation be not very deep, there may be spontaneous recovery (fig. 12). In other cases the ventricle does not escape spontaneously (fig. 14); but it may be caused to contract by rhythmic compression through the chest wall (fig. 15). The arrest of the heart (and, secondarily, of the respiration) is due therefore to an excitation by chloroform of the cardio-inhibitory centre.* ARLOING showed that it does not occur with cut vagi, and this is also emphasised by EMBLEY; moreover, it does not occur if a small dose of atropine has been previously given (see below, p. 328), and it may also fail to be apparent after prolonged anæsthetisation with chloroform in moderate dosage. It fails to occur also in certain individuals, which seem to be less susceptible than others to the cardio-inhibitory effects of the drug. Instances are shown in fig. 22, A, and

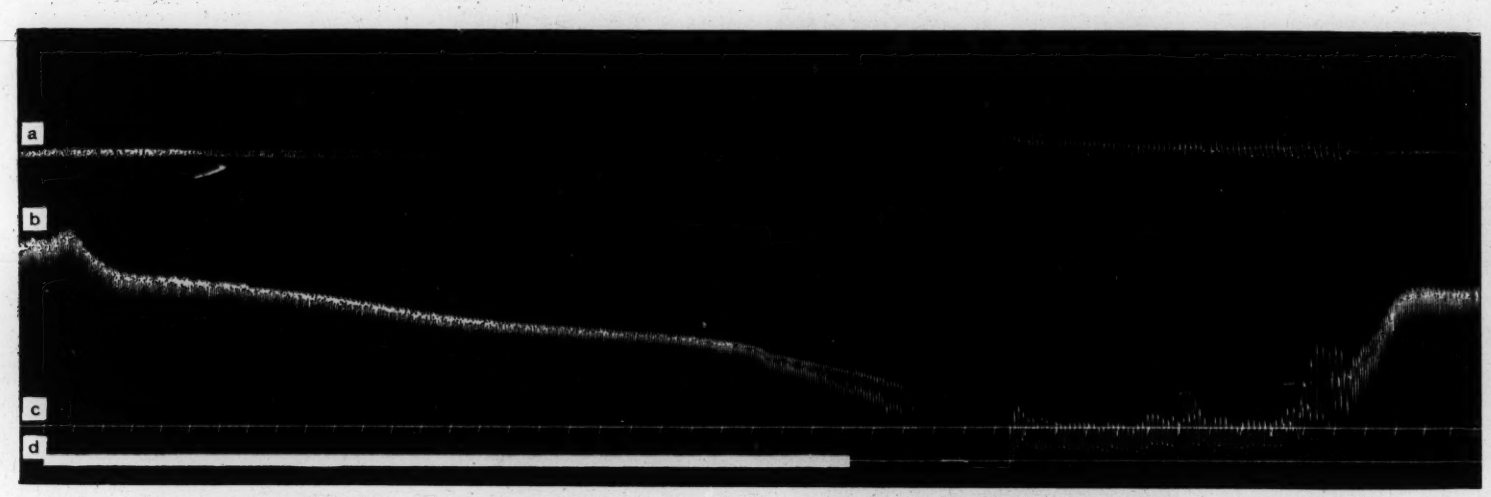

FIG. 15.—Tracing showing (secondary) inhibition of heart from strong chloroform inhalation, with simultaneous cessation of respiration. Recovery, after 30 seconds' arrest, by means of artificial respiration effected by chest compression.
a, respiration; *b*, arterial pressure; *c*, time in 10 seconds; *d*, signal line (2 mm. above abscissa of blood-pressure).

also in fig. 16; the latter from a dog in which the inhalation of air strongly charged with chloroform vapour was pushed until respirations had ceased, the heart continuing to beat with great regularity five minutes longer, but during the last three minutes at a slower rate (probably the result of independent ventricular action).

In connection with this subject, we have investigated the effect of vagal excitation upon the heart in different stages of chloroform anæsthesia, and the effect of small doses of atropine upon the result of vagal excitation. In light anæsthesia, an adequate stimulation of the vagus produces, as in absence of anæsthesia, complete arrest of cardiac movements with a fall of blood-pressure to zero. But even if the excitation be continued, this condition does not last, for although the auricles may

* The above tracings make it abundantly evident that the assertion of LAWRIE (*Lancet*, 1890, vol. i. p. 1393), founded on the report of the Hyderabad Commission, "that sudden death from stoppage of the heart is not a risk of chloroform itself," is completely erroneous.

remain quiescent, the ventricles escape from the inhibition and resume contracting, at first very slowly, afterwards more quickly, although often somewhat irregularly, so that the blood-pressure again rises. On ceasing to excite the vagus, the heart beats more rapidly and more strongly, and there may be a temporary rise of blood-pressure above the normal average just before the excitation. This characteristic effect is shown in fig. 18.

In chloroform anæsthesia the effect upon the heart of vagal excitation is more pronounced and permanent.* The complete arrest of cardiac action may last long enough to cause a concomitant arrest of respiration, and when this occurs, even if the

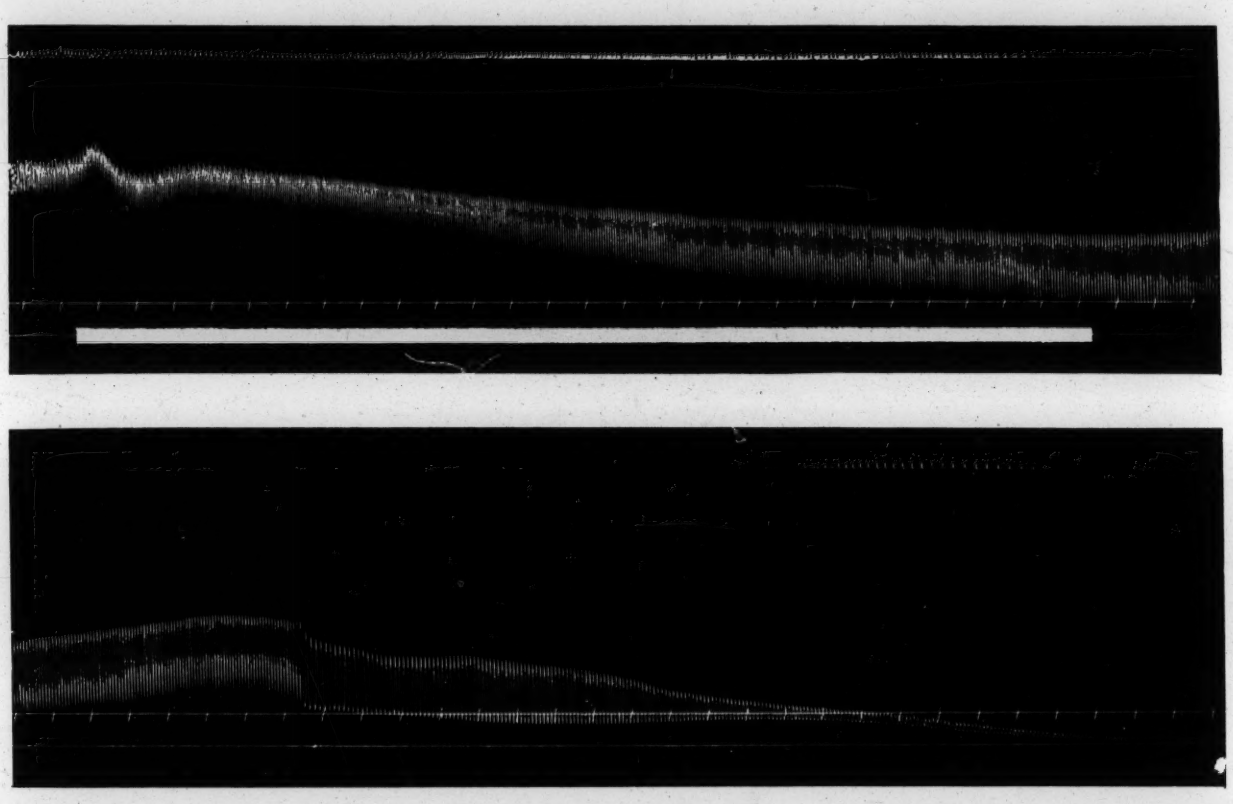

FIG. 16.—Shows a tracing of respiration and blood-pressure under inhalation of air strongly charged with chloroform vapour. This tracing illustrates the type of result obtained when cardiac inhibition does not occur to the extent of causing complete arrest of the heart, but merely a slowing (which may be suddenly increased), the heart failing quite gradually. The chloroform inhalation lasted 4½ minutes, at the end of which period respiration had ceased, and was not again renewed until the heart had nearly stopped, when a 'staircase' group of 25 slow gasping respirations showed themselves—the so-called 'respirations of the death-agony.' The lid reflex, which was present immediately before the inhalation began, disappeared within one minute.

ventricles resume action, their rhythm is very slow, and their force insufficient to raise the blood-pressure much, so that, as a rule, respirations are not spontaneously resumed although artificial respiration may effect recovery. Or it may happen, especially with a strong excitation, that the recovery of the ventricles does not occur at all, and even heart massage, which can be effected by compressing the chest, to which the ventricle may at first respond, may be incapable of causing it permanently to resume its action.

* DASTRE states that this increase of vagal excitability under chloroform was first noted by VULPIAN (*C. r. Soc. Biol.*, 1883, p. 243). According to FRANÇOIS-FRANCK (*ibid.*, p. 255), it disappears with increase of anæsthesia, but this is not in accordance with our experience so far as concerns direct excitation.

If the chloroform be pushed until respirations cease altogether and the blood-pressure is reduced to a few millimetres of mercury, the only sign of life being the slow beat of the heart, adequate vagal excitation will still cause inhibition, which under these circumstances lasts as long as the excitation; the removal of excitation being immediately followed by a resumption of the slow, weak beats. Such inhibition can be obtained as long as there is any perceptible beat (fig. 17).

We have also re-studied the effect and dosage of atropine in preventing inhibition through the vagus. The results obtained are illustrated in the tracings reproduced in fig. 19. (See also fig. 10 for its effect on reflex vagal excitation.)

If a dose of sulphate of atropine of 0·00002 gramme per kilo. be given hypodermically in the dog, the effect upon the vagus is manifest about fifteen to twenty minutes

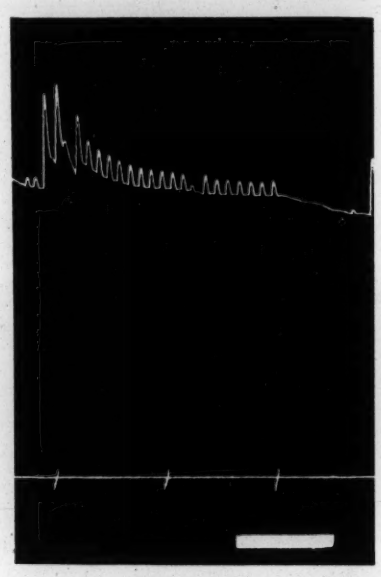

FIG. 17.—Effect of moderate vagal stimulation in the last stage of chloroform anæsthesia, the respiration having long ceased, and the heart beating slowly, feebly, and irregularly. The signal marks the period of vagal stimulation. (The alignment of the signal is a little too much to the left.) It will be seen that excitation of the peripheral vagus still causes arrest of cardiac action, which is at this stage probably entirely ventricular.

after administration, and lasts about three hours. The result of such a dose is in some cases to abolish for a time all vagal influence upon the heart (fig. 19, I.). But in most cases, although there is not complete abolition, nevertheless the strongest vagal excitation is unable to produce, in any stage of chloroform anæsthesia, complete cardiac arrest (fig. 19, II. to VI.). There may be, even with comparatively weak excitation, a slowing of the heart and a consequent fall of blood-pressure; but it is no greater with strong than with weak excitation, and is never accompanied by respiratory arrest, unless in using a very strong excitation there is escape of current to the central end of the nerve. This peculiar condition, in which vagal excitation is unable to cause arrest, but only slowing and diminution in force of the heart, persists for nearly three hours, the

slowing becoming as time proceeds gradually more marked, and the consequent fall of blood-pressure lower. But cardiac arrest does not show itself until the influence of the atropine has completely passed, and the former conditions can be then restored by a similar dose.

There is no reason to believe that the human subject is less susceptible to the influence of atropine than the dog, and the opinion which has been expressed, that an enormous dose would be required to abolish the power of the vagus to cause cardiac arrest, appears therefore to be erroneous.*

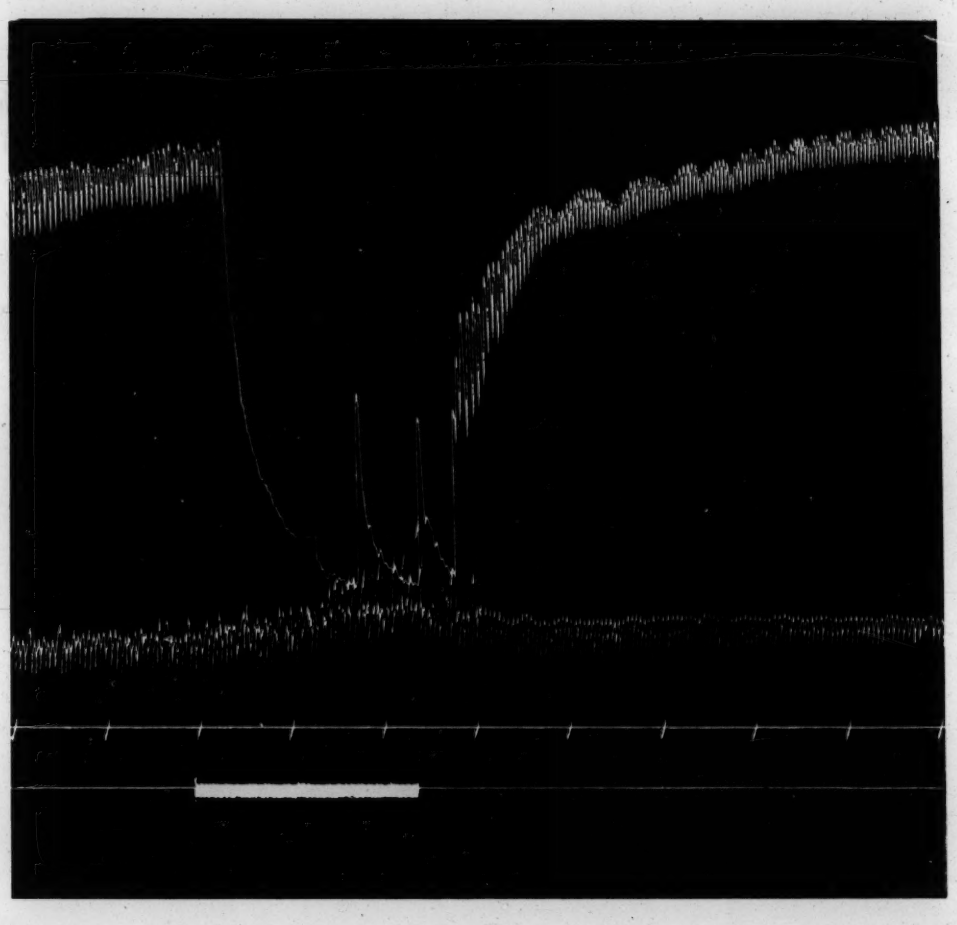

FIG. 18.—Normal effect, in the dog, of vagus stimulation of moderate strength (coil 100 mm.) with light anæsthesia, showing the tendency of the heart to escape from the inhibition. This was taken immediately before and from the same dog as the (reduced) tracings shown in figs. 12, 13, and 14, but chloroform was administered in the interval, and the tendency to inhibition is seen in these to be much more pronounced.

Since abrupt arrest of the heart and of respirations can be absolutely prevented by prior administration of a small dose of atropine, the conclusion forces itself upon us that the precaution of such administration is one that should never be omitted. Atropine cannot, as we shall see, prevent death where a dose of chloroform sufficient to produce paralysis of respiration and complete "paralytic dilatation" of

* See on this subject remarks by H. C. CROUCH and T. G. BRODIE in *Trans. Soc. Anæsth.*, vol. vi. pp. 70 and 81, 1904. J. HARLEY (*Brit. Med. Journ.*, vol. ii., 1868, p. 320) recommended a dose of from $\frac{1}{100}$ grain to $\frac{1}{40}$ grain in man, DASTRE (*Soc. Biol.*, 1883, p. 242) states that a dose of atropine amounting to 0·0015 gramme ($=\frac{1}{43}$ grain) is sufficient for the purpose indicated. LANGLOIS and MAURANGE (*Arch. de Phys.*, 1895, p. 692) recommend the employment of oxy-sparteine in place of atropine.

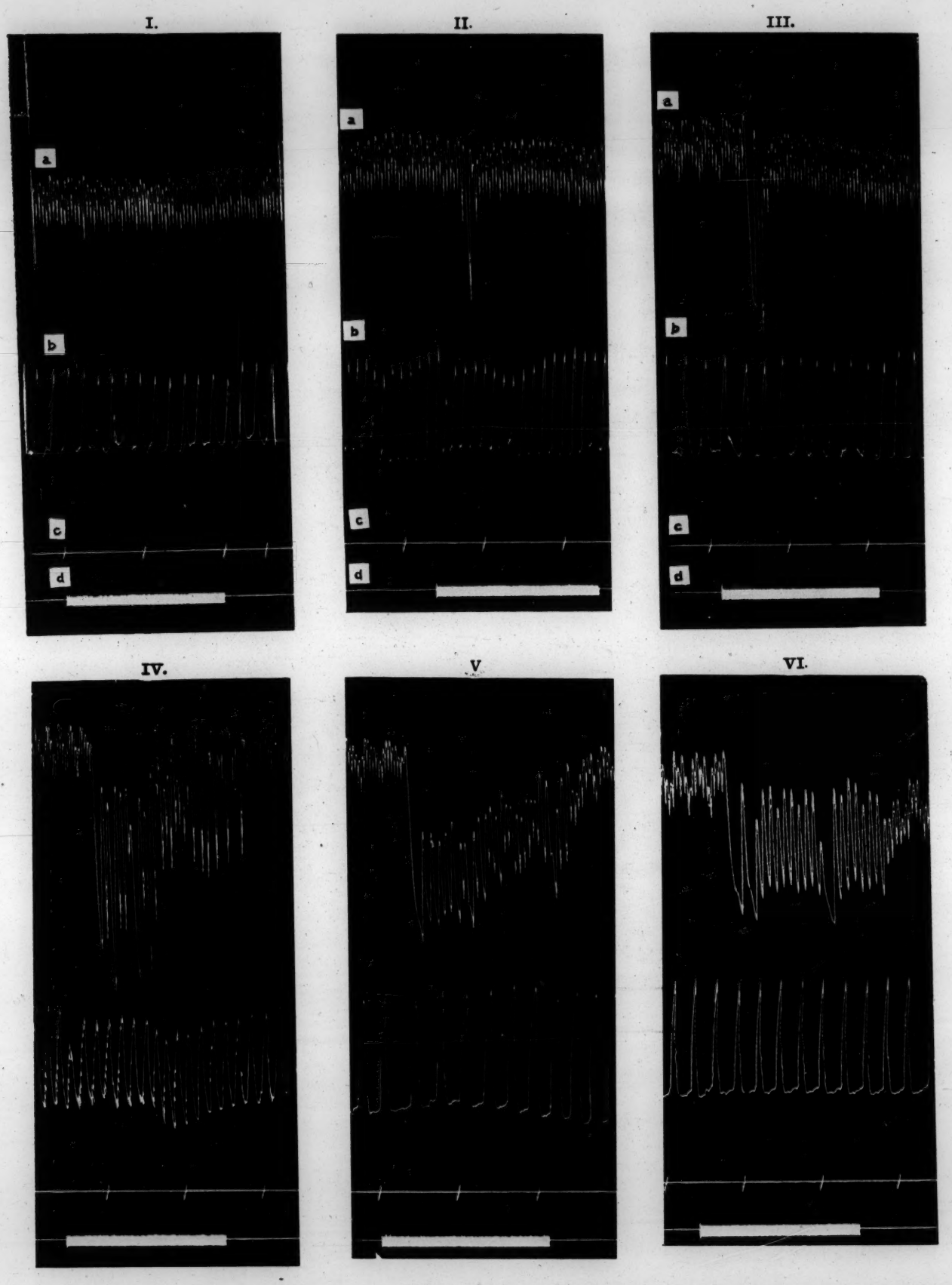

FIG. 19.—Tracings to illustrate the influence on cardiac inhibition by vagus excitation of a small dose of atropine sulphate (·00054 gramme = $\frac{1}{120}$ gr.) administered to a dog weighing 28 kilog. = 61½ lbs.

I., tracing taken 15 minutes after the atropine was administered by injection into the pleural cavity; II., 30 minutes after; III., 45 minutes; IV., 1 hour; V., 1½ hours; VI., 2½ hours after administration.

The blood-pressure and extent of anæsthetisation are approximately the same in all.

The strength of stimulus was the same in all, and was adequate to produce strong inhibition in the absence of atropine. Note that this effect is abolished in 15 minutes, and does not reappear in the same form and extent during at least 2½ hours, although there is a gradually increasing amount of inhibition shown as the atropine is becoming eliminated. But even 2½ hours after the injection the strongest stimulus (coil at 0) failed to produce any more effect than that shown in VI.

a, blood-pressure curve; *b*, respirations; *c*, time in 10 secs.; *d*, signal showing period of vagal excitation.

the heart has been administered, but this is a condition which no anæsthetist has any right or occasion to produce. On the other hand, it can prevent the sudden cardiac arrest which may show itself even at a comparatively early stage of chloroformisation, and which may be produced by the drug itself acting on the cardio-inhibitory centre, with (or without) the assistance of an effect reflexly produced upon that centre by the operation which is proceeding; or by some other concomitant, such as the incidental occurrence of dyspnœa, which is well known also to cause inhibition through the vagus.*

Whether the vagi be cut or their terminal branches blocked by atropine, or whether they be left intact, the ultimate effect of chloroform upon the heart, if its inhalation be pushed, is to produce complete arrest in diastole, a condition being observed which has been termed "paralytic dilatation." The condition is a peculiar one, for the cardiac muscle is not only paralysed and incapable of contracting spontaneously, but is in a *permanently refractory condition*, and incapable of responding to stimuli of any sort.†

To all forms of direct stimulation the heart gives no response, although the muscular tissue is not dead, and it suffices to remove the chloroform by producing a flow of unpoisoned blood or circulating fluid through the coronary vessels to restore its rhythmic contractility and its power of responding to artificial stimuli.‡ This shows that the refractory condition is due to the influence of the drug upon the heart, and it is commonly assumed by writers upon the subject that chloroform enters into combination with the contractile substance of the cardiac muscle and thereby deprives it of irritability. That this assumption is not justified is clear from the fact that no such effect—in doses which are more than sufficient to paralyse the heart—is produced upon either skeletal or upon plain muscular tissue. It is impossible to believe that the chemical constitution of these forms of contractile tissue is so different from that of heart muscle, that the one combines with chloroform and is thereby rendered devoid of irritability, whilst the others show no tendency to combine with or to be materially affected by the drug. It is much more probable that the effect produced is one of excitation of the terminations of the inhibitory nerves, the heart being thereby rendered irresponsive to stimuli. The argument that may be urged against this hypothesis, that if this were so the effect of chloroform in paralysing the heart would be prevented by atropine, is met by the statement that, although atropine blocks the

* An instance of the last-named complication is illustrated in fig. 20. In this animal the breathing was laboured, owing to obstruction of the air-tubes by mucus. There was marked dyspnœa, and the heart-beats were very slow and even arrested whenever the dyspnœa became intense. The violent respiratory efforts succeeded from time to time in clearing the air-passages, and this was followed by partial recovery. This pronounced inhibition was due to asphyxia, which, if more marked than in the instance given, may lead to entire arrest of the heart. Such inhibition from asphyxia does not occur with cut vagi. The condition is one which is not unfamiliar to anæsthetists, who are cognisant of its cause and danger. It is not liable to occur if a prior dose of atropine be administered, partly on account of the effect of this on the vagi and also because atropine tends to prevent the secretion of the mucus which causes the obstruction to respiration. This reason for the administration of atropine will apply equally to ether as to chloroform anæsthesia.

† Sherrington and Sowton (*op. cit.*), in the isolated and perfused cat's heart arrested by chloroform, obtained a renewal of the contractions on stimulation of accelerator nerves. But it is doubtful if this could be obtained with a strong dose of chloroform.

‡ Sherrington and Sowton.

inhibitory path, there is no conclusive evidence that it acts upon the inhibitory end-apparatus in the muscular fibres. According to this view the chloroform-heart—provided that the dose is insufficient to kill the contractile tissues generally—is in a condition of active inhibition rather than in one of passive paralysis. In support of this, it may be stated that although, if the chest be opened immediately after death, the heart may be completely irresponsive to all forms of stimuli, after a little while it often happens that it begins to respond and even exhibits feeble spontaneous contraction,

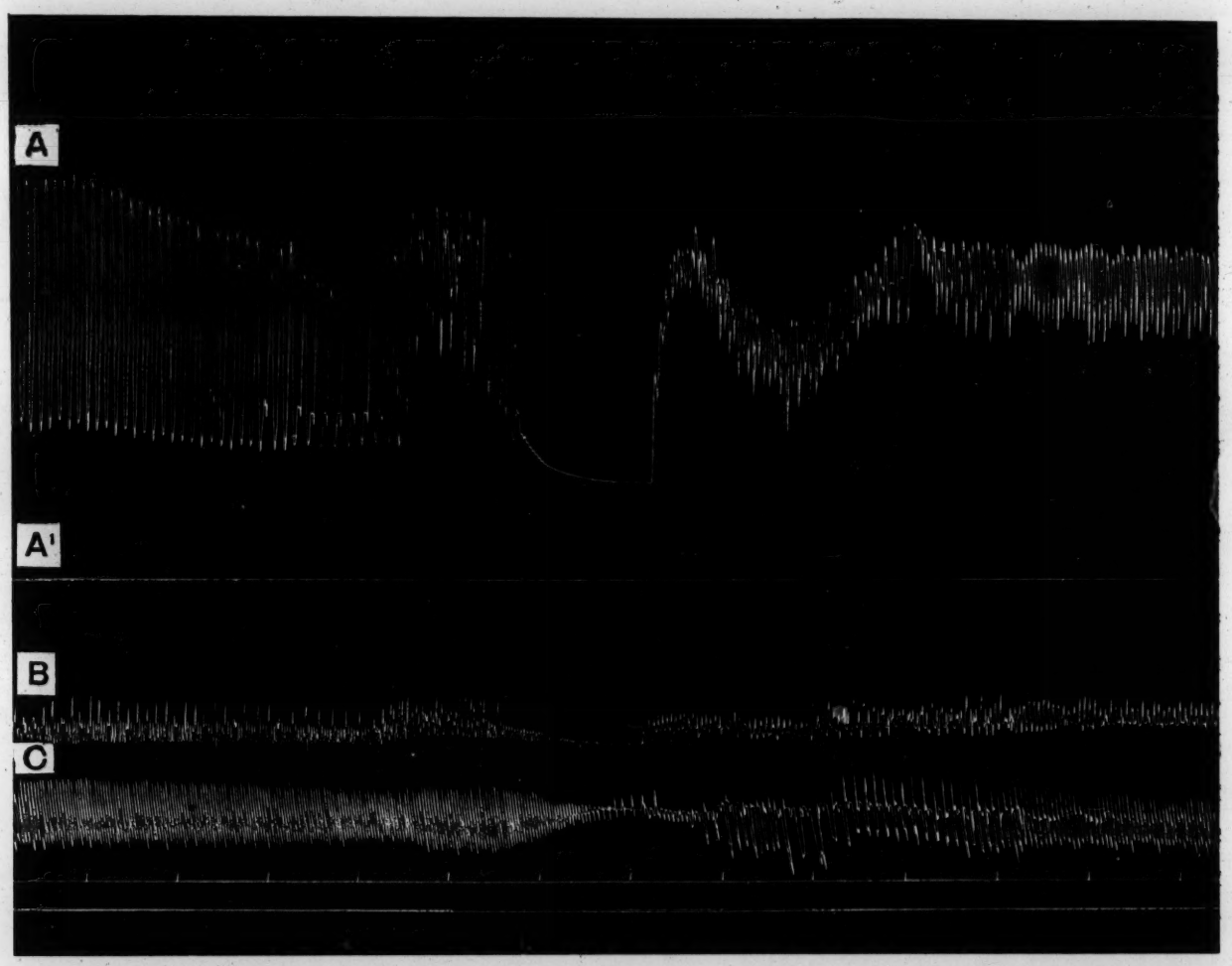

FIG. 20.—Cardiac inhibition produced during chloroform inhalation by dyspnœa resulting from accumulation of mucus in air-passages.

A, blood-pressure; A^1, line 1 centimetre below the zero of blood-pressure; B, costal respiration: C, diaphragmatic respiration. The dyspnœic condition is shown by the extreme rapidity of the respiratory movements at the left hand of tracing. About the middle of the tracing the obstruction to the passage of air was removed, and with the disappearance of the dyspnœa the heart resumed its normal rate of rhythm.

although the chloroform has not been washed away. The phenomenon may be explained if we assume that the inhibitory end-apparatus has died before the contractile substance of the muscular fibres.

To sum up this part of the subject, the conclusions which it appears justifiable to draw regarding the causation of death from the effect of chloroform upon the heart are as follows:—(1) Death *may* be caused in the earliest stage of administration by the action of the drug upon the cardio-inhibitory centre, the stimulation being reflex.

For convenience of description, this may be termed "primary" inhibition. It is probable that it occurs but rarely in man, and some altogether deny that it can produce a fatal result; it is, however, impossible to explain the well-authenticated instances of sudden heart failure at the very commencement of administration without assuming that under certain conditions such primary inhibition, which, when it occurs in other cases, is usually quite evanescent, is occasionally persistent and fatal.*

(2) At a somewhat later stage of administration death is liable to occur from sudden heart failure due to inhibition produced by the action of the drug on the cardio-inhibitory centre, aided by its action on the neuro-muscular end-apparatus of the heart itself, and also by increasing venosity of blood caused by failing respiration. The liability to this form of inhibition, which may be termed "secondary" (as well as to that mentioned under 1), can be removed by the prior exhibition of a moderate dose of atropine, which, by diminishing or abolishing the effect upon the heart of excitation of the cardio-inhibitory centre by chloroform, deprives it of the power to produce sudden cardiac arrest. This precautionary measure was long ago suggested,† and all recent work on the subject emphasises the importance of its adoption.

That the prolonged administration of chloroform itself tends to diminish its excitatory effect upon the cardio-inhibitory centre in the medulla oblongata is probable from the fact that a dosage of chloroform can be given with impunity at the later stages of a long operation which would be highly dangerous if given at earlier stages.

The respiration in these "secondary" cases of inhibition may stop simultaneously with, or shortly before, or immediately after the heart. We have frequently succeeded in effecting resuscitation by artificial respiration in animals, in which both heart and respiration had completely stopped at this stage of poisoning after even a minute or two of cessation of heart-beats, and in two cases as long as three and five minutes respectively after complete cessation; but in other instances we have failed to obtain recovery after three minutes or more of cessation.

The two cases just referred to are of exceptional interest. In the one the animal had been under ether for about an hour (the anæsthetic being inhaled through a Y-shaped trachea tube), when chloroform, at first with considerable intermixture of air, was substituted for the ether. The effect was to produce a gradual fall of blood-pressure from 100 mm. Hg. to about 40 mm., after which both it and the respiration, which was much shallower than under ether, remained nearly constant. After five minutes the lateral air-inlet was cut off, and the dog received a much stronger dose of chloroform. The result of this was immediately apparent in a further fall of blood-pressure, and a slowing and irregularity of the respirations, which ceased altogether about two minutes later, although the heart continued to beat regularly and the blood-pressure was maintained and even rose slightly. About one minute twenty seconds after cessation of respiration the heart suddenly

* This mode of producing inhibition has been especially emphasised by ARLOING (*Thèse, Paris*, 1879), who describes the effect of chloroform in producing heart failure in terms very similar to those which we have employed.

† PITHA (1861, quoted by DASTRE); J. HARLEY, *Brit. Med. Journ.*, vol. i., 1868, p. 320; SCHÄFER, *Brit. Med. Journ.*, vol. ii., 1880, p. 620. FRASER (*Brit. Med. Journ.*, vol. ii., 1880, p. 715), BROWN-SÉQUARD (*C. r. Soc. Biol.*, 1883, p. 289), and DASTRE and MORAT (*Lyon Méd.*, 1882, and *C. r. Soc. Biol.*, 1883, pp. 242 and 259) have made a similar recommendation, but have suggested the addition of morphia, and this combination has often been used (first systematically by AUBERT, *C. r. Soc. Biol.*, 1883, p. 626). But morphia is in some ways antagonistic to atropine, and tends by itself to exalt the irritability of the cardio-inhibitory centre. Without atropine it would undoubtedly increase the danger of heart-arrest in chloroform administration.

stopped and the blood-pressure fell to zero. Artificial respiration (by the pump) was now started, and maintained for five minutes, and two injections of 4 c.c. adrenalin chloride (1 per mille) were meanwhile made into the pleural cavity without the least result being apparent, the heart remaining quiescent and the blood-pressure at zero the whole time. Intermittent compression of the thorax was now substituted for perflation, and the heart responded to this by a pulsation with each compression, although in the intervals the blood-pressure returned to zero. After continuing the compression for a little over a minute, the heart commenced beating spontaneously and the average blood-pressure rose to 50 mm. Hg. It required, however, another three minutes of artificial respiration before the diaphragm began to act and the intermittent compression could be desisted from for a time; but even then (although no more chloroform had been inhaled) the respirations again gradually failed and ceased after ten minutes. A second spell of intermittent compression, lasting two minutes, now, however, effected complete restoration. When this was established and the lid reflex had become brisk, the blood-pressure being 110 mm. Hg., chloroform was again administered in strong form. The blood-pressure gradually fell. In two minutes the lid reflex had disappeared, and in another minute respirations had ceased, followed in twenty seconds by complete cessation of heart beat. Five minutes was now allowed to elapse, during which the animal was to all appearance dead. Artificial respiration by chest compression was then recommenced, and two more doses of 3 c.c. adrenalin chloride solution were successively injected. Five minutes after the artificial respiration was commenced and immediately after the final dose of adrenalin, the heart began to beat spontaneously, and the blood-pressure, at first very low, gradually rose in about four or five minutes, during which artificial respiration was maintained, to about 100 mm. Hg. Natural respiration was, however, not again resumed, the medulla oblongata having to all appearance been deprived for too long a time in this instance of blood.

In the second dog a lethal dose of chloroform vapour was administered twice. The first time both heart and respiration (the latter ten seconds before the heart) had stopped after three and a half minutes' administration. Half a minute later the chloroform was removed, and 4 c.c. of 1 per 1000 adrenalin chloride solution was injected into the pleural cavity. This produced no apparent effect. Three minutes after cessation of heart and respiration, chest compression was begun. Each compression produced a heart response, and the blood-pressure rose from zero to a few millimetres. After four minutes' chest compression another similar dose of adrenalin was injected into the pleural cavity, chest compression being continued. The blood-pressure then began to recover, the heart now beating slightly more rapidly than the chest compression, but natural respiration (diaphragm) was not resumed until another six minutes had elapsed. As in the last case, however, the natural respirations gradually became shallower and slower again, although no more chloroform was given, and fifteen minutes later ceased altogether, the heart and blood-pressure also becoming weak and low; the administration at this stage of a decoction of pituitary, and later of another dose of adrenalin, with the idea of restoring the heart's action, produced no visible effect. After two minutes' cessation of respiration (the heart still beating feebly), recourse was again had to chest compression and then to artificial respiration by the pump. This was very soon followed by recovery of heart and blood-pressure, and a few minutes later natural respirations were resumed and maintained, and artificial respiration was discontinued; recovery was, in fact, complete.

(3) In late stages of administration the heart is paralysed by the direct effect of the drug, acting either upon its muscular tissue (as is usually assumed), or (as we believe) by exciting the neuro-muscular inhibitory end-apparatus, and through this rendering the muscular tissue non-excitable. This effect can probably only occur with a considerable dosage of chloroform in the blood, and the respiratory centre is invariably first paralysed, so that the respirations become slow and shallow and cease before the heart; the time difference between the cessation of heart and respiration being considerably longer than when the cessation occurs early in the administration. This final effect upon the heart is *not* antagonised by atropine. The heart is found to be entirely inexcitable, and no treatment is of any avail short of removal of the poisoned blood from the coronary vessels and the substitution of blood free from chloroform. It is conceivable that this substitution might be done by heart massage, or even by com-

pression of the thorax in attempting artificial respiration; and, as a matter of fact, SCHIFF, BATELLI, and others* have succeeded in restoring the circulation by heart massage combined with artificial respiration (both in dogs and cats, and in one or two rare cases in the human subject), even although a considerable time (fifteen to twenty minutes) had elapsed after complete cessation of the circulation. We ourselves have not succeeded in restoring an animal after the condition here described appeared to be fully established, and we should be disposed to regard the possibility of resuscitation in such cases as remote.†

ANTAGONISING AGENTS.

(1) *Atropine.*—It is apparent from the results obtained by other experimenters, as well as from our own observations, that the chief danger to be guarded against in the administration of chloroform is the inhibitory influence which it produces upon the heart. As we have already pointed out (pp. 328, 329), this influence can be in great measure controlled by the prior administration of a moderate dose of atropine, at least in so far as the primary and secondary instances of inhibition are concerned, and these are the most dangerous because they are apt to occur without the warning which manifests itself in the case of the final heart paralysis, by the prior arrest of respiration. Atropine is therefore to be placed first in the list of antagonising agents; a dose of $\frac{1}{100}$ gr. to $\frac{1}{50}$ gr. for an average man being administered hypodermically half an hour before the administration of chloroform.

(2) *Adrenalin Chloride.*—The employment of this has been suggested in chloroform poisoning by GOTTLIEB.‡ In the two instances which we have recorded on pp. 333 and 334, which were attended by an entirely unusual measure of success so far as resuscitation after apparent death from chloroform had occurred, we administered successive doses of adrenalin, injected into the pleural cavity, as part of the treatment. These happened to be the first two experiments of the series undertaken by us, and we were led to ascribe much of the success which attended them to the use of this drug, and formed high hopes of the value of its administration in cases of chloroform poisoning. Subsequent experience showed, however, that adrenalin by itself is of little or no avail to restore a heart paralysed by chloroform, and even in conjunction with other remedial measures—of which the most important is without doubt artificial respiration by chest compression—we are not in a position, as the result of a number of trials, to affirm that it is able materially to contribute to the process of resuscitation.

(3) *Ammonia Vapour.*—RINGER§ first showed that in the frog's heart ammonia acts as a direct antagonist to chloroform, and may even set in activity a heart which has

* For references see M. BOURCART, *Rev. méd. de la Suisse Romande*, October 20, 1903.

† This is no doubt the condition referred to by RICHET (*Dict. de physiol.*, article "Anesthésie," 1895, p. 523) when he avers that when cardiac syncope occurs artificial respiration never succeeds in effecting restoration; for the statement does not apply to the syncope caused by the secondary inhibition previously referred to.

‡ *Arch. f. Path. u. Pharm.*, Bd. 37, p. 98, 1896. See also BIEDL, *Wien. klin. Wochenschr.*, 1896.

§ *Practitioner*, vol. xxvi. p. 436, 1881.

been arrested by chloroform; and since it also has a stimulating action on the heart and respiratory centre, it is likely that it may prove useful as a restorative in cases in which the pulse and breathing have not altogether ceased. We have investigated the effect in dogs of causing them to inhale a mixture of chloroform vapour and ammonia, made either by dropping chloroform and ammonia upon the cotton-wool of the inhaling bottle, or by mixing chloroform in definite proportions with alcoholic ammonia, using for this purpose a solution of ammonia in absolute alcohol containing 6·8 per cent. of

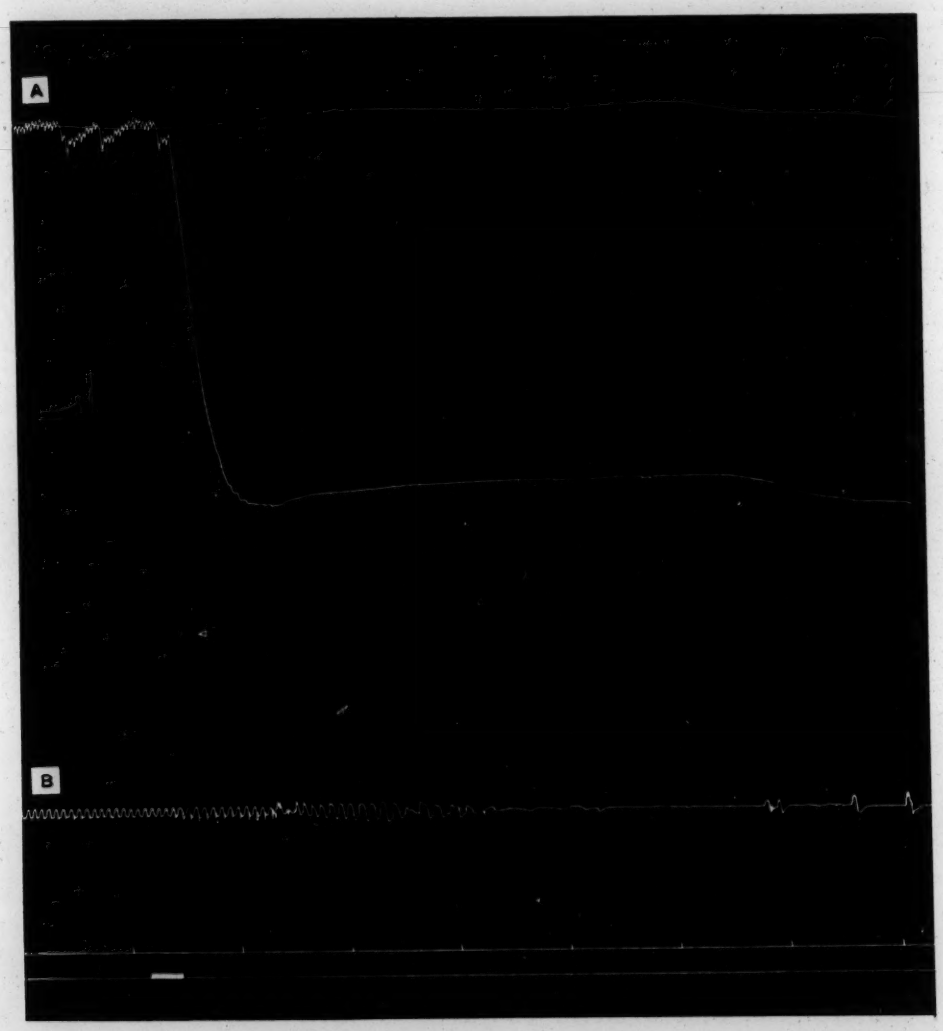

FIG. 21.—Instantaneous heart failure caused by inhalation, at the moment marked by the signal, of air charged with vapour from a mixture of 20 c.c. chloroform and 5 c.c. ammoniated alcohol.

A, blood pressure; B, respirations. The latter continued to show themselves at a slow rate for 3 minutes after the heart had stopped.

ammonia, prepared for us by Messrs Duncan & Flockhart. A mixture of chloroform and ammonia vapours, even if it contain a comparatively small proportion of ammonia, is too pungent to be administered in the first instance, the irritation it causes to the sensory nerves of the mucous surfaces rendering it practically irrespirable. And if the proportion of ammonia be considerable, this excitation may result in powerful cardiac inhibition, and the heart may instantly and permanently stop (fig. 21). If, however,

the proportion of ammonia be less and the animal be already completely under the influence of an anæsthetic, the effect of the addition of the vapour of ammonia or ammoniated alcohol to the chloroform inhaled is strikingly beneficial. The blood-pressure falls but slightly even during a prolonged period of inhalation—the strength of the heart is well maintained—and the tendency to failure of respiration, which is so marked a feature

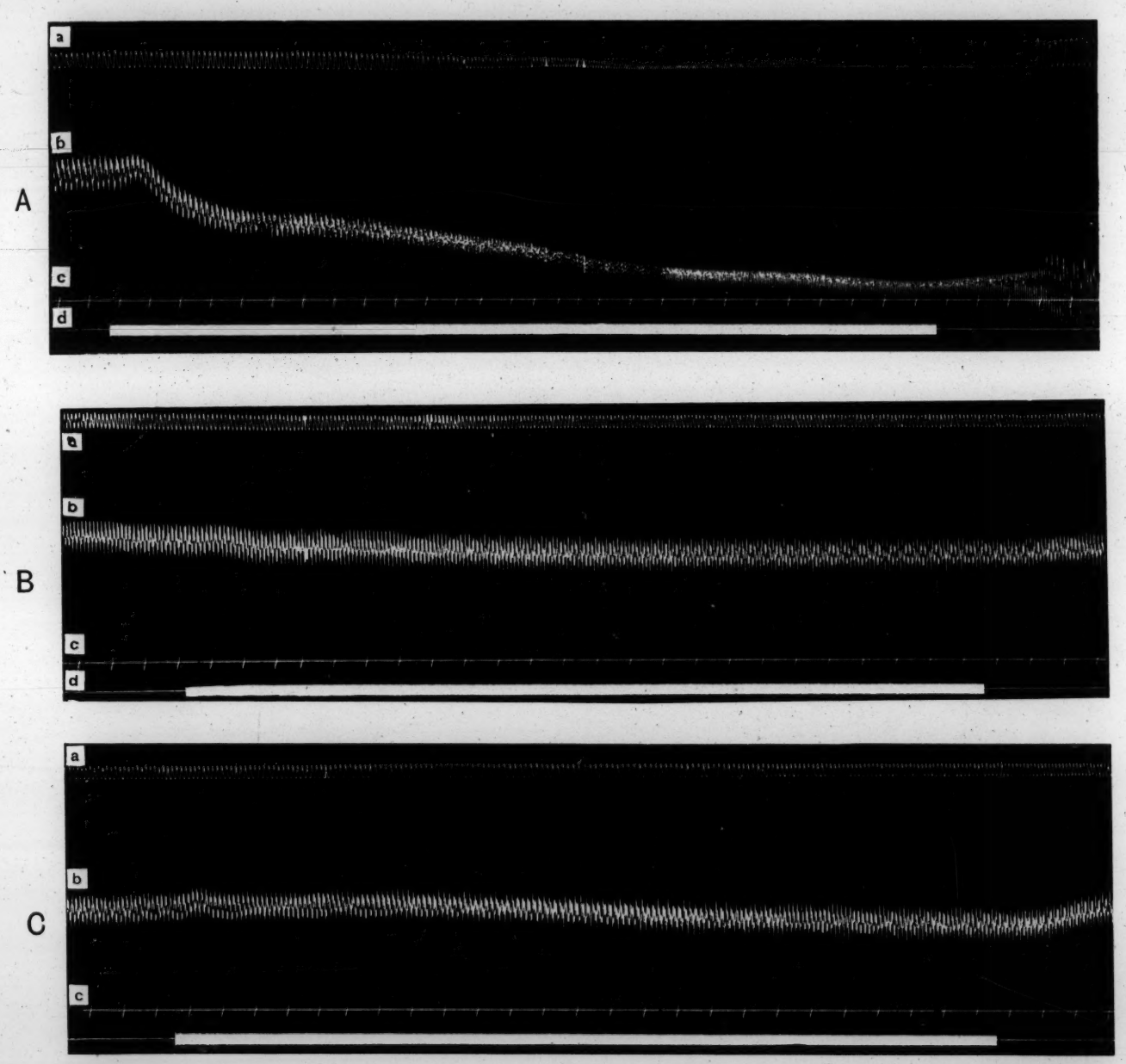

FIG. 22.—Tracings showing in a comparative manner in the same animal the difference of effect between inhalation (A) of pure chloroform, (B) of a mixture of chloroform with ammoniated alcohol (9 to 1), and (C) of a mixture of chloroform and absolute alcohol (9 to 1).

Note in A the rapid fall of blood-pressure and the speedy failure of respiration; in this case the heart continued to beat after the respirations had ceased. After a respiratory arrest of more than a minute, during which the heart showed strong tendency to inhibition, artificial respiration by chest compression was resorted to (the beginning of this is shown): in rather more than a minute the blood-pressure rose—the natural respirations were then resumed. Note in B and C the very slow and slight fall of blood-pressure, and the complete maintenance of respiration during the whole time of administration. In all three cases the air of respiration was charged as strongly as possible, at the ordinary temperature of the laboratory, with the vapour to be inhaled.

a, respirations; *b*, blood-pressure; *c*, time in 10 secs.; *d*, signal showing period of administration. In all cases there was distinct corneal reflex immediately prior to the administration, and this disappeared within 1 minute.

of strong chloroform inhalation, is considerably diminished. The comparative effects of inhalation of chloroform alone and of chloroform *plus* ammoniated alcohol are shown in the tracings A and B reproduced in fig. 22, and the beneficial results of substituting a mixture of chloroform containing alcoholic ammonia for pure chloroform are illustrated in the tracing given in fig. 23.*

(4) *Alcohol Vapour.*—In order to determine how much of the beneficial effect of the mixture of alcoholic ammonia with chloroform was due to the alcohol used as a

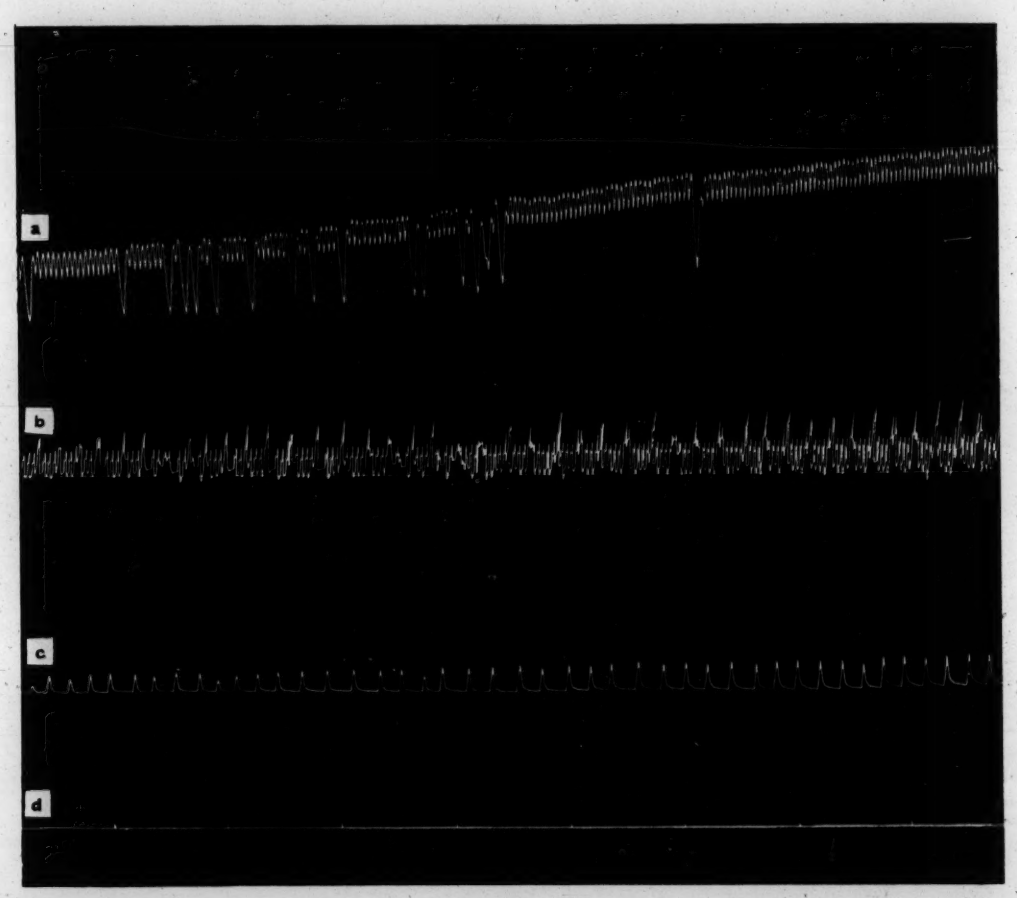

FIG. 23.—Beneficial effect upon blood-pressure, heart, and respiration of substituting ammoniated alcohol and chloroform (1 to 9) for the pure chloroform which was being administered to a dog.

a, blood-pressure; *b*, heart-beats, recorded by a needle passed through chest wall; *c*, respiration; *d*, time in 10 secs. Notice the increasing strength of the heart-beats and of the respiratory movements.

It is also seen that the dropped heart-beats due to vagal inhibition which were occurring during chloroform alone gradually disappear as the result of adding ammoniated alcohol to the chloroform.

vehicle for the ammonia, we next proceeded to investigate the results of using for inhalation the vapour given off from mixtures of chloroform and alcohol. We were somewhat surprised to find that the results were nearly as beneficial when alcohol alone was used, as when alcoholic ammonia vapour was employed. The difference between the effect produced upon blood-pressure and respiration by inhalation of pure chloroform in the one case, and by inhalation of a mixture of chloroform containing 1 part in 10

* The addition of ammonia gas to the chloroform to be used for inhalation was advocated by J. DUNCAN MENZIES (*Brit. Med. Jour.*, vol. ii., 1895, p. 871).

by volume of absolute alcohol, is exemplified in fig. 22 (A and C). It will be observed that the effect of the addition of 1 part absolute alcohol to 9 parts chloroform is to largely prevent the fall of blood-pressure, which is recognised as being one of the most serious dangers attendant on chloroform inhalation, and at the same time to maintain the respirations at a force and frequency very little less than normal. The administration was made in both cases on the same animal by the same method, and using the same quantities of the solutions.

It may also be observed how much more readily recovery takes place after removal of the mixed vapour than after removal of the chloroform. On the other hand, disappearance of the lid reflex occurs a little sooner when pure chloroform is used, but the difference is not great.

We have tried other proportions of alcohol and chloroform, but have obtained no

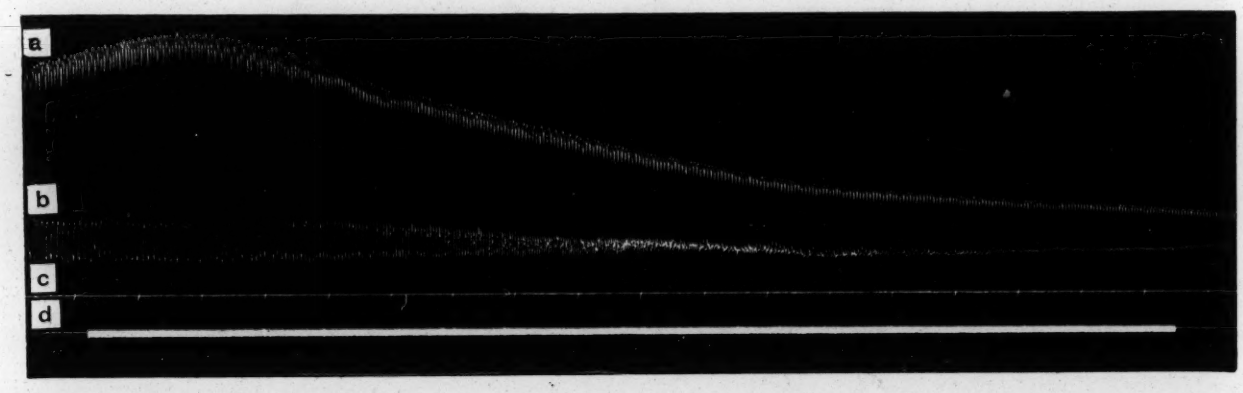

FIG. 24.—Administration by inhalation of air strongly charged with pure chloroform during 3 minutes to dog weighing 11 kilog. which had received (1½ hr. and 1¾ hr. previously) two doses of ·00027 g. ($\frac{1}{120}$ gr. in all) atropine sulphate. The vagus, tested immediately before this tracing was taken, gave no slowing and only a slight fall of pressure, even with coil at 0.

Notice (1) the preliminary rise of blood-pressure due to excitation of vasomotor centre, succeeded by (2) a rapid and regular fall, less steep towards the end; (3) entire absence of slowing of pulse; (4) increase of rapidity, but decrease of excursion of respiratory movements, which became irregular, and eventually hardly perceptible.

a, blood-pressure curve; *b*, respirations; *c*, time in 10 secs.; *d*, signal of chloroformisation and abscissa of blood-pressure. The heart was still beating 4 minutes later, but the blood-pressure was at zero, and the respirations had wholly ceased. The animal was then subjected to artificial respiration by chest compression, and in 1 m. 40 secs. natural respirations were resumed, and the heart and blood-pressure rapidly recovered. The tracing shown in the next figure was taken prior to this one.

better results. Indeed, with a 20 per cent. alcohol chloroform the respirations appeared to be more affected than with the 10 per cent. mixture.

The beneficial effect can hardly be due to the mere dilution of the chloroform vapour by alcohol vapour; moreover, dilution with ether has not this effect, but the result is then practically the same as is obtained with undiluted chloroform. It is therefore to be ascribed directly to the beneficial action of the alcohol on the heart and respiratory centre. We are of opinion that a mixture containing one part by volume of absolute alcohol to nine parts of chloroform should be used when chloroform is indicated as the anæsthetic, since these results show that it is far safer in its action than pure chloroform. There seems reason to believe that the greater safety of the A.C.E. mixture over chloroform depends upon the alcohol it contains, and that the ether is unnecessary; it may further be noted that the alcohol in this mixture is in needlessly

large proportion.* The beneficial effect of adding absolute alcohol to the chloroform used for inhalation is seen in atropinised as well as in normal animals. The respective effects of administering pure chloroform, on the one hand, and chloroform containing 10 parts per cent. of absolute alcohol, on the other, to a dog weighing 24 lbs., which had previously received two successive doses of ·00027 gramme of atropine sulphate, are illustrated by the tracings shown in figs. 24 and 25.

A mixture which is frequently used by anæsthetists in place of the A.C.E. is one of ether and chloroform without alcohol; but from the facts here put forward it would seem better rather to omit the ether than the alcohol. This remark is not to be understood as implying that ether by itself is not a safe anæsthetic—far safer than chloroform, however diluted—but merely that it has not the same antagonising influence as alcohol upon the dangerous tendencies of chloroform.

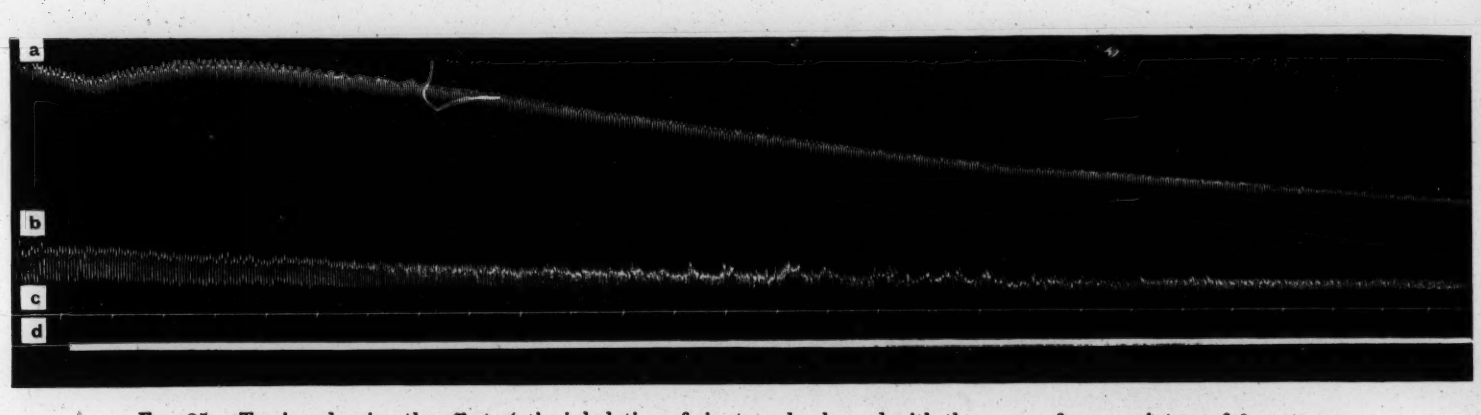

FIG. 25.—Tracing showing the effect of the inhalation of air strongly charged with the vapour from a mixture of 9 parts pure chloroform and 1 part absolute alcohol. The inhalation was continued during nearly 5 minutes. The tracing was begun 10 minutes before that shown in the preceding figure, and is from the same animal (under the influence of atropine sulphate). Notice, as compared with fig. 24, (1) the much more gradual fall of blood-pressure, which even after nearly 5 minutes of administration still keeps fairly high, (2) the effect on the respirations, which are far less influenced than by the pure chloroform, being well maintained during the whole time. On desisting from the inhalation, recovery of blood-pressure was rapid, and the lid reflex, which had disappeared early during the inhalation, was brisk 5 minutes after the chloroform and alcohol mixture had been removed.

a, blood-pressure; *b*, respiration; *c*, time in 10 secs.; *d*, signal.

POST-MORTEM CONDITIONS AFTER DEATH FROM CHLOROFORM.

Although these conditions have been often described, it may not be out of place to add our own experience and observations.

Heart.—In all the cases which we examined immediately after death, all the cavities—with, sometimes, an exception for the left auricle—were distended with blood, the right auricle and great veins of the thorax enormously so. The left ventricle always contained a considerable quantity of blood, but rather less than the right ventricle. If, however, the examination were made some little time after death, the left ventricle was always found empty and firmly contracted. This change from the full flaccid condition to the empty firm condition took place in one case within twenty minutes, while in others it did not show itself for forty-five minutes.

* *Cf.* on this subject, QUINQUARD, *C. r. Soc. Biol.*, 1883, p. 425; and DUBOIS, *ibid.*, p. 441.

Lungs.—The pulmonary arteries are greatly distended with blood. Otherwise the lungs usually present a perfectly normal appearance externally. But in cutting them open we found, in six out of twenty cases examined, a considerable amount of frothy mucus in the bronchial tubes.

Abdominal Viscera.—These exhibited marked venous congestion, especially well seen in the liver, which may be greatly swollen and project beyond the thoracic cage.* It is thus exposed to some risk of rupture if artificial respiration be attempted by the Howard method. This happened in one of our cases, although we were aware of the danger, and always endeavoured to avoid it by compressing the chest well above the liver.

All these appearances are very similar to those which result from asphyxia due to deprivation of air, whether caused by drowning or otherwise. But they are produced independently of any asphyxia caused by paralysis of the respiration by the drug, for they show themselves equally when artificial respiration has been maintained by perflation, and the drug has produced death solely by its action on the heart. Nevertheless, the ultimate effect upon the heart of chloroform and of deprivation of air respectively is strikingly similar. In both cases the final result is a condition of "paralytic dilatation," or, as we have preferred to term it, "inhibition paralysis," in which the heart is absolutely refractory to all kinds of stimuli. In the case of chloroform the exciting cause is doubtless the drug itself; in the case of asphyxia, it is probably the carbon dioxide which has accumulated in the blood and tissues.†

* To observe this condition of the liver and abdominal organs, it is necessary to open the abdomen before the thorax. For if the contents of the latter be first laid bare, and any of the great veins injured, the congestion of the abdominal viscera at once subsides, owing to the escape of blood from their vessels.

† *Cf.* on this subject the Report of the Committee on Suspended Animation, *Trans. Med. Chir. Soc.*, 1904, Suppl., p. 63.

XIII.—Continuants resolvable into Linear Factors. By Thomas Muir, LL.D.

(MS. received August 22, 1904. Read November 7, 1904. Issued separately January 13, 1905.)

(1) It is known that a continuant whose three diagonals are formed of certain equidifferent progressions is resolvable into linear factors, the earliest specimens placed on record being those of SYLVESTER and PAINVIN.* The object of the present paper is to show that there are continuants of quite a different type which are also so resolvable, and to expound a general mode of investigating the subject.

(2) *The continuant of the* nth *order whose main diagonal is*

$$a,\quad a+2\cdot1^2\cdot c,\quad a+2\cdot2^2\cdot c,\quad a+2\cdot3^2\cdot c,\ \ldots\ldots$$

and whose minor diagonals are

$$\begin{array}{lll} 2\cdot(n-1)b, & (n-2)(b+c), & (n-3)(b+2c),\ \ldots\ldots \\ n(b-c), & (n+1)(b-2c), & (n+2)(b-3c),\ \ldots\ldots \end{array}$$

is equal to the product of the n *factors*

$$\begin{aligned} &\{a+2(n-1)b\}\cdot\{a+2(n-3)b+2(2n-3)c\} \\ &\quad\cdot\{a+2(n-5)b+4(2n-5)c\} \\ &\quad\cdot\{a+2(n-7)b+6(2n-7)c\} \\ &\quad\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ &\quad\cdot\{a-2(n-1)b+(2n-2)1c\} \qquad\ldots\qquad (\mathrm{I}) \end{aligned}$$

Taking for the purposes of illustrative proof the case where $n=5$, viz.

$$\begin{vmatrix} a & 2\cdot4b & . & . & . \\ 5(b-c) & a+2c & 3(b+c) & . & . \\ . & 6(b-2c) & a+8c & 2(b+2c) & . \\ . & . & 7(b-3c) & a+18c & 1(b+3c) \\ . & . & . & 8(b-4c) & a+32c \end{vmatrix}$$

and performing the operation

$$\mathrm{col}_1+\mathrm{col}_2+\mathrm{col}_3+\mathrm{col}_4+\mathrm{col}_5$$

we find we can remove the factor $a+8b$ and write the cofactor in the form

$$\begin{vmatrix} a-8b+2c & 3(b+c) & . & . \\ -2b-12c & a+8c & 2(b+2c) & . \\ -8b & 7(b-3c) & a+18c & b+3c \\ -8b & . & 8(b-4c) & a+32c \end{vmatrix}.$$

Performing now on this cofactor the operation

$$\mathrm{col}_1+4\,\mathrm{col}_2+9\,\mathrm{col}_3+16\,\mathrm{col}_4$$

* [SYLVESTER, J. J.] "Théorème sur les déterminants de M. Sylvester," *Nouv. Annales de Math.*, xiii. p. 305.
PAINVIN, . "Sur un certain système d'équations linéaires," *Journ. de Liouville*, 2^e sér., iii. pp. 41–46.
MUIR, THOMAS. "Factorizable Continuants," *Trans. S. Afr. Philos. Soc.*, xiv. pp. 29–33.

we obtain

$$(a+4b+14c)\begin{vmatrix} 1 & 3b+3c & . & . \\ 4 & a+8c & 2b+4c & . \\ 9 & 7b-21c & a+18c & b+3c \\ 16 & . & 8b-32c & a+32c \end{vmatrix},$$

of which the determinant factor is reducible to the three-line form

$$\begin{vmatrix} a-12b-4c & 2b+4c & . \\ -20b-48c & a+18c & b+3c \\ -48b-48c & 8b-32c & a+32c \end{vmatrix}.$$

The next operation

$$\mathrm{col}_1 + 6\,\mathrm{col}_2 + 20\,\mathrm{col}_3$$

gives in like manner

$$(a+20c)\begin{vmatrix} 1 & 2b+4c & . \\ 6 & a+18c & b+3c \\ 20 & 8b-32c & a+32c \end{vmatrix}$$

or

$$(a+20c)\begin{vmatrix} a-12b-6c & b+3c \\ -32b-112c & a+32c \end{vmatrix};$$

and finally the operation

$$\mathrm{col}_1 + 8\,\mathrm{col}_2$$

enables us to change this two-line determinant into

$$(a-4b+18c)\begin{vmatrix} 1 & b+3c \\ 8 & a+32c \end{vmatrix},$$

or

$$(a-4b+18c)(a-8b+8c).$$

The desired result thus is

$$(a+8b)(a+4b+14c)(a+20c)(a-4b+18c)(a-8b+8c).$$

(3) *The continuant of the* nth *order whose main diagonal is*

$$a,\quad a+2(1\cdot3)c,\quad a+2(2\cdot4)c,\quad a+2(3\cdot5)c,\ \ldots$$

and whose minor diagonals are

$$(n-1)b,\quad (n-2)(b+c),\quad (n-3)(b+2c),\ \ldots.$$
$$(n+2)(b-3c),\quad (n+3)(b-4c),\quad (n+4)(b-5c),\ \ldots.$$

is equal to the product of the n *factors*

$$\begin{aligned} &\{a+2(n-1)b\},\\ &\{a+2(n-3)b+2(2n-1)c\},\\ &\{a+2(n-5)b+4(2n-3)c\},\\ &\ldots\ldots\ldots\ldots\\ &\{a-2(n-1)b+6(n-1)c\}. \end{aligned} \qquad \text{(II)}$$

This is established by proceeding in the same way as in § 2, the sets of column-multipliers now being

$$\left.\begin{matrix} 1,\ 2,\ 3,\ 4,\ 5,\ \ldots. \\ 1,\ 4,\ 10,\ 20,\ \ldots. \\ 1,\ 6,\ 21,\ \ldots. \\ 1,\ 8,\ \ldots. \\ 1,\ \ldots. \end{matrix}\right\} \text{ instead of } \left\{\begin{matrix} 1,\ 1,\ 1,\ 1,\ 1\ \ldots. \\ 1,\ 4,\ 9,\ 16\ \ldots. \\ 1,\ 6,\ 20\ \ldots. \\ 1,\ 8\ \ldots. \\ 1\ \ldots. \end{matrix}\right.$$

and the resulting factors greater by

$$0,\quad 4c,\quad 8c,\quad 12c,\quad 16c,\ \ldots\ldots$$

respectively.

(4) Changing a into $a+2c$, n into $n-1$, b into $b+c$ in § 3 we have

$$\begin{vmatrix} a+2c & (n-2)(b+c) & . & . & \ldots. \\ (n+1)(b-2c) & a+8c & (n-3)(b+2c) & . & \ldots. \\ . & (n+2)(b-3c) & a+18c & (n-4)(b+3c) & \ldots. \\ . & . & (n+3)(b-4c) & a+32c & \ldots. \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

$$= \{a+2(n-2)b+2(n-1)c\}\cdot\{a+2(n-4)b+6(n-2)c\}\cdot\{a+2(n-6)b+10(n-3)c\}\ldots\ldots$$

But the continuant here is the complementary minor of the element in the place (1, 1) of the continuant in § 2. Consequently by division we obtain

$$a-\frac{2(n-1)n(b-c)b}{a+2\cdot1^2\cdot c}-\frac{(n-2)(n+1)(b-2c)(b+c)}{a+2\cdot2^2\cdot c}-\ddots-\frac{1\cdot(2n-2)(b-nc+c)(b+nc-2c)}{a+2\cdot(n-1)^2\cdot c}$$

$$=\frac{\{a+2(n-1)b\}\{a+2(n-3)b+2(2n-3)c\}\{a+2(n-5)b+4(2n-5)c\}\ldots.}{\{a+2(n-2)b+2(n-1)c\}\{a+2(n-4)b+6(n-2)c\}\ldots.}\quad .\qquad \text{(III)}$$

(5) *If in the results of* §§ 2, 3 *we annex* f *as a factor to every term on both sides that is independent of* a, *the identity is not interfered with.* . . . (IV)

For, taking (in the fourth order, for shortness' sake) the continuant dealt with in § 2, and putting a/f for a we have

$$\begin{vmatrix} \frac{a}{f} & 3b & . & . \\ 6(b-3c) & \frac{a}{f}+6c & 2(b+c) & . \\ . & 7(b-4c) & \frac{a}{f}+16c & b+2c \\ . & . & 8(b-5c) & \frac{a}{f}+30c \end{vmatrix}$$

$$=\left(\frac{a}{f}+6b\right)\left(\frac{a}{f}+2b+14c\right)\left(\frac{a}{f}-2b+20c\right)\left(\frac{a}{f}-6b+18c\right),$$

whence on multiplying both sides by f^4 there results

$$\begin{vmatrix} a & 3bf & . & . \\ 6(b-3c)f & a+6cf & 2(b+c)f & . \\ . & 7(b-4c)f & a+16cf & (b+2c)f \\ . & . & 8(b-5c)f & a+30cf \end{vmatrix}$$

$$=(a+6bf)(a+2bf+14cf)(a-2bf+20cf)(a-6bf+18cf),$$

as asserted.

(6) *The sum of the elements of the main diagonal of either of the continuants in §§ 2, 3 is equal to the sum of the factors into which the continuant is resolved.* . (V)

This is true of any continuant of the form

$$\begin{vmatrix} a+x_1 & p_1 & . & \dots \\ q_1 & a+x_2 & p_2 & \dots \\ . & q_2 & a+x_3 & \dots \\ \dots & \dots & \dots & \end{vmatrix}_n$$

that is resolvable into factors linear in a. By way of proof we have only to note (1) that since the diagonal term is the only term of the continuant that contains either the n^{th} or $(n-1)^{\text{th}}$ powers of a, it follows that the coefficient of a^{n-1} in the continuant is $x_1+x_2+x_3+ \dots$, or Σx say: and (2) that if $a+\mu_1$, $a+\mu_2$, $a+\mu_3$, . . . be the factors into which the continuant is resolved, the coefficient of a^{n-1} in their product is $\mu_1+\mu_2+\mu_3+ \dots$, or $\Sigma\mu$ say. We thus have

$$\Sigma x = \Sigma\mu,$$

and ∴
$$na + \Sigma x = na + \Sigma\mu,$$

as was to be proved.

(7) The full table of multipliers used in § 2 is found to be

$$1, 1, 1, 1, 1, \dots\dots, 1$$
$$1, 4, 9, 16, \dots\dots, \frac{r}{1}C_{r,1}$$
$$1, 6, 20, \dots\dots, \frac{r}{2}C_{r+1,3}$$
$$1, 8, \dots\dots, \frac{r}{3}C_{r+2,5}$$
$$1, \dots\dots, \frac{r}{4}C_{r+3,7}$$
$$\dots\dots\dots\dots\dots\dots$$

—in other words, each multiplier is of the form

$$\frac{r}{s}C_{r+s-1,\,2s-1};$$

and the question next arises whether the continuant resolved in § 2 is the only one which this set of multipliers is capable of dealing with. In order to make suitable answer we have to ascertain the relations which must exist between the twelve quantities

$$\begin{matrix} \beta_1, & \beta_2, & \beta_3, & \beta_4 \\ p, & q, & r, & s \\ \gamma_1, & \gamma_2, & \gamma_3, & \gamma_4 \end{matrix}$$

in the continuant

$$\begin{vmatrix} a & 2\cdot4\beta_1 & . & . & . \\ 5\gamma_1 & a+p & 3\beta_2 & . & . \\ . & 6\gamma_2 & a+q & 2\beta_3 & . \\ . & . & 7\gamma_3 & a+r & \beta_4 \\ . & . & . & 8\gamma_4 & a+s \end{vmatrix}$$

in order that it may be resolvable into linear factors by means of the operations of § 2.

The performance of the first operation on the columns evolves the equations

$$\begin{aligned} a+8\beta_1 &= 5\gamma_1+a+p+3\beta_2, \\ &= 6\gamma_2+a+q+2\beta_3, \\ &= 7\gamma_3+a+r+\beta_4, \\ &= 8\gamma_4+a+s. \end{aligned}$$

Then the factor $a+8\beta_1$ being removed, the cofactor becomes expressible as a determinant of the next lower order, viz.

$$\begin{vmatrix} a+p-8\beta_1 & 3\beta_2 & . & . \\ 6\gamma_1-8\beta_1 & a+q & 2\beta_3 & . \\ -8\beta_1 & 7\gamma_3 & a+r & \beta_4 \\ -8\beta_1 & . & 8\gamma_4 & a+s \end{vmatrix};$$

and as by hypothesis the performance of the operation

$$\mathrm{col}_1 + 4\,\mathrm{col}_2 + 9\,\mathrm{col}_3 + 16\,\mathrm{col}_4$$

enables us to remove the factor $a+p-8\beta_1+12\beta_2$ we obtain three other conditional equations, to be followed at the next stage by two others, and last of all by one. Of these $4+3+2+1$ equations the four first obtained are obviously best used to determine p, q, r, s in terms of the β's and γ's, the results being

$$\begin{aligned} p &= 8\beta_1 - 3\beta_2 - 5\gamma_1, \\ q &= 8\beta_1 - 2\beta_3 - 6\gamma_2, \\ r &= 8\beta_1 - \beta_4 - 7\gamma_3, \\ s &= 8\beta_1 - 8\gamma_4. \end{aligned}$$

The remaining six form a very interesting set: after simplification they are

$$\begin{aligned} 12\beta_1-18\beta_2+5\beta_3 &= -10\gamma_1+9\gamma_2 \\ 64\beta_1-81\beta_2+7\beta_4 &= -45\gamma_1+35\gamma_3 \\ 15\beta_1-18\beta_2 &= -10\gamma_1+7\gamma_4 \\ 45\beta_2-60\beta_3+14\beta_4 &= -36\gamma_2+35\gamma_3 \\ 24\beta_2-25\beta_3 &= -15\gamma_2+14\gamma_4 \\ \beta_3-\beta_4 &= -\gamma_3+\gamma_4. \end{aligned}$$

Taking the first three and using with them the multipliers 7, -1, 1 respectively, we find, on adding, that

$$5(\beta_1+\gamma_1) - 9(\beta_2+\gamma_2) + 5(\beta_3+\gamma_3) - (_4+\beta\gamma_4) = 0;$$

similarly from the subset of two there results by subtraction

$$3(\beta_2+\gamma_2) - 5(\beta_3+\gamma_3) + 2(\beta_4+\gamma_4) = 0;$$

and the final set, of course, is

$$(\beta_3+\gamma_3) - (\beta_4+\gamma_4) = 0.$$

By means, therefore, of these three derived equations we arrive at the proposition that *in the determinant under discussion the sum of any β and the corresponding γ is constant.*

This being equivalent to only three equations, and other three being still unaccounted for, we put

$$\gamma_1, \gamma_2, \gamma_3, \gamma_4 = \sigma-\beta_1, \sigma-\beta_2, \sigma-\beta_3, \sigma-\beta_4.$$

and learn (1) that one of the equations is not independent of the others, (2) that the β's are connected by the equation

$$\beta_1 - 2\beta_2 + \beta_3 = 7(\beta_2 - 2\beta_3 + \beta_4),$$

and (3) that σ is expressible in terms of any three of the β's, for example,

$$\sigma = -2\beta_1 + 9\beta_2 - 5\beta_3.$$

The conclusion thus is that in the continuant with which we started we can retain any three of the β's, and express in terms of these the fourth β, all the γ's, and p, q, r, s,—thus obtaining a function of *four* variables which is resolvable into linear factors.

(8) Had the determinant operated upon been of the sixth order, we should still have found $\sigma = -2\beta_1 + 9\beta_2 - 5\beta_3$ and the first four β's connected by the same equation as in the preceding case, but there would have been a fresh equation of condition connecting the second set of four consecutive β's, viz.

$$\beta_2 - 2\beta_3 + \beta_4 = 3(\beta_3 - 2\beta_4 + \beta_5).$$

Similarly the case of the seventh order would be found to differ from that of the sixth merely in having the additional equation

$$5(\beta_3 - 2\beta_4 + \beta_5) = 11(\beta_4 - 2\beta_5 + \beta_6);$$

and so on.

As the result of all this we therefore affirm that — *If the continuant*

$$\begin{vmatrix} a & 2(n-1)\beta_1 & . & . & \ldots \\ n\gamma_1 & a+p & (n-2)\beta_2 & . & \ldots \\ . & (n+1)\gamma_2 & a+q & (n-3)\beta_3 & \ldots \\ . & . & (n+2)\gamma_3 & a+r & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}_n$$

be resolvable into linear factors by means of the set of multipliers

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \ldots \\ & 1 & 4 & 9 & 16 \ldots \\ & & 1 & 6 & 20 \ldots \\ & & & 1 & 8 \ldots \\ & & & & 1 \ldots \\ & & & & \ldots \end{array}$$

then (1) *every four consecutive* β*'s are connected by a linear relation,* viz.

$$\begin{aligned} 1\cdot(\beta_1 - 2\beta_2 + \beta_3) &= 7\cdot(\beta_2 - 2\beta_3 + \beta_4), \\ 3\cdot(\beta_2 - 2\beta_3 + \beta_4) &= 9\cdot(\beta_3 - 2\beta_4 + \beta_5), \\ 5\cdot(\beta_3 - 2\beta_4 + \beta_5) &= 11\cdot(\beta_4 - 2\beta_5 + \beta_6), \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

thus making all the β*'s expressible in terms of any three;* (2) *all the* γ*'s are expressible in terms of the same three* β*'s because of the fact that* $\beta_m + \gamma_m = -2\beta_1 + 9\beta_2 - 5\beta_3$ *for all values of* m; *and* (3) p, q, r, . . . *are also so expressible because the sum of the elements of any row of the continuant is constant.* . . . . (VI)

(9) Instead, however, of taking a and three of the β's as variables it is better to take α and β_1, $\beta_1-\beta_2$, $\beta_1-2\beta_2+\beta_3$. Doing this and calling the last three quantities b, c, d,—a change implying the substitution

$$\begin{aligned}\beta_1 &= b,\\ \beta_2 &= b-c,\\ \beta_3 &= b-2c+d,\end{aligned}$$

—we can by using the equations of condition obtain the requisite expressions for σ, β_4, β_5, . . . , γ_1, γ_2, ... in terms of b, c, d. The theorem to which this course ultimately leads is—*The continuant*

$$\begin{vmatrix} A_1 & 2(n-1)\beta_1 & . & . & \cdots\cdots\\ n\gamma_1 & A_2 & (n-2)\beta_2 & . & \cdots\cdots\\ . & (n+1)\gamma_2 & A_3 & (n-3)\beta_3 & \cdots\cdots\\ . & . & (n+2)\gamma_3 & A_4 & \cdots\cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots \end{vmatrix}_n$$

is resolvable into linear factors if

$$\beta_m = b-(m-1)c+\frac{(m-1)(m-2)}{2(2m-1)}\cdot 5d,$$

$$\gamma_m = b+mc-\frac{(m+1)m}{2(2m-1)}\ 5d,$$

$$A_m = a-2(m-1)^2c+\frac{(m-1)^2}{(2m-3)(2m-1)}\{2m(m-2)+3n\}\cdot 5d,$$

the s^{th} *factor being*

$$a+2(n-2s+1)b-2(s-1)(2n-2s+1)c+(n-s+2)(s-1)\cdot 5d \quad . \quad . \quad . \quad \text{(VII)}$$

For example, when $n=4$ we have

$$\begin{vmatrix} a & 2\cdot 3b & . & .\\ 4(b+c-5d) & a-2c+20d & 2(b-c) & .\\ . & 5(b+2c-5d) & a-8c+24d & b-2c+d\\ . & . & 6(b+3c-6d) & a-18c+36d \end{vmatrix}$$

$$=(a+6b)(a+2b-10c+20d)(a-2b-12c+30d)(a-6b-6c+30d).$$

On putting $d=0$ the β's and γ's form equidifferent progressions, and the theorem degenerates into that of § 2.

(10) Out of this effort to obtain greater generality an unexpected and curious result arises; for, whereas at first sight both members of the identity are functions of the four variables a, b, c, d, it is found on careful examination that the right-hand side is expressible as a function of one less. In fact it is readily verified that the s^{th} factor given above can also be written in the form

$$\{a+2(n-1)b\}\ -\ 2(s-1)\{2b-3c\}\ -\ (n-s+2)(s-1)\{4c-5d\}$$

so that the factorial expression for the continuant of the n^{th} order, besides being

$$\begin{aligned}\{a+2(n-1)b\}&\cdot\{a+2(n-3)b-2(2n-3)c+1\cdot n\cdot 5d\}\\ &\cdot\{a+2(n-5)b-4(2n-5)c+2\cdot(n-1)\cdot 5d\}\\ &\cdot\{a+2(n-7)b-6(2n-7)c+3\cdot(n-2)\cdot 5d\}\\ &\quad\cdots\cdots\cdots\cdots\\ &\cdot\{a-2(n-1)b-(2n-2)\cdot 1\cdot c+(n-1)\cdot 2\cdot 5d\},\end{aligned}$$

is also

$$\begin{aligned} &X\cdot\{X \quad 2Y-1\cdot n\cdot Z\}\\ &\quad\cdot\{X-4Y-2\cdot(n-1)\cdot Z\}\\ &\quad\cdot\{X-6Y-3\cdot(n-2)Z\}\\ &\qquad\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ &\quad\cdot\{X-(2n-2)Y-(n-1)\cdot 2Z\} \end{aligned}$$

where

$$X=a+2(n-1)b, \quad Y=2b-3c, \quad Z=4c-5d.$$

And as a, b, c, d cannot conversely be expressed in terms of X, Y, Z alone, the left-hand member of the identity, that is, the continuant, can only be made to appear as a function of X, Y, Z and one of the four a, b, c, d. Consequently, supposing this to be done, and thereafter all terms involving X, Y, Z deleted, we shall obtain a continuant which not only vanishes but which can be viewed as having n vanishing factors.

(11) To obtain this nil-factor continuant there is, however, a better method. For, as it is the special case where X, Y, Z vanish, it must be obtainable by putting

$$\begin{aligned} 4c &= 5d,\\ 2b &= 3c=\frac{15}{4}d,\\ a &= -2(n-1)b=-(n-1)\frac{15}{4}d, \end{aligned}$$

or, therefore, by putting

$$\begin{aligned} d &= \frac{8}{15}e,\\ c &= \frac{2}{3}e,\\ b &= e,\\ a &= -(n-1)2e. \end{aligned}$$

Doing this we find from § 9

$$\begin{aligned} \beta_m &= \frac{1}{2m-1}e,\\ \gamma_m &= -\frac{1}{2m-1}e,\\ A_m &= \frac{2\{n+2(m-1)^2-1\}}{(2m-3)(2m-1)}; \end{aligned}$$

and have the following theorem:—*The value of any continuant of the form spoken of in* § 9 *is not altered by adding to its matrix the matrix of the continuant*

$$\begin{vmatrix} -\frac{2}{1}(n-1)e & 2\cdot(n-1)e & . & . & \ldots\ldots\\ -ne & \frac{2}{1\cdot 3}(n+1)e & (n-2)\frac{1}{3}e & . & \ldots\ldots\\ . & -(n+1)\frac{1}{3}e & \frac{2}{3\cdot 5}(n+7)e & (n-3)\frac{1}{5}e & \ldots\ldots\\ . & . & -(n+2)\frac{1}{5}e & \frac{2}{5\cdot 7}(n+17)e & \ldots\ldots\\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix} \quad . \qquad \text{(VIII)}$$

(12) There is a special case of the theorem (VII) in § 9 which deserves particular attention, viz. the case where σ vanishes, *i.e.* where $5d = 2b + c$. In this case the s^{th} factor

$$= \{a + 2(n-1)b\} - 2(s-1)\{2b - 3c\} + (n-s+2)(s-1)\{2b - 3c\}$$
$$= \{a + 2(n-1)b\} + (n-s)(s-1)(2b - 3c);$$

and the $(n-s+1)^{\text{th}}$ factor

$$= \{a + 2(n-1)b\} - 2(n-s)\{2b - 3c\} + (s+1)(n-s)\{2b - 3c\}$$
$$= \{a + 2(n-1)b\} + (n-s)(s-1)(2b - 3c).$$

This means that *when* $5d = 2b + c$ *in the continuant of* § 9 *the* s^{th} *factor from the beginning is the same as the* s^{th} *factor from the end, and consequently that an even-ordered continuant of this kind is a square.* . . . . . (IX)

(13) The question of the generalisation of the theorem of § 3 may be investigated in a manner perfectly similar to that followed in the preceding paragraphs with regard to the theorem of § 2. The essential point of difference is to be found in the new set of column-multipliers, which are now all of the form $C_{r+2s,\,r-1}$ instead of ${}_{s}^{r}C_{r+s-1,\,2s-1}$. It will suffice merely to enunciate the results. The first is—

If the continuant

$$\begin{vmatrix} a & (n-1)\beta_1 & . & . & \ldots \\ (n+2)\gamma_1 & a+p & (n-2)\beta_2 & . & \ldots \\ . & (n+3)\gamma_2 & a+q & (n-3)\beta_3 & \ldots \\ . & . & (n+4)\gamma_4 & a+r & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

be resolvable into linear factors by means of the set of column-multipliers

$$\begin{array}{rrrrrr} 1, & 2, & 3, & 4, & 5, & \ldots\ldots\ C_{r,1} \\ 1, & 4, & 10, & 20, & \ldots\ldots\ & C_{r+2,3} \\ 1, & 6, & 21, & \ldots\ldots\ & & C_{r+4,5} \\ 1, & 8, & \ldots\ldots\ & & & C_{r+6,7} \\ 1, & \ldots\ldots\ & & & & C_{r+8,9} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \end{array}$$

then (1) *every four consecutive* β*'s are connected by a linear relation, viz.*

$$3(\beta_1 - 2\beta_2 + \beta_3) = 9(\beta_2 - 2\beta_3 + \beta_4),$$
$$5(\beta_2 - 2\beta_3 + \beta_4) = 11(\beta_3 - 2\beta_4 + \beta_5),$$
$$\ldots\ldots\ldots\ldots$$

thus making all the β*'s expressible in terms of any three*; (2) *all the* γ*'s are expressible in terms of the same three* β*'s because of the fact that for all values of* m $\beta_m + \gamma_m = -9\beta_1 + 25\beta_2 - 14\beta_3$; *and* (3) p, q, r, . . . *are also so expressible because of the mode of removing the first factor from the continuant.* . . . . . (X)

It should be noted that the linear relation connecting the first four consecutive β's is that which in § 8 connects the second four,—that, in fact, the relation here is

$$(2r+1)\{\beta_r - 2\beta_{r+1} + \beta_{r+2}\} = (2r+7)\{\beta_{r+1} - 2\beta_{r+2} + \beta_{r+3}\}$$

whereas in § 8 it is

$$(2r-1)\{\beta r - 2\beta_{r+1} + \beta_{r+2}\} = (2r+5)\{\beta_{r+1} - 2\beta_{r+2} + \beta_{r+3}\}.$$

Further, there is a similar difference in the expressions for σ: here the expression is

$$-9\beta_1+25\beta_2-14\beta_3,$$

whereas in § 8 when given in terms of β_2, β_3, β_4 it is

$$-9\beta_2+25\beta_3-14\beta_4.$$

The second theorem is—*The continuant*

$$\begin{vmatrix} A_1 & (n-1)\beta_1 & . & . & \ldots \\ (n+2)\gamma_1 & A_2 & (n-2)\beta_2 & . & \ldots \\ . & (n+3)\gamma_2 & A_3 & (n-3)\beta_3 & \ldots \\ . & . & (n+4)\gamma_3 & A_4 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}_n$$

is resolvable into linear factors if

$$\beta_m = b-(m-1)c + \frac{(m-1)(m-2)}{2(2m+1)}\cdot 7d,$$

$$\gamma_m = b+(m+2)c - \frac{(m+3)(m+2)}{2(2m+1)}\cdot 7d,$$

$$A_m = a-2(m^2-1)c + \frac{m^2-1}{4m^2-1}(2m^2+2+5n)\cdot 7d,$$

the s^{th} *factor being*

$$a+2(n-2s+1)b-2(s-1)(2n-2s+3)c + (n-s+4)(s-1)\cdot 7d,$$

or

$$\{a+2(n-1)b\}-2(s-1)\{2b-5c\}-(n-s+4)(s-1)\{4c-7d\}. \quad . \quad . \quad . \quad (XI)$$

An immediate deduction from this is that *when* $14d=2b+3c$ *the* s^{th} *factor is*

$$a+2(n-1)b+\tfrac{1}{2}(n-s)(s-1)(2b-5c)$$

and is the same as the s^{th} *factor from the end, so that when in addition* n *is even the continuant is a square.* . . . . . (XII)

The third theorem is—*The value of any continuant of the form referred to in* § 3 *is not altered by adding to its matrix the matrix of the continuant*

$$\begin{vmatrix} -\frac{2}{1\cdot 3}(n-1)e & \frac{1}{3}(n-1)e & . & . & \ldots \\ -\frac{1}{3}(n+2)e & -\frac{2}{3\cdot 5}(n-7)e & \frac{1}{5}(n-2)e & . & \ldots \\ . & -\frac{1}{5}(n+3)e & -\frac{2}{5\cdot 7}(n-17)e & \frac{1}{7}(n-3)e & \ldots \\ . & . & -\frac{1}{7}(n+4)e & -\frac{2}{7\cdot 9}(n-31)e & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} \quad . \quad (XIII)$$

(14) The corresponding theorems for the set of column-multipliers

$$\begin{aligned} 1,1,1,1,1,\ldots\ldots, & C_{r-1,0} \\ 1,2,3,4,\ldots\ldots, & C_{r,1} \\ 1,3,6,\ldots\ldots, & C_{r+1,2} \\ 1,4,\ldots\ldots, & C_{r+2,3} \\ 1,\ldots\ldots, & C_{r+3,4} \\ \ldots\ldots & \end{aligned}$$

will be found in the *Trans. S. Afr. Philos. Soc.* referred to above.

When the first theorem there given is attempted to be generalised in the manner employed in the present paper the following is the result :—

The continuant of the nth *order whose main diagonal is*

$$a,\quad a+(b-c)-(n-2)\gamma,\quad a+2(b-c)-2(n-4)\gamma,\quad a+3(b-c)-3(n-b)\gamma,\ \ldots\ .$$

and whose minor diagonals are

$$\begin{array}{lll} (n-1)b, & (n-2)(b+\gamma), & (n-3)(b+2\gamma),\ \ldots\ . \\ c, & 2(c-\gamma), & 3(c-2\gamma),\ \ldots\ . \end{array}$$

is equal to the product of the n *factors*

$$\begin{array}{l} \{a+(n-1)b\} \\ \cdot\{a+(n-2)b-c+1\cdot(n-2)\gamma\} \\ \cdot\{a+(n-3)b-2c+2\cdot(n-3)\gamma\} \\ \quad\ldots\ldots\ldots\ldots \\ \cdot\{a\qquad -(n-1)c\} \end{array} \qquad \ldots\ldots\ldots\ldots\ (\text{XIV})$$

It is seen to degenerate into the original theorem when γ is put equal to 0. If, however, we write X in it for the half-sum of the first and last factors, and Y for $b+c$, the factors may be written

$$\begin{array}{l} \mathrm{X}+\frac{1}{2}(n-1)\mathrm{Y}, \\ \mathrm{X}+\frac{1}{2}(n-3)\mathrm{Y}+1\cdot(n-2)\gamma, \\ \mathrm{X}+\frac{1}{2}(n-5)\mathrm{Y}+2\cdot(n-3)\gamma, \\ \quad\ldots\ldots\ldots\ldots \\ \mathrm{X}-\frac{1}{2}(n-1)\mathrm{Y}\ ; \end{array}$$

thus showing that four variables are not necessary for the expression of the identity. An easier way of reaching the same result is to put

$$\begin{array}{l} a=\alpha-(n-1)\xi, \\ b=\beta+\xi, \\ c=\beta-\xi, \end{array}$$

when it will be found that ξ appears in the continuant but not in its factors; and when there are consequently obtained at one and the same time the case of theorem (XIV) where $b=c$, and an expression for the corresponding nil-factor continuant.

(15) We have thus in all at present three sets of column-multipliers, each of which has associated with it a linearly resolvable continuant of the form

$$\left|\begin{array}{cccc} a & \beta_1 & . & . \\ \gamma_1 & a+p & \beta_2 & . \\ . & \gamma_2 & a+q & \beta_3 \\ . & . & \gamma_3 & a+r \\ . & . & . & . \end{array}\right|$$

Other sets will doubtless be discovered, as the only difficulty is the devising of a set which will not lead to unreasonably complicated expressions for the elements of the continuant. *In all cases if*

$$\xi_1,\ \xi_2,\ \xi_3,\ \ldots\ldots,\ \xi_{n-1}$$

be the diagonal adjacent to the diagonal of units in the set,—that is to say, if the set be

$$\begin{matrix} 1\,,\ \xi_1\,,\ \ldots\ldots\ldots \\ 1\,,\ \xi_2\,,\ \ldots\ldots \\ \ldots\ldots \end{matrix}$$

—and if we denote the elements of the resolvable continuant by their place-names $(1,\,1)$, $(1,\,2)$, . . . , *then the factors of the continuant are*

$$\begin{matrix} \{(1\,,1) & +\xi_1\cdot(1\,,2)\} \\ \cdot\{(2\,,2)-\xi_1\cdot(1\,,2)+\xi_2\cdot(2\,,3)\} \\ \cdot\{(3\,,3)-\xi_2\cdot(2\,,3)+\xi_3\cdot(3\,,4)\} \\ \ldots\ldots\ldots\ldots \\ \{(n\,,n)-\xi_{n-1}\cdot(n-1\,,n)\}\,. \end{matrix} \qquad \text{(XV)}$$

A scrutiny of the procedure connected with the removal of any factor makes this evident. For, firstly, when $s-1$ factors have been removed, the residual determinant has for its first column the line of column-multipliers last used, viz.

$$1\,,\ \xi_{s-1}\,,\ \ldots\ldots\,;$$

secondly, this determinant when reduced to the next lowest order has

$$(s\,,s)-\xi_{s-1}\cdot(s-1\,,s)$$

for the first element of its diagonal; thirdly, the employment of the next line of column-multipliers, viz.

$$1\,,\ \xi_s\,,\ \ldots\ldots\,,$$

changes the said element into

$$(s\,,s)-\xi_{s-1}\cdot(s-1\,,s)+\xi_s(s\,,s+1)\,;$$

and this, in virtue of the character of the process, is the next factor ready for removal.

It may be noted in corroboration of § 6 that the sum of the factors thus expressed is

$$(1\,,1)+(2\,,2)+(3\,,3)+\ \ldots\ +(n\,,n)\,.$$

(16) Observing from the foregoing that

$$(n\,,n)-\xi_{n-1}(n-1\,,n)$$

is the last factor, we have suggested to ourselves the obtaining of the factors in the reverse order by the use of a set of *row*-multipliers, the first operation being

$$\mathrm{row}_n\ -\ \xi_{n-1}\cdot\mathrm{row}_{n-1}+\ \ldots\,.$$

An interesting result is thus reached, viz., that corresponding to each set of column-multipliers for the resolution of a continuant there is an equally effective set of row-multipliers.

Thus returning to the continuant of § 2 and performing the operation

$$\mathrm{row}_5\ -\ 8\,\mathrm{row}_4\ +\ 28\,\mathrm{row}_3\ -\ 56\,\mathrm{row}_2\ +\ 35\,\mathrm{row}_1$$

we find we can remove the factor $a-8b-8c$, and write its cofactor in the form

$$\begin{vmatrix} a & 2\cdot4b & . & . & . \\ 5(b+c) & a-2c & 3(b-c) & . & . \\ . & 6(b+2c) & a-8c & 2(b-2c) & . \\ . & . & 7(b+3c) & a-18c & 1(b-3c) \\ 35 & -56 & 28 & -8 & 1 \end{vmatrix},$$

and thence in the form

$$\begin{vmatrix} a & 2\cdot 4b & . & . \\ 5(b+c) & a-2c & 3(b-c) & . \\ . & 6(b+2c) & a-8c & 2(b-2c) \\ -35(b-3c) & 56(b-3c) & -21(b-5c) & a+8b-42c \end{vmatrix}.$$

Similarly the operation

$$\text{row}_4 - 6\,\text{row}_3 + 15\,\text{row}_2 - 10\,\text{row}_1$$

now enables us to remove the factor $a-4b-18c$; and the operations

$$\text{row}_3 - 4\,\text{row}_2 + 3\,\text{row}_1,$$
$$\text{row}_2 - 1\,\text{row}_1$$

the remaining factors. The set of column-multipliers

$$\begin{array}{ccccc} 1, & 1, & 1, & 1, & 1 \\ & 1, & 4, & 9, & 16 \\ & & 1, & 6, & 20 \\ & & & 1, & 8 \\ & & & & 1 \end{array}$$

is thus equivalent to the set of row-multipliers

$$\begin{array}{ccccc} 1, & -8, & 28, & -56, & 35 \\ & 1, & -6, & 15, & -10 \\ & & 1, & -4, & 3 \\ & & & 1, & -1 \\ & & & & 1. \end{array}$$

(17) The general result is that *the table of row-multipliers suitable for the resolution of the continuant of* § 2 *is*

$$\left.\begin{array}{ccccccc} 1, & -C_{2n-2,1}, & C_{2n-2,2}, & -C_{2n-2,3}, & \ldots\ldots, & (-)^{n+1}\tfrac{1}{2}C_{2n-2,n-1} \\ & 1, & -C_{2n-4,1}, & C_{2n-4,2}, & \ldots\ldots\ldots\ldots & \\ & & \ldots\ldots\ldots\ldots\ldots\ldots & & & \end{array}\right\} \quad . \qquad \text{(XVI)}$$

Similarly it is found after a little investigation that *the table of row-multipliers suitable for the resolution of the continuant of* § 3 *is*

$$\left.\begin{array}{ccccc} 1, & -\frac{2}{1}(n-1)C_{2n-1,0}, & \frac{2}{2}(n-2)C_{2n-1,1}, & -\frac{2}{3}(n-3)C_{2n-1,2}, & \ldots\ldots \\ & 1, & -\frac{2}{1}(n-2)C_{2n-3,0}, & \frac{2}{2}(n-3)C_{2n-3,1}, & \ldots\ldots \\ & & \ldots\ldots\ldots\ldots\ldots\ldots & & \end{array}\right\} \quad . \qquad \text{(XVII)}$$

the general form of the multiplier being $2^{\frac{r-s}{s+1}}C_{2r+1,s}$.

Lastly, *the table of row-multipliers suitable for the resolution of the continuant of* § 13 *is*

$$\left.\begin{array}{ccccc} 1, & -C_{n-1,1}, & C_{n-1,2}, & -C_{n-1,3}, & \ldots\ldots \\ & 1, & -C_{n-2,1}, & C_{n-2,2}, & \ldots\ldots \\ & & \ldots\ldots\ldots\ldots & & \end{array}\right\} \quad . \quad . \qquad \text{(XVIII)}$$

—that is to say, may be got by a rearrangement of the column-multipliers: for

example, in the case of the 5th order the equivalent tables of column-multipliers and row-multipliers are

$$\begin{array}{ccccc} 1, & 1, & 1, & 1, & 1 \\ & 1, & 2, & 3, & 4 \\ & & 1, & 3, & 6 \\ & & & 1, & 4 \\ & & & & 1, \end{array} \qquad \begin{array}{ccccc} 1, & -4, & 6, & -4, & 1 \\ & 1, & -3, & 3, & -1 \\ & & 1, & -2, & 1 \\ & & & 1, & -1 \\ & & & & 1. \end{array}$$

(18) There falls now to be noted a set of theorems regarding resolvable continuants of a totally different form but connected with and derivable from those of §§ 2, 3, 13. If in any one of these latter theorems we put 0 for the element in the place (1, 1), the continuant is expressible as the negative product of the elements in the places (1, 2), (2, 1), and a continuant of the lower order $n-2$: further, one of the said elements is contained in the first factor of the original continuant and the other in the last factor: in this way, therefore, the resolution of the new continuant of order $n-2$ is secured. Thus, taking the five-line continuant dealt with in § 2 and putting $a=0$ we obtain

$$-2\cdot4\cdot5b(b-c)\begin{vmatrix} 8c & 2(b+2c) & . \\ 7(b-3c) & 18c & 1\cdot(b+3c) \\ . & 8(b-4c) & 32c \end{vmatrix}$$
$$= 8b(4b+14c)(20c)(-4b+18c)(-8b+8c),$$

and therefore

$$\begin{vmatrix} 8c & 2(b+2c) & . \\ 7(b-3c) & 18c & 1\cdot(b+3c) \\ . & 8(b-4c) & 32c \end{vmatrix} = \frac{8}{5}(4b+14c)\cdot 20c\cdot(-4b+18c).$$

The general theorems thus obtained are

$$\left.\begin{array}{llll} \mathrm{A}_1 & (n-1)\beta_1 & . & \ldots\ldots \\ (n+4)\gamma_1 & \mathrm{A}_2 & (n-2)\beta_2 & \ldots\ldots \\ . & (n+5)\gamma_2 & \mathrm{A}_3 & \ldots\ldots \\ \ldots\ldots & & & \end{array}\right| = 2\cdot\frac{n-1}{n+2}\cdot\left\{\begin{array}{l} 2(n-1)b-2(2n+1)c+1\cdot(n+2)\cdot 5d \\ \{2(n-3)b-4(2n-1)c+2\cdot(n+1)\cdot 5d\} \\ \{2(n-4)b-6(2n-3)c+3\cdot n\cdot 5d\} \\ \ldots\ldots\ldots\ldots \end{array}\right\} \quad \text{(XIX)}$$

if $$\beta_m=b-(m+1)c+\frac{m(m+1)}{2(2m+3)}5d,\quad \gamma_m=b+(m+2)c-\frac{(m+2)(m+3)}{2(2m+3)}5d,$$

and $$\mathrm{A}_m=(m+1)^2\left\{-2c+\frac{2m(m+2)+6+3n}{(2m+1)(2m+3)}5d\right\}.$$

$$\left.\begin{array}{llll} \mathrm{A}_1 & (n-1)\beta_1 & . & \ldots\ldots \\ (n+6)\gamma & \mathrm{A}_2 & (n-2)\beta_2 & \ldots\ldots \\ . & (n+7)\gamma_2 & \mathrm{A}_3 & \ldots\ldots \\ \ldots\ldots & & & \end{array}\right|_n = 4\cdot\frac{n+1}{n+4}\cdot\left\{\begin{array}{l} 2(n-1)b-2(2n+3)c+1\cdot(n+4)\cdot 7d \\ \{2(n-3)b-4(2n+1)c+2\cdot(n+3)\cdot 7d\} \\ \{2(n-5)b-6(2n-1)c+3\cdot(n+2)\cdot 7d\} \\ \ldots\ldots\ldots\ldots \end{array}\right\} \quad \text{(XX)}$$

if $$\beta_m=b-(m+1)c+\frac{m(m+1)}{2(2m+5)}7d,\quad \gamma_m=b+(m+4)c-\frac{(m+4)(m+5)}{2(2m+5)}7d,$$

and $$\mathrm{A}_m=(m+1)(m+3)\left\{-2c+\frac{2m^2+8m+10+5n}{(2m+3)(2m+5)}7d\right\}.$$

$$\begin{vmatrix} 2(b-c-n\gamma+2\gamma) & (n-1)(b+2\gamma) & . & \ldots\ldots \\ 3(c-2z) & (3b-c-n\gamma+4\gamma) & (n-2)(b+3\gamma) & \ldots\ldots \\ . & 4(c-3\gamma) & 4(b-c-n\gamma+6\gamma) & \ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & \end{vmatrix}_n \quad \text{(XXI)}$$

$$= (n+1)\cdot\{n(\beta+\gamma)-c\}\{(n-1)(b+2\gamma)-2c\}\{(n-2)(\beta+3\gamma)-3c\}\ldots\ldots$$

Of these three results it would be interesting to obtain independent proofs.

(19) In a paper of HEINE's on Lamé's functions (*Crelle's Journ.*, lvi. pp. 77–99) there occurs the continued fraction

$$z - c_0 - \frac{c_0c_1}{z-c_1-c_2} - \frac{c_2c_3}{z-c_3-c_4} - \ldots$$

where the c's are m in number with the values

$$\tfrac{1}{2}m(m+1),\ \tfrac{1}{4}(m-1)(m+2),\ \tfrac{1}{4}(m-2)(m+3),\ \ldots\ldots,\ \tfrac{1}{4}\cdot 1\cdot 2m,$$

the fractional factor being $\frac{1}{4}$ in every case except the first. From extraneous considerations the value of the continued fraction was given

$$\frac{z(z-2^2)(z-4^2)\ \ldots\ldots\ (z-m^2)}{(z-1^2)(z-3^2)\ \ldots\ldots\ (z-\overline{m-1}^2)} \quad \text{for } m \text{ even},$$

and

$$\frac{(z-1^2)(z-3^2)\ \ldots\ldots\ (z-m^2)}{(z-2^2)\ \ldots\ldots\ (z-\overline{m-1}^2)} \quad \text{for } m \text{ odd};$$

but the author added, "Einen directen Beweis für diese Summirung des Kettenbruches habe ich noch nicht aufgefunden." This, however, is readily obtained by writing the value of the continued fraction as the quotient of two determinants, viz.

$$\frac{\begin{vmatrix} z-c_0 & c_0 & . & \ldots\ldots \\ c_1 & z-c_1-c_2 & c_2 & \ldots\ldots \\ . & c_3 & z-c_3-c_4 & \ldots\ldots \\ \ldots\ldots\ldots\ldots & & & \end{vmatrix}}{\begin{vmatrix} z-c_1-c_2 & c_2 & \ldots\ldots \\ c_3 & z-c_3-c_4 & \ldots\ldots \\ \ldots\ldots\ldots\ldots & & \end{vmatrix}},$$

and then using two of the foregoing theorems. Thus, taking the case where $m=6$, and where therefore the c's are

$$21,\ 10,\ 9,\ 7\tfrac{1}{2},\ 5\tfrac{1}{2},\ 3,$$

we have to evaluate the quotient

$$\begin{vmatrix} z-21 & 21 & . & . \\ 10 & z-19 & 9 & . \\ . & 7\frac{1}{2} & z-13 & 5\frac{1}{2} \\ . & . & 3 & z-3 \end{vmatrix} \div \begin{vmatrix} z-19 & 9 & . \\ 7\frac{1}{2} & z-13 & 5\frac{1}{2} \\ . & 3 & z-3 \end{vmatrix},$$

the dividend of which, changed into

$$\begin{vmatrix} z-21 & 6\cdot 3\frac{1}{2} & . & . \\ 4(3\frac{1}{2}-1) & z-21+2\cdot 1^2 & 2\cdot(3\frac{1}{2}+1) & . \\ . & 5(3\frac{1}{2}-2) & z-21+2\cdot 2^2 & 1\cdot(3\frac{1}{2}+2) \\ . & . & 6(3\frac{1}{2}-3) & z-21+2\cdot 3^2 \end{vmatrix},$$

is seen to be one of the simplest cases of the determinant of § 2, and thus to have for its value

$$z(z-4)(z-16)(z-36).$$

In like manner the divisor may be written

$$\begin{vmatrix} z-19 & 2\cdot 4\frac{1}{2} & . \\ 3(4\frac{1}{2}-2) & z-19+6 & 1(4\frac{1}{2}-1) \\ . & 4(4\frac{1}{2}-3) & z-19+16 \end{vmatrix}$$

and is then recognised to be a very special case of the continuant of § 3, and therefore to be equal to

$$(z-1)(z-9)(z-25).$$

The continued fraction in question is consequently equal to

$$\frac{z(z-4)(z-16)(z-36)}{(z-1)(z-9)(z-25)}.$$

XIV.—The Igneous Geology of the Bathgate and Linlithgow Hills. By J. D. Falconer, M.A., B.Sc. *Communicated by* Professor James Geikie, LL.D., D.C.L., F.R.S. (With a Map.)

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Introduction.

The Bathgate and Linlithgow Hills occupy a well-defined belt of rising ground stretching S.S.W.–N.N.E. from Bathgate to Bo'ness, and included in the marginal portions of sheets 31 and 32 of the 1-in. maps of the Ordnance and Geological Surveys. Throughout the range the steeper slopes face the west, while the eastern flanks are deeply buried in drift. The glaciated contours are well seen from the east, and the sky-line is in several places deeply indented by glacial grooves. The physical geography is throughout intimately dependent upon the geological structure, but the latter is simplicity itself when contrasted with the complicated structure of the shalefields to the east. Alternating zones of volcanic and sedimentary rocks strike parallel with the direction of elongation of the range from Bathgate to Bonnytoun Hill, and these are cut by a later connected series of dykes and sills of intrusive igneous rock. The latter, as a rule, are more resistant than the lavas, and form the more prominent features of the landscape, while the sedimentary intercalations may frequently be traced, even where no rock is visible, by the trough-like depressions which have been hollowed out of them between the zones of lava.

As convenient boundary-lines for the study of these hills, I have chosen as lower limit the final western outcrop of the Houston Coal from Drumcross to Champfleurie, and thence a line drawn directly north to the shore of the Firth at Stacks, and as upper limit the outcrop of the Castlecary Limestone. The accompanying map represents on a reduced scale the 6-in. sheets ix. N.E., v. S.E., v. N.E, i. S.E., i. N.E. of Linlithgowshire, with the eastern halves of ix. N.W., v. S.W., v. N.W., i. S.W., i. N.W. of the same county. I have thought it unnecessary to continue the map to the south of the Bathgate Railway, beyond which the volcanic series rapidly runs out, while the country is so much covered with drift that little rock is visible. The geology below the Houston Coal and above the Castlecary Limestone is simply sketched in from the Survey sheets, and has not been personally verified in detail.

The Houston Coal.

The outcrop of the Houston Coal, crossed by numerous dip faults, has been well proved from Deans to Drumcrosshall. Thence it strikes north by west to the neighbourhood of Blackcraig and West Binny, having been worked many years ago at both of these places. In the vicinity of Ochiltree Mill the outcrop is uncertain, the ground being much faulted and pierced by many intrusions. North of Peace Knowe, however, it reappears and strikes north by west to the Haugh Burn fault, by which it is shifted to the east beyond the limits of the present map.

Between the Houston Coal and the first volcanic zone come the Houston shales and marls, with thin sandstones and occasional beds of ash and agglomerate. Good sections are found in the Mains Burn and the Brox Burn and its tributaries.

The First Volcanic Zone, or *The Brox Burn Ash.*

This zone can be traced, with interruptions, from Drumcrosshall to the Haugh Burn. It consists throughout of stratified volcanic ash, varying much in texture and usually green in colour, but weathering yellow or brown at Chapelhill and Bankhead. Interbedded lava is nowhere found in this zone, although in places the compact ash weathers spheroidally, and presents a deceptive resemblance to a decomposing crystalline igneous rock. Characteristic sections are found in the Brox Burn and its tributary to the south. On account of the absence of exposures, this zone cannot be traced to the east of the Longmuir plantation. It reappears, however, on the same horizon on the Riccarton road, immediately to the north.

Between this ash and the second volcanic zone there appear some thinly bedded sandstones and shales, frequently ashy themselves, and interstratified with thin bands of ash. These are well seen in the Brox Burn and in a streamlet in the northern angle of the Longmuir plantation.

The Second Volcanic Zone, or *The Longmuir and Riccarton Lavas.*

This zone is taken to include two apparently distinct groups of lavas. The lower or Longmuir group extends from a point a little to the east of Broomyknowes, through the Longmuir and Balditop plantations, to Drumcross. The upper or Riccarton group is of greater thickness but of less extent, stretching only from the Riccarton Burn to the Rigghead plantation, a little to the south of Tartraven. The two groups are separated by a series of sandstones, shales, and thin ashbeds, well seen in the streamlets and quarries on the northern slope of the Riccarton Hills. Petrographically, the Longmuir lavas are throughout fine-grained olivine-basalts, while the Riccarton lavas can be separated into a lower zone of coarse-grained olivine-dolerites* extending the full length of the group, and an upper zone of olivine-basalts stretching from the Riccarton Burn to North Mains.

This subdivision of the zone is everywhere apparent to the north of the Mains Burn, but to the south so little rock is exposed that the lines are to some extent hypothetical. The sedimentary intercalation between the two groups of lavas cannot be traced by means of actual exposures of sandstones and shales. In consequence, more than usual reliance must be placed upon the prolongation to the south of the petrographical variations established above. The coarse-grained olivine-dolerites of the lower zone of the Riccarton lavas are nowhere found to the south of the Rigghead plantation, beyond which the exposures are all of olivine-basalts similar to the Longmuir lavas to the north. The sedimentary band, apparently reduced in thickness, is therefore drawn, as nearly as possible, between the basalts and the dolerites. Marginal sedimentary intercalations are probably numerous throughout this zone. Two such are shown on the map, one of shales and sandstones found in drains on the farms of Drumcross and Quarter, the other of sandstones, shales, fine-grained green ash, and agglomerate, exposed in the Brox Burn at the Balditop plantation. The sections exposed during the construction of the Bangour reservoir indicated considerable disturbance of the strata at that point. Another thin bed of sandstone, interstratified with the lavas, may be seen immediately to the east of the neck in the Riccarton Hills.

This second volcanic zone cannot be definitely traced to the south of the Galabraes fault, although the section at Starlaw indicates the occurrence of scattered lava-flows. North of Broomyknowes also the lavas rapidly run out, and their place is taken by interstratifications of sandstone, shales, and thin ashbeds, with at least one thin band of shelly limestone exposed in the burn below Riccarton Mill.

Between the second and third volcanic zones there comes another sedimentary intercalation, represented by shales and thin ashbeds at Craigs, limestone and shale at Tartraven, shales and ash in the Riccarton Burn west of Beecraigs, sandstone, shale, and limestone at Whitebaulks, sandstone at Hillhouse, sandstone and limestone at

* The terms "basalt" and "dolerite" are used throughout to denote macroscopic distinctions, "basalt" implying a fine-grained, compact, and usually porphyritic rock, "dolerite" a coarse-grained rock, not evidently porphyritic.

Carsie Hill, east of Cauldhame, and at Peat Hill, on the north side of the Haugh Burn fault.

The Third Volcanic Zone, or *The Kirkton and Hillhouse Lavas.*

This zone includes the Kirkton, Tartraven, and Hillhouse lavas. A very characteristic ash, with black matrix and yellow lapilli, lies in several places at the base of this zone. It is well seen at Whitelaw, Craigs, Whitebaulks, and Hillhouse. The lavas of this zone can be studied with ease in numerous exposures from Kirkton Mains and Boghall to The Knock. Porphyritic olivine-basalts predominate, but a thin band of dolerites strikes N.W. from the Raven Craig. The limestones and accompanying shales and ash of the east and west Kirkton quarries, so well described in the Survey Memoir accompanying sheet 32, occur as isolated lenticular patches between successive lava-flows, and evidently occupy a much higher horizon than the Tartraven Limestone. A similar intercalation of sandstone may be seen on the eastern slope of the Knock Hill. From the Knock to Tartraven little rock is visible, but numerous exposures are found in the Tartraven Hills where the road cuts through a series of dark-blue lustrous limburgitic olivine-basalts. The lavas of this zone probably run out to the north of the Mains Burn, for in a streamlet to the east of Balvormie the only representative of the zone is the basal ash noted above. This apparently swells out by Whitebaulks to Hillhouse, where it dips below a group of coarse-grained olivine-dolerites, which, after suffering displacement by the Haugh Burn fault, runs out to the north of Parkly Place.

The Hurlet Limestone.

This well-marked horizon can be traced from Glenbare quarry, east of Bathgate, to the North Mine quarry on the Tartraven road. For a mile to the north of this point the outcrop is conjectural, no trace of the limestone being found at the surface. It reappears, however, in characteristic sections in the Hillhouse and Hiltly quarries. North of Hiltly the outcrop must be shifted to the east by the Haugh Burn fault, and probably strikes north from the vicinity of Parkly Place to Linlithgow Poorhouse, and thence north-west to the shore at Stacks. The Hurlet Limestone is throughout associated with sandstones, shales, and thin ashbeds; and detailed descriptions of sections, formerly better visible than now, may be found in the Survey Memoirs and Mr Cadell's papers.

To the east of Linlithgow no volcanic rocks are found below the Hurlet Limestone, within the limits of the present map, and, other than the thick sandstone formerly quarried at Kingscavil, little rock of any kind is visible.

The Fourth Volcanic Zone, or *The Hilderston and Hiltly Lavas.*

This zone reaches its greatest thickness immediately to the south of Linlithgow, but even here the apparent thickness is greater than the actual thickness, on account of

the effect of the Haugh Burn fault. Petrographically, the zone is composed of a number of alternating bands of olivine-basalts and olivine-dolerites. Transverse traverses in the neighbourhood of Clarendon, Hiltly, Wairdlaw, or Hilderston readily reveal this structure. As a rule, however, these bands cannot be traced far in a north and south direction. A stratified intercalation of ash and ashy shales is found in Preston Glen, and another of limestone and shales at Wairdlaw. The basalt overlying this limestone is noteworthy, both for its platy jointing and for the fact that it is the only lava throughout the whole volcanic series which contains phenocrysts of felspar in any abundance. The same rock can be traced on the south side of the valley at Wairdlaw, but the limestone below is nowhere visible. A line of springs behind Craigmailing probably marks the prolongation to the south of this sedimentary intercalation, and indications of its presence are also found on the eastern slope of Cathlaw Hill. Towards the south interbedded sediments probably become more abundant. Two bands are exposed in the Mavis Glen, and these can be traced for a considerable distance to the north and to the south by means of the shallow depressions between the lavas to which they give rise.

The lavas of this zone cannot be traced to the north of Linlithgow Loch. In all probability they rapidly run out, and their place appears to be taken in part by a thick bed of volcanic ash found in a bore in Bonnytoun farm. Farther north the sandstones, shales, and thin limestones of Carriden probably occupy approximately the same horizon.

The Index Limestone and the Bo'ness Lavas.

Between the fourth and fifth volcanic zones there occurs in the Bathgate Hills an important belt of sedimentary rock, which includes the lower Bathgate coals and the Index limestone. It retains a fairly uniform thickness and character from Bathgate to Kipps. North of Kipps, however, the thickness gradually increases, and volcanic material becomes mingled with the sedimentary. On the eastern slope of Cockleroy a bed of ash appears not far below the probable position of the Index limestone, and at Kettlestoun fine-grained volcanic mudstones, passing into ashy sandstones and shales, are found overlying the lavas of the fourth volcanic zone. Between Kettlestoun and Linlithgow Bridge the only exposure is in the river Avon at the railway viaduct, where a very vesicular basalt lies a few feet below the Index limestone. The journals of bores quoted by Mr Cadell seem to indicate that a considerable proportion of the rock below the glacial gravels of this district is of volcanic origin. This change in the character of the strata makes it very doubtful whether the Kipps coals and the Index limestone are continued across Cockleroy to Kettlestoun. It is quite possible, however, that the coals do exist, but almost certainly in an attenuated form, of no commercial value, and much destroyed by intrusive rock.

North of the Edinburgh and Glasgow Railway the Index limestone is repeatedly exposed in the river Avon, while the sedimentary zone, as a whole, opens out rapidly

to include the Bo'ness coalfield and its intercalated volcanic rocks. In one sense, therefore, since mere thickness of strata is in this case no index to rate of deposition, the whole of the Bo'ness coalfield may be considered the equivalent of the lower Bathgate coalfield. On the other hand, from the position of the igneous material where it first appears at Cockleroy and Linlithgow Bridge, the lower Bo'ness coals might possibly be considered more nearly the equivalent of the lower Bathgate coals in point of time. Further, the lavas of the Bo'ness coalfield apparently form a group by themselves entirely distinct from the volanic zones of the Bathgate Hills to the south. It is highly probable, as Mr CADELL has suggested, that the conditions of sedimentation were entirely different on either side of a volcanic orifice somewhere in the vicinity of Little Mill.

It is unnecessary to describe in detail the stratigraphy of the Bo'ness coalfield. That has been admirably done already by Mr CADELL, and the Little Mill district alone still remains more or less of a puzzle. Petrographically, the lavas of Bonnytoun Hill, south of the Roman road, can be readily subdivided into three zones—a lower zone of coarse-grained olivine-dolerites between the Red Coal and the Wandering Coal; a middle zone of porphyritic olivine basalts between the Wandering Coal and the Western Main Coal; and an upper zone of coarse-grained dolerites between the Western Main Coal and the Muirhouse coals. The middle zone alone can be traced below the ash of Little Mill to Pepper Hill and Linlithgow Bridge. To the north the lower zone can be traced continuously to the Bonhead fault, but the two upper zones, north of the Roman road, apparently pass into finer-grained doleritic basalts, which persist throughout the remainder of the coalfield. The rock exposed above Bonsyde is similar to the lavas of the middle zone to the west, and is probably a displaced portion of the other lavas of the hill.

The Fifth Volcanic Zone, or *The Kipps and Bishopbrae Lavas.*

This zone lies a short distance above the Index limestone, and may be traced, with interruptions, from Linlithgow to Bathgate. Between the Avon Paper Mills and the vicinity of Cockleroy, where this zone reaches its greatest thickness, the only exposure is found in the Cauld Burn at East Belsyde. It is highly probable, however, that this zone is continuous throughout. Petrographically, the lavas belong mostly to types of olivine-basalt, and limburgitic varieties may be studied with ease in the neighbourhood of Kipps. Coarser-grained doleritic types occur here towards the summit of the series, and appear also in the river Avon at Linlithgow.

Above the lavas and below the Dykeneuk limestone evidence of the continuance of volcanic action is found in Carriber Glen, where a thick series of ashy sandstone and volcanic mudstones, in places fossiliferous, are exposed in the gorge of the river Avon. Similar ashy sandstones are found at Threegables, east of Bowden Hill; and in a streamlet between Lochcote and Gormyre a bed of ash occurs on approximately the same horizon, overlaid by a peculiar blue mudstone, which in places much resembles a decomposed igneous rock.

The Dykeneuk and Castlecary Limestones.

The Dykeneuk limestone, though proved in many bores, is seen at the surface in three exposures only—at Dykeneuk, Woodcockdale, and Carribber. The Castlecary or Levenseat limestone is exposed at Craigenbuck, at the Birkhill viaduct, the Avon aqueduct, Carribber, Bowden Hill, and Lochcote. Both limestones are probably continuous across Bowden Hill, though in places cut out by intrusive rock. South of Bishopbrae their presence beneath the surface has been repeatedly proved in bores. The strata between the two limestones consist of sandstones and shales, thickest towards the north and thinning out towards the south, with the effect of bringing the limestones closer together from Carribber southwards. No trace of volcanic activity is found above the Dykeneuk limestone.

The Volcanic Necks.

These are found in the eastern part only of the volcanic area. A small neck full of green ash pierces the Riccarton Hills south-east of Belcraigs. Another, full of coarse agglomerate, breaks through the stratified ash of the first volcanic zone to the west of Wester Ochiltree. A group of seven small necks is found in the neighbourhood of Hiltly and Parkly Place, some filled with coarse agglomerate, and others with fine-grained ash similar to that at the base of the third volcanic zone. The Necks of Pilgrim's Hill and Carriden are also included in the accompanying map.

The Intrusive Rocks.

The intrusive rocks of this district are readily separated into two groups according to their microscopical characters,—a smaller group of olivine-basalts and dolerites, and a larger group of augite diabases, with little or no olivine, but with frequently abundant hypersthene. An intersertal microlitic or micropegmatitic groundmass is usually present in the latter, in greater or less abundance. This difference in mineral composition seems to be most easily explained on the assumption that the two groups are the products of different periods of igneous activity. The smaller and more basic group might readily have been produced from the consolidation of igneous material similar to that which produced the lavas. They may therefore be regarded as more or less contemporaneous intrusions. The large group, however, is of a more acid character, and is certainly the product of a later period of igneous activity.

(*a*) *Contemporaneous intrusions.*—Small intrusions of olivine-basalt are found in many of the Necks enumerated above, but, other than these, few contemporaneous intrusions have been recognised throughout the volcanic zones. A small intrusion of olivine-basalt cuts the ash of the first volcanic zone in the Brox Burn near the Bangour reservoir. A short dyke of similar material occurs in the third volcanic zone north-

east of Wairdlaw, and another at Haddie's Walls in the Kipps coalfield. The small dyke piercing the lava-cliff to the west of Hiltly is probably of contemporaneous origin. Much also of the intrusive rock in the vicinity of Ochiltree Mill, as well as the Walton and Carriden intrusions, may be referred, from petrographical characters, to the first period of igneous activity.

(*b*) *The later intrusions.*—These may be subdivided into two groups:—

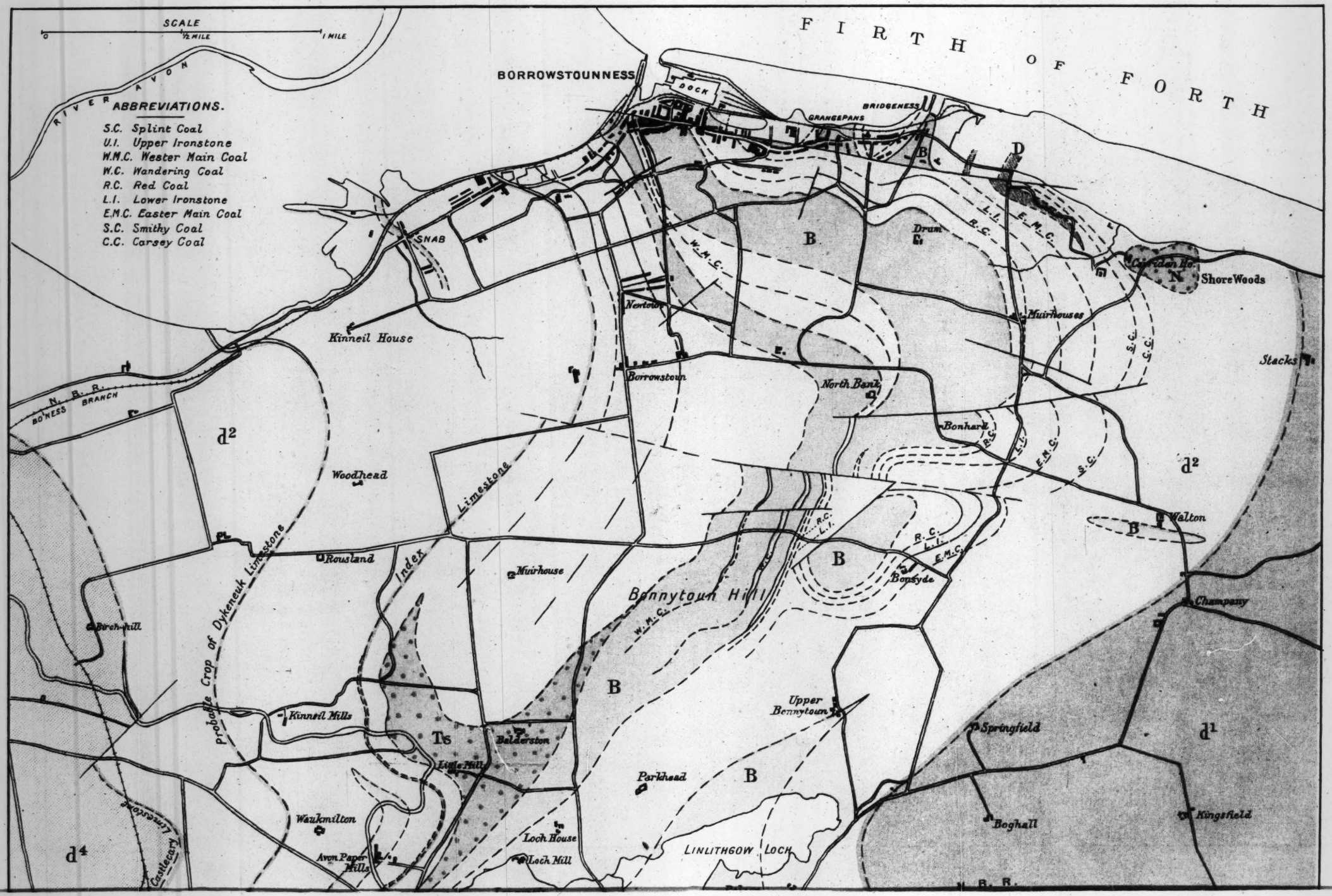
1. Dyke-like intrusions with a vertical or highly inclined junction—
   E.g. The Raven Craig, the Knock, the Witch Craig, the Wairdlaw and Cockleroy intrusions, and the E.–W. dykes.
2. Sills or laccolitic intrusions—
   E.g. The Kettlestoun Hills, the Belsyde Hills, and the Torphichen Hills.

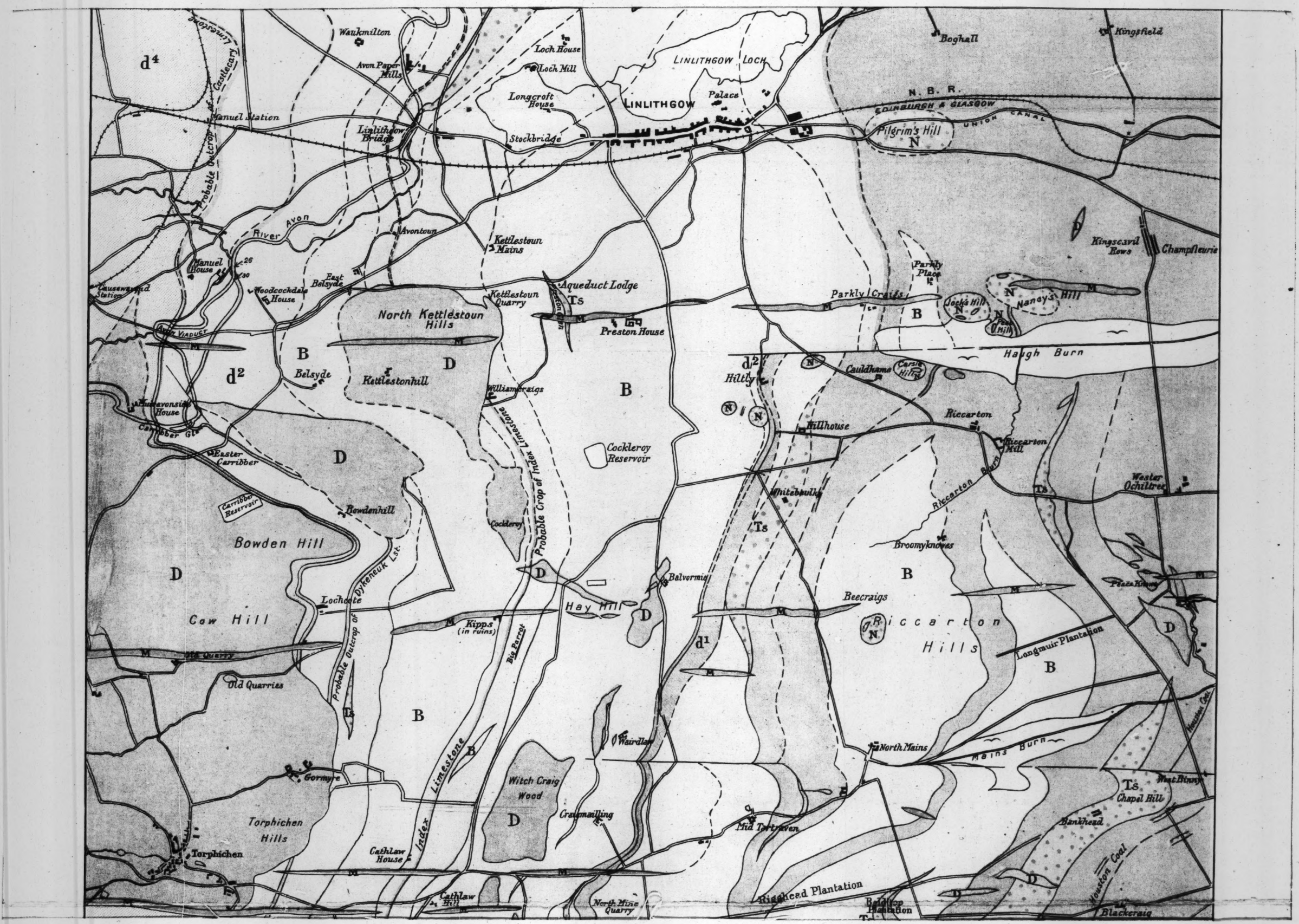
The similarity in petrographical type of these intrusions indicates contemporaneity of origin in late or post-carboniferous times. Strict contemporaneity, however, is not implied, the intrusions having evidently been inserted in succession during the period. Those intrusions with a N.–S. elongation are probably oldest, being cut, as in the case of the Raven Craig, by the E.–W. faults, along which dykes have usually risen. These dykes are apparently the feeders of the sills, the lowest and oldest of which are sometimes covered by the feeders of the higher and younger. The upper limits of the dykes themselves were probably irregular, different portions of the same dyke rising to different levels. The upward termination of one of these dykes is particularly well seen at Broomyknowes and Belcraigs. This may explain in part the discontinuity of outcrop of some of these dykes when traced from east to west. In other cases, however, as in the Parkly Craigs dyke, the different portions seem to run out, the dyke being continued on a parallel line a few yards to the north or south.

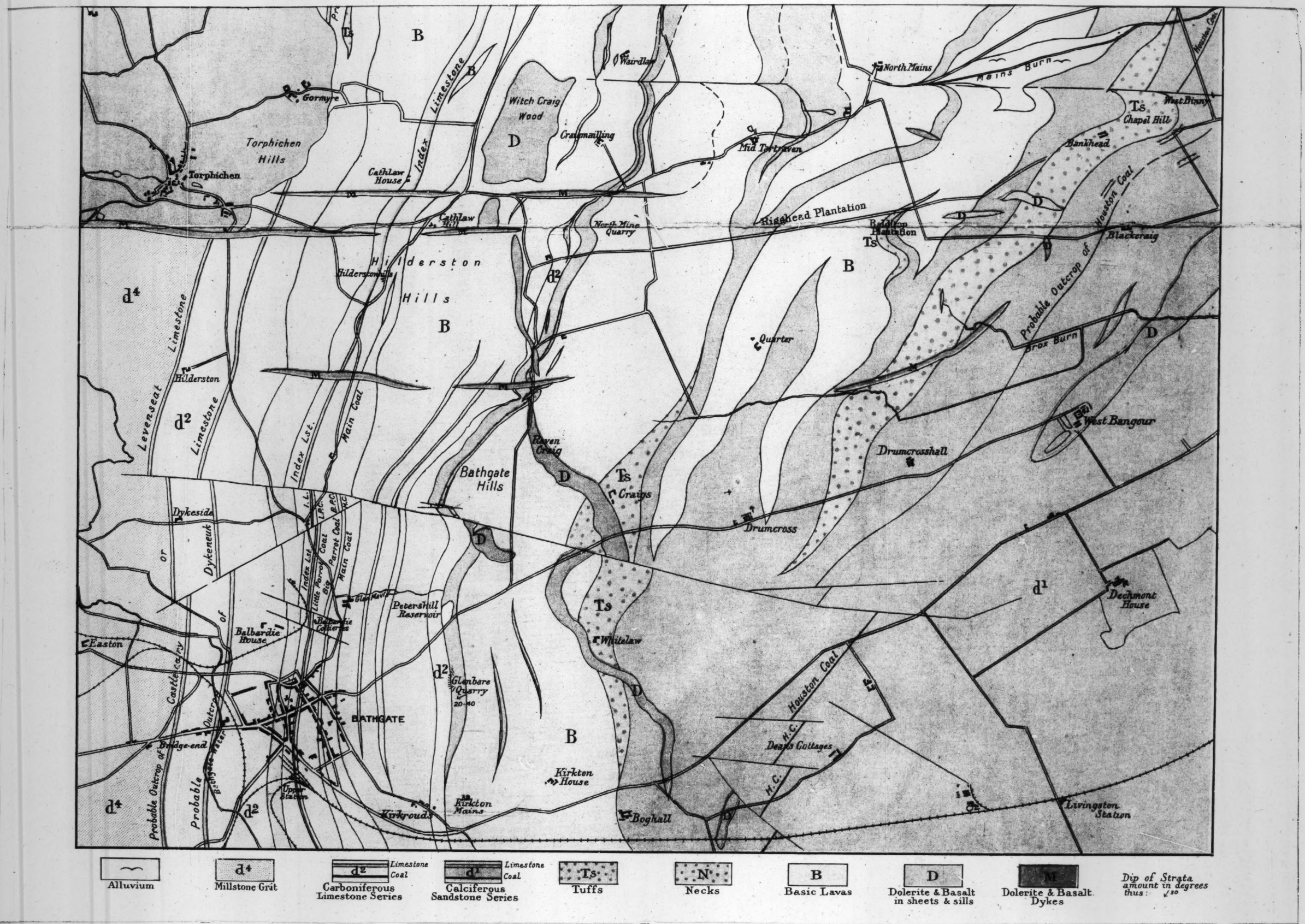
General Results.

1. The lavas of the Bathgate Hills are olivine-bearing from base to summit of the series, and are pierced by a few contemporaneous intrusions of similar material.
2. The Bo'ness lavas form a group entirely distinct from the lava-zones of the Bathgate Hills.
3. The volcanic zones are crossed by a later-connected series of dykes and sills, probably of Palæozoic age.

The detailed results of the microscopical and chemical analysis of the rocks are reserved for a future paper.


SCALE
0 ½ MILE 1 MILE
RIVER AVON
ABBREVIATIONS.
S.C. Splint Coal
U.I. Upper Ironstone
W.M.C. Wester Main Coal
W.C. Wandering Coal
R.C. Red Coal
L.I. Lower Ironstone
E.M.C. Easter Main Coal
S.C. Smithy Coal
C.C. Carsey Coal
FIRTH OF FORTH
BORROWSTOUNNESS
DOCK
BRIDGENESS
GRANGEPANS
SNAB
Kinneil House
N. B. R. BO'NESS BRANCH
Newtown
Borrowstoun
North Bank
Drum
Muirhouses
Shore Woods
Stacks
Bonhark
Woodhead
Limestone
Rousland
Index
Muirhouse
Bonnytoun Hill
Boneyde
Walton
Champany
Birch-hill
Probable Crop of Dykeneuk Limestone
Kinneil Mills
Balderston
Little Hill
Upper Bonnytoun
Springfield
Parkhead
Boghall
Kingsfield
Waukmilton
Loch House
Loch Mill
Avon Paper Mills
LINLITHGOW LOCH
Castlecary
Limestone
B
D
N
Ts
d1
d2
d4


Waukmilton
Avon Paper Mills
Loch House
Loch Mill
LINLITHGOW LOCH
Longcroft House
LINLITHGOW
Palace
Boghall
Kingsfield
N. B. R.
EDINBURGH & GLASGOW
UNION CANAL
Pilgrim's Hill
Manuel Station
Linlithgow Bridge
Stockbridge
Probable Outcrop of Castleary Lst.
River Avon
Avontoun
Kettlestoun Mains
Kingscavil Rows
Champfleurie
Manuel House
Causewayend Station
Woodcockdale House
East Belsyde
Aqueduct Lodge
Kettlestoun Quarry
Preston Cut
Preston House
Parkly Place
Parkly Crags
Jock's Hill
Nancy's Hill
Avon Viaduct
North Kettlestoun Hills
Belsyde
Kettlestonhill
Haugh Burn
Hillhouse
Cauldhame
Carsie Hill
Riccarton
Williamcraigs
Cockleroy Reservoir
Hillhouse
Riccarton Mill
Riccarton Burn
Easter Carribber
Carribber Reservoir
Bowdenhill
Cockleroy
Whitebulk
Western Ochiltree
Bowden Hill
Probable Outcrop of Index Lst.
Broomyknowes
Balvormie
Lochcote
Hay Hill
Beecraigs
Riccarton Hills
Longauir Plantation
Kipps (in ruins)
Big Parrot
Old Quarries
Old Quarry
Wairdlaw
North Mains
Mains Burn
Houston Cast.
Gormyre
Witch Craig Wood
Limestone
Index
Craigmailling
Mid Torrhaven
Torphichen Hills
Torphichen
Cathlaw House
Cathlaw Hill
North Mine Quarry
Riggfoot Plantation
Chapel Hill
Banktead
Houston Coal
Blackcraig
d4
d2
d1
B
D
Ts


Ts
B
Witch Craig Wood
D
Wairdlaw
Craigmailling
North Mains
Mains Burn
Houston Coal
Ts
Chapel Hill
West Binny
Bankhead
Torphichen Hills
Gormyre
Torphichen
Cathlaw House
Index Limestone
Mid Torbraven
Riggshead Plantation
Bridge Mansion
Blackcraig
Cathlaw Hill
North Mine Quarry
Hilderston Hills
d4
d2
B
Hilderston
Levenseat Limestone
d2 Limestone
Index Lst.
Main Coal
Quarter
Probable Outcrop of Houston Coal
Brox Burn
D
West Bangour
Drumcrosshall
Raven Craig
Bathgate Hills
Ts
Craigs
Drumcross
Dykeside
Dykeneuk
Index Lst.
Little Parrot Coal
Big Parrot Coal
Main Coal
Petershill Reservoir
Glenbare Quarry
20-40
Balbardie House
Easton
Castle Carry
Outcrop of
Bridge-end
BATHGATE
Probable Outcrop of Bathgate Main Coal
Upper Station
Kirkton Mains
Kirkroods
Kirkton House
Houston Coal
H.C.
Deans Cottages
Boghall
Livingston Station
d1
Dechmont House
Whitelaw
Alluvium
Millstone Grit
Limestone
Coal
Carboniferous Limestone Series
Calciferous Sandstone Series
Tuffs
Necks
Basic Lavas
Dolerite & Basalt in sheets & sills
Dolerite & Basalt Dykes
Dip of Strata amount in degrees thus: 30

XV.—On a New Family and Twelve New Species of Rotifera of the Order Bdelloida, collected by the Lake Survey.

By James Murray. *Communicated by* Sir John Murray, K.C.B., F.R.S. (With Seven Plates.)

(MS. received January 13, 1905. Read January 23, 1905. Issued separately March 3, 1905.)

Introduction.

The new species here described were found in the course of the work of the Lake Survey on Loch Ness and other Highland Lochs. Half of the number were found in lakes, though they are not exclusively lacustrine, three in ponds, two from moss growing on the shores of Loch Ness, and one in a stream running into Loch Ness.

Structure.—A short account of the structure of a typical Bdelloid will be necessary, in order to render intelligible the terms used in the descriptions. A Bdelloid is a Rotifer which can creep like a leech or caterpillar. The body is segmented, and consists of head, neck, trunk, and foot. The head, neck, and foot are telescopic, and can be completely withdrawn into the trunk. The normal number of segments is sixteen, but there may be more or less, the variation being chiefly in the foot. The head consists of three segments, the neck of three, the trunk of six. It is believed that the number of segments in each of those portions of the body is invariable, but two or more segments may be so united as to be indistinguishable. The foot is more variable, the number of segments, normally four, varying from one to six. Beginning at the anterior end, the first and second segments of the head form the rostrum. The first has an inverted tip, from which rise the two rostral lamellæ, numerous motile cilia, and sometimes larger tactile setæ. The third segment is the *oral*, and bears the mouth, and the corona when present. The *first cervical* bears the antenna, and frequently a number of prominences. The second and third *cervical* have no appendages. The first, second, third, and fourth segments of the *trunk* are called the *central*, and form the broadest part of the body. The next two segments of the trunk, the *pre-anal* and the *anal*, together form the *rump*, which is generally clearly marked off both from the central part of the trunk and from the foot. At the end of the anal segment is the *anus*. The segments beyond the anus constitute the *foot*. The first joint of the foot commonly has the skin on the dorsal surface thickened, and often bears a rounded *boss* or other processes. The *penultimate* joint bears the spurs. The *last* joint bears the toes, or the perforate disc which takes their place. The segmentation is superficial, and affects only the skin. When the animal is fully extended, the various organs usually occupy definite segments, though the arrangement is not invariable. The *brain*, generally somewhat triangular, occupies the *second cervical*, but when large may extend

through all three *cervical* segments. The *eyes*, when present, are in the rostrum, near its tip, or on the back of the brain. The *mastax*, containing the jaws with their teeth, is in the *third cervical* or *first central*. The *stomach* extends through all four *central* segments, which also contain the *ovaries* and the eggs or young. The *pre-anal* contains the intestine, the *anal* the *contractile cloaca*. The *foot-glands* occupy all the joints of the foot, and may extend into the *anal*, or even into the *pre-anal*. The *toes* are three or four in number, or they are united to form a *disc*, which is perforate with pores for the passage of the *mucus*.

The *corona* of the Philodinadæ consists mainly of two nearly circular *discs*, borne on *pedicels*. The *principal wreath* borders the discs. The *secondary wreath* runs round the bases of the pedicels from back to front, and merges in the cilia of the mouth. Near the centre of each disc is in many species a seta, or pencil of setæ, or several short motile cilia, which usually rise from a small papilla or a larger process. The *lower lip* is the central portion of the under side of the mouth, and is shaped like a V. The *upper lip* is the space between the bases of the pedicels and the front of the rostrum. Its form is very characteristic for each species. Its most important structures are two folds of skin, which continue those prominences at the sides of the mouth known as the *collar*. These folds run round the bases of the pedicels, close to the secondary wreath, and may meet in the middle line just in front of the rostrum, or may terminate at some distance apart, in processes of various form. In the middle line, between the pedicels, is often found a peg-like process, known as a *ligule*. The water-vascular system, with its *vibratile tags*, usually about six pairs, is difficult to observe. The number of pairs of tags seen is always noted, though this may not be the full number present.

The skin may be smooth and hyaline, stippled with pellucid dots, viscous, papillose, or variously warted or spiny. The back and sides of the trunk are longitudinally plicate. The ventral side is obscurely transversely plicate. In a few species the ventral transverse folds are numerous and deep.

Habits.—The great majority of the known species are free and independent animals. None are truly pelagic. Even those which are in the habit of swimming only do so for short distances, and in the shelter afforded by mosses and other water plants. Those which do not swim creep in caterpillar fashion on the plants among which they live or on the mud of ponds. When feeding they anchor themselves by the foot. The Adinetadæ and some *Callidinæ* can also glide forward by the action of certain cilia—in the Adinetadæ those of the corona, in the *Callidinæ* those of the rostrum. Several species secrete protective cases; others accumulate irregular tubes of débris.

Parasitism.—A number of species are ectoparasites upon other animals. None are internal parasites or feed upon their host. They are commensals or messmates. They attach themselves by the foot to some crustacean, insect larva, or other animal. They seem to desire from their host only protection, conveyance from place to place, possibly a share of food. *Asellus* is a favourite host, and often carries several species together. All the Bdelloids known to me which have taken to the parasitic mode of life are large

animals. They are distinguished by bulk of body, long and powerful foot, large strong spurs, and usually ample corona. The life agrees with them. Signs of degeneration are not lacking, however. Their affinities are with genera which normally possess eyes, but most of these parasites are blind, and have smaller brains than are possessed by free-living species of the same genera.

Symbiosis.—A number of Bdelloids have the foot of a peculiar type, in which there are no distinct toes, their place being taken by a disc, which is perforated by numerous pores for the passage of the mucus. It has been suggested that the species having this kind of foot live in symbiotic relationship with certain Hepatics, such as *Frullania*, which have some of their leaves in the form of little pitchers. The suggestion is plausible, inasmuch as such Hepatics are seldom found without the pitchers occupied by the Rotifers. There is, however, something to be said against the belief that the relation is one of symbiosis. There is no evidence of any advantage to the plants, though it has been guessed that the animals are in some way beneficial to them, and that to obtain this benefit the pitchers have been developed as an attraction to Rotifers. It is more probable that the pitchers of the Hepatics serve the sole purpose of retaining moisture, and that many species of Rotifers have found and taken advantage of those little reservoirs. In that case there is no symbiosis, only a mild form of parasitism. Bdelloids having the discoid foot are not only found on Hepatics. They abound in many other situations. Nor are they the only Bdelloids which frequent *Frullania* cups. Many species with the ordinary foot are commonly found there. It might be supposed that a Bdelloid would have less need of a strongly adhesive foot in the shelter of a *Frullania* pitcher than in many more exposed situations. The discoid foot is an advantage to a species in any situation, and it is to be noted that all the species having it are large, powerful animals.

Two species having the discoid foot are here described. One is from a lake, the other from a pond, and neither has yet been found on Hepatics.

Formation of food-pellets.—Four of the species described in this paper belong to that section of the genus *Callidina* in which the food is moulded in the œsophagus into pellets. All the animals having this characteristic agree in many other points of structure, such as the small size of the corona, and form a very natural group. Most of them have the neck very long and the gullet correspondingly elongated, and forming a large loop when the neck is contracted. The pellets differ greatly in consistence in different species. Some are loosely put together, and quickly disintegrate when passed into the stomach. In some species they seem to be mixed, while in the œsophagus, with something which gives them coherence. Such pellets maintain their size and form unchanged during the whole of their passage through the alimentary canal, and are finally voided entire. When first passed into the stomach they are granular, and often dark from the admixture of food particles. They gradually lose the granular character as they move through the stomach, till when passed out they are clear spheres.

BDELLOIDA.

All the Bdelloid Rotifera hitherto known have been included in two families, distinguished by different types of corona. The Philodinadæ have the corona divided into two discs, which bear the primary and secondary wreaths of cilia. The Adinetadæ have no discs, the corona consisting of a flat surface, furred with short cilia, divided by a non-ciliated space in the middle line, which may correspond to the space separating the discs in the Philodinadæ.

An animal discovered in Loch Vennachar in 1902, in the course of the work of the Lake Survey there, could not be referred to either of the known families. After prolonged study, continued for more than two years, it is now proposed to constitute a new family for its reception.

Microdinadæ, n. fam.

No corona, the ciliated alimentary tract ceasing at the mouth; jaws intermediate between the *ramate* type of all other Bdelloida and the *malleo-ramate* type of *Melicerta*.

The only species at present known is a Philodinoid animal. It resembles the genus *Philodina* in general form, in the rostrum, and in having four toes. The absence of corona would not of itself have justified the establishment of a new family. It might have been regarded as a degenerate *Philodina* which had taken to a different mode of feeding, and lost its corona from disuse. It was only after the peculiar structure of the jaws was understood that it became evident that the definition of the family Philadinadæ could not be modified to include it.

As now understood, the new family is seen to differ more from the other two families of the order than they do from one another. The Adinetadæ differ from the Philodinadæ only in the form of the corona, and in the partly adnate rostrum, free at the tip. They have the same form of jaws and of all other structures.

The Microdinadæ differ from both, not only in the lack of corona, but in the shape of the jaws. It comes nearer to the Philodinadæ in the free rostrum and the number of toes. On the other hand, the form of mouth might more readily be derived from that of *Adineta*. In Plate II. are shown heads of Philodinadæ (fig. *a*), Adinetadæ (fig. *c*), and Microdinadæ (fig. *b*). They are drawn from the ventral side in order to show the similar form of lower lip in all. On the same plate are drawn three pairs of jaws:— fig. *d* shows the ramate jaws of Philodinadæ and Adinetadæ, fig. *e* those of *Microdina*, fig. *f* those of *Melicerta*. It will be seen that the jaws of *Microdina* differ about as much from the *ramate* as from the *malleo-ramate* type, and sufficiently approach the latter, in the anterior position of the teeth and the less rigid union of the various parts, to constitute in some degree a link between the Bdelloida and the Rhizota.

Microdina, n. gen.

Toes, four. Yolk-mass with four nuclei. Gullet very short. Teeth, three or four on each side, at anterior end of jaws.

The terminal cilia of the gullet, which project a little way and assist in seizing the food, might be regarded as constituting a rudimentary corona. It is not usual, however, to regard the end of the gullet in a Philodine as part of the corona, that term being restricted to the discs and the two wreaths. It seems, therefore, more correct to consider *Microdina* as having no corona.

The parts of the jaws are movably articulated, not rigidly united as in other Bdelloids. The rami have large curved processes on the ventral side. The manubrii may have no loops, or may have from one to three, more or less distinct.

In Philodinadæ and Adinetadæ the larger teeth cross about the middle of the jaw, and there are finer striæ towards each end. In Microdinadæ the large teeth are all at the anterior end, and fine striæ only cross the posterior half of the jaws.

Owing to the shortness of the gullet, the jaws are close to the mouth.

Microdina paradoxa, n. sp. (Plates I. and II.)

Specific characters.—Of moderate size, stout, enlarged at level of mastax and at posterior part of trunk, contracted between those parts. Œsophagus and large granular mass connected with it, of a bright crimson colour. Stomach voluminous, its walls filled with coloured globules. Foot of three joints. Spurs short, stout, somewhat bottle-shaped, a broader basal portion contracting into a narrower apical portion, obtuse, separated by narrow convex interspace. Toes large and thick, the ventral pair much larger than the dorsal. Foot-glands forked. Antenna two-jointed, flattened. Oviparous.

General description.—Greatest length $\frac{1}{100}$ to $\frac{1}{80}$ inch. Always fiddle-shaped, owing to the narrowing between the head and the enlargement of the trunk. The position of the posterior enlargement varies. When an egg is carried, the broadest part will be in the second or third *central* segment. When there is no egg and the ovaries are undeveloped, the *fourth central* or *pre-anal* may be broadest. The rostrum differs in no way from that of a typical Philodine. The mouth is small and somewhat trifoliate. The lower lip is of the V-shape usual in the order. The upper part is obscurely two-lobed. The sides of the mouth are finely longitudinally striate. The whole animal is sometimes pale rose-colour or purple. More generally it is colourless, except for the crimson gland and œsophagus, the stomach and the egg. The globules in the stomach walls vary greatly in colour. They have been seen red, yellow, greenish, orange, magenta, sienna, or umber. The two last colours are commonest, and are used in the illustrations. The egg is of a tawny yellow. A clear fleshy mass fills the head

between the rostrum and the œsophagus. The posterior portion of this mass is, from its position and its connection with the antenna, regarded as the brain, but its outline could not be traced. Between the mastax and the stomach are two clear gastric glands, which meet on the ventral side.

Habits.—Of tireless activity. It creeps without ceasing on the stems of algæ and mosses, feeding all the time. Its mode of feeding is unlike that of any other Bdelloid, though *Adineta* resembles it in some respects. A biting action is continually repeated. In this the rostrum takes part. The food is caught between the rostrum and the lower lip, and pushed close to the mouth by the bending down of the rostrum. The cilia of the mouth, working downwards, catch the food that is thus brought near and sweep it into the gullet. The brush of cilia on the rostrum contributes to the action by sweeping downwards also, and to some extent compensates for the lack of discs. It was never seen to pause or rest, as other Bdelloids do occasionally.

The deposition of the egg was on one occasion seen. The animal was fully contracted. When the egg was almost completely passed out, the end which still remained in the aperture was seen to be surrounded by a circlet of clear spherical bodies. Most of these adhered to the egg when it separated, but a few remained attached to the aperture (Plate I. fig. *d*).

Variation.—Only one species of the family is known with certainty. The lack of corona deprives us of several characters of great service in distinguishing species of Philodinadæ. Various forms of spurs have been seen in Microdinadæ, but it is not yet clear whether any of these belong to distinct species (Plate II. figs. *g* to *j*).

The jaws also differ in different examples. While agreeing in general features, the degree of development of the loops of the manubrium varies greatly in different individuals. Some show no trace of any loops, while others have three well developed, two on the outer side and one on the inner, passing behind the teeth.

Habitat.—At the margins of large lakes and of clear hill lochs, also occasionally in pure running water.

Discovered in Loch Vennachar, 20th May 1902, on the occasion of the visit of the Scottish Natural History Society, as guests of Sir JOHN MURRAY; Loch Ness and Loch Morar, 1903; hill lochs on Carnahoulin, Fort-Augustus, 1904; Loch Treig, December 1904. Very abundant in Loch Vennachar and frequent in Loch Ness.

PHILODINADÆ.

Classification.—The Ehrenbergian division of the Philodinadæ into genera distinguished by the presence or absence of eyes, and by the position of the eyes when present, has long been recognised as artificial. In those genera unrelated species are brought together, and closely related species are separated. Suggestions for a more natural classification have been made, notably by MILNE, but none have been generally

accepted. The number of known species belonging to this family is now so great that some subdivision of the genera would be desirable, even if those genera were natural. Many of the new species show the artificial character of the old genera, and render a revision imperative.

I understand that a revision of the genera is now being prepared by Mr BRYCE, who, from his long experience of the order, is so well qualified to do so. This being so, I shall here only amend the definitions of the genera *Philodina* and *Callidina* so as to render them more natural. The classification based upon the eyes having proved defective, other characters of a more reliable nature have been sought. The number of toes has been suggested by MILNE * as a basis for classification. The mode of reproduction was thought of. It was found that large groups of species agreeing in the number of toes, also agreed in the mode of reproduction. One or two exceptions, however, lessen the value of the mode of reproduction as a generic character, and it must be abandoned in the meantime.

PHILODINA.

Generic character.—Toes, four. MILNE'S suggestion is adopted, though it is recognised that the genus will have to be divided. Thus defined, the genus does not differ greatly from that of EHRENBERG. All the species having eyes in the neck (*i.e.* seated on the brain) are found, with one exception, to have four toes. The main result of the alteration will be the transfer to *Philodina* of several species hitherto included in *Callidina*.

CALLIDINA.

Generic characters.—Toes, three; or foot ending in a disc. Normally oviparous.

This definition is simply provisional. It is unsatisfactory, in that it includes a character, viz., the mode of reproduction, which is not quite invariable. It is only by using this character that the genus *Rotifer* could be kept separate.

As formerly distinguished by a single negative character, viz., the absence of eyes, the genus *Callidina* became the receptacle for all the homeless wanderers of the family, till it now includes a host of species, many of which have little affinity one with another. It is with this genus that a revision of the family will be mainly concerned. Four of the new species here described belong to that very natural section of the genus in which the food is moulded into pellets. Two have the discoid 'symbiotic' foot. This type of foot might be made the basis of a genus, were it not that it is in some cases impossible to determine whether there are separate toes or not. It is, moreover, suspected that the discoid foot may have been independently acquired by unrelated animals.

* *Proc. Phil. Soc. Glasgow*, vol. xvii. p. 134, 1886.

Callidina angusticollis, n. sp. (Pl. III. figs. 2*a* to 2*k*.)

Specific characters.—Small, colourless; form pitcher-shaped in lateral view, the lower lip large, elevated, spout-like. Discs small, close together, inclined obliquely towards the mouth. Oral segment elongate, encircled about midway by a series of four thickenings. Food moulded into pellets. Foot minute, not obviously segmented; spurs short, acute, decurved, meeting at base. Dental formula 2/2. Secretes a brown flask-shaped protective case.

General description.—Greatest length $\frac{1}{90}$ to $\frac{1}{100}$ inch when feeding. Head laterally compressed, elongate from front to back. Discs sloping downward and outward from middle line as well as forward towards the mouth. Lower lip larger, relatively to the size of the animal, than in any other species known. Thickenings on oral segment diagonally placed, as shown in section, fig. 2*e*. Rostrum of moderate length, with fairly large lamellæ. Antenna of two joints, length equal to $\frac{2}{5}$ diameter of neck. Neck with large rounded thickenings at each side of antenna, and ventral thickening. Neck very long and slender. Gullet correspondingly elongated. Stomach voluminous, filled with round, clearly-outlined pellets of uniform size. These are coherent, and do not disintegrate in their passage through the alimentary canal. They are voided whole.

No eyes. Reproduction oviparous. Case oval, slightly flattened on ventral side, pale yellow when young, dark brown when old. Neck of case long, with annular striæ, mouth slightly expanded.

The foot, being apparently useless inside the case, which the animal never seems to leave, is very small. It can only be seen when the animal is forced out of its case. No separation of the first and second joints can be distinguished. The rudiments of toes probably exist, as the spur-bearing joint is not closed at its lower end, but they were not seen.

Habits.—Trusting apparently to the protection afforded by its shell, it is not at all shy, and usually resumes feeding very soon after being disturbed. When feeding, the neck is bent backward. Before beginning to feed, the head is often put out and the neck bent sharply over the edge of the case till the rostrum touches the outside of the case. The case is believed to be secreted from the skin, but the process has not been observed. The animal may occupy empty (or even inhabited) shells of Rhizopods, such as *Difflugia* or *Nebela*. Careful examination has always revealed a normal case inside the shell thus occupied. On one occasion the *Callidina* was seen in a shell of *Difflugia* which was shorter than its case. The projecting neck of the case was viscous, as shown by adherent matter, and nearly colourless. It had probably been just completed. Old animals show no viscosity, either of skin or case. The case is thin, smooth, and brown, and does not adhere to the animal. It is a cleanly animal. The pellets, which at first contain the food, are eventually passed out as clear spheres. After voiding them it clears them out by fully contracting its body and rolling about from side to side of the case till they are forced out through the neck.

Habitat.—On the leaves of mosses and hepatics in a great variety of situations, in peat bogs, on the ground, walls, or trees, most frequently seen on *Fontinalis* growing at the margins of lakes. First seen in Loch Morar, common about Fort-Augustus; occurs also in North Uist. Probably widely distributed. Before the animal came under my notice Mr BRYCE had made some studies of it, and suggested the specific name. Dr PENARD has also studied it in Switzerland.

Callidina annulata, n. sp. (Plate III. figs. 3*a* to 3*f*.)

Specific characters.—Small, colourless, in lateral view pitcher-shaped, the lower lip spout-like. Discs inclined towards mouth, their surfaces parallel to long axis of body. Oral segment much elongated, marked by annular plicæ, which are stronger towards the base. First neck segment with similar plicæ. Antenna very small, its length equal to $\frac{2}{5}$ of the diameter of the neck. Teeth, seven or eight in each jaw. Food moulded into pellets. Foot short, of three joints. Spurs, short cones, meeting at base. Reproduction oviparous.

General description.—Length about $\frac{1}{100}$ inch when feeding. Oral segment twice as long as broad. Discs reniform, separated by very narrow sulcus. Neck and gullet very long. Rostrum short and broad, with small lamellæ. Stomach large, nearly filling the trunk, containing clear rounded pellets of uniform size.

Resembling *C. angusticollis* in size and general form, it may be easily distinguished from that species by the smaller lower lip, greater forward inclination of the discs, longer oral segment, with annular plicæ and without thickenings, numerous teeth, larger foot, and lack of protecting case. Some examples carried large oval eggs. Intestine, glands, and vibratile tags were not observed.

Habits.—Being unable to secrete a case for itself, as is done by *C. eremita* and other species, it seeks shelter, like the hermit crab, in the empty shells of other animals. Shells of *Difflugia*, *Nebela*, and other Rhizopods are commonly occupied. It was first observed in cases of *C. angusticollis*, the original occupants of which had died, leaving only the tough jaws behind. The presence of those jaws, with their pairs of teeth, led to the two species being confused for some time. It also frequently takes cover in the pitchers of *Frullania* and other Hepatics. It is often found creeping about without protection of any sort, but it has never been seen to feed unless when in a shelter of some kind. When feeding it is not timid. It may frequently be observed, in detached pitchers of *Frullania*, whirling rapidly about, regardless of collisions.

Habitat.—Among aquatic mosses growing in Loch Morar, October 1903, Loch Ness, 1904. Not confined to lakes. Common on Hepatics, Fort-Augustus, Blantyre Moor.

Callidina crenata, n. sp. (Plate IV. figs. 6*a* to 6*d*.)

Specific characters.—Small, colourless. Trunk and foot papillose. Head and neck smooth. Neck with a prominence on each side of the antenna. Teeth, seven or eight in each jaw. Foot of three joints. A crenate boss on first joint. Spurs short, tapering, acuminate, divergent. Toes, three. Food moulded into pellets. Posterior margin of pre-anal segment with a rounded prominence, free from papillæ, on each side. Oviparous.

Description.—Length $\frac{1}{80}$ inch when fully extended. Rostrum short, with lamellæ of moderate size. About twelve longitudinal folds on the trunk, at equal distances apart, not fainter dorsally. Papillæ not crowning the folds, as in *C. aspera*, Bryce, but regularly distributed over the whole surface of the trunk, smaller than in *C. aspera*, rounded, without pits or pores, diminishing in size on the foot. Spurs dotted. Toes short, blunt. Egg elongate, narrowed at anterior end.

This description is incomplete, as the animal was never seen to feed. Seen in the retracted state the discs are small and close together. The only other species which moulds the food into pellets, and at the same time has the skin papillose, is *C. aspera*, Bryce. From that it is distinguished by the more numerous teeth, smaller papillæ, and pre-anal processes.

Habits.—Although fairly abundant in several collections, nothing could be learned as to its habits. All the examples studied were very sluggish in their motions. They crept about very slowly; and though some of them were watched for long periods, they showed no disposition to feed.

Habitat.—Among ground moss and hepatics from the shores of Loch Ness and elsewhere near Fort-Augustus, February 1904, frequent; not yet found anywhere else.

Callidina pulchra, n. sp. (Plate IV. figs. 5*a* to 5*f*.)

Specific characters.—Small, colourless. Trunk very broad, strongly stippled. Corona narrower than neck or collar, with central setæ on discs. First neck segment with the anterior edge turned outwards like a rim all round. Rostrum short and broad, with a large brush of long cilia. Teeth, three to five in each jaw. Food moulded into pellets. Foot short, of three joints. Spurs short, divergent, acuminate. Toes, three.

General description.—Length about $\frac{1}{100}$ inch when creeping, $\frac{1}{120}$ inch when feeding. Very short and broad. Skin not papillose, but covered with uniform large clear dots. Trunk longitudinally plicate; dorsal folds faint, lateral deep. Stomach very voluminous, filled with large pellets.

Very similar to *C. lata*, Bryce, to which it is closely related. It agrees with that species in the breadth of trunk, the shape of the corona, the central setæ on the discs, and the dental formula. It differs in the oval rather than ovate trunk, the stippled skin, the projecting edge of the first neck segment, and the shorter spurs. The shape of the

trunk makes it a more elegant animal than *C. lata*, the great posterior breadth of which imparts a clumsiness of gait as it moves.

Habits.—In its steady, deliberate motions and mode of feeding it resembles *C. lata*. When creeping it goes steadfastly forward, increasing the length of each step by a gliding movement produced by the cilia of the rostrum. It feeds for shorter periods than *C. lata*.

Habitat.—In ponds near Fort-Augustus, February 1904. Found among the sediment obtained by washing aquatic mosses. It was very abundant in some ponds. When a portion of the sediment was put into a bottle with some water and tightly corked the animal continued to abound, and increased in numbers for some months, though the water was never changed.

Callidina muricata, n. sp. (Plate V. figs. 7*a* to 7*h*.)

Specific characters.—Of moderate size, narrow. Trunk with strong longitudinal plicæ, covered with low rounded tubercles. Corona narrower than neck. Discs large, separated by very small interstice. Rostrum narrow, with large lamellæ, which project laterally. Antenna slender, length equal to half diameter of neck. Neck with large thickenings on each side of antenna. Brain large, elongate; no eyes. Dental formula 2/2; border of jaws crenate. Food not moulded into pellets. Foot short, of four joints. Spurs slender, tapering, meeting at base, divergent, incurved. Toes, three; large, tapering. Reproduction oviparous.

General description.—Greatest length $\frac{1}{70}$ to $\frac{1}{100}$ inch. Skin of trunk yellowish, viscous, with little extraneous matter adhering. Stomach large, its walls containing small dark-greyish globules. Yolk-mass with eight nuclei. Egg large, oval. Discs nearly touching. Border of jaws brown.

The tubercles are of equal size, rounded, and disposed in transverse and longitudinal rows. They are probably permanent, and not mere hardened secretions as in *C. incrassata*, but this is not proven. On the back they are hidden by the deep longitudinal plicæ. The transverse rows, about nine on the trunk, give a false appearance of close segmentation. The tubercles are more obvious on the ventral side, and all over when fully retracted. The glands, intestine, and cloaca were normal. Vibratile tags not seen. Apart from the tubercles, the species may be known by the close approximation of the large discs and by the caliper-like spurs.

Habits.—Very slow in its motions. It extends itself with studied deliberation, like *Rotifer tardus*, and is not often willing to feed. It feeds steadily, but only for a short time. On all the occasions when it was seen feeding the ventral side was uppermost, so that the details of the upper lip could not be seen.

Habitat.—In the sediment of ponds, Fort-Augustus, January 1904, frequent; Blantyre Moor.

Callidina crucicornis, n. sp. (Plate V. figs. 8*a* to 8*g*.)

Specific characters.—Large, slender, elongate. Rostrum very long, of two conspicuous joints, with very large, spreading lamellæ. Antenna very small. Brain large; no eyes. Jaws relatively very small; dental formula, 2/2. Stomach voluminous; food not moulded into pellets. Foot short, of three joints, very prominent dorsal boss on first joint. Spurs long, tapering, with distinct shoulder on inner side at base, capable of being brought together at the points or crossed over one another. Last joint of foot long, with three very large toes.

General description.—Greatest length $\frac{1}{50}$ to $\frac{1}{60}$ inch. Every part elongate except the foot. Colour dull yellow or greyish. Longitudinal plicæ few, fainter on back. Salivary glands well developed, one long narrow pair extending beyond the mastax to the upper part of the stomach. Walls of stomach thick, filled with larger and smaller dark yellow globules. Intestine oval, its long axis transverse, partly covered in dorsal view by stomach. Yolk-mass large, with eight small nuclei. Space between spurs straight or convex, according to position of spurs. Terminal toes long, slender, two-jointed. Dorsal toe as long as the others, but usually extended to only half the length. Foot-boss pointing backwards.

Owing to its disinclination to feed, the description cannot be completed. In the retracted state the discs are large and elongate. The species has a superficial resemblance to *Callidina longirostris* in the long rostrum and spurs, and also to *Philodina macrostyla* and its allies. It is believed to have no close affinity with any of those species, all of which are viviparous, while this is oviparous. The rostrum tapers gradually from the oral segment, and is not abruptly narrowed as in *C. longirostris*.

Habits.—Although it has been known for more than two years, and has been under constant observation for nearly one year, and thousands of examples have been carefully studied by three or four observers, little is known of its habits, as it has never once been seen to feed. It creeps slowly and deliberately, examining everything it encounters with its rostrum, which appears to be a very delicate organ of touch. It is very mobile, and can be bent backwards and forwards and from side to side. The lamellæ, which are only inferior in size to those of *C. cornigera*, are waved about in the way characteristic of that species, and which has led to the supposition that they are organs of smell.

When washed out of the mosses among which it lives, and allowed time to settle down among the sediment, it is found that it takes up its position, not on the surface of the sediment, but a little way down in it. The stomach is often seen to be well filled with food. These facts, together with its disinclination to feed, lead me to suppose that it may have an aversion to light, and will not feed unless in darkness If this is so, it may be impossible to complete the description of the head. Against the

suggestion is the fact that though it will not feed, it does not appear to be uneasy in the glare of the microscope lamp.

Habitat.—In lakes and ponds. Bottom of Loch Rannoch, at depth of 9 or 10 feet, April 1902. Abundant in pond near Fort-Augustus, January 1904. It thrives well in tightly corked bottles, and may continue in them for months.

Callidina armata, n. sp. (Plate VI. figs. 10*a* to 10*h*.)

Specific characters.—Large, massive. Corona broader than trunk. Rostrum short, broad; lamellæ small. Antenna as long as diameter of neck, clavate. A pair of tooth-like processes close below the mouth. Jaws relatively small, with two teeth on each. Stomach voluminous, reddish. Foot short, of three joints. A broad rounded fold at base of first joint. Spurs small, acuminate, incurved and decurved, interspace equal to diameter of base of spur. Foot ends in round perforated disc.

General description.—Greatest length when creeping, $\frac{1}{65}$ inch. Trunk with few longitudinal folds, dorsal faint, lateral deeper. All colourless, except alimentary canal. Food not moulded into pellets. Walls of stomach containing small reddish globules. Intestine roundish. Foot-glands of few cells, terminal cell largest. First foot-joint marked with annular striæ. Terminal joint long, disc perforated by many pores, ducts in common sheath. Four pairs of vibratile tags were seen.

Resembling *C. symbiotica* and allied species in massive build and discoid foot, it may be distinguished from all other species by the ventral processes below the mouth, the heavy antenna, and the dorsal fold at the base of the foot.

Habits.—Strong and active, like all the 'symbiotic' species. As it creeps rapidly about, the disc is exposed for an instant. It is a steady feeder. The function of the processes below the mouth could not be gathered from its actions.

Habitat.—On water weeds growing in Loch Ness and the Caledonian Canal at Fort-Augustus; although abundant during November and December 1903, it was not again found till December 1904, when it once more became common. The same beds of weeds, chiefly *Myriophyllum* and *Fontinalis*, were frequently examined during the intervening months without the species being once found. This may indicate that it has only a short season, though it is unusual for Bdelloids to have any seasonal limits.

Callidina incrassata, n. sp. (Plate VI. figs. 9*a* to 9*f*.)

Specific characters.—Large, stout. Trunk protected by thick plates formed of a hardened secretion. Rostrum short and very broad, with small ciliate lamellæ. Antenna considerably longer than diameter of neck. Neck with large process at each side of antenna. Corona as wide as trunk, discs large, interstice equal to half diameter of disc. Central papillæ on discs. Foot very short, of three joints. Spurs small, twice

as long as broad, acuminate, divergent, obtuse, incurved. Jaws with broad, brown, pectinate border, dental formula 5/4. Food not moulded into pellets. Oviparous.

General description.—Greatest length $\frac{1}{60}$ to $\frac{1}{80}$ inch. Trunk dark yellow. Anterior row of tubercles more prominent than the others, sometimes so long that they hang down over the next two rows. Third segment of neck, close to tubercled trunk, viscous, and with a little extraneous matter adhering. Rostrum slightly broader towards apex, ciliated cup usually quite everted, the lamellæ then standing far apart. Papillæ on discs, only once seen, like little curved thorns. Viscera difficult to see through the dark, thickened skin. Under strong pressure stomach seen, with its walls filled with small clear globules. Brain large. Glands, intestine, cloaca, and vibratile tags not seen. Arrangement of teeth unusual. Three large teeth in one jaw fit into the spaces between four large teeth in the other. There is an additional thinner tooth at each end of the row of three.

In the contracted state the tubercled trunk is so similar to that of *Philodina macrostyla*, variety *tuberculata*, that it might be passed over for that species. When it extends itself it is found to differ in everything else. Every part of the Philodine is long and slender, of the Callidine short and broad, except the antenna. This is straight, not elbowed as in *P. macrostyla*. The foot appears to end in a disc, as in the 'symbiotic' *Callidinæ*. It is a very small and obscure disc, and no perforations could be seen. It may yet be found to have short, broad toes. The tubercles could be removed by rolling the animal under the coverslip.

Habits.—Very slow and cautious. After being disturbed it may remain fully contracted and motionless for a long time. It puts out its head very gradually, feeling carefully about with its long antenna before venturing out. When it has gained confidence it walks forward rather briskly for an animal so heavily armoured. The very short foot is only momentarily seen, the disc not at all, unless it happens to be walking upside down. It was not eager to feed, and when it tried to do so was evidently annoyed by the débris surrounding it, and soon desisted.

Habitat.—In the sediment of one or two ponds at Fort-Augustus, February 1904. It was pretty abundant for a time in one pond.

Philodina laticornis, n. sp. (Plate VII. figs. 12*a* to 12*c*.)

Specific characters.—Very large. Foot and rump together about $\frac{2}{5}$ of greatest length when creeping. Corona narrower than trunk, discs with small central papillæ. Rostrum short, broad, with very small ciliate lamellæ. Antenna stout, length equal to $\frac{3}{5}$ diameter of neck. Brain fairly large, with pair of large, oblique, yellowish-red eyes. Two teeth in each jaw. Foot of three joints. Spurs large, broad, divergent, interstice slightly exceeding diameter of spur at base. Dorsal toes small, ventral long,

incurved. Reproduction viviparous. Swims free, with spurs brought close together and discs also approximated.

General description.—Greatest length $\frac{1}{50}$ to $\frac{1}{40}$ inch when creeping. All colourless except the alimentary canal. Longitudinal folds of trunk few, dorsal faint, lateral stronger. Brain elongate, triangular. Stomach ample. Intestine elongate, elliptical. Rump and foot of about equal length. First foot-joint long, with faint annular striæ. Second joint with stronger striæ, crossed above the spurs by two oblique folds of skin, which nearly meet in the middle line, and give the appearance of an extra joint. Spurs with obscure shoulder on inner side at base, then slightly contracted and expanded again nearer tip. Foot-glands rather small, with very long ducts. Most examples with two well-grown young, showing teeth and corona, and one younger fœtus. Vibratile tags, five on each side seen.

Habits.—The large size, lanky form, and large spurs and toes, suggest that the animal is a parasite, but it has not yet been found attached to any host. On the other hand, its readiness to swim, and its characteristic attitude when swimming, spurs and discs being brought together as though to lessen the resistance, are like the actions of a free-living animal. Several species of parasitic Bdelloids have small brains and are blind, and there is some reason to believe that this reduction is a consequence of the mode of life. The power of swimming might be of advantage to an ectoparasite by enabling it to change its host if necessary. When swimming, the rostrum is kept fully extended. When creeping, the toes are often kept out during the whole of the step.

Habitat.—Among aquatic mosses growing at the margin of Loch Ness, at Fort-Augustus, April 1904; in the Caledonian Canal, Fort-Augustus, December 1904.

Philodina laticeps, n. sp. (Plate VII. figs. 11*a* to 11*h*.)

Specific characters.—Very large, elongate, yellowish. Corona very large, much wider than trunk, discs broad, concave, separated by space nearly equal to diameter of disc. On each disc an elevated conical papilla, with broad apex bearing several short motile cilia. Rostrum short and broad, with minute lamellæ. Antenna short, length equal to $\frac{1}{3}$ diameter of neck. Brain a minute triangle, no eyes. Teeth, two on each jaw, with one thinner tooth. Foot and rump together just under half of total length. Foot of four joints. Spurs large, broad, blade-shaped, divergent, interstice equal to diameter of spur. Dorsal toes small, ventral long, incurved. Parasitic on insect larvæ. Oviparous.

General description.—Greatest length $\frac{1}{50}$ to $\frac{1}{40}$ inch when creeping. All hyaline except alimentary canal. Trunk longitudinally plicate, central segments covered with a hair-like growth, which is probably a vegetable parasite. Corona broadest and discs largest known in the order. Yolk-mass with eight or nine nuclei; the large egg pointed at anterior end. Intestine long, elliptical. Foot-glands long, with very long ducts. Four vibratile tags on each side seen.

Habits.—Parasitic on insect larvæ which live in running water. It has been found on larvæ of several species, adhering to the thorax, between the bases of the legs. When separated from its host it is little disturbed, immediately begins to creep actively about, and readily feeds. When feeding it is very restless, and sweeps the great corona from side to side and all over the field. The apparent breadth of the corona is often increased by a peculiar habit the animal has of pulling in the sides of the trunk till it resembles a stem supporting a large flower. It is then more like one of the large-headed Rhizota, such as *Œcistes velatus*, than a typical Philodine (fig. 11*h*). When feeding it draws the rostrum in till it is depressed below the surrounding surface of the head.

Comparison of P. laticeps *with* P. laticornis.—The two species resemble one another very closely in some characters, and differ greatly in others. The agreement is so close that it is difficult to avoid the conclusion that they are related animals. On this supposition an interesting comparison of the differences of structure in relation to the different modes of life may be made. *P. laticeps* is a parasite; *P. laticornis* has only been found free. They agree in general form, in the rostrum, spurs, and long curved ventral toes, so closely that but for the longer foot of *P. laticeps* the same drawing of the extended animal could represent both. *P. laticornis* has a large brain and eyes, small papillæ on discs, larger antenna, and shorter three-jointed foot. *P. laticeps* has much larger corona, very large papillæ on the discs, shorter antenna, longer four-jointed foot, much smaller brain, and no eyes. If the parasite *P. laticeps* has been derived from the free-living *P. laticornis*, it is interesting to note that while it has gained a larger mouth, it has lost its eyes and most of its brain. Should *P. laticornis*, as is possible, prove to be also parasitic, the force of the comparison is diminished, but not altogether lost. The habit of swimming might enable a parasite to change its host when necessary, and so render it less dependent, and the retention of the large brain and eyes may be attributed to this habit.

P. laticeps is oviparous, *P. laticornis* viviparous. This is the only instance known to me of closely related Bdelloids differing in the mode of reproduction.

Habitat.—In a little stream entering Inchnacardoch Bay, Loch Ness. Very abundant during the winter of 1903–4. Any handful of *Fontinalis* taken from this stream and shaken in water yielded thousands of examples. Early in the summer of 1904 the stream dried up, and remained in this condition till October. When the water returned to the channel insects and rotifers had disappeared, and up till the end of November neither had again been found. Similar streams in the same district were searched, but though larvæ were found, there were no rotifers upon them.

Philodina humerosa, n. sp. (Plate IV. figs. 4*a* to 4*g*.)

Specific characters.—Small, dull grey, strongly plicate on trunk. Ventral transverse folds, fourteen or fifteen. Central setæ on discs spring from large conical

prominences. Space between discs equal to half diameter of disc. Rostrum short, shaped like an acorn, basal joint papillose. At back of oral segment, on each side of rostrum, a large rounded papillose prominence. Length of antenna equal to diameter of neck. Neck with rounded prominence at each side of antenna. Foot of three joints, stippled. Spurs small, tapering, divergent. Dorsal toes tapering; ventral larger, obtuse. Two teeth on each jaw. Oviparous.

General description.—Greatest length when creeping, $\frac{1}{100}$ to $\frac{1}{130}$ inch. Skin of trunk dull yellowish-gray, opaque, finely stippled, foot more strongly stippled. The prominence from which the central seta arises occupies nearly the whole of the upper surface of the disc. The great papillose bosses on the back of the oral segment are unique. They are conspicuous when the animal is creeping as well as when feeding. The skin of the first foot-joint is thickened dorsally, but does not form a boss.

Related species.—This species is closely related to *Philodina alpium* (*Callidina alpium*, Ehr.) and *P. brycei*, (*C. brycei*, Weber). The three species form a very natural group. They are semi-loricated. The skin of the trunk is thick. Its anterior edge is cut into definite forms and bears six knobs or processes. Its ventral surface is crossed by deep transverse folds, 9 to 15 in number. Though not quite rigid, it alters little in shape. When the animal is fully retracted the deep longitudinal folds allow the anterior edge of the trunk to be closed. In *P. alpium* and *P. brycei* the two anterior dorsal processes of the trunk form a fork which receives the antenna, as in *Anuræa* and *Brachionus*. In all three species the central setæ rise from large conical processes. There are four toes.

Habits.—Like its relatives *P. alpium* and *P. brycei*, it is very slow in its movements. When it has been left undisturbed for a time it feeds with confidence. It ceases feeding at short intervals, but resumes again at the same spot.

Habitat.—Found in ground moss and *Frullania* growing on stones. Old pier at the Monastery, Fort-Augustus, 7th February 1904. At several spots near Fort-Augustus. Not yet seen anywhere else.

EXPLANATION OF PLATES.

The drawings of the complete animals are all made to a uniform scale, with the exception of *Microdina*, which is drawn larger. The separate details are drawn of any convenient size. In the descriptions the only measurement given is the greatest length of the animal. All other measurements obtained are put into the drawings. Whenever possible the width of the corona, collar, neck, trunk, rump, and foot are measured, also the length of head, neck, trunk, foot, spurs, and jaws. These sizes, expressed in figures, convey but a vague impression of the appearance of an animal unless accompanied by a drawing. They are therefore omitted from the text.

The form of the upper lip is carefully drawn, but is not included in the descriptions, as no common names for its various parts have yet been agreed upon.

Plate I.

1. *Microdina paradoxa*, n. sp.

a, dorsal view, example from L. Vennachar, 1902.
b, lateral view, another L. Vennachar example.
c, ventral view, variety from L. Treig, 1904.
d, deposition of egg.
e, *f*, antenna in different degrees of extension.
g, foot, showing toes and glands under pressure.
h, toes, dorsal view.
i, rostrum, ventral side.

Plate II.

a, head of *Callidina papillosa*, ventral side.
b, head of *Microdina paradoxa*, „ „
c, head of *Adineta barbata*, „ „
d, jaws of *Philodina brycei*.
e, jaws of *Microdina*, form with three loops.
f, jaws of *Melicerta*.
g, *h*, *i*, *j*, four varieties of spurs of *Microdina*.
k, *l*, *m*, *n*, *o*, jaws of *Microdina*, five views of same pair.
k, direct ventral.
l, oblique ventral.
m, direct dorsal.
n, dorsal, under pressure, rami turned on side.
o, lateral.

Plate III.

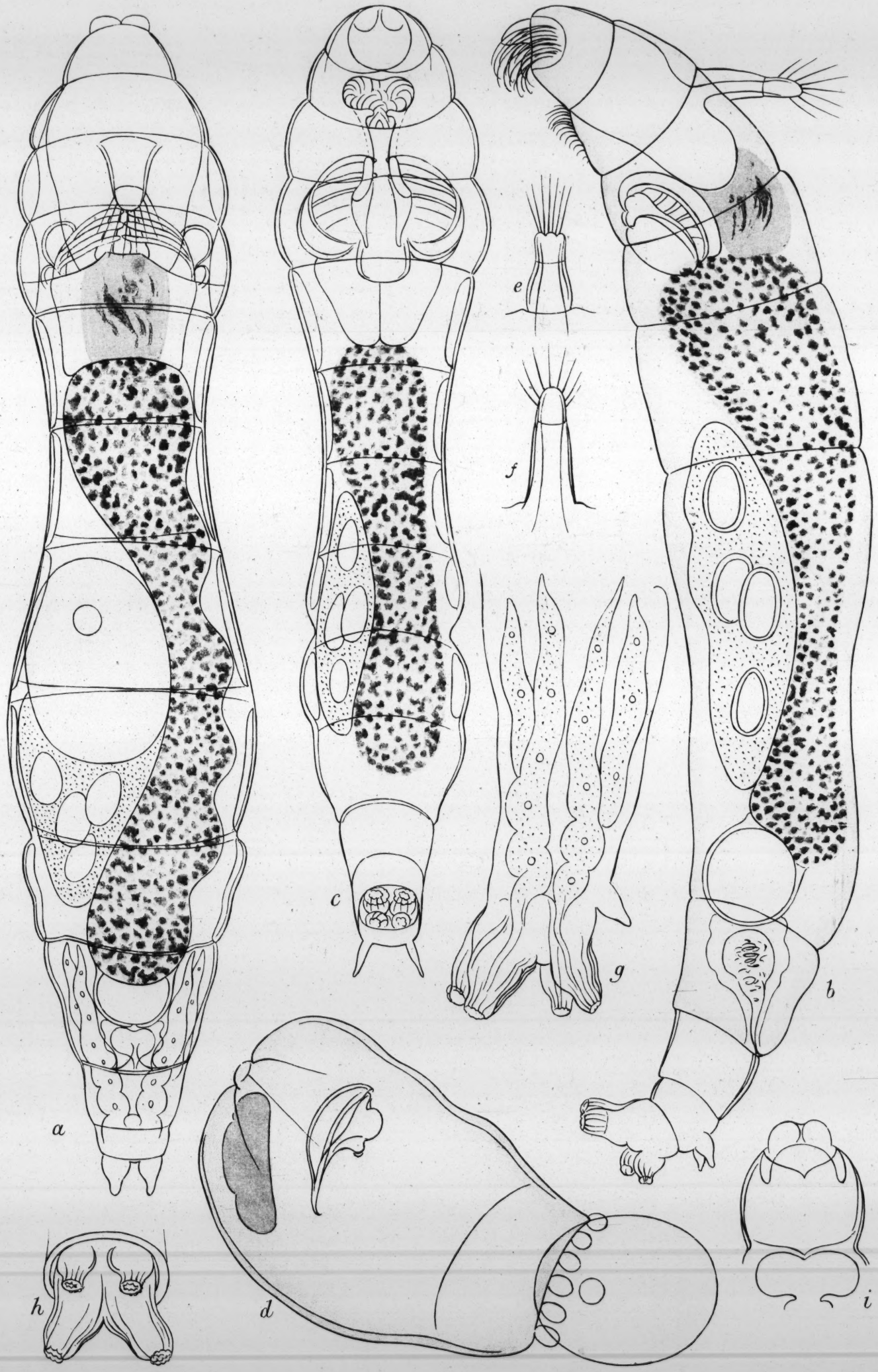
2. *Callidina angusticollis*, n. sp.

2*a*, animal in case, feeding, dorsal.
2*b*, side of head, feeding.
2*c*, jaws.
2*d*, head seen from above.
2*e*, section of oral segment at thickenings.
2*f*, section of neck.
2*g*, spurs.
2*h*, side of foot and rump.
2*i*, animal in case, in characteristic attitude.
2*j*, side of rostrum.
2*k*, front of rostrum.

3. *Callidina annulata*, n. sp.

3*a*, animal in *Frullania* cup, feeding, dorsal.
3*b*, side of head.
3*c*, front of rostrum.
3*d*, jaws.
3*e*, antenna.
3*f*, spurs.

MURRAY: A NEW FAMILY AND TWELVE NEW SPECIES OF BDELLOIDA.—PLATE I.



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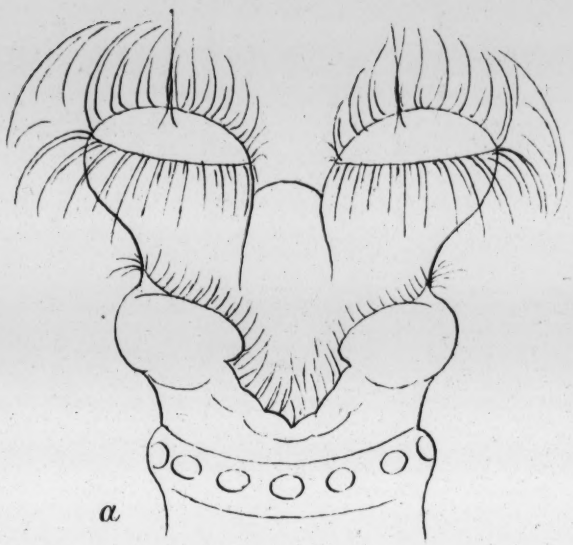
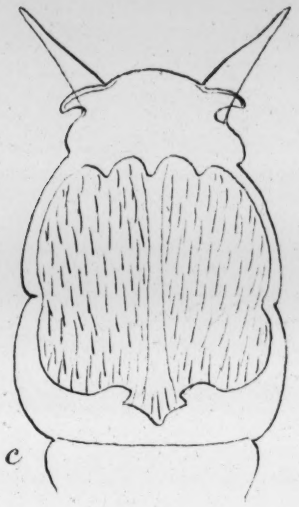
MICRODINA PARADOXA, n. sp.

MURRAY: A NEW FAMILY AND TWELVE NEW SPECIES OF BDELLOIDA.—PLATE II.


a

b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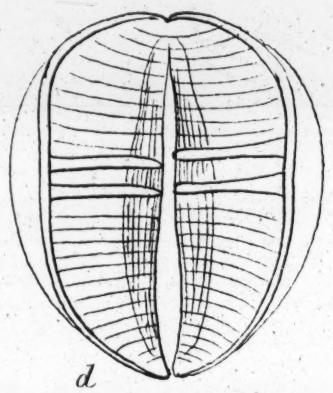
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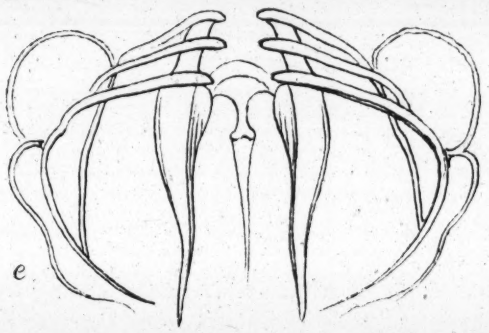
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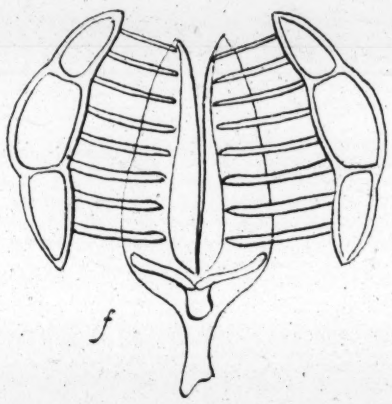
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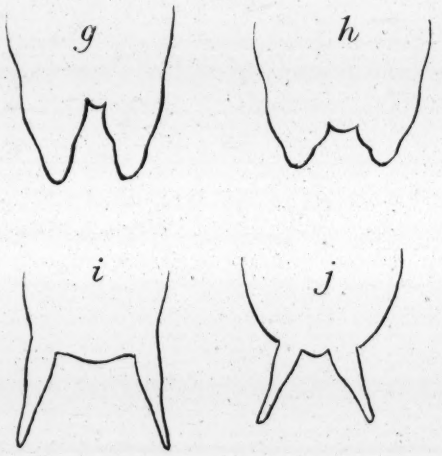
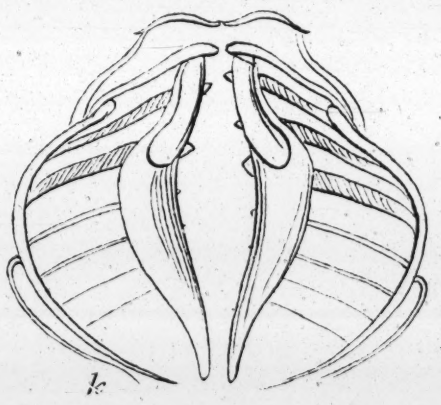

g h

i j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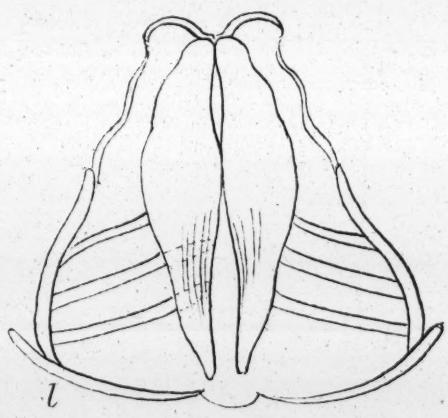
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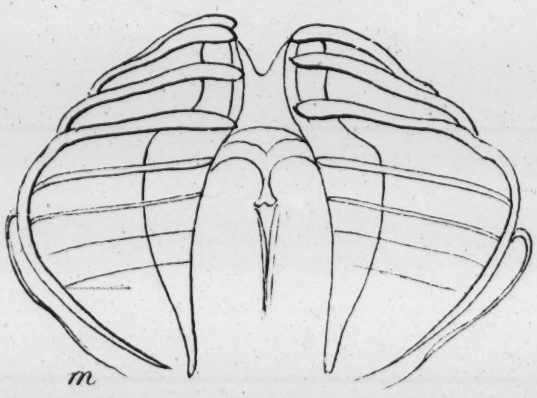
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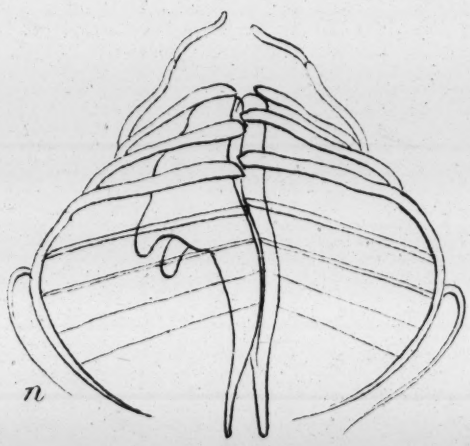
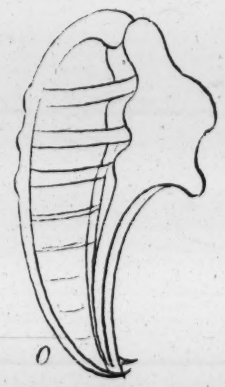

n


o

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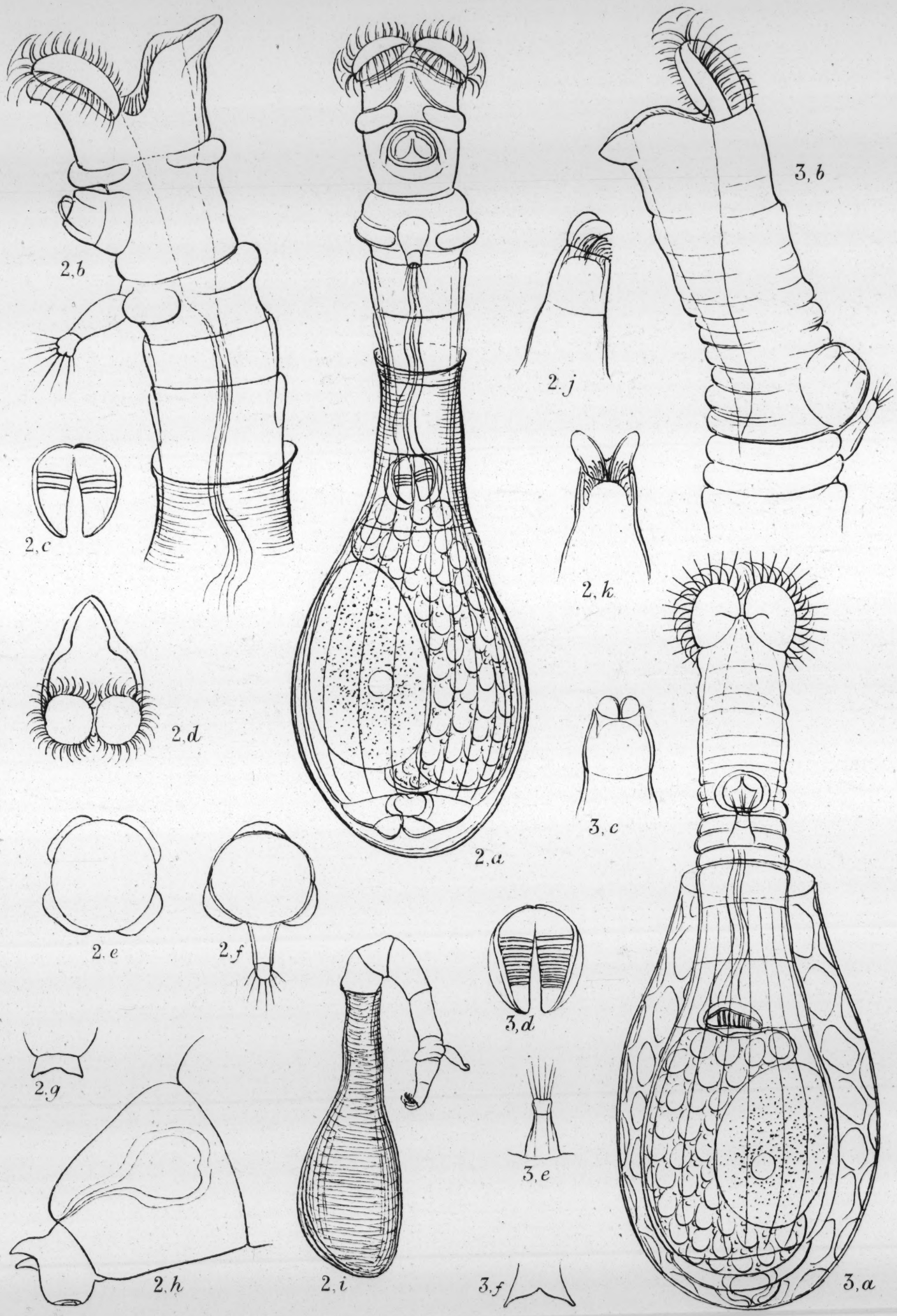

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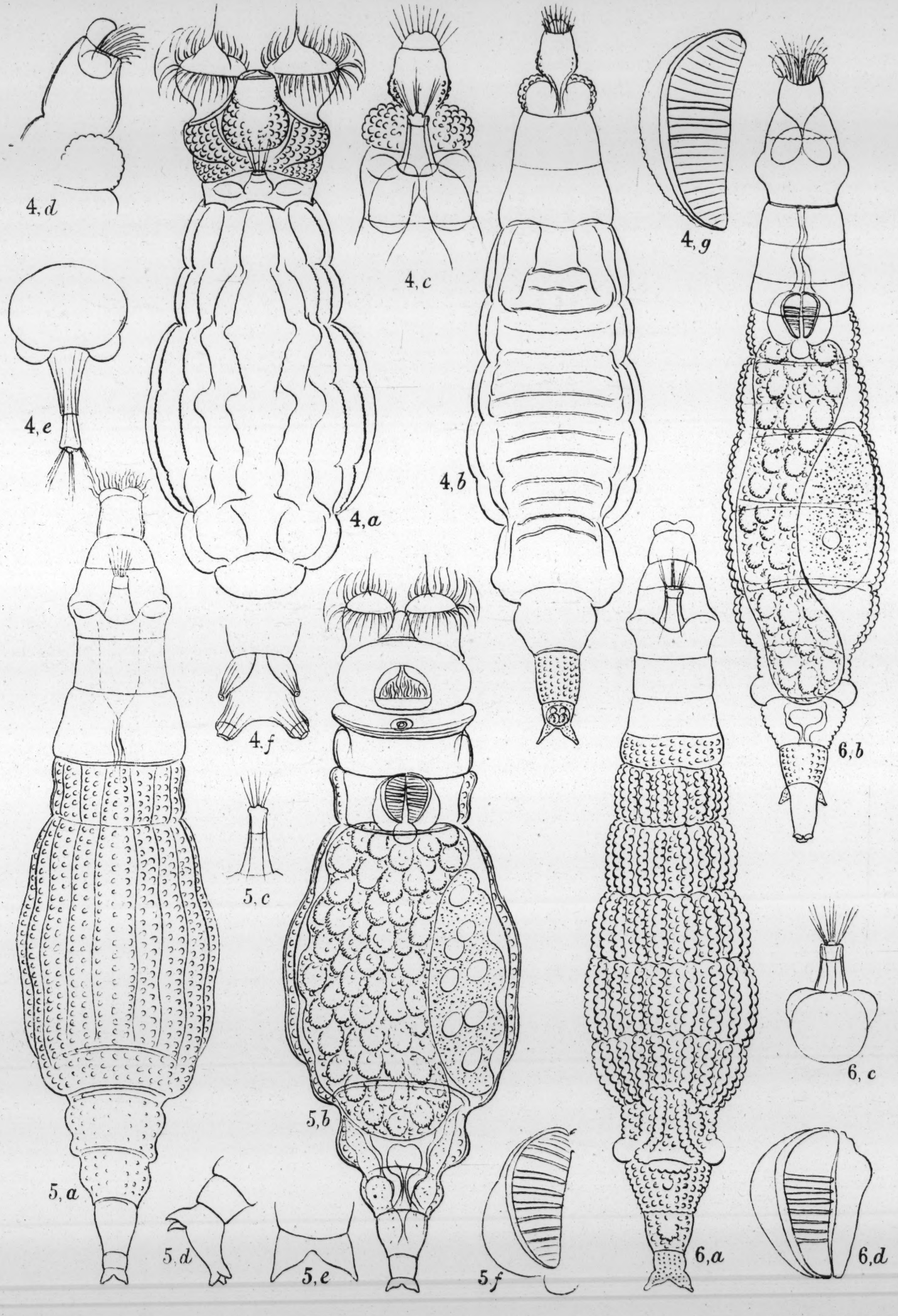
2, CALLIDINA ANGUSTICOLLIS, n. sp. 3, CALLIDINA ANNULATA, n. sp.

MURRAY: A NEW FAMILY AND TWELVE NEW SPECIES OF BDELLOIDA.—PLATE IV



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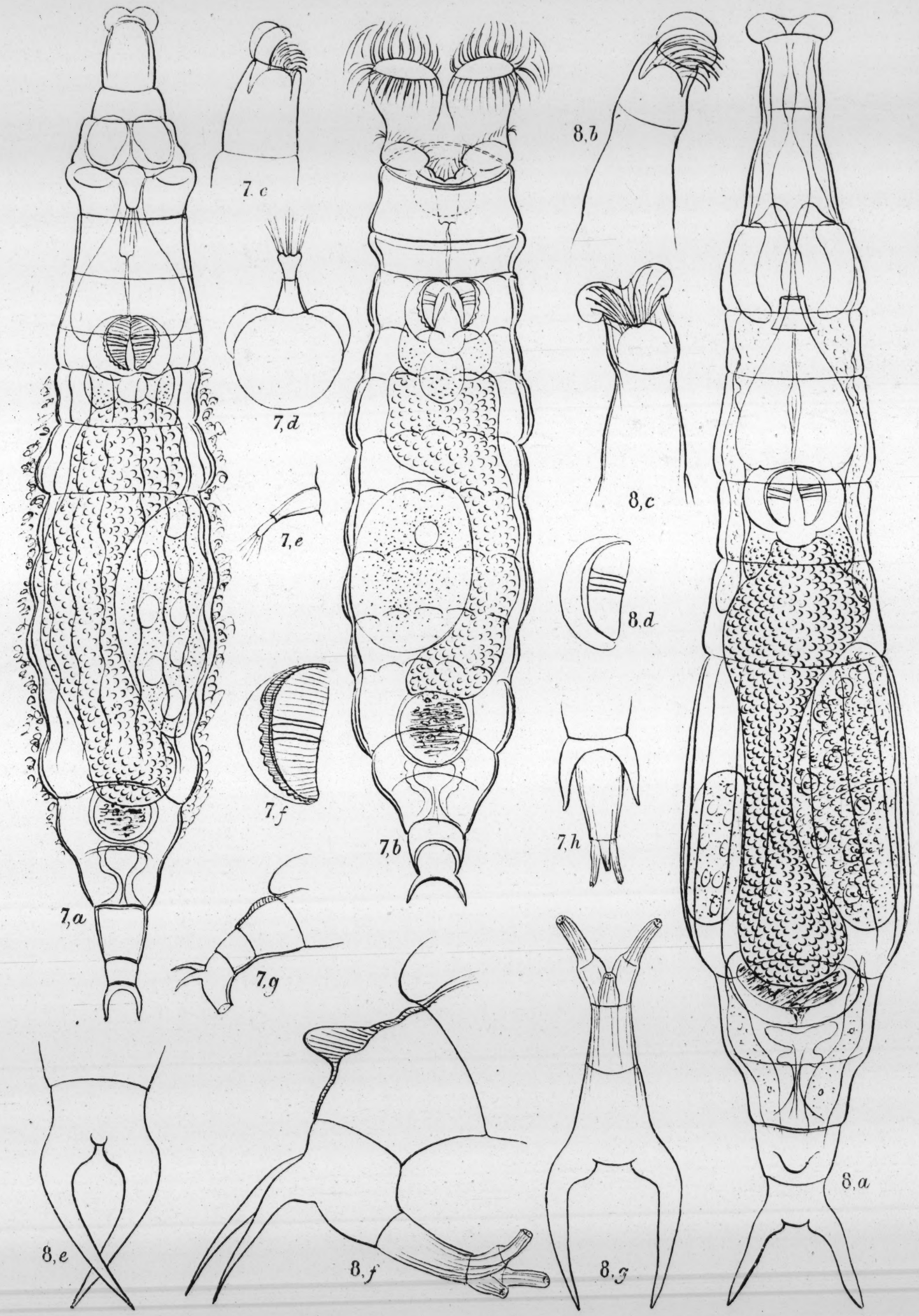
4. PHILODINA HUMEROSA, n. sp. 5. CALLIDINA PULCHRA, n. sp. 6. CALLIDINA CRENATA, n. sp.

MURRAY: A NEW FAMILY AND TWELVE NEW SPECIES OF BDELLOIDA.—PLATE V.



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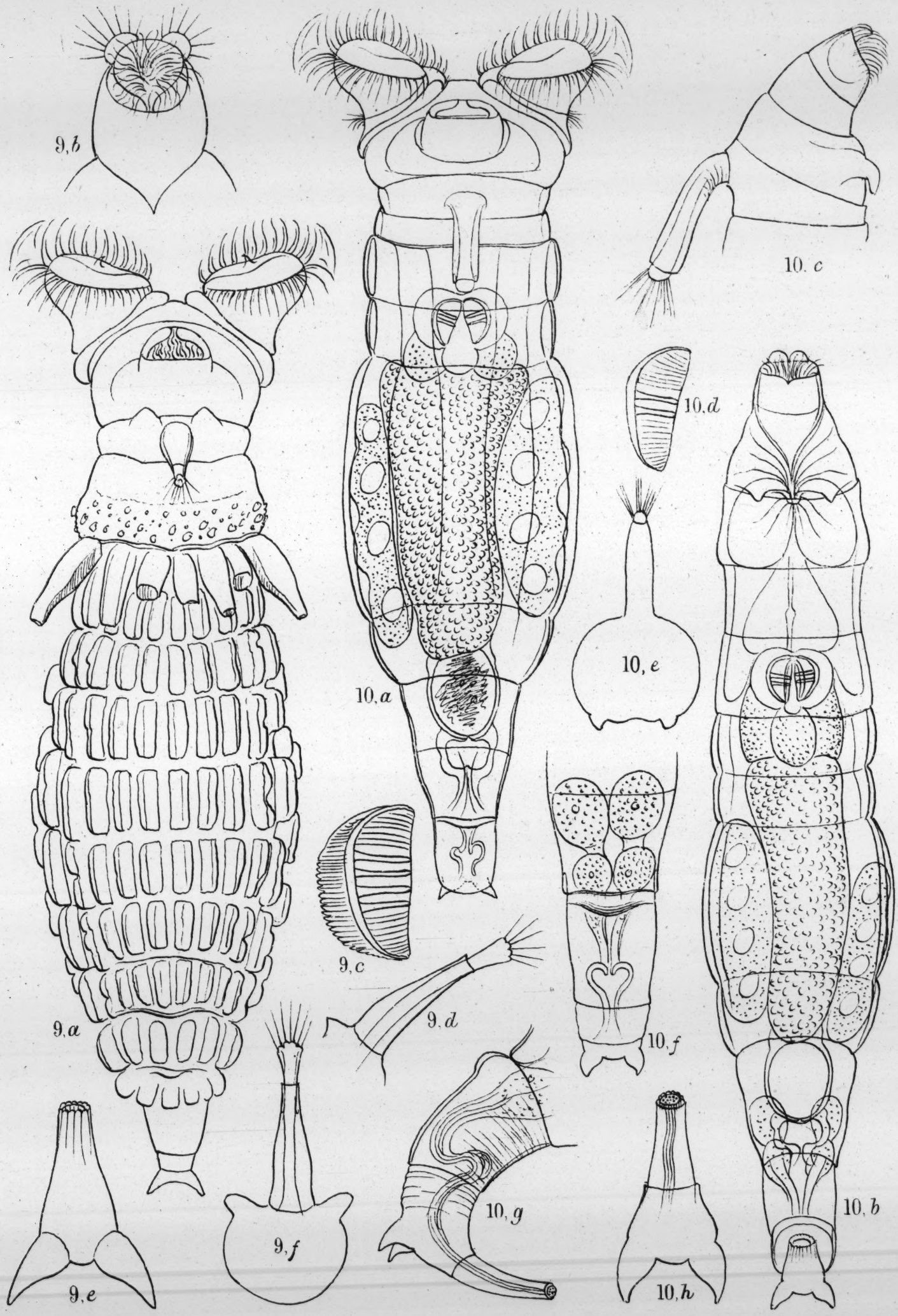
7. CALLIDINA MURICATA, n.sp. 8. CALLIDINA CRUCICORNIS, n.sp.

MURRAY: A NEW FAMILY AND TWELVE NEW SPECIES OF BDELLOIDA.—PLATE VI.



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9, CALLIDINA INCRASSATA, n. sp. 10, CALLIDINA ARMATA, n. sp.

Plate IV.

4. *Philodina humerosa*, n. sp.

4*a*, dorsal view, feeding.
4*b*, ventral view, creeping.
4*c*, back of head.
4*d*, side of rostrum.
4*e*, section of neck.
4*f*, toes.
4*g*, jaws.

5. *Callidina pulchra*, n. sp.

5*a*, dorsal view, showing stippling.
5*b*, dorsal view, feeding.
5*c*, antenna.
5*d*, side of foot.
5*e*, spurs.
5*f*, jaws.

6. *Callidina crenata*, n. sp.

6*a*, dorsal view, creeping, showing papillæ.
6*b*, ventral view, creeping, showing internal structure.
6*c*, section of neck.
6*d*, jaws.

Plate V.

7. *Callidina muricata*, n. sp.

7*a*, dorsal view, creeping.
7*b*, ventral view, feeding.
7*c*, side of rostrum.
7*d*, section of neck.
7*e*, side of antenna.
7*f*, jaws.
7*g*, side of foot.
7*h*, spurs and toes.

8. *Callidina crucicornis*, n. sp.

8*a*, dorsal view, creeping
8*b*, side of rostrum.
8*c*, front of rostrum.
8*d*, jaws.
8*e*, spurs crossed.
8*f*, side of foot.
8*g*, spurs and toes.

Plate VI.

9. *Callidina incrassata*, n. sp.

9*a*, dorsal view, feeding.
9*b*, front of rostrum.
9*c*, jaw.
9*d*, side of antenna.
9*e*, spurs and disc.
9*f*, section of neck.

10. *Callidina armata*, n. sp.

10*a*, dorsal view, feeding.
10*b*, ventral view, creeping.
10*c*, side of head.
10*d*, jaw.
10*e*, section of head, showing tooth-like processes.
10*f*, dorsal view of foot, showing fold and glands.
10*g*, side of foot.
10*h*, spurs and disc.

PLATE VII.

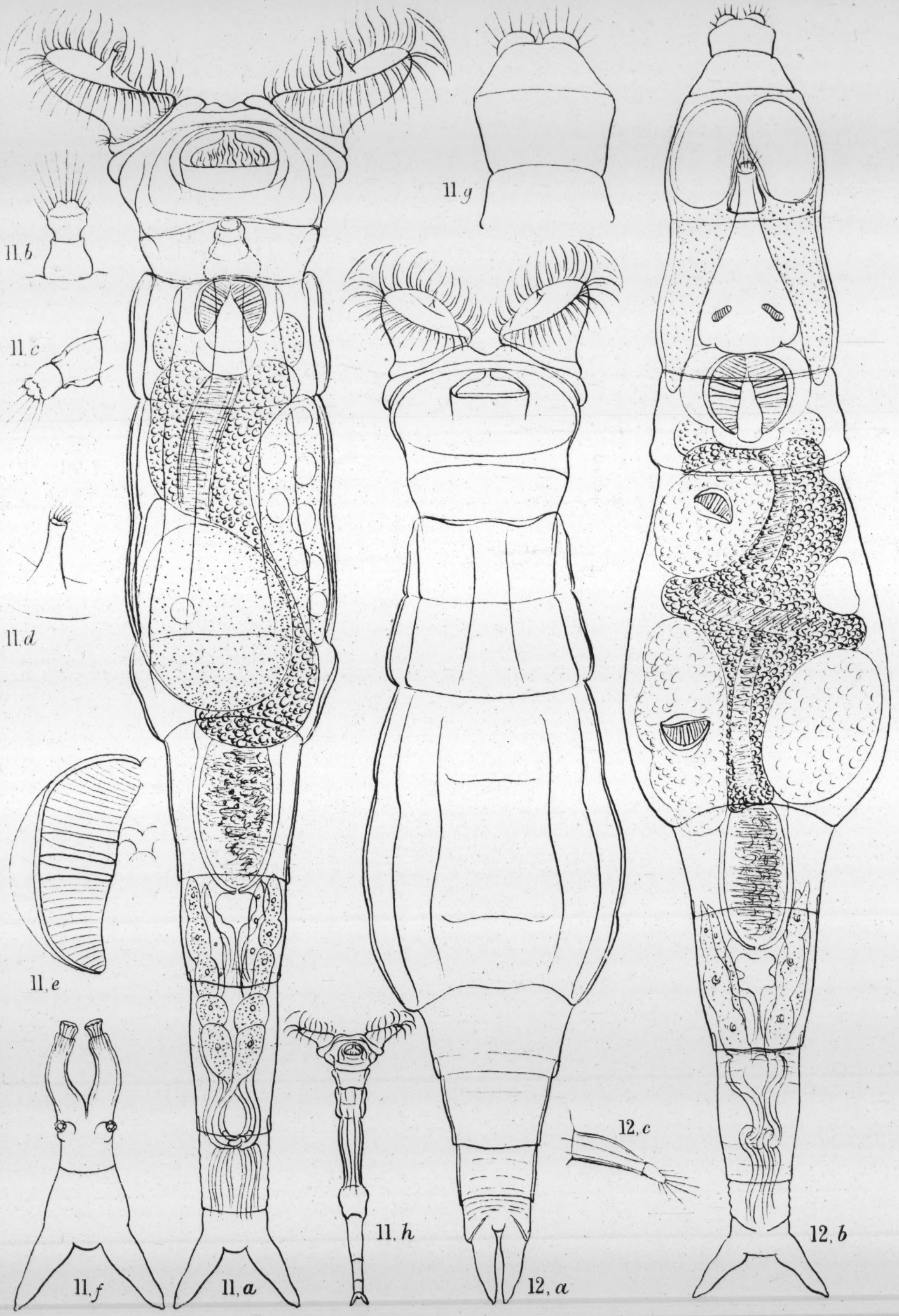
11. *Philodina laticeps*, n. sp.

11*a*, dorsal view, feeding.
11*b*, front of antenna.
11*c*, side of antenna.
11*d*, papilla on disc.
11*e*, jaw.
11*f*, spurs and toes.
11*g*, back of rostrum.
11*h*, characteristic attitude, with trunk narrowed.

12. *Philodina laticornis*, n. sp.

12*a*, dorsal view, swimming.
12*b*, dorsal view, creeping, showing viscera.
12*c*, side of antenna.

MURRAY: A NEW FAMILY AND TWELVE NEW SPECIES OF BDELLOIDA.—PLATE VII.



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11, PHILODINA LATICEPS, n. sp. 12, PHILODINA LATICORNIS, n. sp.

XVI.—The Eliminant of a Set of General Ternary Quadrics.—(Part III.)

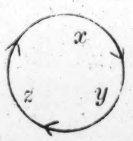
By Thomas Muir, LL.D.

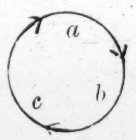
(MS. received December 12, 1904. Read January 23, 1905. Issued separately April 15, 1905.)

(41) The invariance of the equations

$$\left.\begin{aligned} a_1x^2 + b_1y^2 + c_1z^2 + f_1yz + g_1zx + h_1xy &= 0, \\ a_2x^2 + b_2y^2 + c_2z^2 + f_2yz + g_2zx + h_2xy &= 0, \\ a_3x^2 + b_3y^2 + c_3z^2 + f_3yz + g_3zx + h_3xy &= 0, \end{aligned}\right\}$$

with regard to the group of cyclical substitutions,





and the consequent invariance of the eliminant with regard to the reduced group consisting of the last two substitutions, has been already referred to. When the eliminant is expressed in terms of the three-line determinants formable from the array of coefficients, the invariance in question is self-evident, as each of the twenty-eight parts composing the expression is invariant by itself. For convenience this form may be repeated from § 31 with a slightly improved notation. It is

0000	$+\Sigma 1'4'8'9$
$+\Sigma 0077'$	$+\Sigma 4457'$
$-2\Sigma 0016$	$+\Sigma 1'558'$
$+\Sigma 07'8'9'$	$-\Sigma 1688'$
$-2\Sigma 0147'$	$-\Sigma 44'99'$
$+\ 0456$	$-\Sigma 11'88'$
$+2\Sigma 0159$	$-\Sigma 1'6'88$
$+\ 0123$	$+\Sigma 1137$
$+\Sigma 0125'$	$+\Sigma 115'7'$
$+\Sigma 04'5'6'$	$-\Sigma 4468$
$+\ 0'7'8'9'$	$-\Sigma 1'448$
$-\ 0'456$	$+\Sigma 1166$
$+\Sigma 45'7'9'$	$-\Sigma 1489$
$+\Sigma 13'7'8'$	$-\Sigma 114'9$

where

$$\begin{array}{ccc} & 0 & \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array} \qquad \begin{array}{ccc} & 0' & \\ 1' & 2' & 3' \\ 4' & 5' & 6' \\ 7' & 8' & 9' \end{array}$$

are used for

$$\begin{array}{ccccc} & | a_1b_2c_3 |, & & | f_1g_2h_3 |, & \\ | a_1b_2f_3 |, & | b_1c_2g_3 |, & | c_1a_2h_3 |, & | c_1g_2h_3 |, & | a_1h_2f_3 |, & | b_1f_2g_3 |, \\ | a_1b_2g_3 |, & | b_1c_2h_3 |, & | c_1a_2f_3 |, & | c_1h_2f_3 |, & | a_1f_2g_3 |, & | b_1g_2h_3 |, \\ | a_1b_2h_3 |, & | b_1c_2f_3 |, & | c_1a_2g_3 |, & | c_1f_2g_3 |, & | a_1g_2h_3 |, & | b_1h_2f_3 |, \end{array}$$

respectively.

(42) There is, however, a second form of invariance which it is convenient now to consider. Looking at the equations we at once see that the performance of the interchange

$$\begin{pmatrix} x & a & f \\ y & b & g \end{pmatrix}$$

leaves them unaltered, and that the same is true of either of the interchanges

$$\begin{pmatrix} y & b & g \\ z & c & h \end{pmatrix}, \quad \begin{pmatrix} z & c & h \\ x & a & f \end{pmatrix}.$$

From this it follows that the eliminant is invariant to each one of the three interchanges

$$\begin{pmatrix} a & f \\ b & g \end{pmatrix}, \quad \begin{pmatrix} b & g \\ c & h \end{pmatrix}, \quad \begin{pmatrix} c & h \\ a & f \end{pmatrix}.$$

Taking the first of these and observing its effect on our twenty determinants of the third order we find that it is equivalent to the substitution

$$\begin{pmatrix} 0, & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9; & 0', & 1', & 2', & 3', & 4', & 5', & 6', & 7', & 8', & 9' \\ -0, & -4, & -6, & -5, & -1, & -3, & -2, & -7, & -9, & -8; & -0', & -4', & -6', & -5', & -1', & -3', & -2', & -7', & -9', & -8' \end{pmatrix}.$$

In an expression, like the special form of eliminant given in the preceding §, where each term is the product of four of these determinants, and where, therefore, the sign of each determinant may be changed with impunity, this substitution has the same effect as the simpler substitution

$$\begin{pmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 8, & 9, & 1', & 2', & 3', & 4', & 5', & 6', & 8', & 9' \\ 4, & 6, & 5, & 1, & 3, & 2, & 9, & 8, & 4', & 6', & 5', & 1', & 3', & 2', & 9', & 8' \end{pmatrix}$$

or the interchange

$$\begin{pmatrix} 1, & 2, & 3, & 8, & 1', & 2' & 3', & 8' \\ 4, & 6, & 5, & 9, & 4', & 5' & 6', & 9' \end{pmatrix}.$$

Similarly we have

$$\begin{pmatrix} b & g \\ c & h \end{pmatrix} = \begin{pmatrix} 0, & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9; & 0', & 1', & 2', & 3', & 4', & 5', & 6', & 7', & 8', & 9' \\ -0, & -6, & -5, & -4, & -3, & -2, & -1, & -9, & -8, & -7; & -0', & -6', & -5', & -4', & -3', & -2', & -1', & -9', & -8', & -7' \end{pmatrix}$$

and therefore in the case of products of four

$$= \begin{pmatrix} 1, 2, 3, 4, 5, 6, 7, 9; & 1', 2', 3', 4', 5', 6', 7', 9' \\ 6, 5, 4, 3, 2, 1, 9, 7; & 6', 5', 4', 3', 2', 1', 9', 7' \end{pmatrix},$$

$$= \begin{pmatrix} 1, 2, 3, 7; & 1', 2', 3', 7' \\ 6, 5, 4, 9; & 6', 5', 4', 9' \end{pmatrix};$$

and, lastly, in the same circumstances

$$\begin{pmatrix} c & h \\ a & f \end{pmatrix} = \begin{pmatrix} 1, 2, 3, 7; & 1', 2', 3', 7' \\ 5, 4, 6, 8; & 5', 4', 6', 8' \end{pmatrix}.$$

(43) A comparison of the three interchanges, which, in the case of a four-factor product, we have thus found to be equivalent to

$$\begin{pmatrix} a & f \\ b & g \end{pmatrix}, \quad \begin{pmatrix} b & g \\ c & h \end{pmatrix}, \quad \begin{pmatrix} c & h \\ a & f \end{pmatrix}$$

respectively, leads at once to the further observation that if the expression in which the interchanges have to be made be invariant to the cyclical substitution, the three interchanges are not essentially different. So far, therefore, as the above eliminant is concerned, we need only consider one of the interchanges, say the interchange

$$\begin{pmatrix} 1, 2, 3, 8; & 1', 2', 3', 8' \\ 4, 6, 5, 9; & 4', 6', 5', 9' \end{pmatrix},$$

it being borne in mind that this implies that the determinants

$$0, 7; 0', 7'$$

are invariant to the interchange. The determinants $0, 0'$, which are invariant to the cyclical substitution as well, we shall therefore speak of as being doubly-invariant.

(44) Turning then to the eliminant and applying this interchange to each of its twenty-eight parts, we find that twelve of them, viz., the

1st, 2nd, 3rd, 4th, 5th, 7th, 11th, 15th, 18th, 21st, 26th, 27th

are doubly-invariant; that twelve others may be grouped as six binomials which are doubly-invariant, either term of each binomial being produced from the other term, viz.

6th and 8th,
13th and 14th,
16th and 22nd,
17th and 23rd,
19th and 20th,
25th and 28th;

and that the four remaining parts (the 9th, 10th, 12th, 24th) are

$$\Sigma 0125', \quad 04'5'6', \quad -0'456, \quad -\Sigma 4468.$$

Now we can show (see § 39) that $04'5'6' - 0'456$ is expressible as the difference of two terms which are each doubly-invariant, viz., the difference $07'8'9' - 0'789$. Further, since

$$\Sigma 0125' = -\Sigma 1129 + \Sigma 1246$$

the other two terms are expressible in the form

$$-(\Sigma 1129 + \Sigma 4468) \quad + \quad \Sigma 1246,$$

where the binomial and the single term which follows it are both doubly-invariant. There is thus finally obtained an expression for the eliminant which shows its property of double-invariance, the constituent parts being fourteen single terms and seven binomials, viz.

	0000	$+\ 0123 \quad +\ 0456$
$-$	$2\cdot\Sigma 0016$	$-\Sigma 1129 \quad -\Sigma 4468$
$+$	$\Sigma 0077'$	$+\Sigma 1137' \quad +\Sigma 4457'$
$-$	$2\cdot\Sigma 0147'$	$-\Sigma 114'9 \quad -\Sigma 1'448'$
$+$	$2\cdot\Sigma 0159$	$+\Sigma 115'7' \quad +\Sigma 1'558'$
$+$	$4\cdot\ 07'8'9'$	$-\Sigma 11'88 \quad -\Sigma 44'99'$
$-$	$0'789$	$+\Sigma 13'7'8' \quad +\Sigma 45'7'9'$
$+$	$0'7'8'9'$	
$+$	$\Sigma 1166$	
$-$	$\Sigma 1489$	
$+$	$\Sigma 1'4'8'9'$	
$-$	$\Sigma 1688'$	
$-$	$\Sigma 1'6'88'$	

(45) Let us return now to the eliminant of the 10th order as obtained in § 9, viz.

$$\begin{vmatrix}
a_1 & . & . & h_1 & . & c_1 & . & g_1 & b_1 & f_1 \\
. & b_1 & . & a_1 & f_1 & . & c_1 & . & h_1 & g_1 \\
. & . & c_1 & . & b_1 & g_1 & f_1 & a_1 & . & h_1 \\
a_2 & . & . & h_2 & . & c_2 & . & g_2 & b_2 & f_2 \\
. & b_2 & . & a_2 & f_2 & . & c_2 & . & h_2 & g_2 \\
. & . & c_2 & . & b_2 & g_2 & f_2 & a_2 & . & h_2 \\
a_3 & . & . & h_3 & . & c_3 & . & g_3 & b_3 & f_3 \\
. & b_3 & . & a_3 & f_3 & . & c_3 & . & h_3 & g_3 \\
. & . & c_3 & . & b_3 & g_3 & f_3 & a_3 & . & h_3 \\
. & . & . & 4 & 5 & 6 & 4' & 5' & 6' & 0+0'
\end{vmatrix}.$$

By transposition of rows and of columns, and by altering its sign, this is readily changeable into

$$\begin{vmatrix}
a_1 & b_1 & c_1 & h_1 & . & . & g_1 & . & . & f_1 \\
a_2 & b_2 & c_2 & h_2 & . & . & g_2 & . & . & f_2 \\
a_3 & b_3 & c_3 & h_3 & . & . & g_3 & . & . & f_3 \\
. & h_1 & . & a_1 & b_1 & c_1 & . & f_1 & . & g_1 \\
. & h_2 & . & a_2 & b_2 & c_2 & . & f_2 & . & g_2 \\
. & h_3 & . & a_3 & b_3 & c_3 & . & f_3 & . & g_3 \\
. & . & g_1 & . & . & f_1 & a_1 & b_1 & c_1 & h_1 \\
. & . & g_2 & . & . & f_2 & a_2 & b_2 & c_2 & h_2 \\
. & . & g_3 & . & . & f_3 & a_3 & b_3 & c_3 & h_3 \\
. & 6' & 6 & 4 & . & 4' & 5' & 5 & . & 0+0'
\end{vmatrix},$$

—a form more suitable for obtaining the cofactors of the elements of the last row in terms of the familiar three-line determinants 0, 1, 2, ... Taking first the cofactor of $0+0'$, and using LAPLACE's expansion-theorem, we find it

$$\begin{aligned} &= 0(00-16)-7(-89')+4(02-17')-3(05+69')+9(78)-8'(25-7'9'),\\ &= 000-\Sigma 016-\Sigma 147'+2{\cdot}789+7'8'9'. \end{aligned}$$

Similarly the cofactor of 4 is found to be

$$\begin{aligned} &-002+038+056+03'6-189-1'48+224\\ &-255'+3'47'-4'99'-559+5'7'9'+669+6'89, \end{aligned}$$

and the cofactor of $4'$ to be

$$\begin{aligned} &-004+013+036'+067-119-11'4+1'8'9'\\ &+244+26'8'-25'7-2'99'+339'+3'79-579. \end{aligned}$$

The full eliminant is thus

$$\begin{aligned} &(0+0')[000-\Sigma 016-\Sigma 147'+2{\cdot}789+7'8'9']\\ &+\overset{\circ}{\Sigma}4\begin{bmatrix}-002+038+03'6+056-1'48-189+224\\ -255'+3'47'-4'99'-559+5'7'9'+669'+6'89\end{bmatrix}\\ &+\overset{\circ}{\Sigma}4'\begin{bmatrix}-004+013+036'+067-119-11'4+1'8'9'\\ +244+26'8'-25'7-2'99'+339'+3'79-579\end{bmatrix}. \end{aligned}$$

This does not, of course, differ from the form used in § 41. As a matter of fact, it will be found on examination that nineteen of its thirty-eight terms agree with terms in the expression of § 41, and that the other nineteen can be changed without much difficulty so as to establish the identity of the two expressions.

(46) As may be supposed, however, the importance of the new result does not consist in its affording a verification of that previously obtained. It is more interesting, in fact, in its unsimplified state; for it has now to be noted that each of the three lengthy expressions found in it as the cofactors of $0+0'$, 4, $4'$ can be put in the form of a simple three-line determinant. For example,

$$\begin{vmatrix} 0+\frac{58'}{4} & 8 & \frac{59}{4}\\ \frac{67}{5} & 0+\frac{69'}{5} & 9\\ 7 & \frac{48}{6} & 0+\frac{47'}{6}\end{vmatrix}$$

$$\begin{aligned} &= \left(0+\frac{58'}{4}\right)\left(0+\frac{69'}{5}\right)\left(0+\frac{47'}{6}\right)+\frac{67}{5}\cdot\frac{48}{6}\cdot\frac{59}{4}+789-\overset{\circ}{\Sigma}7\left(0+\frac{69'}{5}\right)\frac{59}{4},\\ &= 000+00\left(\frac{58'}{4}+\frac{69'}{5}+\frac{47'}{6}\right)+0\left(\frac{47'9'}{5}+\frac{58'7'}{6}+\frac{69'8'}{4}\right)+7'8'9'+2{\cdot}789-\overset{\circ}{\Sigma}07\frac{59}{4}-\overset{\circ}{\Sigma}7\frac{699'}{4},\\ &= 000+\overset{\circ}{\Sigma}05\left(\frac{08'-79}{4}\right)+\overset{\circ}{\Sigma}69\left(\frac{08'-79}{4}\right)+7'8'9'+2{\cdot}789,\\ &= 000-\overset{\circ}{\Sigma}053-\overset{\circ}{\Sigma}369'+7'8'9'+2{\cdot}789,\\ &= 000-\overset{\circ}{\Sigma}016-\Sigma 147'+7'8'9'+2{\cdot}789,\\ &= \text{cofactor of } 0+0'. \end{aligned}$$

Similarly it may be shown that

$$-\begin{vmatrix} 8 & \frac{59}{4} & 2-\frac{55'}{4} \\ 0+\frac{69'}{5} & 9 & 3-\frac{66'}{5} \\ \frac{48}{6} & 0+\frac{47'}{6} & 1-\frac{44'}{6} \end{vmatrix}$$

$$= -002+038+03'6+056-1'48-189+224-255'+3'47'-4'99'-559+5'7'9'+669'+6'89$$

$$= \text{cofactor of } 4\,;$$

and

$$-\begin{vmatrix} 0+\frac{28'}{3} & \frac{27}{3} & 5-\frac{22'}{3} \\ 9 & 0+\frac{39'}{1} & 6-\frac{33'}{1} \\ \frac{19}{2} & 7 & 4-\frac{11'}{2} \end{vmatrix}$$

$$= -004+013+\ \ldots\ .$$

$$= \text{cofactor of } 4'.$$

It consequently follows that the eliminant may be put in the form

$$(0+0')\begin{vmatrix} 0+\frac{58'}{4} & 8 & \frac{59}{4} \\ \frac{67}{5} & 0+\frac{69'}{5} & 9 \\ 7 & \frac{48}{6} & 0+\frac{47'}{6} \end{vmatrix} - \mathring{\Sigma}4\begin{vmatrix} 8 & \frac{59}{4} & 2-\frac{55'}{4} \\ 0+\frac{69'}{5} & 9 & 3-\frac{66'}{5} \\ \frac{48}{6} & 0+\frac{47'}{6} & 1-\frac{44'}{6} \end{vmatrix} - \mathring{\Sigma}4'\begin{vmatrix} 0+\frac{28'}{3} & \frac{27}{3} & 5-\frac{22'}{3} \\ 9 & 0+\frac{39'}{1} & 6-\frac{33'}{1} \\ \frac{19}{2} & 7 & 4-\frac{11'}{2} \end{vmatrix}$$

or, say, for convenience of future reference,

$$(0+0')\Delta_1 - \mathring{\Sigma}4\Delta_2 - \mathring{\Sigma}4'\Delta_3.$$

(47) A glance at the cofactor of $0+0'$ as first obtained, viz., $000-\mathring{\Sigma}016-\ldots$ shows that it is invariant to the interchange of § 42. Since, therefore, the performance of this interchange on the *determinant* form of it, Δ_1,—where the invariance is not in evidence,—cannot make any alteration in substance, we shall obtain thereby a second determinant form ∇_1, thus arriving at the unexpected identity

$$\begin{vmatrix} 0+\frac{58'}{4} & 8 & \frac{59}{4} \\ \frac{67}{5} & 0+\frac{69'}{5} & 9 \\ 7 & \frac{48}{6} & 0+\frac{47'}{6} \end{vmatrix} = \begin{vmatrix} 0+\frac{39'}{1} & 9 & \frac{38}{1} \\ \frac{27}{3} & 0+\frac{28'}{3} & 8 \\ 7 & \frac{19}{2} & 0+\frac{17'}{2} \end{vmatrix}.$$

Further, since the whole eliminant is invariant to the interchange, the removal from it of the invariant portion $(0+0')\,\Delta_1$ must leave a portion which in substance, if not in form, is likewise invariant. This consideration gives us the identity

$$\overset{\circ}{\Sigma}4 \begin{vmatrix} 8 & \frac{59}{4} & 2-\frac{55'}{4} \\ 0+\frac{69'}{5} & 9 & 3-\frac{66'}{5} \\ \frac{48}{6} & 0+\frac{47'}{6} & 1-\frac{44'}{6} \end{vmatrix} + \overset{\circ}{\Sigma}4' \begin{vmatrix} 0+\frac{28'}{3} & \frac{27}{3} & 5-\frac{22'}{3} \\ 9 & 0+\frac{39'}{1} & 6-\frac{33'}{1} \\ \frac{19}{2} & 7 & 4-\frac{11'}{2} \end{vmatrix}$$

$$= \overset{\circ}{\Sigma}1 \begin{vmatrix} 9 & \frac{38}{1} & 6-\frac{33'}{1} \\ 0+\frac{28'}{3} & 8 & 5-\frac{22'}{3} \\ \frac{19}{2} & 0+\frac{17'}{2} & 4-\frac{11'}{2} \end{vmatrix} + \overset{\circ}{\Sigma}1' \begin{vmatrix} 0+\frac{69'}{5} & \frac{67}{5} & 3-\frac{66'}{5} \\ 8 & 0+\frac{58'}{4} & 2-\frac{55'}{4} \\ \frac{48}{6} & 7 & 1-\frac{44'}{6} \end{vmatrix},$$

or, say,

$$\overset{\circ}{\Sigma}4\,\Delta_2 + \overset{\circ}{\Sigma}4'\,\Delta_3 = \overset{\circ}{\Sigma}1\,\nabla_2 + \overset{\circ}{\Sigma}1'\,\nabla_3.$$

If we denote the first portion of the eliminant, viz. $(0+0')\,\Delta_1$ by A, and the second portion by $-B$, and the alternative forms of these by A′ and $-B'$, we can thus express the eliminant in four different ways, viz.

$$A-B, \quad A'-B, \quad A-B', \quad A'-B'.$$

(48) Using the cyclical substitution on the right-hand member of the immediately preceding identity, we see that we can put in place of it

$$\overset{\circ}{\Sigma}2 \begin{vmatrix} 7 & \frac{19}{2} & 4-\frac{11'}{2} \\ 0+\frac{39'}{1} & 9 & 6-\frac{33'}{1} \\ \frac{27}{3} & 0+\frac{28'}{3} & 5-\frac{22'}{3} \end{vmatrix} + \overset{\circ}{\Sigma}2' \begin{vmatrix} 0+\frac{47'}{6} & \frac{48}{6} & 1-\frac{44'}{6} \\ 9 & 0+\frac{69'}{5} & 3-\frac{66'}{5} \\ \frac{59}{4} & 8 & 2-\frac{55'}{4} \end{vmatrix}.$$

But the determinants here are those occurring in the left-hand member: consequently we deduce

$$\overset{\circ}{\Sigma}(4-2'\,\Delta_2 = \overset{\circ}{\Sigma}(2-4')\,\Delta_3.$$

(49) Returning to § 46, and noting that Δ_2 has two columns in common with Δ_1, and that the result of the cyclical substitution on Δ_2 is simply to change this pair of columns into another pair of Δ_1, the third column remaining all the while unaltered, we see that

$$(0+0')\,\Delta_1 - \overset{\circ}{\Sigma}4\,\Delta_2$$

can be put in the form

$$\begin{vmatrix} 0+\frac{58'}{4} & 8 & \frac{59}{4} & 2-\frac{55'}{4} \\ \frac{67}{5} & 0+\frac{69'}{5} & 9 & 3-\frac{66'}{5} \\ 7 & \frac{48}{6} & 0+\frac{47'}{6} & 1-\frac{44'}{6} \\ 4 & 5 & 6 & 0+0' \end{vmatrix}.$$

A new expression for the eliminant is thus

$$\begin{vmatrix} 0+\frac{47'}{6} & 7 & \frac{48'}{6} & 1-\frac{44'}{6} \\ \frac{59}{4} & 0+\frac{58'}{4} & 8 & 2-\frac{55'}{4} \\ 9 & \frac{67'}{5} & 0+\frac{69'}{5} & 3-\frac{66'}{5} \\ 6 & 4 & 5 & 0+0' \end{vmatrix} - \sum 4' \begin{vmatrix} 0+\frac{28'}{3} & \frac{27}{3} & 5-\frac{22'}{3} \\ 9 & 0+\frac{39'}{1} & 6-\frac{33'}{1} \\ \frac{19}{2} & 7 & 4-\frac{11'}{2} \end{vmatrix}.$$

This can be further improved by removing from the four-line determinant the terms containing $0'$, and associating them in the form $0'\nabla_1$ with $-\Sigma 4'\Delta_3$ just as $0\Delta_1$ has been associated with $-\Sigma 4\Delta_2$, the final result being

$$\begin{vmatrix} 0+\frac{47'}{6} & 7 & \frac{48}{6} & 1-\frac{44'}{6} \\ \frac{59}{4} & 0+\frac{58'}{4} & 8 & 2-\frac{55'}{4} \\ 9 & \frac{67}{5} & 0+\frac{69'}{5} & 3-\frac{66'}{5} \\ 6 & 4 & 5 & 0 \end{vmatrix} + \begin{vmatrix} 0+\frac{17'}{2} & \frac{19}{2} & 7 & 4-\frac{11'}{2} \\ 8 & 0+\frac{28'}{3} & \frac{27}{3} & 5-\frac{22'}{3} \\ \frac{38}{1} & 9 & 0+\frac{39'}{1} & 6-\frac{33'}{1} \\ 4' & 5' & 6' & 0' \end{vmatrix}.$$

From this, by making the now familiar interchange, we obtain the alternative form

$$\begin{vmatrix} 0+\frac{17'}{2} & 7 & \frac{19}{2} & 4-\frac{11'}{2} \\ \frac{38}{1} & 0+\frac{39'}{1} & 9 & 6-\frac{33'}{1} \\ 8 & \frac{27}{3} & 0+\frac{28'}{3} & 5-\frac{22'}{3} \\ 2 & 1 & 3 & 0 \end{vmatrix} + \begin{vmatrix} 0+\frac{47'}{6} & \frac{48}{6} & 7 & 1-\frac{44'}{6} \\ 9 & 0+\frac{69'}{5} & \frac{67}{5} & 3-\frac{66'}{5} \\ \frac{59}{4} & 8 & 0+\frac{58'}{4} & 2-\frac{55'}{4} \\ 1' & 3' & 2' & 0' \end{vmatrix}.$$

Here, however, if in both determinants we transpose the second and third rows, and thereafter the second and third columns, we find that either determinant has three rows in common with one of the determinants of the form from which this was derived. Subtraction is thus readily accomplished, the result being

$$\begin{vmatrix} 0+\frac{47'}{6} & 7 & \frac{48}{6} & 1-\frac{44'}{6} \\ \frac{59}{4} & 0+\frac{58'}{4} & 8 & 2-\frac{55'}{4} \\ 9 & \frac{67}{5} & 0+\frac{69}{5} & 3-\frac{66'}{5} \\ 6-1' & 4-2' & 5-3' & . \end{vmatrix} = \begin{vmatrix} 0+\frac{17'}{2} & \frac{19}{2} & 7 & 4-\frac{11'}{2} \\ 8 & 0+\frac{28'}{3} & \frac{27}{3} & 5-\frac{22'}{3} \\ \frac{38}{1} & 9 & 0+\frac{39'}{1} & 6-\frac{33'}{1} \\ 2-4' & 3-5' & 1-6' & . \end{vmatrix}.$$

This is a verification of the identity

$$\mathring{\Sigma}(4-2')\Delta_2 = \mathring{\Sigma}(2-4')\Delta_3$$

already obtained in § 48. The latter form of it shows that both of the four-line determinants are invariant to the cyclical substitution; and as the interchange of § 42 transforms the one into the other, it follows that both are doubly-invariant.

(50) The two new forms of eliminant just reached make clear the fact that if one of the four sets of determinants

$$4, 5, 6 \; ; \; 4', 5', 6' \; ; \; 1, 2, 3 \; ; \; 1', 2', 3' \; ;$$

vanishes, the eliminant takes the form of a single four-line determinant. For example, if 4, 5, 6 have each the value zero, the eliminant is

$$\begin{vmatrix} 0+\frac{17'}{2} & \frac{19}{2} & 7 & -\frac{11'}{2} \\ 8 & 0+\frac{28'}{3} & \frac{27}{3} & -\frac{22'}{3} \\ \frac{38}{1} & 9 & 0+\frac{39'}{1} & -\frac{33'}{2} \\ 4' & 5' & 6' & 0+0' \end{vmatrix}.$$

We are thus brought to consider the problem of finding the set of four equations whose coefficients are the elements of this determinant. In the quest for a solution we are not without a lead, since for one of the very special cases brought into notice by SYLVESTER the desired set of equations has already been obtained.*

(51) From the fundamental set of equations there can be deduced (§ 33)

$$\begin{aligned} |\, u_1 c_2 h_3 \,| &\equiv -9x^2 + 2y^2 - 7'yz + 1'xy = 0, \\ |\, u_1 a_2 b_3 \,| &\equiv 0z^2 + 1yz + 4zx + 7xy = 0, \end{aligned}$$

and from these by multiplication by z and y respectively we obtain two equations involving the desirable facients yz^2, zx^2, xy^2, xyz, together with the undesirable y^2z. On eliminating the last mentioned there results

$$(02+17')yz^2 + 19zx^2 + 27xy^2 + (24-11')xyz = 0,$$

and by cyclical substitution

$$\begin{aligned} 38yz^2 + (03+28')zx^2 + 27xy^2 + (35-22')xyz &= 0, \\ 38yz^2 + 19zx^2 + (01+39')xy^2 + (16-33')xyz &= 0. \end{aligned}$$

* *Proc. Roy. Soc. Edin.*, xx. p. 377.

From these we obtain

$$\begin{aligned}
\left(0+\tfrac{1}{2}7'\right)yz^2 \quad & \tfrac{1}{2}9zx^2 + \quad 7xy^2 + \left(4-\tfrac{1}{2}1'\right)xyz = 0 \\
8yz^2 + \left(0+\tfrac{2}{3}8'\right)zx^2 + \quad & \tfrac{2}{3}7xy^2 + \left(5-\tfrac{2}{3}2'\right)xyz = 0 \\
\tfrac{3}{1}8yz^2 + \quad 9zx^2 + \left(0+\tfrac{3}{1}9'\right)xy^2 + & \left(6-\tfrac{3}{1}3'\right)xyz = 0.
\end{aligned}$$

The performance of the interchange of § 42 gives the companion set

$$\begin{aligned}
\left(0+\tfrac{5}{4}8'\right)x^2y + \quad 8y^2z + \quad & \tfrac{5}{4}9z^2x + \left(2-\tfrac{5}{4}5'\right)xyz = 0 \\
\tfrac{6}{5}7x^2y + \left(0+\tfrac{6}{5}9'\right)y^2z + \quad & 9z^2x + \left(3-\tfrac{6}{5}6'\right)xyz = 0 \\
7x^2y + \quad \tfrac{4}{6}8y^2z + \left(0+\tfrac{4}{6}7'\right)z^2x + & \left(1-\tfrac{4}{6}4'\right)xyz = 0.
\end{aligned}$$

The necessary fourth equation for the cases alluded to in the preceding paragraph is got from the cubic M_3 of § 9,

$$4x^2y + 5y^2z + 6z^2x + 4'yz^2 + 5'zx^2 + 6'xy^2 + (0+0')xyz = 0,$$

or from its companion M'_3,

$$2'x^2y + 3'y^2z + 1'z^2x + 2yz^2 + 3zx^2 + 1xy^2 + (0+0')xyz = 0,$$

by putting three appropriate coefficients equal to zero.

(52) Another special case of similar type is still more interesting, viz., the case where $7'$, $8'$, $9'$ vanish. The Jacobian of the given set of equations, viz.

$$-\Sigma(8'x^3) + \Sigma(2\cdot4+2')x^2y + \Sigma(2\cdot2+4')yz^2 + (4\cdot0+0')xyz = 0,$$

then loses three of its terms; and as the operation $2M_3 + M'_3$ gives

$$\Sigma(2\cdot4+2')x^2y + \Sigma(2\cdot4'+2)yz^2 + 3(0+0')xyz = 0,$$

it is clear that there follows

$$\Sigma(2-4)yz^2 + (0-2\cdot0')xyz = 0,$$

—an equation which can be used to complete the first set of three in § 51. Since the vanishing of $7'$, $8'$, $9'$ makes $11'$, $22'$, $33'$ identical, the resulting eliminant is

$$\begin{vmatrix}
0 & \frac{19}{2} & 7 & 4-2' \\
8 & 0 & \frac{27}{3} & 5-3' \\
\frac{38}{1} & 9 & 0 & 6-1' \\
2-4' & 3-5' & 1-6' & 0-20'
\end{vmatrix}.$$

Had we used the operation $M_3 + 2\,M'_3$ we should have obtained the alternative form

$$\begin{vmatrix} 0 & 8 & \frac{59}{4} & 2-4' \\ \frac{67}{5} & 0 & 9 & 3-5' \\ 7 & \frac{48}{6} & 0 & 1-6' \\ 4-2' & 5-3' & 6-1' & 0-2\cdot0' \end{vmatrix}.$$

In this case, however, the two forms are not even superficially different, the one being obtainable from the other by changing rows into columns and attending to the identities

$$26=89 \quad , \quad 34=97 \quad , \quad 15=78:$$

or,

$$159 = 267 = 348 = 789.$$

XVII.—Theorems relating to a Generalization of Bessel's Function. II.

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1.

Introduction.

The theory of the functions commonly known as q functions might perhaps be greatly developed, if investigators were to work on lines suggested by the functional notation of well-known analytic functions. For instance, the analysis connected with the circular functions $\sin x$, $\cos x$, . . . , might be regarded as the theory of certain infinite products without using any special functional notation. It need not be explained however, how great was the gain to elementary algebra by the introduction of the exponential function (regarded as the limit of a certain infinite product, or as the limit of a certain infinite series) denoted e^x, with certain characteristic properties, enabling the worker to make transformations easily and quickly. Of course, the vast store of interesting and in many cases useful results connected with the elementary functions of analysis might have been obtained without the introduction of any notation capable of rapid and easy transformations, but I think it unlikely that they would have been obtained.

In chapter xi. of Cayley's *Elliptic Functions* the identity

$$\frac{1}{1-q^2.1-q^4\ldots.1-q^{2n}}\left\{1+\frac{q^2}{1-q^2}.\frac{q^{2n}}{1-q^{2n+2}}+\frac{q^8}{1-q^2.1-q^4}.\frac{q^{4n}}{1-q^{2n+2}.1-q^{2n+4}}+\ldots.\right\}$$
$$=\prod_{m=1}^{\infty}\frac{1}{(1-q^{2m})} \qquad (1)$$

is used in order to express Jacobi's Θ function, in the well-known form

$$1-2q\cos 2x+2q^4\cos 4x-2q^9\cos 6x+\ldots.$$

The likeness of the series (1) to Bessel's series is very obvious. It is a very special case of the series which I have denoted $J_{[n]}$ in previous papers, and in itself might have suggested a theory of q functions analogous to Bessel's functions. In the discussion

of q functions a great variety of notations has been used. I propose in this paper to bring before the Society a series of formulæ relating firstly to a function $\mathrm{E}_p(x)$ analogous to exp (x). These formulæ are supplementary to those given in *Trans. Roy. Soc. Edin.*, vol. xli. pp. 105–118, and will lead to one or two interesting properties of a function $\mathrm{J}^p_{m,n}(x)$, which may be termed a generalized Bessel-function of double order, and to various novel expressions of elliptic functions in terms of the generalized Bessel-function. For example, JACOBI's Θ function is expressed by the form

$$\Theta\left(\frac{2\mathrm{K}x}{\pi}\right) = q_0 \frac{\mathfrak{J}_{[n]}(u)\mathfrak{J}_{[n]}(v)}{\mathrm{J}_{[n]}(u)\mathrm{J}_{[n]}(v)} \quad . \quad . \quad . \quad . \quad (\mathrm{A})$$

in which

$$q_0 = \prod_{m=1}^{\infty}(1-q^{2m})$$

$$u = \frac{i\sqrt{q}}{q-1}e^{ix}$$

$$v = \frac{i\sqrt{q}}{q-1}e^{-ix}$$

$$p = q = e^{-\pi\frac{\mathrm{K}'}{\mathrm{K}}}$$

It is noteworthy that n (the order of the J functions) is in (A) an arbitrary number. It appears only in the expression on the right side of that equation. A definite integral expression for the functions J will also be given.

2.

FUNCTION $\mathrm{E}_p(x)$.

The series

$$1 + \frac{x}{p-1} + p\frac{x^2}{p-1\cdot p^2-1} + \ldots .$$

and its equivalent product

$$(1-x)(1-px)(1-p^2x) \ldots \ldots$$

are well known: we derive a function analogous to the exponential function. (Cf. *Trans. R.S.E.*, xli. p. 116; and *Proc. L.M.S.*, series 2, vol. ii. p. 194.)

$$\mathrm{E}_{\frac{1}{p}}(x) = 1 + \frac{x}{[1]} + p\frac{x^2}{[2]!} + \ldots . + p^{r(r-1)/2}\frac{x^r}{[r]!} + \ldots . .$$

$$\mathrm{E}_p(x) = 1 + \frac{x}{[1]} + \frac{x^2}{[2]!} + \ldots \ldots + \frac{x^r}{[r]!} + \ldots \ldots \ldots$$

$$[r] = (p^r-1)/(p-1)$$

The function $\mathrm{E}_p(x)$ may be regarded, like the exponential function, either as the limit of a certain infinite series or a certain infinite product. The results numbered (2) . . . (26) are either easily obtained or are known in other forms.

$$\mathrm{E}_p(x)\cdot\mathrm{E}_{\frac{1}{p}}(-x)=1 \qquad . \quad (2)$$

$$\mathrm{E}_p(x)\cdot\mathrm{E}_p(-x)=1+\frac{x^2}{1-p^2}(1-p)^2+\frac{x^4}{1-p^2\cdot 1-p^4}(1-p)^4+\ldots\ldots\ldots \quad (3)$$

$$=\mathrm{E}_{p^2}\left(x^2\frac{p-1}{p+1}\right) \qquad . \quad (4)$$

which reduce, when $p=1$, to $\exp(x)\times\exp(x)=1$,

$$\mathrm{E}_p\left(\frac{x}{p-1}\right)\cdot\mathrm{E}_p\left(-\frac{x}{p-1}\right)=1+\frac{x^2}{1-p^2}+\frac{x^4}{1-p^2\cdot 1-p^4}+\ldots\ldots \quad (5)$$

$$=\mathrm{E}_{p^2}\left(-\frac{x^2}{p^2-1}\right) \qquad . \quad (6)$$

$$=\prod_{m=1}^{\infty}(1-x^2p^{-2m}) \qquad . \quad (7)$$

The product is absolutely convergent if $|\,p\,|>1$. The series are convergent, however, if $|\,p\,|\geqslant 1$, and also for $|\,p\,|<1$ provided $x<\frac{1}{1-p}$.

It follows that

$$\mathrm{E}_p(\omega x)\mathrm{E}_p(-\omega x)\mathrm{E}_p(\omega^2 x)\mathrm{E}_p(-\omega^2 x)\ldots\ldots\mathrm{E}_p(\omega^n x)\mathrm{E}_p(-\omega^n x)=\mathrm{E}_{p^{2n}}\left(x^{2n}\frac{(p-1)^{2n}}{p^{2n}-1}\right) \quad (8)$$

$$\text{if}\quad |\,p\,|>1,\quad \omega=(-1)^{\frac{1}{n}}$$

The corresponding theorem in case $p<1$ is easily obtained by inversion of the base p.

$$\mathrm{E}_{\frac{1}{p}}\left(\frac{px}{p-1}\right)\cdot\mathrm{E}_{\frac{1}{p}}\left(-\frac{px}{p-1}\right)=\sum_{n=0}^{\infty}p^{n^2+n}\frac{x^{2n}}{(p^2-1)(p^4-1)\ldots\ldots(p^{2n}-1}} \quad (9)$$

$$=\prod_{m=1}^{\infty}\left\{1-x^2p^{2m}\right\} \qquad . \quad (10)$$

$$\mathrm{E}_{\frac{1}{p}}\left(\frac{x\sqrt{p}}{p-1}\right)\cdot\mathrm{E}_{\frac{1}{p}}\left(-\frac{x\sqrt{p}}{p-1}\right)=\prod_{m=1}^{\infty}\left\{1-x^2p^{2m-1}\right\} \qquad . \quad (11)$$

$$=\mathrm{E}_{\frac{1}{p^2}}\left(-\frac{px^2}{1-p^2}\right) \qquad . \quad (12)$$

$$=\sum_{0}^{\infty}p^{n^2}\frac{x^{2n}}{(p^2-1)(p^4-1)\ldots(p^{2n}-1)} \qquad . \quad (13)$$

$$\mathrm{E}_{\frac{1}{p^2}}\left(\frac{px^2}{1-p^2}\right)\cdot\mathrm{E}_{\frac{1}{p^2}}\left(\frac{px^{-2}}{1-p^2}\right)=\frac{1}{\prod_{m=1}^{\infty}(1-p^{2m})}\left\{1-2p(x^2+x^{-2})+2p^4(x^4+x^{-4})-2p^9(x^6+x^{-6})+\ldots\right\} \quad (14)$$

On putting $x=e^{iu}$, the series on the right becomes JACOBI's function

$$\Theta\left(\frac{2\mathrm{K}u}{\pi}\right)$$

in which $p=e^{-\pi\frac{\mathrm{K}'}{\mathrm{K}}}$

$$\mathrm{E}_p(x)\mathrm{E}_{\frac{1}{p}}(y)=1+\frac{(x+y)}{[1]}+\frac{(x+y)(x+py)}{[1][2]}+\ldots\ldots \quad (15)$$

$$p<1,\ x<\frac{1}{1-p},\ y \text{ unrestricted}$$

(Cf. *Proc. Edin. Math. Soc.*, vol. xxii. (8).) or $p>1,\ y<\frac{1}{p-1},\ x$ unrestricted.

From this we derive

$$\mathrm{E}_p\left(\frac{x}{1-p}\right).\mathrm{E}_{\frac{1}{p}}\left(\frac{p^n x}{p-1}\right)=1+[n]x+\frac{[n][n+1]}{[2]!}x^2+\ldots\ldots\ldots \quad (16)$$

$$=f_n^p(x) \quad (17)$$

$$\mathrm{E}_p\left(\frac{x}{1-p}\right).\mathrm{E}_{\frac{1}{p}}\left(\frac{px}{p-1}\right)=1+x+x^2+\ldots \quad (18)$$

$$=\frac{1}{1-x} \qquad (x<1) \quad (19)$$

In this expression we notice that inversion of the base p simply interchanges the E functions in the product on the left side of the equation (18).

$$\mathrm{E}_p\left(\frac{p^n x}{1-p}\right)\mathrm{E}_{\frac{1}{p}}\left(\frac{x}{1-p}\right)=1-[n]x+p\frac{[n][n-1]}{[2]!}x^2-p^3\frac{[n][n-1][n-2]}{[3]!}x^3+\ldots \quad (20)$$

$$=\phi_n^p(-x) \quad (21)$$

Hence

$$f_n^p(x)=f_{-n}^p(p^n x) \quad (22)$$

$$f_n^p(x)\times f_n^p(-x)=f_n^{p^2}(x^2) \quad (23)$$

$$\phi_n^p(x)\times\phi_n^p(-x)=\phi_n^{p^2}(x^2) \quad (24)$$

$$f_n^p(x)\times\phi_n^p(-x)=1 \quad (25)$$

The equations

$$e^x\cdot e^{-x}=1$$

$$\mathrm{E}_p(x)\cdot\mathrm{E}_{\frac{1}{p}}(-x)=1$$

are special cases of (25).

If n be infinite

$$f_n^p(x)=\mathrm{E}_{\frac{1}{p}}\left(\frac{x}{p-1}\right) \quad (26)$$

Function $\mathrm{I}_{[n]}(x)$.

It is well known in the theory of Bessel's function that

$$e^{-x}\cdot\mathrm{I}_n(x)=\frac{x^n}{2^n\cdot\Gamma(n+1)}\left\{1-x+\frac{2n+3}{2(2n+2)}x^2-\frac{2n+5}{2\ 3(2n+2)}x^3+\ldots\ldots\right\}$$

$$\mathrm{I}_n(x)=i^{-n}\mathrm{J}_n(ix).$$

In a paper on Basic numbers applied to Bessel's function (*Proc. Lond. Math. Soc.*, series 2, vol. iii., 1905), I have extended this theorem in the form

$$\begin{aligned} &\mathrm{E}_p(-x)I_{[n]}(x) \\ &= \mathrm{E}_{\frac{1}{p}}(-x)\mathrm{I}_{[n]}(x) \\ &= \frac{x^n}{\{2n\}!}\left\{1 - x + \frac{[2n+3]}{[2][2n+2]}x^2 - \frac{[2n+5]}{[2][3][2n+2]}x^3 + \ldots\ldots\right\} \quad . \quad . \quad (27) \end{aligned}$$

$$\mathrm{I}_{[n]}(x) = i^{-n}\mathrm{J}_{[n]}(ix)$$
$$I_{[n]}(x) = i^{-n}\mathfrak{J}_{[n]}(ix)$$

the conditions for convergence being as follows:

Case i $\quad |p|>1$ $\quad$ $\mathrm{E}_p(x)$ and $\mathrm{I}_{[n]}(x)$ are absolutely convergent for all values of x.

$\mathrm{E}_{\frac{1}{p}}(x)$ and $I_{[n]}(x)$ are absolutely convergent if $x<\frac{p}{p-1}$.

Case ii $\quad |p|<1$ $\quad$ $\mathrm{E}_{\frac{1}{p}}(x)$ and $I_{[n]}(x)$ are absolutely convergent for all values of x.

$\mathrm{E}_p(x)$ and $\mathrm{I}_{[n]}(x)$ are absolutely convergent if $x<\frac{1}{1-p}$.

The series (27) is convergent for all values of p.
It is easily deduced that

$$\mathrm{J}_{[n]}(x) = \mathrm{E}_p(ix)\mathrm{E}_p(-ix)\mathfrak{J}_{[n]}(x) \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$\mathfrak{J}_{[n]}(x) = \mathrm{E}_{\frac{1}{p}}(ix)\mathrm{E}_{\frac{1}{p}}(-ix)\mathrm{J}_{[n]}(x) \quad . \quad . \quad . \quad . \quad . \quad (29)$$

From these relations some interesting expressions for various elliptic functions may be found.

3.

Relations with Elliptic Functions.

By means of equation (29) we are able to write

$$\frac{\mathfrak{J}_{[n]}(xt)\mathfrak{J}_{[n]}(xt^{-1})}{\mathrm{J}_{[n]}(xt)\mathrm{J}_{[n]}(xt^{-1})} = \mathrm{E}_{\frac{1}{p}}(ixt)\mathrm{E}_{\frac{1}{p}}(-ixt)\mathrm{E}_{\frac{1}{p}}(ixt^{-1})\mathrm{E}_{\frac{1}{p}}(-ixt^{-1}) \quad . \quad . \quad . \quad (30)$$

Replacing x by u, $(u = ix\sqrt{p}/(p-1))$, we obtain by means of result (11)

$$\frac{\mathfrak{J}_{[n]}(ut)\mathfrak{J}_{[n]}(ut^{-1})}{\mathrm{J}_{[n]}(ut)\mathrm{J}_{[n]}(ut^{-1})} = \prod_{m=1}^{\infty}\left\{(1-x^2t^2p^{2m-1})(1-x^2t^{-2}p^{2m-1})\right\} \quad . \quad . \quad . \quad (31)$$

Using result (12), the right side of this equation may be written

$$\mathrm{E}_{\frac{1}{p^2}}\left(\frac{px^2t^2}{1-p^2}\right)\mathrm{E}_{\frac{1}{p^2}}\left(\frac{px^2t^{-2}}{1-p^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (32)$$

or

$$= \mathrm{E}_{\frac{1}{p^2}}\left(\frac{1}{[2]}\cdot\frac{px^2t^2}{1-p}\right) \times \mathrm{E}_{\frac{1}{p^2}}\left(\frac{1}{[2]}\cdot\frac{px^2t^{-2}}{1-p}\right) \quad . \quad . \quad . \quad . \quad (33)$$

This expression, when expanded in a LAURENT series of ascending and descending powers of t, takes the form (*Trans. R.S.E.*, vol. xli. p. 117 (μ)),

$$I_{[0]}\left(\frac{x^2}{p-1}\right)+\sum_{m=1}^{\infty}(-1)^m(t^{2m}+t^{-2m})p^{m^2}I_{[m]}\left(\frac{x^2}{p-1}\right) \quad . \quad . \quad . \quad (34)$$

In case $x=1$, the product $\prod_{m=1}^{\infty}\left\{(1-x^2t^2p^{2m-1})(1-x^2t^{-2}p^{2m-1})\right\}$ may be expressed as

$$\frac{1}{\prod_{m=1}^{\infty}(1-p^{2m})}\left\{1-p(t^2+t^{-2})+p^4(t^4+t^{-4})-\ldots\ldots\right\} \quad . \quad . \quad . \quad (35)$$

(*Cf.* CAYLEY'S *Elliptic Functions*, p. 297, ed. 1876.)
We see incidentally that

$$\frac{1}{\prod_{m-1}^{\infty}(1-p^{2m})}=I_{[n]}\left(\frac{1}{p-1}\right) \quad . \quad . \quad . \quad . \quad . \quad (36)$$

for all positive integral values of n. Denoting the nature of the base by an index, we write

$$I^{p}_{[0]}\left(\frac{1}{p-1}\right)=I^{p}_{[1]}\left(\frac{1}{p-1}\right)=I^{p}_{[2]}\left(\frac{1}{p-1}\right)=\ldots\ldots \quad . \quad . \quad . \quad (37)$$

$$I^{\frac{1}{p}}_{[0]}\left(\frac{p}{1-p}\right)=p^{-1}\,I^{\frac{1}{p}}_{[1]}\left(\frac{p}{1-p}\right)=p^{-4}\,I^{\frac{1}{p}}_{[2]}\left(\frac{p}{1-p}\right)=\ldots\ldots \quad . \quad . \quad . \quad (38)$$

which is the expression in generalized Bessel-function-notation of the well-known result

$$\frac{1}{1-q^2\cdot 1-q^4\ldots 1-q^{2m}}\left\{1+\frac{q^{2n+2}}{(1-q^2)(1-q^{2n+2})}+\ldots\ldots\right\}=\prod_{m=1}^{\infty}\frac{1}{(1-q^{2m})}.$$

On replacing t by $e^{i\theta}$, the equation (31) becomes

$$\frac{\mathfrak{J}_{[n]}\left(ix\frac{p^{\frac{1}{2}}e^{i\theta}}{p-1}\right)\mathfrak{J}_{[n]}\left(ix\frac{p^{\frac{1}{2}}e^{-i\theta}}{p-1}\right)}{\mathrm{J}_{[n]}\left(ix\frac{p^{\frac{1}{2}}e^{i\theta}}{p-1}\right)\mathrm{J}_{[n]}\left(ix\frac{p^{\frac{1}{2}}e^{-i\theta}}{p-1}\right)}=\prod_{m=1}^{\infty}(1-2x^2\cos 2\theta p^{2m-1}+x^4p^{4m-2}) \quad . \quad . \quad (39)$$

Using now JACOBI'S notation, and writing $u=\frac{i\sqrt{p}}{p-1}e^{ix}$, $v=\frac{i\sqrt{p}}{p-1}e^{-ix}$, $q_0=\prod_{m=1}^{\infty}(1-q^{2m})$, $q=p$, we obtain

$$q_0\frac{\mathfrak{J}_{[n]}(u)\mathfrak{J}_{[n]}(v)}{\mathrm{J}_{[n]}(u)\mathrm{J}_{[n]}(v)}=\Theta\left(\frac{2\mathrm{K}x}{\pi}\right) \quad . \quad . \quad . \quad . \quad . \quad (40)$$

$$2q_0q^{\frac{1}{4}}\sin x\cdot\frac{\mathfrak{J}_{[n]}(p^{\frac{1}{4}}u)\mathfrak{J}_{[n]}(p^{\frac{1}{4}}v)}{\mathrm{J}_{[n]}(p^{\frac{1}{4}}u)\mathrm{J}_{[n]}(p^{\frac{1}{4}}v)}=\mathrm{H}\left(\frac{2\mathrm{K}x}{\pi}\right) \quad . \quad . \quad . \quad . \quad (41)$$

$$2q^{\frac{1}{4}}k^{-\frac{1}{4}}\sin x\frac{\mathrm{J}_{[n]}(u)\mathfrak{J}_{[m]}(p^{\frac{1}{4}}u)\mathrm{J}_{[n]}(v)\mathfrak{J}_{[m]}(p^{\frac{1}{4}}v)}{\mathrm{J}_{[m]}(p^{\frac{1}{4}}u)\mathfrak{J}_{[n]}(u)\mathrm{J}_{[m]}(p^{\frac{1}{4}}v)\mathfrak{J}_{[n]}(v)}=sn\left(\frac{2\mathrm{K}x}{\pi}\right) \quad . \quad . \quad . \quad (42)$$

$$2q^{\frac{1}{4}}\cos x\sqrt{\frac{k'}{k}}\cdot\frac{\mathrm{J}_{[n]}(u)\mathfrak{J}_{[m]}(ip^{\frac{1}{4}}u)\mathrm{J}_{[n]}(v)\mathfrak{J}_{[m]}(ip^{\frac{1}{4}}v)}{\mathrm{J}_{[m]}(ip^{\frac{1}{4}}u)\mathfrak{J}_{[n]}(u)\mathrm{J}_{[m]}(ip^{\frac{1}{4}}v)\mathfrak{J}_{[n]}(v)}=cn\left(\frac{2\mathrm{K}x}{\pi}\right) \quad . \quad . \quad . \quad (43)$$

$$\sqrt{k'}\frac{\mathrm{J}_{[n]}(u)\mathfrak{J}_{[m]}(iu)\mathrm{J}_{[n]}(v)\mathfrak{J}_{[m]}(iv)}{\mathrm{J}_{[n]}(iu)\mathfrak{J}_{[m]}(u)\mathrm{J}_{[n]}(iv)\mathfrak{J}_{[m]}(v)}=dn\left(\frac{2\mathrm{K}x}{\pi}\right) \quad . \quad . \quad . \quad . \quad (44)$$

We notice that in the expressions for sn, cn, dn, two arbitrary constants (orders of the functions) n, m appear:

JACOBI'S function $Z(x) = \dfrac{\Theta'(x)}{\Theta(x)}$ is related to the J functions as follows,

$$\frac{2K(p-1)}{i\pi\sqrt{p}} \cdot Z\left(\frac{2Kx}{\pi}\right) = e^{ix}\left\{\frac{\mathfrak{J}'_{[n]}(u)}{\mathfrak{J}_{[n]}(u)} - \frac{J'_{[n]}(u)}{J_{[n]}(u)}\right\} - e^{-ix}\left\{\frac{\mathfrak{J}'_{[n]}(v)}{\mathfrak{J}_{[n]}(v)} - \frac{J'_{[n]}(v)}{J_{[n]}(v)}\right\} \quad . \quad . \quad (45)$$

It is plain that WEIERSTRASS'S functions σ, ζ, $\wp$, may be expressed by similar formulæ.

For convenience of printing, the order of the functions will sometimes be expressed by n instead of $[n]$. The known formulæ of JACOBI'S functions will, it is evident, give rise to corresponding forms in the case of the J functions: for example

$$sn^2 + cn^2 = 1$$

gives rise to

$$\sin^2 x \frac{\mathfrak{J}^2_m(p^{\frac{1}{4}}u)\mathfrak{J}^2_m(p^{\frac{1}{4}}v)}{J^2_m(p^{\frac{1}{4}}u)J^2_m(p^{\frac{1}{4}}v)} + k'\cos^2 x \frac{\mathfrak{J}^2_m(ip^{\frac{1}{4}}u)\mathfrak{J}^2_m(ip^{\frac{1}{4}}v)}{J^2_m(ip^{\frac{1}{4}}u)J^2_m(ip^{\frac{1}{4}}v)} = \frac{k}{4q^{\frac{1}{4}}}\frac{\mathfrak{J}^2_n(u)\mathfrak{J}^2_n(v)}{J^2_n(u)J^2_n(v)} \quad . \quad . \quad (46)$$

$$= u\frac{\sqrt{p}}{p-1}e^{ix},\ v = \frac{\sqrt{p}}{p-1}e^{-ix},\ p = q = e^{-\pi\frac{K'}{K}}.$$

Using (11) and (12) it is easily found by the method of § 10, p. 116, vol. xli., *Trans. R.S.E.*, that

$$\prod_{m=1}^{\infty}\left\{1 - 2ap^{2m-1}\cos x + a^2p^{4m-2}\right\} = I_{[0]}\left(\frac{a}{p-1}\right) + 2\sum_{n=1}^{\infty}p^{n^2}\cos nx I_{[n]}\left(\frac{a}{p-1}\right).$$

By FOURIER'S theorem we write therefore

$$p^{n^2}I_{[n]}\left(\frac{a}{p-1}\right) = \frac{1}{2\pi}\int_0^{2\pi}\left\{\prod_1^{\infty}(1 - 2ap^{2m-1}\cos x + a^2p^{4m-2})\right\}\cos nx \cdot dx \quad . \quad . \quad (47)$$

In case $a = 1$, this reduces to

$$p^{n^2}I_{[n]}\left(\frac{1}{p-1}\right) = \frac{1}{2\pi q_0}\int_0^{2\pi}\cos nx \cdot \Theta\left(\frac{Kx}{\pi}\right)dx \quad . \quad . \quad . \quad . \quad (48)$$

4.

FUNCTION $J^p_{n,m}(x)$.

Forming in a series, according to powers of t, the product

$$J_{[n]}(xt) \times J_{[n]}(xt^{-1})$$

we obtain

$$J^p_{n,}(x) + \sum_{m=1}^{\infty}(-1)^m(t^{2m} + t^{-2m})J^p_{n,m}(x) \quad . \quad . \quad . \quad . \quad (49)$$

in which

$$J^p_{n,m}(x) = \sum_{r=0}^{\infty}(-)^r\frac{x^{2m+2n+4r}}{\{2m+2n+2r\}!\{2m+2r\}!\{2n+2r\}!\{2r\}!} \quad . \quad . \quad . \quad (50)$$

$$\{2r\}! = [2][4] \ldots . [2r]$$

In the same way if we take

$$\mathfrak{J}^{p}_{n,m}(x)=\sum_{r=0}^{\infty}(-1)^{r}\frac{x^{2m+2n+4r}}{\{2m+2n+2r\}!\{2m+2r\}!\{2n+2r\}!\{2r\}!}p^{4r(m+n+r)} \quad . \quad . \quad (51)$$

which is related to $J_{n,m}$ by inversion of the base p, since

$$J^{\frac{1}{p}}_{n,m}(x)=p^{2(m^2+n^2+mn)}\mathfrak{J}^{p}_{n,m}(x)$$

Taking the product of two $\mathfrak{J}$ series we find

$$\mathfrak{J}_{[n]}(xt)\times\mathfrak{J}_{[n]}(xt^{-1})=\mathfrak{J}^{p}_{n,0}(x)+\sum_{m=1}^{\infty}(-1)^{m}p^{2m(m+n)}(t^{2m}+t^{-2m})\mathfrak{J}^{p}_{n,m}(x) \quad . \quad . \quad . \quad (52)$$

In a previous paper (*Trans. R.S.E.*, vol. xli. p. 106) it has been shown that

$$J_{[n]}(xt)\times\mathfrak{J}_{[\nu]}(xt)=\sum_{0}^{\infty}(-1)^{r}\frac{\{2n+2\nu+4r\}!}{\{2n+2\nu+2r\}!\{2n+2r\}!\{2\nu+2r\}!\{2r\}!}x^{n+\nu+2r} \quad . \quad . \quad (53)$$

There is a certain similarity of form among the series (50), (51), (53).

Consider now the product of four J functions

$$J_{[n]}(xt)\cdot J_{[n]}(xt)\cdot\mathfrak{J}_{[\nu]}(xt^{-1})\cdot\mathfrak{J}_{[\nu]}(xt^{-1}) \quad . \quad . \quad . \quad . \quad (54)$$

This expression may be written in two other forms. Firstly, by (49) and (52) we write it

$$\left\{J^{p}_{n,0}(x)+\sum_{1}^{\infty}(-1)^{m}(t^{2m}+t^{-2m})J^{p}_{n,m}(x)\right\}\cdot\left\{\mathfrak{J}^{p}_{\nu,0}(x)+\sum(-1)^{m}p^{2m(m+\nu)}(t^{2m}+t^{-2m})\mathfrak{J}^{p}_{\nu,m}(x)\right\} \quad . \quad (55)$$

Secondly, by means of (53) we express (54) as

$$\left\{\sum_{r=0}^{\infty}(-1)^{r}\frac{\{2n+2\nu+4r\}!}{\{2n+2\nu+2r\}!\{2n+2r\}!\{2\nu+2r\}!\{2r\}!}(xt)^{n+\nu+2r}\right\}\cdot$$
$$\left\{\sum_{r=0}^{\infty}(-1)^{r}\frac{\{2n+2\nu+4r\}!}{\{2n+2r+2\nu\}!\{2n+2r\}!\{2\nu+2r\}!\{2r\}!}(xt^{-1})^{n+\nu+2r}\right\} \quad . \quad . \quad . \quad (56)$$

Equating coefficients of powers of t in (55) and (56), we find from the terms independent of t

$$\sum_{r=0}^{\infty}\left(\frac{\{2n+2\nu+4r\}!}{\{2n+2\nu+2r\}!\{2n+2r\}!\{2\nu+2r\}!\{2r\}!}\right)^{2}x^{2n+2\nu+4r}=J^{p}_{n,0}\mathfrak{J}^{p}_{\nu,0}+2\sum_{m=1}^{\infty}p^{2m(m+\nu)}J^{p}_{n,m}\mathfrak{J}^{p}_{\nu,m} \quad . \quad (57)$$

The terms in the series on the left side of (57) are the squares of the terms in (53). Generally

$$\sum_{r=0}^{\infty}\{2m+2n+2\nu+2r\}!\{2m+2\nu+2r\}!\{2m+2n+2r\}!\frac{\{2n+2\nu+4r\}!\{2n+2\nu+4m+4r\}!x^{2m+2\nu+2n+4r}}{\{2n+2\nu+2r\}!\{2m+2n+2r\}!\{2n+2\nu+2r\}!\{2m+2r\}!\{2n+2r\}!\{2\nu+2r\}!\{2r\}!}$$
$$=\sum_{r=1}^{\infty}p^{2m(m+\nu)}\left\{J^{p}_{n,m+r}\mathfrak{J}^{p}_{\nu,m}+p^{4mr+2r(r+n)}J^{p}_{n,m}\mathfrak{J}^{p}_{\nu,m+r}\right\} \quad . \quad . \quad . \quad . \quad (58)$$

5.

In this section of the paper I propose to state briefly some results which may be deduced by means of (53), (28), (29),

$$\left.\begin{aligned} &\mathrm{E}_p(ix)\mathrm{E}_p(-ix) \\ &= 1 - \frac{x^2}{1-p^2}(1-p)^2 + \frac{x^4}{1-p^2\cdot 1-p^4}(1-p)^4 - \ldots\ldots \\ &= \mathrm{J}_{[0]}(px)\mathrm{J}_{[0]}(x) + 2p\mathrm{J}_{[1]}(px)\mathrm{J}_{[1]}(x) + 2p^4\mathrm{J}_{[2]}(px)\mathrm{J}_{[2]}(x) + \ldots\ldots \end{aligned}\right\} \quad . \quad . \quad (59)$$

Indicating the nature of the base of each function by an index, we write

$$\mathrm{J}^p_{[0]}\left(\frac{x}{p+1}\right)\cdot \mathfrak{J}^{p^2}_{[0]}\left(\frac{x^2}{p^2+1}\right) + 2\sum_{m=1}^{\infty} \mathfrak{J}^{p^2}_{[m]}\cdot \mathrm{J}^p_{[2m]} = \mathfrak{J}^p_{[0]}\left(\frac{x}{p+1}\right) \quad . \quad . \quad . \quad (60)$$

whence by (29)

$$\left.\begin{aligned} &\mathrm{J}^p_{[0]}\left(\frac{x}{p+1}\right)\mathrm{J}^{p^2}_{[0]}\left(\frac{x^2}{p^2+1}\right) + 2\sum_{m=1}^{\infty}\mathrm{J}^p_{[2m]}\cdot \mathrm{J}^{p^2}_{[m]} = \mathrm{E}_{p^2}\left(\frac{ix^2}{p^2+1}\right)\mathrm{E}_{p^2}\left(-\frac{ix^2}{p^2+1}\right)\mathfrak{J}^p_{[0]}\left(\frac{x}{p+1}\right) \\ \text{by (29)} \quad &= \mathrm{E}_{p^2}\left(\frac{ix^2}{p^2+1}\right)\mathrm{E}_{p^2}\left(-\frac{ix^2}{p^2+1}\right)\mathrm{E}_{\frac{1}{p}}\left(\frac{ix}{p+1}\right)\mathrm{E}_{\frac{1}{p}}\left(-\frac{ix}{p+1}\right)\mathrm{J}^p_{[0]}\left(\frac{x}{p+1}\right) \\ \text{by (4) and 12)} \quad &= \mathrm{E}_{p^4}\left(x^4\frac{1-p^2}{(p^2+1)^3}\right)\mathrm{E}_{\frac{1}{p^2}}\left(x^2\frac{p-1}{(1+p)^3}\right)\mathrm{J}^p_{[0]}\left(\frac{x}{p+1}\right) \end{aligned}\right\} \quad . \quad (61)$$

From (19) we find

$$\frac{1}{(1-x^2t^{-2})(1-x^2t^2)} = \left\{ \mathrm{I}_{[0]}\frac{x^2}{1-p} + \sum (t^{2m}+t^{-2m})\mathrm{I}_{[m]}\left(\frac{x^2}{1-p}\right)\right\} \left\{ I_{[0]}\left(\frac{px^2}{1-p}\right) + \sum (-1)^m p^{m^2}(t^{2m}+t^{-2m})I_{[m]}\left(\frac{px^2}{1-p}\right)\right\} \quad . \quad . \quad (62)$$

$$\frac{1}{1-x^4} = \mathrm{I}_{[0]}I_{[0]} + 2\sum (-1)^r p^{r^2}\mathrm{I}_{[r]}I_{[r]} \quad . \quad . \quad . \quad . \quad (63)$$

$$\frac{\mathrm{E}_{p^2}\left(\frac{px^2}{1-p}\right)\mathrm{E}_{p^2}\left(\frac{px^2}{p-1}\right)}{1-x^4} = \mathrm{I}_{[0]}\left(\frac{x^2}{1-p}\right)\mathrm{I}_{[0]}\left(\frac{px^2}{1-p}\right) + 2\sum (-1)^r p^{r^2}\mathrm{I}_{[r]}\left(\frac{x^2}{1-p}\right)\mathrm{I}_{[r]}\left(\frac{px^2}{1-p}\right) \quad . \quad (64)$$

$$\frac{\mathrm{E}_{p^4}\left(x^4\frac{(1+p)^2}{p^4-1}\right)}{1-x^4} = \mathrm{I}_{[0]}\left(\frac{x^2}{1-p}\right)\mathrm{I}_{[0]}\left(\frac{px^2}{1-p}\right) + 2\sum (-1)^r p^{r^2}\mathrm{I}_{[r]}\left(\frac{x^2}{1-p}\right)\mathrm{I}_{[r]}\left(\frac{px^2}{1-p}\right) \quad . \quad . \quad (65)$$

$$\frac{1}{1-2x^2\cos 2\theta + x^4} = \left\{ \mathrm{I}_{[0]} + 2\cos 2\theta \mathrm{I}_{[1]} + \ldots \right\}\left\{ I_{[0]} - 2p\cos 2\theta I_{[1]} + \ldots \right\} \quad . \quad . \quad (66)$$

$$\left\{ \mathrm{J}_{[1]} \right\}^2 = \frac{[4]}{[2]}\mathrm{J}_{[0]}\mathrm{J}_{[2]} + p^2\frac{[8]}{[4]}\mathrm{J}_{[1]}\mathrm{J}_{[3]} + \ldots\ldots + p^{r(r-1)}\mathrm{J}_{[r-1]}\mathrm{J}_{[r+1]} + \ldots\ldots \quad . \quad . \quad (67)$$

$$\left\{ \mathfrak{J}_{[1]} \right\}^2 = \frac{[4]}{[2]}\mathfrak{J}_{[0]}\mathfrak{J}_{[2]} + p^2\frac{[8]}{[4]}\mathfrak{J}_{[0]}\mathfrak{J}_{[3]} + \ldots\ldots + p^{r(r-1)}\mathfrak{J}_{[r+1]}\mathfrak{J}_{[r-1]}\ldots\ldots \quad . \quad . \quad (68)$$

$$\mathrm{E}_p(ix)\mathrm{E}_p(-ix) = \left\{ \mathrm{J}_{[0]} \right\}^2 + \frac{[4]}{[2]}\left\{ \mathrm{J}_{[1]} \right\}^2 + \ldots\ldots + p^{r(r-1)}\frac{[4r]}{[2r]}\left\{ \mathrm{J}_{[r]} \right\}^2 + \ldots\ldots \quad . \quad (69)$$

$$\mathrm{E}_{\frac{1}{p}}(ix)\mathrm{E}_{\frac{1}{p}}(-ix) = \left\{ \mathfrak{J}_{[0]} \right\}^2 + \frac{[4]}{[2]}\left\{ \mathfrak{J}_{[1]} \right\}^2 + \ldots\ldots + p^{r(r-1)}\frac{[4r]}{[2r]}\left\{ \mathfrak{J}_{[r]} \right\}^2 + \ldots\ldots \quad . \quad (70)$$

$$\left\{ \mathrm{J}_{[0]} \right\}^2 - 2p \left\{ \mathrm{J}_{[1]} \right\}^2 + 2p^4 \left\{ \mathrm{J}_{[2]} \right\}^2 - \ldots\ldots = \left\{ \sum (-1)^r \frac{x^{2r}}{(1-p^2)(1-p^4)\ldots(1-p^{2r})} (p-1)^{2r} \right\}$$
$$\left\{ \sum \frac{(1+p)(1+p^3)\ldots(1+p^{2r-1})x^{2r}}{(1-p^2)(1-p^4)\ldots(1-p^{2r})} (p-1)^{2r} \right\} \qquad (71)$$

$$\left\{ \mathrm{J}_{[0]} \right\}^2 - \frac{[4]}{[2]} \left\{ \mathrm{J}_{[1]} \right\}^2 + \ldots\ldots + p^{r(r-1)} \frac{[4^2]}{[2r]} \left\{ \mathrm{J}_{[r]} \right\}^2 + \ldots\ldots$$
$$= \left\{ \sum (-1)^r \frac{x^{2r}}{(1-p^2)(1-p^4)\ldots(1-p^{2r})} (p-1)^{2r} \right\}$$
$$\left\{ \sum \frac{2(1+p^2)(1-p^4)\ldots(1+p^{2r-r}) \cdot (1+p^2)(1+p^4)\ldots(1+p^{2r})}{\{(1-p^2)(1-p^4)\ldots(1-p^{2r})\}^2} x^{2r} (p-1)^{2r} \right\} \qquad (72)$$

(Cf. *Trans. R.S.E.*, p. 110 (26).)

$$\mathrm{J}_{[0]}(x)\mathrm{J}_{[0]}(y) - 2p\mathrm{J}_{[1]}(x)\mathrm{J}_{[1]}(y) + 2p^4\mathrm{J}_{[2]}(x)\mathrm{J}_{[2]}(y) - \ldots\ldots$$
$$= \left\{ 1 - \frac{y^2}{1-p^2}(p-1)^2 + \frac{y^4}{1-p^2 \cdot 1-p^4}(p-1)^2 - \ldots \right\} \left\{ 1 - \frac{(x+py)^2}{[2]^2} + \frac{(x+py)^2(x+^3y)^2}{[2]^2[4]^2} - \ldots \right\} \qquad (73)$$

$$\mathrm{J}_{[n]}\mathrm{J}_{[-n]} + \mathrm{J}_{[-n]}\mathrm{J}_{[-1+n]} = \frac{1}{x\Gamma_{p^2}([n])\Gamma_{p^2}([1-n])} \left\{ [2]\mathrm{J}^2_{[0]} + [4]\mathrm{J}^2_{[1]} + [8]\mathrm{J}^2_{[2]} + \ldots\ldots \right\} \qquad (74)$$

$$\mathrm{E}_p(ix)\mathrm{E}_p(-ix)\frac{x^{2n-1}}{\{2n\}!\{2n-2\}!} = \mathrm{J}_{[n-1]}\mathrm{J}_{[n]} + \frac{[4n+2]}{[2]}\mathrm{J}_{[n]}\mathrm{J}_{[n+1]}$$
$$+ p^{s(s-1)}[4n+4s-2]\frac{[4n+2][4n+4]\ldots\ldots[4n+2s-2]}{[2][4]\ldots[2s]}\mathrm{J}_{[m+s]}\mathrm{J}_{[m+s-1]} \qquad (75)$$

and a similar form for x^{2n}. (Cf. *Proc. Lond. Math. Soc.*, series 2, vol. iii.)

$$\sum_{r=0}^{\infty} (-1)^r \frac{[4r+2]}{[2]} p^{r(r-1)} \mathrm{J}^2_{\left[\frac{2r+1}{2}\right]} = \frac{1}{[2]^2 \cdot \Gamma_{p^2}\left(\frac{3}{2}\right)} \left\{ \sum (-1)^r \frac{x^{2r}}{(1-p^2)(1-p^4)\ldots(1-p^{2r})} (p-1)^{2r} \right\}$$
$$\left\{ \sum \frac{2(1+p^2)\ldots(1+p^{2r-2}) \cdot (1+p)\ldots(1+p^{2r-1})}{[2r+1]!} x^{2r-1} \right\} \qquad (76)$$

(Cf. *Proc. Edin. Math. Soc.*, Theorem of LOMMEL, vol. xxii.)

It is plain that great numbers of such theorems may be found and expressed in various forms by means of the transformations belonging to $\mathrm{E}_p(x)\cdot$, but the examples given above will suffice to illustrate the notation.

XVIII.—On Pennella balænopteræ: a Crustacean, parasitic on a Finner Whale, *Balænoptera musculus*. By Sir William Turner, K.C.B., D.C.L., F.R.S. (With Four Plates.)

(Read February 6, MS. received February 8, 1905. Issued separately May 26, 1905.)

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Introduction.

In September 1903 I received a bottle containing twelve specimens of a large parasite presented to me by Mr Chr. Castberg, the manager of a Norwegian whaling company which has established a fishing station at Ronasvoe in the north of Shetland.* In his letters Mr Castberg stated that the parasites were attached to a Finner whale, which, from its size, the mottled character of the whalebone and the pointed head, was obviously a Razorback—*Balænoptera musculus*. The parasites were numerous, and were fixed to the back of the whale, and the attached end penetrated through the skin into the blubber. Although Mr Castberg had seen many hundred whales, this is the first occasion on which he had met with this form of parasite.

From the characters of the specimens I concluded that they were a giant species of a parasitic Crustacean, of the family Lernæidæ, and on further investigation I associated them with the genus Pennella (Oken).

This genus is now regarded as including those members of the Lernæidæ which, as studied in the females, have the head stunted and club-shaped, with horn-like arms radiating from its base; the body elongated, cylindriform, not bent into a sigmoid shape; the anterior part of the body attenuated, but widening further back; a pair of genital openings with depending ova strings; the terminal part of the body caudate, giving origin to the characteristic bristle-like pennate appendages; pairs of minute rudimentary feet springing from the ventral surface of the body close to the base of the head.

From the time of Aristotle, naturalists had recognised that the Tunny and Swordfish were infested by worm-like parasites, fastened to the skin near the fin. Rondeletius,

* I am indebted to my valued correspondent, Mr Thomas Anderson, merchant, of Hillswick, Shetland, for putting me into communication with Mr Castberg.

Gesner and Salvianus, in their respective treatises, written in the sixteenth century, described such parasites, and Rondeletius and Gesner figured specimens from the tunny.

Boccone published in 1674 an account of parasites found on the swordfish, Xiphias, implanted in its flesh, which he named Sangsue or "Hirudo cauda utrinque pinnata," and he gave a figure. It would seem as if this animal was different from that described by Rondeletius and Gesner. Boccone had figured a very interesting object, named by him a "poux" or "pediculus," as big as a pea, attached to the ventral surface of the parasite, immediately in front of the genital openings. He stated that it was fixed as firmly to the parasite as a limpet was to a rock. I am disposed to regard this so-called "pediculus" as the male of the female parasite to which it was attached. Its small size compared with that of the female, and its position and attachment close to the genital openings, corresponded with that of the male of the parasitic crustacean, *Chondracanthus lophii*, described and figured by Dr H. S. Wilson and myself in 1862.

Linnæus, in the *Systema Naturæ*, 1758, classed amongst the Vermes Zoophyta the genus Pennatula or Sea Pens, and he named the parasite described by Boccone, which infests Xiphias, *Pennatula filosa*. In 1759 J. L. Odhelius contributed to the *Amœnitates Academicæ* of Linnæus, a dissertation entitled "Chinensia Lagerstromiana,"* in which he gave the characters of *Pennatula sagitta* (p. 257, and fig. 13), a parasite infesting *Lophius histrio*, the sea-bat of the China Sea. John Ellis reproduced in 1764 Boccone's figure of *P. filosa* and Odhelius's figure of *P. sagitta*. In 1802 Holten recognised a parasite on the flying fish, *Exocætus volitans*, which he named *P. exocæti*, specimens of which, burrowing into the abdominal cavity of that fish, have been recently described, 1901, by Mr Andrew Scott.

Oken classed the Lernæidæ amongst the Mollusca, removed these parasites from the Sea Pens, Pennatula, and placed them in a distinct genus, Pennella, whilst De Blainville suggested Lerneopenna as the generic name. Cuvier and naturalists generally had adopted Oken's term, though some preferred the spelling *Penella*.

Additional species were discovered from time to time. Chamisso and Eysenhardt described *Penella diodontis* from the branchiæ of *Diodontis mola*, captured in the Pacific; Dekay named *P. sagitta* as adhering to *Diodon pilosus*, and von Nordmann, in his description of *P. sagitta* from *Lophius marmoratus*, thought that it and Dekay's specimen were the same species. Angus found a parasite on a species of *Coryphæna* near the gills, which William Baird named *Penella pustulosa*. Milne Edwards stated that *Pennella sultana* had been found in the mouth of *Carenx ascensorius*. Steenstrup and Lütken gave an account of *P. varians* which infested a "Dolphin," the species of which was not determined. E. Perceval Wright described *Pennella orthagorisci* from specimens obtained from *Orthagoriscus mola* caught in Cork harbour in 1869. They were implanted in the skin on either side of the dorsal fin, and the total length of the parasite from the head to the anal opening was 7 inches. He

* Named after the Swedish Councillor, Magnus Lagerström.

also stated that BAIRD referred a Pennella from a sunfish captured in Cornwall to *P. filosa*. G. M. THOMSON gave an account (1889) of a Pennella found on a swordfish (*Histiophorus herschelii*), which he named *P. histiophori*. RAMSAY H. TRAQUAIR has called my attention to two specimens of Pennella in the Collection of the Royal Scottish Museum, which he had provisionally named *P. orthagorisci*. Possibly they may have been included in the Natural History Museum of the University, which was transferred many years ago to the Royal Scottish Museum, but nothing definite is known of the animal on which they were parasitic, or when they were obtained. One specimen was deprived of the head and arms; the other had a head and two lateral arms, but no dorsal arm, and it was about 5 inches long.*

Observations on the Lernæidæ during the first quarter of the last century induced naturalists to consider that these parasites were not to be regarded as Worms, Molluscs, or Zoophytes, but that they had an affinity to the Crustacea. Their position was finally adjusted in 1832 by ALEXANDER VON NORDMANN, who, from the young having the non-parasitic character of Cyclops, from the segmented structure of the male, which is a free swimming animal, though it may become attached to the female, and from the position and characters of the feet, definitely placed these curious animals amongst the Crustacea, in which they are now generally regarded by naturalists as forming a family of parasitic Copepoda.

An important extension of our knowledge of the hosts to which different species of Pennella may become attached was made when it was ascertained that specimens had been obtained imbedded in the skin of species of whales frequenting the North Atlantic Ocean. STEENSTRUP and LÜTKEN published in 1861 a memoir in which a Pennella was described as attached to a *Hyperoodon rostratus* captured in 1855 south of the Faroe Islands; they named the parasite *Pennella crassicornis*. They referred to an observation made some years previously by VON DÜBEN that a Pennella, species not named, had been obtained from a Finner whale. In 1866 G. O. SARS stated that specimens of a Pennella with the head buried in the blubber were seen attached to *Balænoptera musculus*. In 1877 KOREN and DANIELSSEN published a memoir on a Pennella found on *Balænoptera rostrata*, and preserved in the museum at Bergen, which they had named *Pennella balænopteræ* twenty years previously. Other specimens from *B. rostrata*, buried with the head and horn-like arms in the blubber in the vicinity of the external organs of generation, had subsequently been added to this museum. VAN BENEDEN, in his memoirs on the natural history of the Cetacea, referred to these Balænopteræ as serving as hosts for a Pennella; and he further stated, though without giving very definite authorities, that this parasitic crustacean had also been found on *Balænoptera sibbaldii*, and probably on *B. borealis*.

* Dr TRAQUAIR showed at the meeting of the Royal Society at which this memoir was read two dried specimens of *Pennella exocæti*, which he had received in November 1904 from Captain PATER. It appears that when Captain PATER was on a voyage in the South Pacific a flying fish flew on to the ship; and deeply rooted in the wall of its abdomen, behind the pectoral fin, were the two specimens of Pennella, which he removed and sent to the Royal Scottish Museum.

As the memoir of KOREN and DANIELSSEN contains a description of the external characters with observations on the internal anatomy of the female *Pennella balænopteræ*, and is illustrated by a plate with nine figures, I have made a careful comparison of my specimens with their description and drawings.

EXTERNAL CHARACTERS OF THE FEMALE.

As the specimens in my possession, like those studied by KOREN and DANIELSSEN, were not uniform in length, I have measured the longest and the shortest in order to show the variation, and in the following table I have recorded their chief dimensions, alongside of the corresponding measurements of two of the specimens described by the Norwegian naturalists.

	K. & D.		TURNER.	
	A.	B.	A.	B.
Whole length of parasite	320 mm.	300	294 mm.	206
Length of head	7 "	6	5 "	4
Breadth of head	8 "	7	5 "	4
Longest horn-like arm	15 "	14	33 "	23
Greatest thickness of arm	2 "	2	3 "	3
Length of thoracico-abdominal part	315 "	294	289 "	202
Greatest thickness of same	6 "	6	4·5 "	4
Length of pennated abdominal part	45 "	42	30 "	25

It is obvious from these measurements that the females varied considerably in length; and as my shortest specimen had a pair of long ova strings attached to the ventral surface, it may be assumed to be adult equally with the longest. It will be noticed that neither of the two specimens is so long as the shortest of those recorded by KOREN and DANIELSSEN, whilst their longest specimen was 320 mm. ($12\frac{1}{2}$ inches). *P. balænopteræ* is therefore a giant amongst the Copepoda.

The head, both in length and breadth, slightly smaller than in their examples, had a stunted, club-shaped appearance. Its colour, that of the arms and of the upper part of the so-called thoracic region, was brownish-yellow, whilst the lower part of that region and the entire extent of the abdomen was of a dark purplish hue with a shade of green, even after the specimens had been for several months in spirit. The head, arms, and upper part of the thorax were imbedded in the skin and blubber, on the juices in which the parasite lived. The greenish-purple-tinted part of the body floated in the sea-water, and was more or less in contact with the skin of the whale. Seen through the medium of the water, it would approximate to the colour of the skin, and would furnish an example of protective mimicry.

The summit of the head was studded with numerous shallow, papilla-like tubercles; they also surrounded the cleft-like opening of the mouth, which formed a deep mesial groove extending for a small distance on the ventral surface of the head. A short groove was present on the dorsal surface, which had, at its upper end, a blunt, hook-like tubercle at each margin, but in no instance did I see a pair of pointed, claw-

like antennæ, relics of the free Cyclops stage of development, such as are represented by KOREN and DANIELSSEN in their figure 9, tab. xvi.

From the base of the head three horn-like arms arose, which extended almost horizontally outwards; they were the anchors of attachment implanted in the blubber of the whale. One sprang from the mesial dorsal surface, whilst the others were right and left lateral. They varied in length in the same specimen, and the dorsal arm was usually the shortest. They differed also in thickness and were irregular on the surface; the free end was blunt (Plate I. figs. 1, 2), and in one specimen a lateral arm was bifurcated.

The body of the parasite extended from the base of attachment of the arms to the free end of the pennated portion. It varied materially in thickness in different parts of its length. Immediately below the arms its transverse diameter was 3 to 4 mm.; it was somewhat flattened on both dorsal and ventral surfaces, and on the ventral surface, close to the mesial line, most of the specimens showed pairs of appendages. They were so minute as to be scarcely visible to the naked eye. In two specimens four pairs were seen, as had been figured by KOREN and DANIELSSEN. In others, two pairs, or even a single pair, only were recognised, and in a few they were not visible. Their recognition was assisted by the presence of a spot of dark pigment. Four is without doubt the typical number of these feet-like appendages, though it would seem as if this number was not always preserved in the process of transformation from the embryonic cyclopoid form to their retrograde condition in the adult (Plate I. fig. 2).

Eight mm. from the base of the arms the transverse diameter of the body diminished to 1·5 mm., and for a considerable distance it preserved this diameter; it was cylindrical in shape, smooth on the surface, and not unlike in form and colour a steel knitting-needle. It was an elongated neck-like division of the body, very characteristic of the parasite, and may be regarded as the thoracic segment.

The body was prolonged into the abdomen, which increased in bulk, measured 4 mm. in breadth, lost its smooth appearance, and was marked by numerous transverse constrictions, between which minute bead-like projections were arranged in rows. The abdomen was the widest and most deeply coloured part of the body; as it contained both alimentary canal and the female genital organs, it may appropriately be named the genito-abdominal segment. At the lower end two genital openings were seen on the ventral surface, from which depended the pair of ova strings. Immediately above these openings was a small rounded eminence, to which probably the male parasite may attach itself when engaged in impregnation.

The ova strings were a pair of very slender threads, yellowish-brown in colour, and of remarkable length; in one parasite each string measured 400 mm. (15·7 in.). They floated free in the surrounding medium; they were sometimes almost straight, but at others they had an undulating character.

The terminal part of the body was prolonged behind the genital openings from 25 to 30 mm., varying in the different specimens; it was only 2 mm. in transverse diameter

and came to a free end. It had a caudate appearance; but as it contained the intestinal end of the alimentary canal, it should be regarded as the caudate segment of the abdomen. The anal orifice was situated in a cleft at its free end. Its dorsal surface was marked by transverse constrictions, and from its ventral surface a number of bristle-like structures arose, which gave to the terminal part of the body the pennate character which has decided the generic name.

Chitinous Coat.

The chitinous coat of the parasite was translucent, firm, and so tough as to turn the edge of the razor. It was for the most part homogeneous throughout its substance, but in places delicate lines, parallel to each other and to the plane of the surface, gave it a laminated appearance, as if it had been formed by superposition of layers. It varied in thickness in different regions, as was seen both in longitudinal and transverse sections. In the head, this coat was about $\frac{1}{3}$rd of a mm. thick, but at the origin of the arms it was about $\frac{2}{3}$rds of a mm. In the arm itself the thickness varied in different parts. In proximity to the head it formed about $\frac{2}{3}$rds of the diameter of a transverse section, in the middle of the arm about $\frac{1}{3}$rd, and near the free end about $\frac{1}{2}$. In the attenuated thoracic region the proportion was about $\frac{1}{2}$, in the genito-abdominal part it was less, and it was a little thinner on the ventral than on the dorsal aspect. In the pennated abdomino-caudate segment it represented about $\frac{1}{3}$rd of the transverse diameter of the parasite.

On the outer surface of the chitinous envelope a layer of cuticle was present, which was usually closely adherent to the chitin, but in places it was partially detached, and had probably been drawn off in cutting the sections. When examined microscopically it was seen to be striated in a direction perpendicular to the plane of the surface; higher magnification showed this appearance to be due to short columnar cells, which were arranged parallel to each other. In sections where the displaced cuticle had been turned over so as to expose its free surface, the broader ends of the columnar cells were seen to be at that surface, and by their close apposition to each other to form a continuous layer.

The chitinous wall was lined by a membrane, which in various localities, to be subsequently referred to, was richly pigmented (figs. 17, 24, 26).

In the papilla-like tubercles, in the parts of the head not occupied by the muscles, in the thickened part of the body immediately below the head and in the arms, an areolated tissue was situated within the membranous lining of the wall of chitin.

Structure of the Head.

The internal structure of the head was examined in a series of transverse and longitudinal sections from its summit to the base of attachment of the arms. The

papilla-like tubercles formed the most marked feature of the summit. Each had a definite chitinous envelope, which inclosed an areolated tissue, the areolæ of which varied materially in size, and corresponded in character with the tissue in the axis of the arms to be subsequently described.

Within the tuberculated summit numerous transversely striped muscular fibres occupied a large proportion of the space dorsally and laterally inclosed by the chitinous envelope. They arose from the inner surface of the envelope, which in transverse section had a ridge and furrow-like character. The muscular fibres in this region situated laterally to the mesial plane converged from their origin and seemed to end in a common tendon, which was attached to the papilla-like tubercles situated on the side of the cleft which formed the oral aperture (figs. 5, 7, 8). Their apparent function was to draw the sides of the cleft asunder, widen the aperture, and by successive contractions and relaxations to convert the cleft into a suctorial mouth. In transverse sections of the head below the tubercles the muscular fibres were less numerous; those situated in proximity to the mesial plane converged on the dorsal wall of the alimentary canal, on which they could act directly as dilators. The fibres situated further from the mesial plane reached the dorsal aspect of a pair of bodies, to be immediately described, which stained readily with carmine. The striped muscular fibres were seen as low down as the origin of the arms, but they were absent immediately below these appendages, and their place was to a large extent taken by the areolated tissue.

I have more than once referred to a tissue, which I have named 'areolated,' situated in the head, in the part of the body immediately below the head, and in the arms into which it was prolonged at their base of attachment. In a subsequent section I shall have to call attention to a similar tissue in the abdomen. Koren and Danielssen described a layer of adipose matter, in most places not very thick, though it could form isolated fatty agglomerations; in the head, arms and the upper thoracic division of the body it formed a thick stuffing, and corresponded in its position to the areolated tissue seen in my specimens: the adipose tissue was composed of fat cells, which, they say, had one or more ramifications on the cell.

In its general characters the areolated tissue consisted of a meshwork of connective tissue, continuous with the membranous lining of the chitinous wall of the parasite. In the strands of this meshwork, more especially in its peripheral part, nucleated cells were seen in places in considerable numbers, which in size and general appearance were not unlike leucocytes. The areolæ of the meshwork varied in size, the largest being just visible to the naked eye, whilst the smallest required a magnification of two hundred to three hundred diameters. In specimens taken from the head, when the tissue was teased with needles and examined in glycerine, the areolæ were seen to contain rounded or ovoid cells, which, like fat cells, refracted the light strongly, and showed the characteristic reaction of fat with osmic acid; in the act of teasing, many of the fat cells were ruptured and oil globules escaped. In

sections through the head and arms, which had been treated with nitric acid in order to soften the chitin previous to making the section, subsequently soaked in alcohol, and then mounted in Canada balsam, the tissue was modified in appearance. Although some of the cells retained the ovoid form and to some extent the refracting character, the majority had more or less irregular outlines, and their contents had generally the appearance of a granular cell-plasm, not usually staining strongly with carmine; though sometimes the granules were relatively large, and stained more deeply with carmine, as if they had a nuclear character. It would seem as if, with the disappearance of the fat, the cell-plasm had come into view.

In certain localities the areolated tissue showed characters deserving of more detailed notice. In the arms, where they adjoined the head and where the areolated tissue was small in proportion to the thickness of the arm, two large areolæ, each containing granular cell-plasm with a nucleus, were very distinct (fig. 11). About the middle of the arm, also, a pair of areolæ, containing granular cell-plasm, similar in size and in close relation to the wall, were present; but as the areolated tissue in this part of the arm was much more abundant than near the head, a cluster of large areolæ also occupied the central area of the tissue (fig. 12). A somewhat similar appearance was seen in the relatively smaller amount of this tissue near the tip of the arm.

In some sections the areolated tissue in the arms was modified in a peculiar manner. Whilst in some of the areolæ the refracting character of fat cells was distinctive, many others, especially those of large size, were crowded with nuclei, which stained deeply with carmine. The nuclei were so closely set that the amount of cell-plasm associated with each nucleus was extremely small, and the latter dominated in quantity and distinctness over the cell-plasm. It seemed as if an extensive proliferation of the nuclei had taken place (fig. 13).

In sections through the head in proximity to the arms, where the areolated tissue was relatively abundant, the largest areolæ with their contained cells occupied the mid-area of the tissue, whilst the smaller areolæ formed its peripheral part (fig. 9). The tissue which constituted the axis of the papilla-like tubercles of the head consisted of the smaller type of areolæ, though they were not uniform in size, as some were four or five times larger than others.

It should be noted that the part of the parasite immediately below the arms had on the ventral surface the pairs of limb-like appendages already referred to. They were so extremely rudimentary that it was difficult to recognise them with the naked eye, and sometimes even they were absent. It is within this part of the body that the areolated tissue was most abundant. Had the limbs been functionally active, one cannot doubt but that an adequate amount of striped muscle would have been developed in this region as their motor apparatus; but, under the changed conditions, it was no longer required, and its place had been taken by a passive, areolated tissue containing fat cells.

In addition to the œsophagus, the muscular fibres, and the areolated tissue, the

chitinous wall of the head inclosed three objects—a pair placed laterally, which were readily coloured by carmine (fig. 8, *g*), and one placed mesially next the ventral surface, which did not take the carmine dye (fig. 8, V).

The red stained bodies were recognised in sections through the head as high as the sides of the oral chink, and were obviously nerve ganglia. At their upper end they were separated from each other by the mesial oral chink, the tubercles connected with its walls and the areolated tissue associated with the tubercles. Each was placed close to the common tendon of attachment of the bundle of striped muscular fibres already described on each side of the head. In the upper part of a ganglion not more than six to twelve characteristic cells could be seen in the plane of section, but opposite the lower end of the oral chink the ganglion increased in size and the cells were much more numerous. Immediately below the oral cleft the ganglia were relatively large, and were situated partly to the side of the œsophagus and partly ventrally to it, but they were not continuous with each other on the ventral surface, as they were separated by the mid-ventral object which did not take the carmine stain. The ganglia were traced in successive sections as far as opposite the origins of the arms, but they were not visible in the sections immediately below the arms, where their place was occupied by areolated tissue. It was noticed that where each ganglion had a wide transverse diameter, it was not unusual for the cells in its centre to show signs of disintegration; and sometimes this was so extensive that a cavity had formed, the wall of which was irregular and showed no sign of a lining membrane (fig. 8, *g*, *g*).

When examined under a high magnifying power the structure of the ganglion cells was readily recognised. The nuclei were large and oval in shape, and as they stained a deep red with carmine, they were very distinct, and an intranuclear network of fibrillæ was present in them. The cell-plasm was granulated. The best-marked cells were considerably larger than the motor cells in the lumbar enlargement of the human spinal cord, though others were very much smaller. The bodies of the cells were polygonal, and from the angles delicate processes of the cell-plasm projected. As a rule, the cells were closely aggregated, and it was difficult to trace these processes for any distance, but they were sufficiently distinct to leave no doubt of the multipolar character of the cells. In places minute intercellular intervals were visible, and the outlines of the cells were defined by a distinct wall. Although the relative proportion of the nucleus to the cell-plasm varied in the cells, it was evident that in the largest cells the cell-plasm exceeded three or even four times in quantity the size of the nucleus (fig. 15).

From the character of the cells there can, I think, be no doubt that the red stained bodies were a pair of nerve ganglia. Their position in the head, their relation to its ventral surface and to the œsophagus, localise them as œsophageal ganglia, situated laterally and ventrally to the gullet, though not united to each other on the ventral aspect of the œsophagus. When portions of these ganglia were removed, teased with

needles, and stained with picrocarmine, delicate fibres were seen to lie between the cells and to emerge from the ganglia, which, from their association with the nerve cells, were obviously nerve fibres. Some were non-medullated; others, again, apparently contained a medullary substance, which had aggregated into little clumps within the neurilemma.

The mid-ventral object above referred to, when examined in relatively thick sections and under low magnification, seemed to be a solid cord-like body, lying mesially in the long axis of the head. It was situated between the ventral aspect of the alimentary canal and the inner surface of the ventral chitinous wall of the head. In longitudinal sections it was traced as high as the muscles of the head, the fibres of which arched above it, from their origin from the envelope of chitin to the side of the oral cleft. Its transverse diameter was greater than the antero-posterior, and it was bounded by a distinct capsule of fibrous tissue, which gave it a definite outline and differentiated it from the surrounding structures. The œsophageal ganglia were in relation to its sides, and in places even encroached on its ventral surface, and their upper ends were in the same transverse plane as the upper limit of its investing capsule. Below the ganglia it was bounded by the areolated tissue which was so abundant at and immediately below the arms. From its position it might have been taken for an axial nerve cord associated with the œsophageal ganglia, but no fibres could be detected in it, and it did not stain with carmine (fig. 8, V).

When thin sections were examined with a Zeiss lens × 250 the capsule was seen to be lined by a layer of rounded cells; in favourable sections they formed a continuous lining, but not unfrequently they were arranged in patches, separated by intervals. The cells were much smaller than the nuclei of the nerve cells in the adjoining ganglia, they were nucleated, and the cell-plasm was dimly granular. The material generally inclosed by the capsule had a granular character, and, as a rule, showed no trace of structure, and was possibly a coagulated substance. Sometimes, however, nucleated cells of great translucency were interspersed in the granular material, and fatty-looking globules were occasionally present.

In sections through the body of the parasite in the thoracic segment the corresponding arrangement, interposed between the alimentary canal and the ventral wall of chitin, was the ventral mesial space, so that the mid-ventral object above described was obviously a prolongation upwards into the head of the ventral space of the cœlom.

In some of the transverse sections through the parasite made a little above the attachment of the arms a special appearance was seen. It consisted in the presence of a band or column of chitin, almost circular in outline, lying in relation to the dorsal space and interposed between the œsophagus and the inner surface of the dorsal wall of the chitinous envelope, and apparently quite independent of it. It was difficult to give a satisfactory explanation of the part which the band played in the economy of the parasite (figs. 8, 10, *Ch*).

Alimentary Canal.

The canal extended in a direct line from mouth to anus, and had no convolutions in any part of its course. The oral cleft passed deeply into the substance of the ventral surface of the head, and was continued at its lower part into a relatively wide œsophagus, down which a bristle could readily be passed.

In transverse sections through the upper part of the œsophagus, the diameter from side to side was seen to be much greater than in the dorsi-ventral direction, and the opposite walls were almost in contact. The ventral wall of the canal was in close relation with the capsule of the mid-ventral space of the cœlom, which lay between it and the chitinous wall of the head, the dorsal wall was in relation to the musculature of the head, and the sides were in contact with the œsophageal ganglia (fig. 8).

In the lower part of the head, where the muscular fibres were replaced by areolated tissue, the dorsal wall of the canal was separated from the chitinous envelope by the dorsal space, which contained a granulated material, possibly a coagulum. The space was bounded by a fibrous membrane, which was lined by nucleated cells, though frequently they were in patches and did not form a continuous layer. These cells were about the size of leucocytes, and not unlike them in appearance. The muscular wall of the alimentary canal was attached to the areolated tissue at its sides by bands, formed of connective tissue and non-striped muscle, which constituted short lateral mesenteries; between these bands were narrow channels, in which blood or other nutritive fluid may have circulated.

Transverse sections through the body immediately below the arms showed the alimentary canal in the axis of the section, with a space in relation to both its dorsal and ventral surfaces. The lumen of the canal was not so compressed dorsi-ventrally as in the head. Well-marked areolated tissue surrounded the canal with its dorsal and ventral spaces, and closely packed the whole area between them and the inner surface of the chitinous wall (fig. 9). As it efficiently supported the canal, the lateral mesenteries were short and their fibres were continued into the meshwork of the areolæ, which again was continuous with the membrane lining the inner surface of the wall. A few scattered pigment cells were seen in this membrane, though not nearly so abundant as lower down in the thoracic segment of the body.

In sections through the attenuated thoracic segment the areolated tissue was no longer present, and the space inclosed by the chitinous wall was occupied by the alimentary canal and the dorsal and ventral spaces. The canal was in the axis of the section and was reniform in shape; its lateral angles were in such close relation to the lining membrane of the chitin that the mesenteries were practically absent (fig. 16). The dorsal and ventral spaces were proportionally large, almost equal in size, and were situated between the lining membrane and the corresponding wall of the alimentary canal. Each space was inclosed by a definite wall of fibrous membrane, the inner surface of which was lined by a layer of nucleated cells; the cell-plasm in some was granular

in character, though in others it was more translucent. The spaces were frequently devoid of contents, though in some sections irregular fragments, granular in appearance and possibly a coagulated substance, were present. The dorsal and ventral spaces, not only in relation to this, but to other divisions of the alimentary canal, formed the cœlom or body cavity. KOREN and DANIELSSEN named the dorsal space the dorsal canal, and stated that during life it was full of red thinly-flowing blood.

The chitinous wall was lined by a definite membrane, in which was a layer of large stellate cells, full of a rich purplish-black pigment.

The alimentary canal and the associated spaces retained the characters just described as far down the body as where the attenuated thoracic part was continued into the genito-abdominal segment, in which the chitinous wall also possessed a lining membrane with large richly-pigmented cells. The alimentary canal was in the axis of the segment, and its transverse section was almost round, and so capacious that it may properly be regarded as the stomach. Each lateral aspect was attached to the adjoining pigmented membrane by a mesentery. The dorsal and ventral spaces were relatively small. Between the canal and the sides of the chitinous wall the upper ends of the two ovaries were situated (fig. 17).

Somewhat lower in the genito-abdominal segment the alimentary canal had a reniform outline in transverse section. In proximity to the genital orifices it was compressed dorsi-ventrally, and the opposite walls were almost in contact. In some sections the canal gave origin at a lateral angle to one and occasionally more diverticular prolongations, the lumen in which was continuous with that of the canal (figs. 20, 21, 22). At its lateral angles the wall of the canal was attached to the pigmented lining of the chitinous wall by fibres, apparently non-striped muscle, which formed lateral mesenteries, and the fibres formed a loose network, in the meshes of which, as well as in the interspaces of the pigmented membrane, were nucleated cells, some scattered, others in clusters, many of which resembled leucocytes, though others were elongated, caudate, and stellate, not unlike the corpuscles of connective tissue.

In the genito-abdominal segment, in relation to the lateral mesenteries and to the sides of the dorsal space, the areolated tissue was present in abundance, and the cells in the areolæ were distinctly fatty; the pigment of the pigmented lining membrane was prolonged into the strands of the meshwork, and caused them to contrast strongly with the light-refracting contents of the fat cells which they surrounded (figs. 25, 26, 27).

In the terminal caudate segment of the abdomen the intestinal division of the canal had a similar compressed appearance; the wall of chitin was lined by a membrane associated with characteristic pigment cells; lateral mesenteries and adipose areolated tissue corresponded with the arrangement described in the genito-abdominal segment (fig. 32).

In the genito-abdominal and terminal segments the dorsal and ventral spaces were well marked, and the dorsal was much more capacious than the ventral. The membrane which bounded them was lined by a layer of cells, sometimes continuous,

though at others in patches, similar in character to those previously described in the spaces of the thoracic segment, whilst the contents consisted of an indefinite granulated, possibly a coagulated, material. The existence of these spaces in front of and behind the alimentary canal permits an expansion of the walls and an increase in the size of the lumen when the animal is feeding, and their greater size in the lower end of the intestine leads one to infer that the ejecta accumulate in it prior to expulsion.

Two coats were readily recognised in the alimentary canal in its entire length —a muscular and a mucous. In favourable sections an intermediate sub-mucous coat was seen. The muscular coat consisted of the usual non-striped form of fibre arranged in two layers—an external longitudinal and an internal circular or transverse. In the abdominal and caudate divisions this coat was thickened, and had a crenulated appearance in the sections. When sections were made either longitudinally or obliquely through the canal, to enable one to obtain a view of the free surface of the mucous membrane, numerous slender, closely-set rugæ, lying parallel to each other, were seen to extend longitudinally along its surface (fig. 6). In transverse sections they were cut across, and they then had the appearance of villous processes projecting into the lumen. It was observed in these sections that the sub-mucous coat formed the core of the projections, whilst the free surface was formed of the mucosa; obviously, therefore, they were not true villi, but were permanent rugæ, like the circular valvulæ conniventes in the small intestine of the mammalia. At the lower end of the canal the rugæ were more elongated and thicker than in the thoracic segment of the body. The mucous membrane was covered by a layer of epithelium, the cells of which in favourable specimens were seen to be short columns. In longitudinal or oblique sections through the canal in which the inner surface of the mucous membrane could be seen, the broader ends of the cells were recognised as forming the free surface of the mucosa. The lumen of the intestine contained epithelial and other débris.

In proximity to the anus the intestine and the structures around it were specially modified. A short distance above the anus the intestine in transverse section was flask-shaped, the stalk of which was attached to the dorsal wall of chitin by a narrow mesial dorsal mesentery, composed of non-striped muscle, which divided the dorsal space into two lateral halves. The ventral space had not at first a corresponding division. In addition, muscular fibres on each side passed from the chitinous wall to the sides of the intestine: these fibres had the form of striped muscle, but were not definitely striated. The wall of chitin was lined by a strongly pigmented membrane, in which numerous leucocyte cells were seen, either scattered or in groups. The proper muscular wall of the intestine was thicker than in the upper part of the caudate segment, and the parallel ridges of the mucous membrane were closely set together.

A little nearer the anus the section through the intestine was ellipsoidal, with the long axis directed dorsi-ventrally. The ventral space was now divided into two lateral

halves by a broad mesial mesentery formed of non-striped muscle. The walls of the dorsal and ventral spaces were lined by cells like those previously described. Strong striated muscles were situated laterally to the intestine; they arose from the wall of chitin, and were inserted by tendinous bands into the wall of the gut. Processes of the highly pigmented lining membrane passed between bundles of these fibres, and differentiated them into distinct muscles.

At the anus itself the dorsal and ventral spaces were scarcely to be recognised; the lumen of the intestine was small and laterally compressed. The submucous coat was greatly thickened and the mucous membrane showed no parallel ridges. The lateral striped muscles were well marked. The dorsal and ventral mesial bands of non-striped muscle were prolonged on to the sides of the intestine, external to its proper coat, and were arranged in an ellipse. At and near therefore the anal orifice the intestine was provided with transversely striped muscles, situated laterally, which acted as dilators; and with non-striped muscular fibres, distinct from the proper muscular coat, which formed a sphincter muscle (fig. 19).

No specially differentiated Vascular System was recognised, and no structures that could be regarded as heart, blood- or lymph-vessels. The dorsal and ventral spaces associated with the alimentary canal, and the intervals between the bundles of fibres of the mesenteries and of the lining membrane of the chitinous wall, provided channels for the distribution of a nutritive fluid.

Nervous System.

In the section on the structure of the head I have described the pair of œsophageal ganglia, which, from their size, constituted the most important divisions of the nervous system. Their relation and structure having already been narrated, it is unnecessary to repeat them; but it may be stated that the position of the ganglia enabled them readily to supply nerve fibres to the wall of the œsophagus and to the striped muscular fibres, which formed important constituent parts of the head. Associated with the ganglia was a relatively large nervous cord, composed of numbers of delicate nerve fibres.

In transverse sections through the elongated thoracic segment clusters of cells were seen at intervals in close relation to the pigmented lining membrane of the ventral part of the wall of chitin and to the ventral space. The cells coloured readily with carmine, and the nuclei stained deeply and were relatively large. In some sections at least one process could be seen to arise from the cell body; in others a process arose from opposite aspects of the cell body, and the cells appeared to be fusiform or bipolar; other cells, again, were multipolar, and with delicate processes extending for a recognisable distance. In one specimen a process could be traced so far undivided as to be obviously the axon of the cell. Each cluster of cells formed a small nerve ganglion, the cells in which were smaller than in

the œsophageal ganglia. From cells of this character being seen so frequently in the transverse sections, it was clear that a chain of ganglia extended longitudinally along the ventral aspect of the thoracic segment of the parasite, immediately internal to the chitinous envelope, and that from the ganglia nerves could readily be distributed to the wall of the adjoining parts of the alimentary canal.

In transverse sections through the genito-abdominal segment collections of cells were seen immediately internal to the pigmented lining of the ventral part of the wall of chitin. They were arranged in a crescentic row, which followed the curvature of the wall, and the concavity of the crescent was directed towards the cement ducts, but was separated from them by a definite interval. The cells were nucleated, the cell-plasm was granulated, and two or three times greater in amount than the nucleus, which, again, was as big or even somewhat larger than the leucocytes, so abundant in the lateral mesenteries of the alimentary canal. Some of the cells were globular, others were elongated and rounded at the ends; occasionally I saw a multipolar cell, or one with a single pole, and frequently the cells were fusiform, with attenuated poles. From the position of the groups of cells in relation to the ventral wall of the parasite, and from their size and general character, I am of opinion that they form the abdominal chain of the nervous axis, and are engaged in the innervation of the organs contained in the genito-abdominal segment.

In many of the sections a cell was situated beyond the termination of each horn of the crescent, which was greatly elongated, and its outer pole was prolonged as far as the wall of the cement duct, which protruded a pointed process to meet it. In some sections I observed that this pole bifurcated, and its limbs embraced and were prolonged into the wall of the cement duct. This cell was placed at the side of the ventral space and seemed to be in its wall.

Pennate Appendages.

These appendages, which constituted one of the most characteristic features of the genus, grew from the ventral surface of the terminal caudate segment of the abdomen, whilst an occasional one sprang from the sides of the genito-abdominal segment near the genital openings. They formed a closely-set brush-like arrangement, the bristles of which varied in length and projected from 4 to 7 mm. from the base of their attachment, which was continuous with the wall of the segment, and had the character of a papillary outgrowth of the wall. Branches arose from the stunted basal papillæ, and these almost immediately again divided, so that from six to ten secondary branches might proceed from a common stem. Each branch had a pigmented core, inclosed in a translucent wall of chitin (Pl. I. fig. 4).

In sections made through the caudate segment the knife sometimes passed through the chitinous wall at the spot where the base of a papilla sprang from it (fig. 18). The chitin of the segment was prolonged into the wall of the bristles, and the pigmented membrane lining the wall of the segment was continued directly into their

axis, and thence into the branches. In transverse sections through the bristles the pigmented core was frequently partially or wholly divided into two portions, which were either close together, or were partially fused and formed a dumbbell-like figure.

Reproductive Organs.

Koren and Danielssen, in their account of *Pennella balænopteræ*, figured a dissection of the genito-abdominal segment. They described as present in it a pair of ovaries with oviducts, a pair of cement glands with excretion canals, the latter of which were nearer to the ventral surface than the oviducts, and two short canals. No mention is made of the receptacula, and the ova strings were wanting in most of their specimens.

Ovaries.—The ovaries were situated in the upper part of the genito-abdominal segment. At and near its junction with the thoracic segment, where the alimentary canal was dilated, and the dorsal and ventral spaces were relatively small, short lengths of a divided tube were seen in transverse sections to occupy a relatively large region on each side of the canal, dorsal to the lateral mesentery, and laterally to the dorsal space; the portions of each tube were scattered in the region, and were, I believe, the upper end of the ovary, for they were occupied by nucleated cells which resembled rudimentary ova. The oviducts and cement ducts were not present (fig. 17).

In sections a little lower down the parts of the divided ovary were in greater lengths and more continuous with each other, the tube was cylindriform in shape, and had reached or almost reached the mesial plane of the parasite, so as to lie immediately internal and parallel to the pigmented lining of the chitinous wall, but separated from the alimentary canal by the dorsal space. The part of each tube which lay next the wall followed its curvature. Somewhat lower down the wall of the tube next to the dorsal space bulged into diverticula and lost its cylindriform character. Whilst each ovary was in many sections situated entirely on its own side of the mesial plane of the parasite, in others the inner ends of the tubes from the opposite sides crossed the mesial plane and slightly overlapped each other. In all these sections the oviducts and cement ducts were present and were transversely divided (figs. 20, 21).

The wall of the tube was formed of a delicate membrane, and the lumen contained unfertilised ova (fig. 24). In many instances they were so closely packed together that the outlines of the individual cells were obscure. The ova were larger and more precisely differentiated when in proximity to the wall of the tube, which, from its surface being slightly crenulated, and from the passage of slender processes from the membranous wall into the lumen, seemed to be partially divided into compartments, in each of which an ovum was lodged. Each ovum contained a relatively large, well-defined germinal vesicle, situated at or near the centre of the cell-plasm, and in each vesicle, about its centre, was at least one germinal spot; not unfrequently two spots were present, and in some instances I saw three spots in a germinal vesicle.

Cement Glands.—The pair of cement glands were situated in the upper part of the genito-abdominal segment. In a considerable part of their extent they were alongside the ovaries, and corresponded in their general relations; but in some of the transverse sections ovarian tubes were present without adjoining cement glands, and in others portions of the cement glands were seen with no ovarian tubes in proximity to them. The ovaries and the cement glands were therefore not quite equal in extent. When both were present in the same transverse section the ovaries lay across the mesial plane, between the two cement glands, the latter of which were placed in close relation to the dorsal surface of the lateral mesenteries. When the cement glands alone were present they lay close to the mesial plane of the parasite, and were separated from each other by a prolongation of the dorsal space (figs. 22, 23).

Each cement gland consisted of a coiled tube inclosed in a membranous capsule, from the inner surface of which fibrous processes passed between the coils of the tubes. Lying close to the outer surface of that part of each capsule which was next the interposed dorsal space, well-marked nucleated cells, which possessed one or more processes, were seen; and as similar cells were present in the membranous wall of the space where it was next the pigmented lining membrane of the chitinous envelope, these cells should be regarded as belonging to the wall of the space rather than to the capsule of the cement gland. The coiled tube of the cement gland, in making the section, had been cut into short pieces, transversely, obliquely and longitudinally. It had a well-defined wall, which was lined by a layer of short cubical cells. The lumen of the tube contained a dimly granular substance which stained with carmine and with hæmatoxylin. No sign of an ovum could be seen in the tube.

Oviducts, Cement ducts, Receptacula, Ova strings.—Each ovary and cement gland had a characteristic duct. In the numerous transverse sections made through the genito-abdominal segment, except at the highest part of the ovary, an oviduct and a cement duct were seen on each side of the mesial plane. They were placed ventrally to the lateral mesenteries, and in relation to the sides of the ventral mesial space. They had evidently emerged near the upper ends of their respective organs, and had passed forward into the ventral region of the segment, down which they ran to the receptacula. The cement duct on each side was in front of the oviduct, and was separated from it in the upper part of their course by a slight interval. A slender intermediate band passed from the wall of one to that of the other, and at its junction with the cement duct the wall of that duct was much thinner than in other parts of its circumference. Lower down in the segment the walls of the two ducts came in contact with each other and fused together. Before their lumina became continuous with each other, the oviduct diminished in its calibre, and the cement duct became elongated antero-posteriorly. The intermediate part of the common wall then disappeared, and the receptaculum was formed a little above the genital openings (figs. 20–23, 25, 26).

The ducts were readily distinguished from each other, both by their relative position and their characters. The wall of the oviduct was much the thinner, and was

generally cylindriform, though it showed in each transverse section from four to ten or twelve bulgings on the outer surface of its wall. In many sections each bulging seemed to contain a nucleated cell, but under higher powers this was not so evident. In some sections a sharp line, apparently the lining membrane of the duct, was continued round the wall, as if to shut off the bulgings with their contents from the lumen. In others, again, the bulgings projected into the lumen, and were not shut off by a lining membrane. When the wall of the oviduct had fused with that of the cement duct, a layer of nucleated cells was traced from the outer surface of one duct to that of the other. In some transverse sections through the oviducts the lumen contained a delicate network of fibres which radiated from the centre to the periphery. The wall of the cement duct was several times thicker than that of the oviduct, except at the spot where it was joined either by the intermediate band or by the wall of the oviduct. The lumen of the cement duct, especially near its lower end, frequently contained a plug of cement which almost filled the tube (figs. 20–23).

Receptacula.—Each receptaculum seminis was situated at the side of the ventral space near its anterior part. Its antero-posterior diameter was longer than the transverse, and the wall lying next the ventral space was sometimes thinner, at others thicker, than the opposite wall; the anterior part both of wall and lumen had frequently a tortuous appearance, as if slightly convoluted, and from its lower end a short canal arose, which ended in the genital orifice. The lumen almost invariably contained a plug of cement, antero-posteriorly elongated like the receptaculum itself (figs. 29, 30).

Associated with the lower part of the genito-abdominal segment was a distinct muscular arrangement, the fibres of which were transversely striped, but in addition some bands belonging to the lateral mesenteries consisted of unstriped fibre. The striped muscles arose from the chitinous wall, some bundles ventral to the mesentery, others within its substance; they passed downwards and inwards, and in the transverse sections the fibres were usually cut through transversely or obliquely. Some fibres were attached to the wall of the oviducts; but the greater number reached the posterior end and outer wall of the receptacula, to which they were attached by tendon-like structures (fig. 28). Owing to this arrangement, the wall could be drawn outwards and the lumen of the receptaculum made larger, a condition which doubtless prevailed when the ova, the cement, and probably the spermatic fluid also, passed into it.

Ova strings.—These were about the thickness of fine sewing-thread. They began at the genital orifices and floated in the sea in which the animal lived. The outer part of each string was formed of cement, and the space which it inclosed would have contained the ova had they been ripe for extrusion. When examined microscopically, transverse lines closely set together were seen to pass from one to the other side of the inclosing cement. In my specimens I saw no ova in the ova strings. The ovarian ova were unripe, and there was an absence of ova in the oviducts and receptacula. When sections were made through the ova strings, the space inclosed by the cement was seen

to contain a quantity of minute fatty particles, the products of degeneration. It should be remembered that the parasites were taken in the autumn, after they had, in all probability, shed a crop of ova, and before the next crop was ripe for impregnation.

The Male.

It is well known that in the parasitic Copepoda the male is insignificant in size as compared with the female. In Chondracanthus and some other genera it has been ascertained that the male is attached to the female, close to the apertures for the ova strings. I consequently made a close examination of the ventral surface of the body of all my specimens of Pennella, with the object of observing if a male were present in any of them, but I failed to recognise one. Koren and Danielssen stated definitely in their memoir on *P. balænopteræ* that they had not seen any males attached to their specimens, so that the male of *P. balænopteræ* is as yet unknown. It would, indeed, appear that the recognition of the male in any species of Pennella is a rare occurrence. I have stated in the introductory section that Boccone, so long ago as 1674, figured, firmly affixed to the Pennella, which is now regarded as *P. filosa*, a small object which he spoke of as a "pediculus" or louse. I have no doubt, from its relative size and the place of attachment, that it was the male of the species. Boccone, therefore, should have the credit of being the first to see and figure a male Pennella, though he did not realise its sexual significance. In *Pennella exocœti* and in *P. varians* the male has also been recognised and figured.

The habitat of the male Copepod, as in the case of Pennella, when not attached to the female, is uncertain. In a species like *Lernæa branchialis* affixed to the gills of the Gadidæ and flounders, males have been found within the gill-chamber, some attached independently to the branchiæ, others to the bodies of the females. In *P. balænopteræ* the females were affixed to the extensive surface of the smooth back of a great whale, to which they had doubtless attached themselves in the Cyclops phase, through which the female passes before she becomes adult and assumes relatively gigantic proportions, though in many respects retrograde characters. If the males of Pennella be provided with hooked antennæ, like those found in the male Lernæa, as there is no adjoining chamber for their lodgment, they may become directly attached to the skin of the whale in proximity to the females, until the time arrives when they are required to affix themselves to the females for the purpose of fertilising the ova when these are ripe for impregnation. If the male Pennella, as is very probable, is insignificant in size, when unattached though perhaps in close proximity to the female, it would easily be overlooked.

In considering the question where and when the ovarian ova are impregnated in the parasitic Copepoda, it has to be kept in mind that whilst the female is fixed to its host, the male retains for a considerable time the character of a free swimming Crustacean, though subsequently it affixes itself in many of, and possibly in all, the species

to the female, close to the genital openings, with probably the power of again detaching itself when insemination is completed. It is possible, as has been shown by ANDREW SCOTT in Lernæa, that fertilisation of the female may take place in the Cyclops stage when she has become attached to the host. Naturalists have expressed different opinions on the locality where the spermatic fluid comes in contact with the ova and impregnates them. Some have thought that as they pass out of the genital openings the proximity of the male, from its attachment close to the openings, allows the sperm to bathe the wall of the ovum and the spermatic threads to penetrate it. This, however, is doubtful, as the sea-water in which the parasite lives would, possibly, in such a case have washed away the sperm and impeded or prevented impregnation.

Others, again, consider that the sperm enters the genital orifices, passes into the receptacula, distends them and ascends the ovarian ducts, so as to meet the ova in their descent. It is obvious that penetration of the ovum by the sperm filaments, which is essential to impregnation, must occur before the ovum meets with the cement and is coated by it. This can only take place in the oviduct, for the secretion of the cement gland flows into the receptaculum and can envelop the ova as they enter it. The observations recorded in this memoir show that the receptaculum and its short excretory canal, even in the unimpregnated female, contained cement, so that the ova could have been coated by it before they had passed out of the genital opening to form, along with the cement, the ova string. It would seem, therefore, that fertilisation of the ovum must occur in the oviduct. The non-attachment of males to my female specimens should be associated with the unripe condition of the ova, and the consequent impossibility of fertilisation being effected at the time when the parasites were collected.*

COMPARISON WITH OTHER SPECIES.

A careful comparison of the characters of my specimens, with the description and figures by KOREN and DANIELSSEN in their excellent memoir, has satisfied me of their identity with the species which they have named *Pennella balænopteræ*. This species therefore infests both *Balænoptera musculus* and *B. rostrata*.

I have not seen the species which STEENSTRUP and LÜTKEN described and delineated as found on *Hyperoodon rostratus*, and which they named *Pennella crassicornis*. KOREN and DANIELSSEN, however, after comparing original specimens, considered that their *Pennella balænopteræ* was quite distinct from *Pennella crassicornis*, as it was half as long again; had a broader and longer head; the horns were nearly horizontal and very slender, whilst in *Pennella crassicornis* the dorsal horn was inclined almost perpendicularly downwards.

When compared with the species of Pennella infesting fish, *P. balænopteræ* is very much longer. ODHELIUS regarded the length of *P. sagitta* as equal to the breadth of

* Ova which had developed to the nauplius stage were seen by H. S. WILSON and myself in *Chondracanthus lophii* collected in August, and in *Lerneopoda dalmanni* collected early in the year. See our Memoirs in *Trans. R.S. Edinburgh*, 1862.

the thumb. DEKAY's specimen of *P. sagitta* was little more than half an inch long, but it is obvious from his figure that the head and arms had been torn off: the length was probably one inch. V. NORDMANN stated (Heft 2, S. 122) that *P. sagitta* found on *Lophius marmoratus* was only ten lines long, and with the ova strings 1 inch 4 lines. STEENSTRUP and LÜTKEN measured nine specimens of *P. varians* and found them to vary in length from 18 to 27·5 mm. (0·7 to 1·1 inch). These species are the most diminutive known. BOCCONE stated that his specimens were usually about 4 inches long. BAIRD gave 4 inches as the length of his *Penella pustulosa*, CHAMISSO and EYSENHARDT's specimen *P. diodontis*, THOMSON's *P. histiophori*, and also *P. exocœti*, were about the same length. PERCEVAL WRIGHT's specimen, *P. orthagorisci*, was 7 inches long.

Species differed in the number of horn-like arms which radiated from the base of the head. In BOCCONE's figure neither head nor arms were depicted, and I am disposed to think that they had been broken off in the process of detaching the parasite from the Swordfish, in the flesh of which they were buried; the abdominal segment of the body was shown to be longer than the short thoracic segment. In ODHELIUS' figure of *P. sagitta* neither head nor arms were represented, probably for the same reason. In the *Pennella sagitta* described by VON NORDMANN, and in *P. pustulosa*, *diodontis*, and *orthagorisci*, two arms were said by their respective describers to have been present. THOMSON states that *P. histiophori* had two lateral arms, but projecting between them posteriorly was a rounded protuberance; which, without doubt, was a rudimentary dorsal arm. *P. sultana*, *exocœti*, *crassicornis*, and *balænopteræ* had each three arms. In eighteen specimens of *P. varians* examined by STEENSTRUP and LÜTKEN, one-third were said to have had three arms, two-thirds only two arms. The lateral arms were constant, but the dorsal arm was variable in the same species. The presence of a dorsal arm is not therefore a constant element in the establishment of specific differences. A specific character, which is obviously of importance, is the relative length of the thoracic and genito-abdominal segments. In *P. balænopteræ* the thoracic was twice the length of the conjoined genito-abdominal and caudate abdominal segments, and nearly three times the length of the genito-abdominal segment. By way of contrast in VON NORDMANN's *P. sagitta* and in THOMSON's *P. histiophori* the thoracic and genito-abdominal segments did not seem to be distinctly differentiated from each other; in PERCEVAL WRIGHT's *P. orthagorisci* and in *P. exocœti* the genito-abdominal and thoracic segments were about equal in length, and in CHAMISSO and EYSENHARDT's *P. diodontis* the genito-abdominal segment was nearly twice the length of the thoracic. Although there seems to be no doubt that *P. balænopteræ* is a species quite distinct from those that are parasitic on fish, it is difficult to say definitely, in the absence of the type specimens for comparison, if all the fish-infesting forms of Pennella that have been described by different specific names have a true claim to this distinction, though it is, I think, probable, if a careful comparison were made, that the number of so-called species would be diminished.

Conchoderma.

Several naturalists have observed that species of Pennella have occasionally attached to them animals belonging to the sub-class Cirripedia. Chamisso and Eysenhardt seem to have been amongst, if not the first naturalists to describe a Cirriped attached to a Pennella. In 1821 they stated that a *Lernæa* (*Pennella*) *diodontis*, from the branchiæ of *Diodontis mola*, captured in the Pacific, had *Lepas anatifera* affixed to it. G. O. Sars described, 1865, a Pennella, with the head buried in the blubber of *Balænoptera musculus*, to which *Cineras vittata* (*Conchoderma virgata*) was attached. Koren and Danielssen figured two specimens of *Conchoderma virgata* affixed close to the genital orifices of *Pennella balænopteræ*, and they stated that in another example as many as seven specimens were attached to the thin thoracic part. Paul Meyer saw in the collection at Naples six examples of a Pennella from *Xiphias gladius*, to one of which *Conchoderma virgata* was affixed: owing to the Pennella being imperfect, the species was not determined. To one of the examples of *Pennella orthagorisci* in the Royal Scottish Museum, already referred to, three specimens of *Conchoderma virgata* were cemented at or near the junction of the thoracic and abdominal segments. Also the *P. exocœti* in the same museum were similarly infested.

It is interesting to note that cases have been recorded of a direct attachment of Conchoderma to the skin of whales. Thus, Charles Darwin, p. 66, stated that he had seen the basal end of the peduncle of *Conchoderma aurita* sunk into the skin of Cetacea. G. O. Sars had described the same species attached to the humpbacked whale, *Megaptera boops*, and a similar attachment had also been noticed by Sophus Hallas.

One of my specimens of *P. balænopteræ* had an example of *Conchoderma virgata* cemented at the junction of the thoracic and genito-abdominal segments (Plate I. fig. 3). It is unnecessary to describe the generic characters of Conchoderma, or the specific characters of *C. virgata*, as they have been so fully narrated in the classical treatise of Charles Darwin; but in order to identify the species, I may briefly refer to the external appearance of my specimen. It measured 46 mm. (1·8 in.) in extreme length, 15 mm. in the greatest dorsi-ventral diameter, and 15 mm. in greatest breadth. Though the peduncle blended with the capitulum they could be differentiated, and the former was found to be slightly longer than the latter. The dorsal carinal plate was 16 mm. long and 3 wide, and reached the anterior end of the capitulum. The scutal plate was three-lobed and 7 mm. in length. The tergal plate was 5 mm. long and only 1 mm. in width. The interval between the upper lobe of the scutum and the carina was 8 mm., and between the anterior lobe of the scutum and the tergum 6 mm. The coat in the intervals between the plates was not calcified, and was yellowish-grey in colour, with three purple bands on each side extending antero-posteriorly. The highest band on each side reached the dorsal border behind the carina, where it blended with its fellow. The other bands ran independently the whole length of the animal, and did not branch. A pair of stunted processes at the anterior end of the carina represented the pair of ear-

like appendages so characteristic of *Conchoderma aurita.* My specimen in general form and characters resembled that figured by DARWIN in plate iii. fig. 2, but it had three, and not four, purple stripes in the mantle.

In conclusion, I would express my acknowledgments and thanks to Mr JOHN HENDERSON, Assistant Keeper of the University Anatomical Museum, for the aid which he has given me in preparing the numerous sections examined in the course of my research, and for photographing those which illustrate the internal anatomy of the parasite, many of which have been reproduced in three of the plates.

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EXPLANATION OF PLATES.

PLATE I.

The drawings in this plate were made from nature by MR JAMES T. MURRAY.

Fig. 1. *Pennella balænopteræ* seen from its ventral surface. Natural size.

Fig. 2. Head, arms, and upper part of thoracic segment, to show the four pairs of rudimentary feet-like appendages; enlarged.

Fig. 3. Abdomen, showing the attachment of *Conchoderma virgata* to the upper end of the genito-abdominal segment. Natural size.

Fig. 4. Pennate bristles detached from the caudate abdominal segment. × 10. p. 423.

PLATE II.

Fig. 5. Longitudinal dorsi-ventral section through the head and upper end of the body of Pennella. The arrangement of the striped muscles *s.m.* in the head, the relations of the œsophagus Œ. to the mid-ventral space V., to an œsophageal ganglion *g.* and to the areolated tissue *ar.* are shown. × 8. p. 419.

Fig. 6. A more highly magnified section through a part of the œsophagus Œ., in which a portion of the free surface of its mucous lining with the longitudinal folds parallel to each other, also the fibres of the non-striped muscular wall, *n.m.*, passing into the surrounding areolated tissue, *ar.*, and forming mesenteries, are shown. × 14. p. 421.

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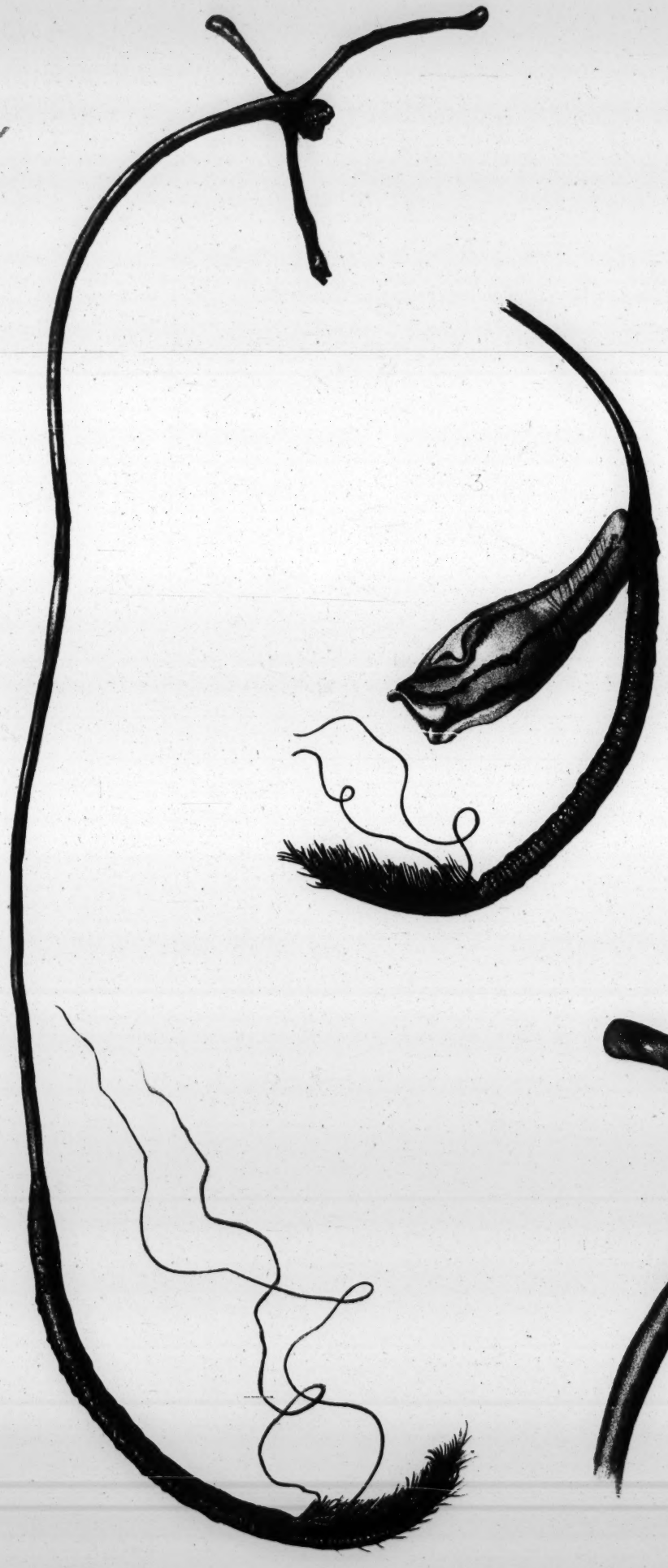
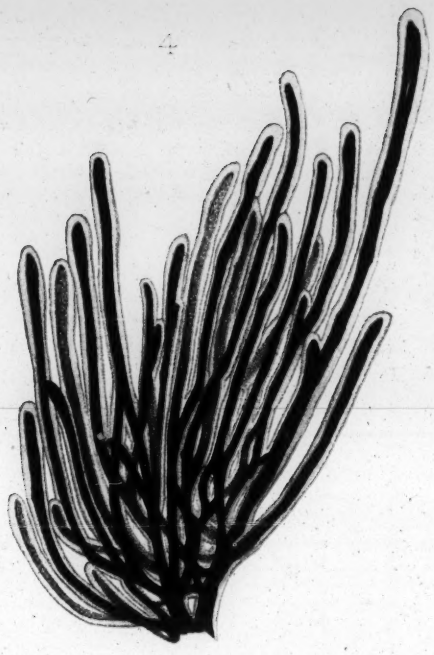
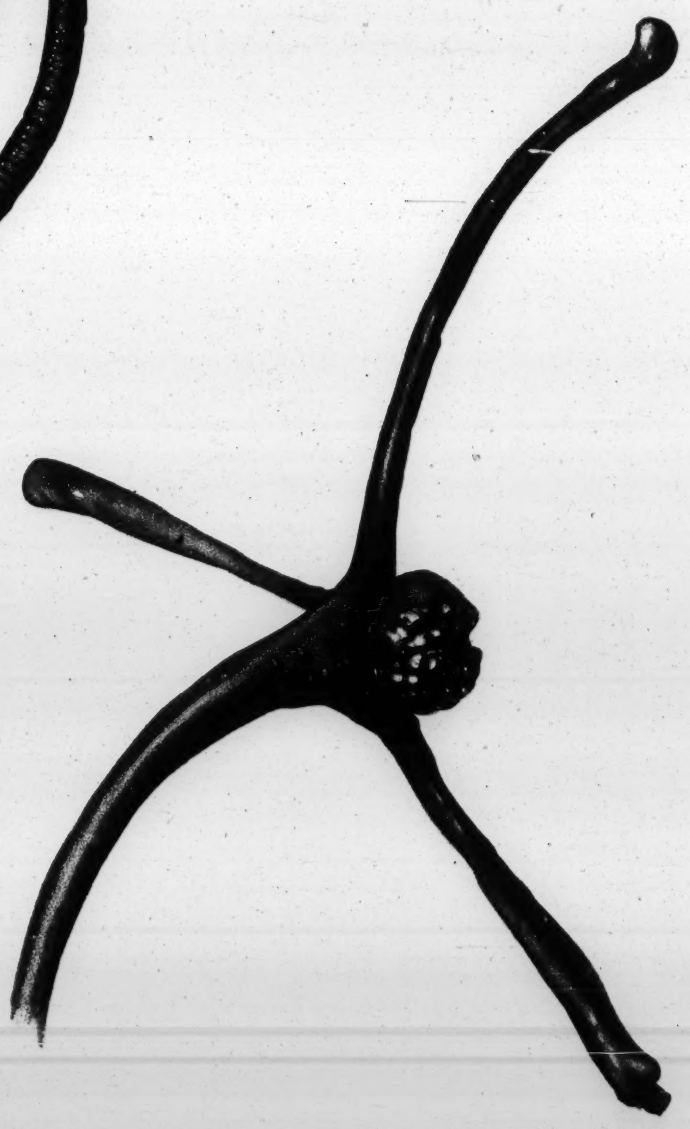






Sir Wm. Turner on *Pennella balænopteræ*.—Plate II.

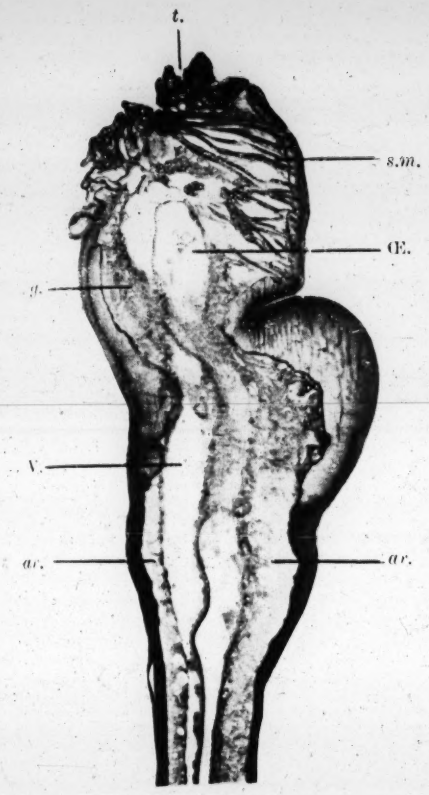
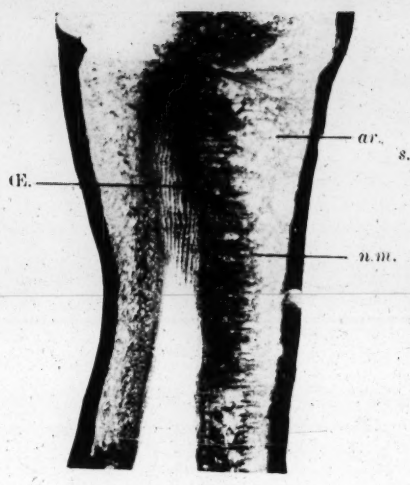


Fig. 5.



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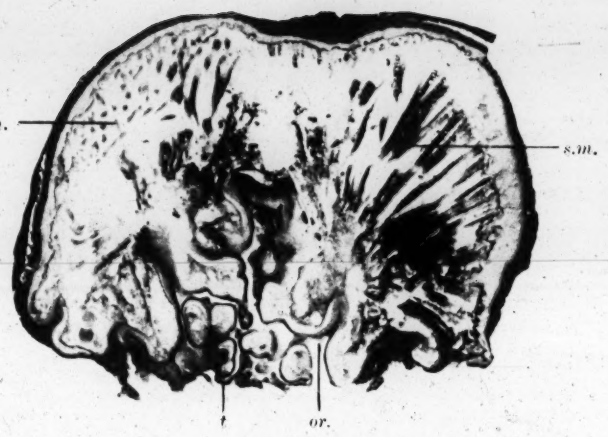

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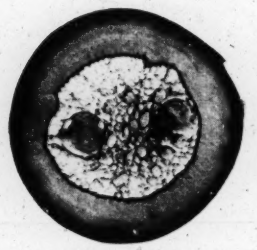

Fig. 11.

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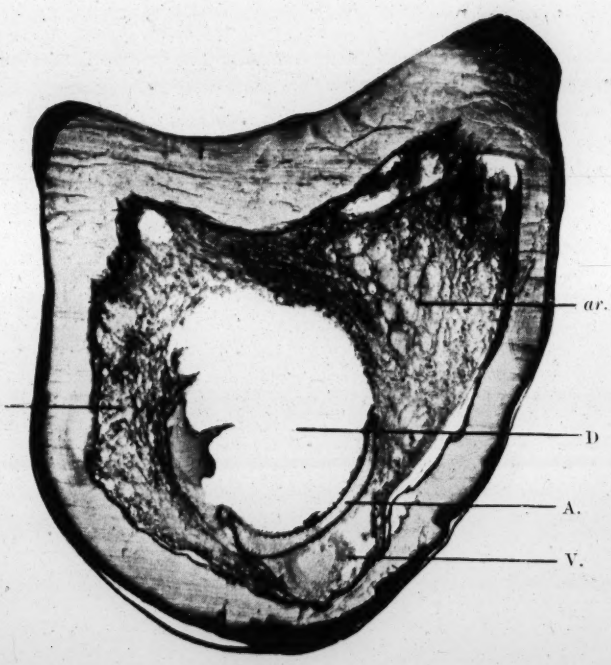

Fig. 9.

Fig. 12.

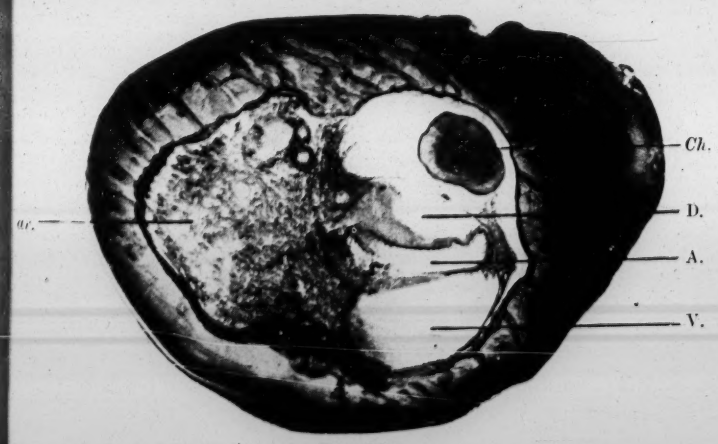

Fig. 10.

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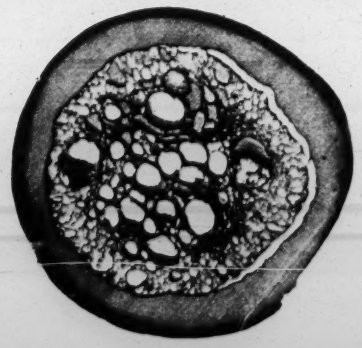


Fig. 13.

Sir Wm. Turner on *Pennella balænopteræ*.—Plate III.

Fig. 15.

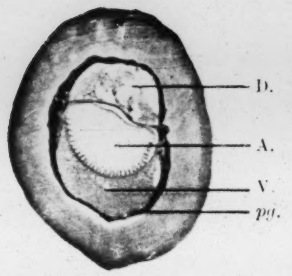
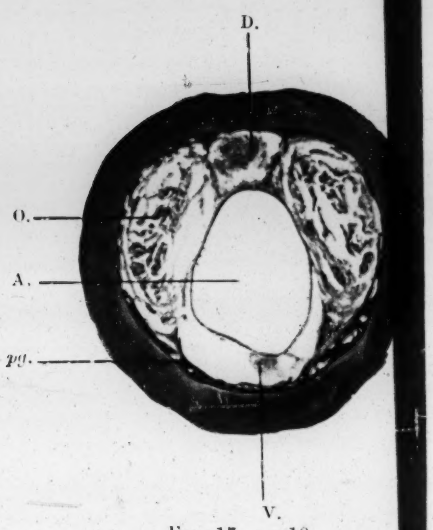


Fig. 16. ×13.



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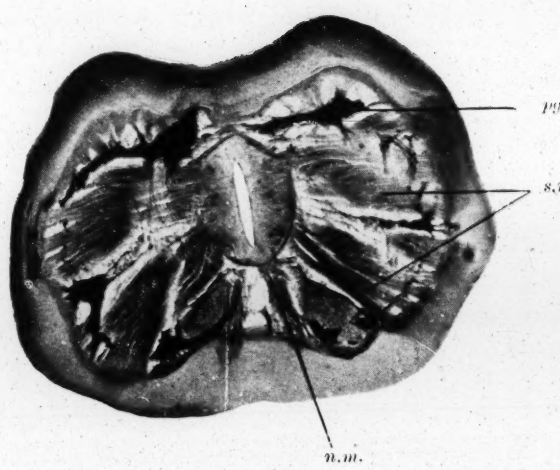

Fig. 19

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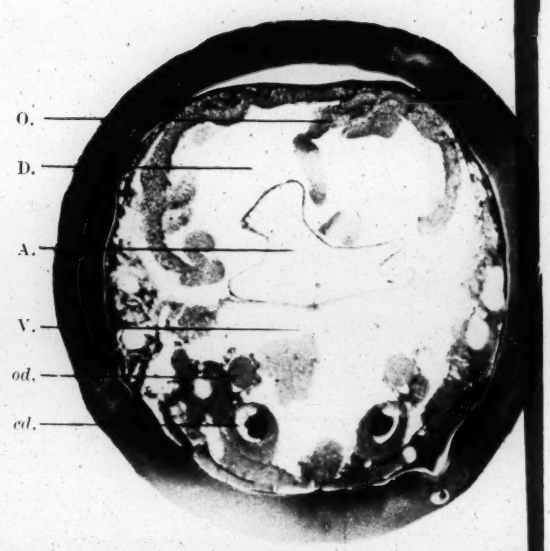

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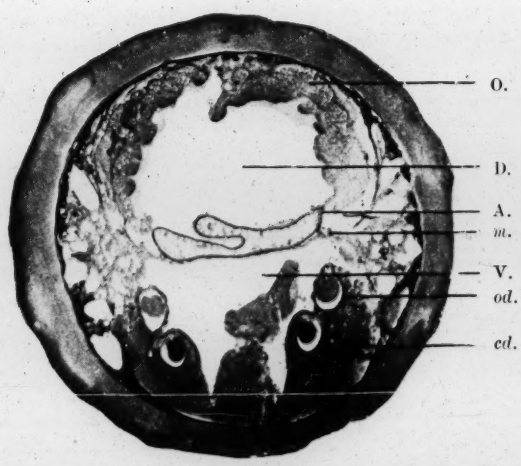

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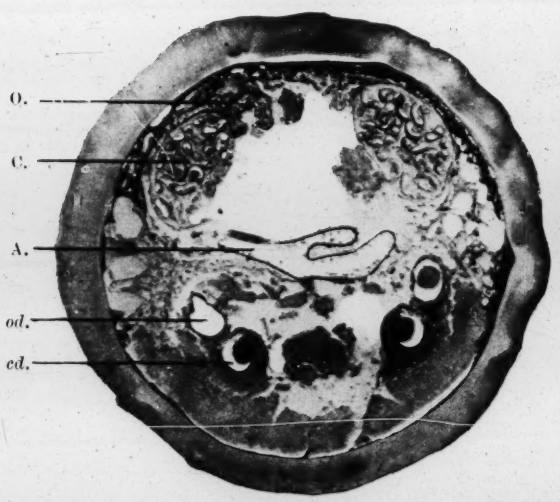

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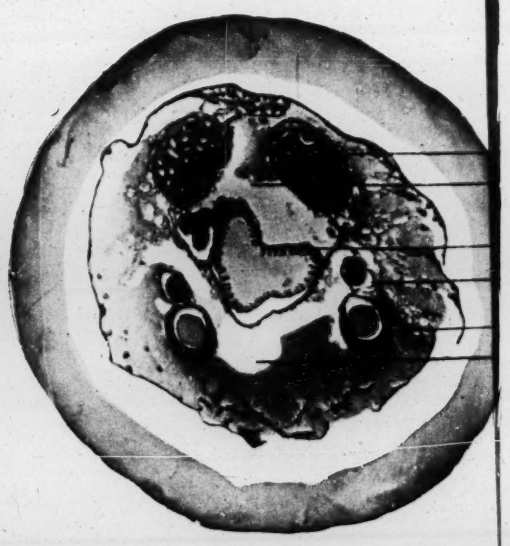
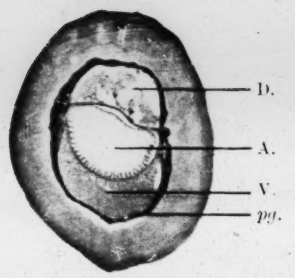


Fig. 23. ×13.

Sir Wm. Turner on *Pennella balænopteræ*.—Plate III.

Fig. 15.



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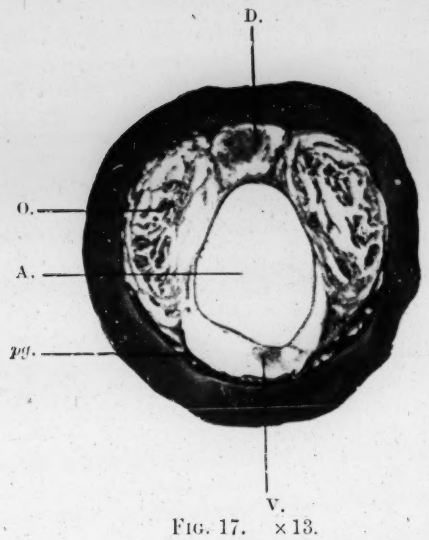

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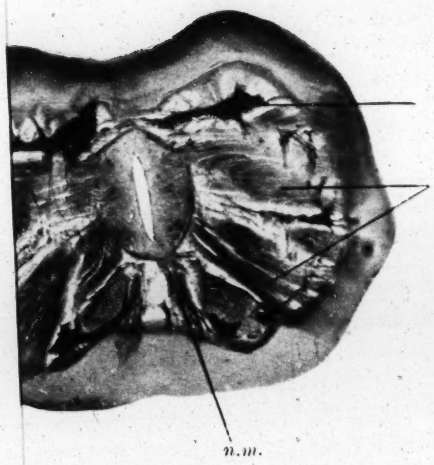
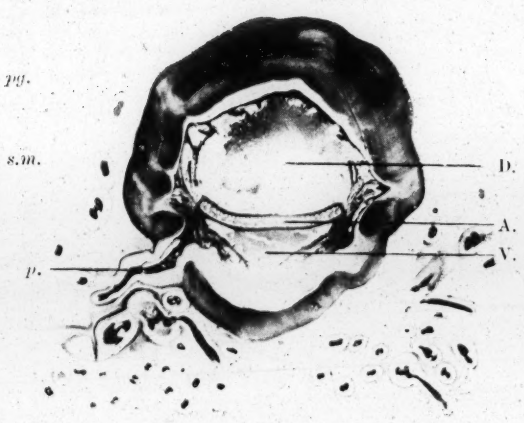


Fig. 19



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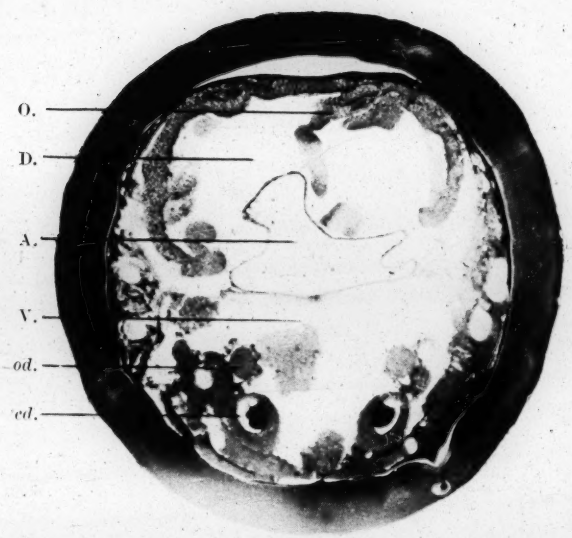

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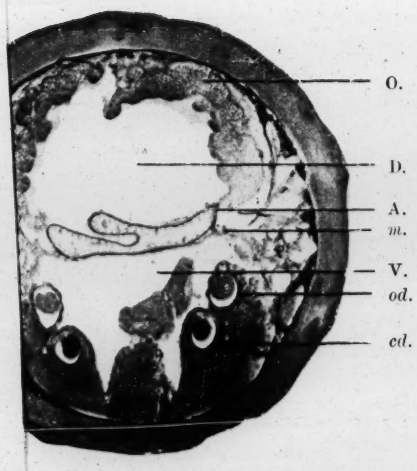

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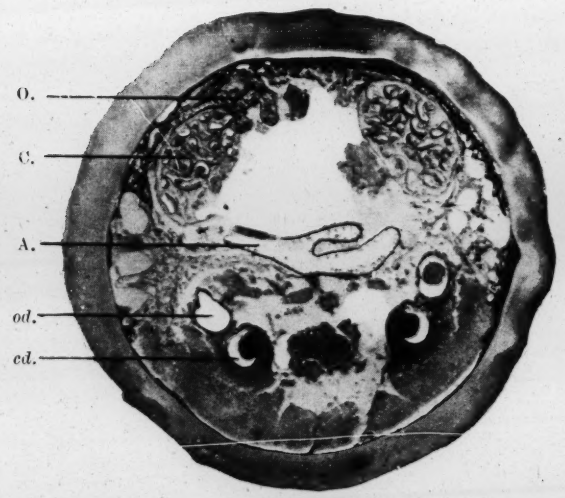

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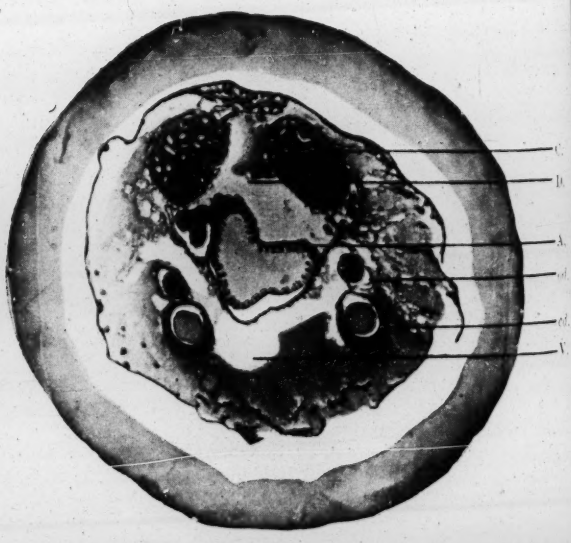


Fig. 23. ×13.

Sir Wm. Turner on *Pennella balænopteræ*.—Plate IV.

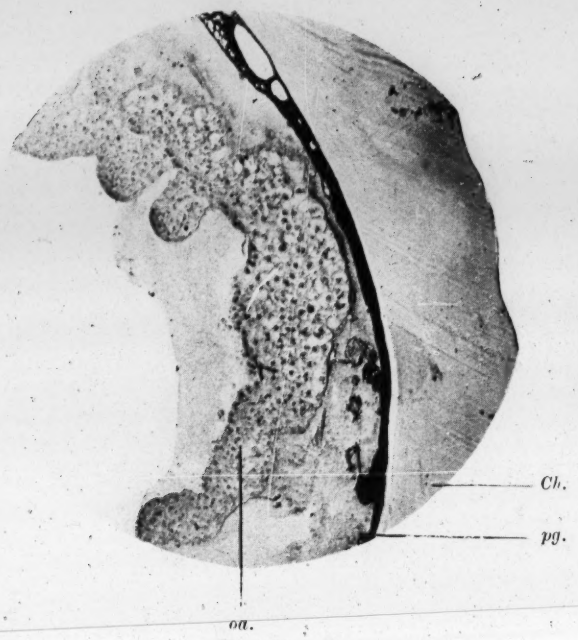
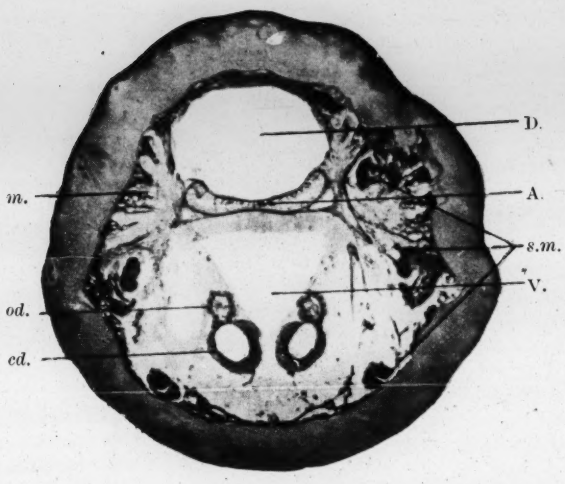


Fig. 24.



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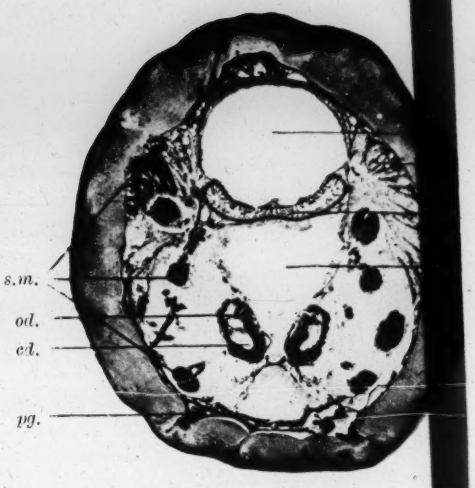

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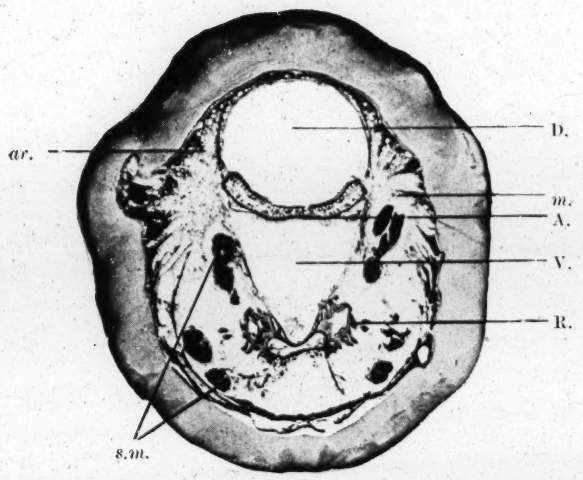


Fig. 27. ×13.

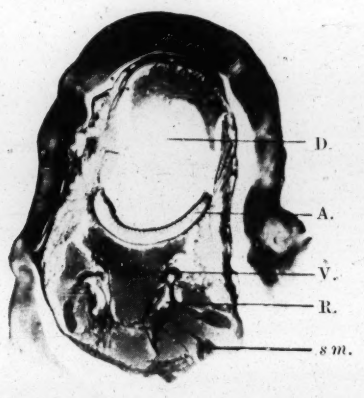


Fig. 28. ×9.

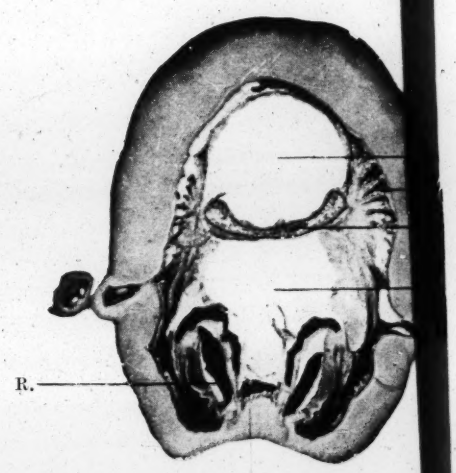


Fig. 29. ×13.

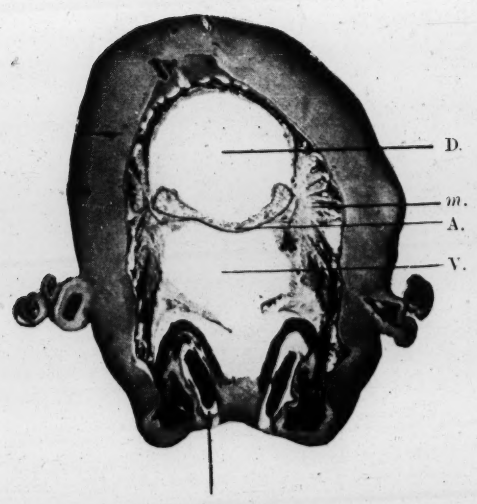


Fig. 30 ×13.

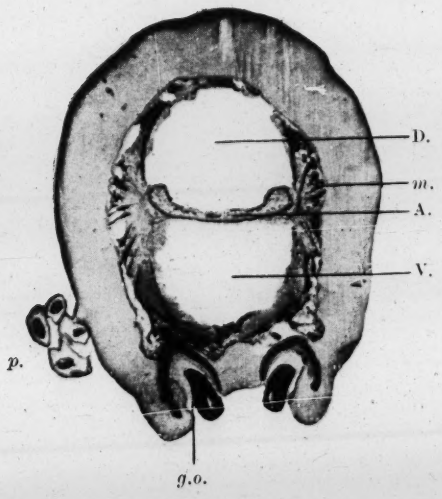


Fig. 31. ×13.

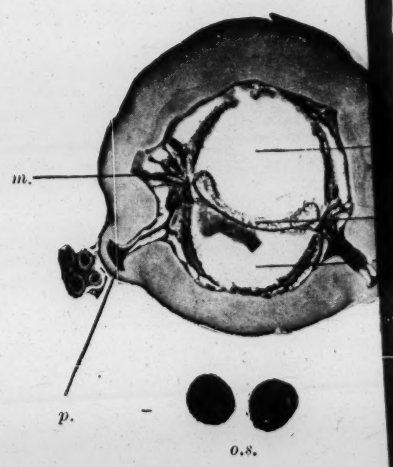


Fig. 32. ×13.

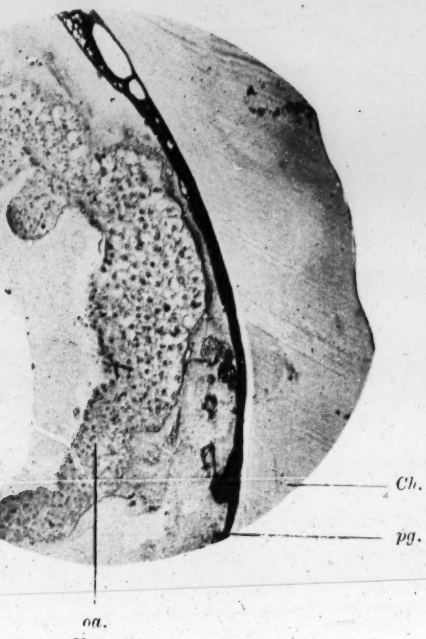


Fig. 24.

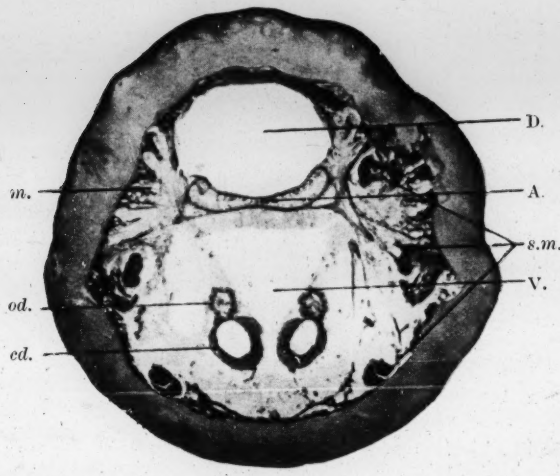


Fig. 25. ×13.

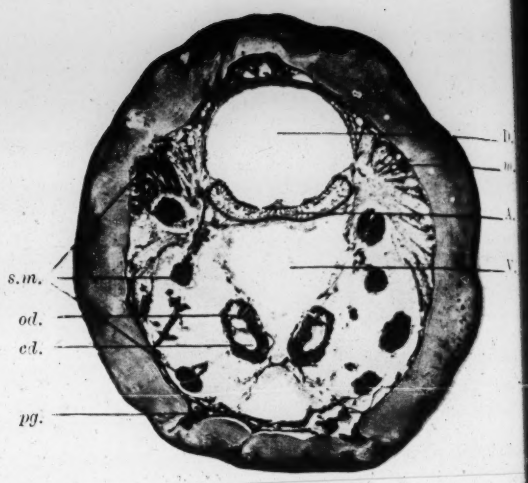
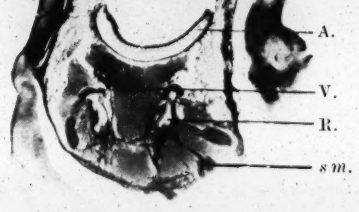


Fig. 26. ×13.

Fig. 27. ×13.



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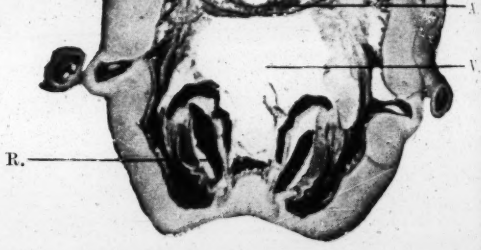


Fig. 29. ×13.

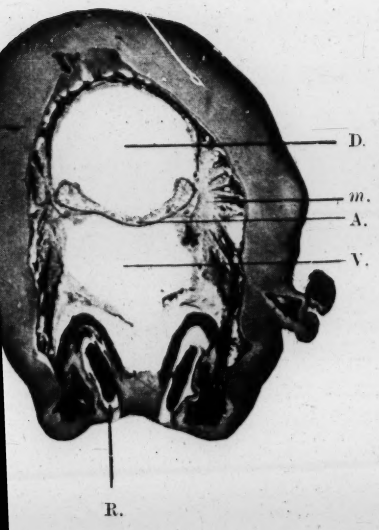


Fig. 30 ×13.

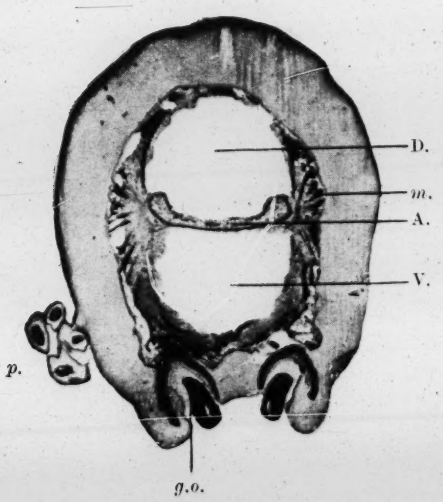


Fig. 31. ×13.

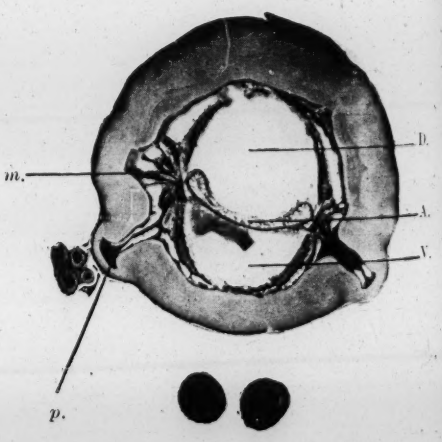


Fig. 32. ×13.

Fig. 7. Transverse section through the upper part of the head. The oral cleft on the ventral surface, the tubercles and the arrangement of the striped muscles are shown. × 13. p. 415.

Fig. 8. Transverse section through the head below the oral cleft, showing the relation of the alimentary canal A. to the midventral and dorsal spaces, the pair of œsophageal ganglia, each of which has a cavity in the centre, and the striped muscles. *Ch.* marks a transversely divided column of chitin, lying in relation to the dorsal space. × 13. pp. 418, 419.

Fig. 9. Transverse section through the parasite close to the origin of the arms. The relations of the alimentary canal with the ventral and dorsal spaces and the large amount of areolated tissue are shown. × 13. p. 416.

Fig. 10. An oblique section through the parasite at the origin of an arm, showing the relations of the alimentary canal to the dorsal and ventral spaces, and to a mass of areolated tissue. The position of the inner column of chitin, *Ch.*, is also shown. × 13. p. 418.

Fig. 11. Transverse section through the arm close to its origin to show the two large areolæ and those of smaller size. × 13. p. 416.

Fig. 12. Transverse section through the arm about its middle. × 13. p. 416.

Fig. 13. Another transverse section through the arm, in which many of the areolæ were crowded with nuclei. × 13. p. 416.

Fig. 14. Transverse section through the body a short distance below the arms, showing the areolated tissue which surrounded the alimentary canal and dorsal and ventral spaces. × 13. p. 419.

Plate III.

Fig. 15. Section through an œsophageal ganglion, showing the nucleated nerve cells. p. 417.

Fig. 16. Transverse section through the attenuated thoracic region, showing the alimentary canal and the dorsal and ventral spaces. × 13. p. 419.

Fig. 17. Transverse section through the body at the junction of the thoracic and genito-abdominal segments. The upper end of the pair of ovaries can be seen at the sides of the alimentary canal, the mucous lining of which was torn off in making the section. × 13. p. 420.

Fig. 18. Transverse section through the caudate abdominal segment, showing the relation of the alimentary canal to the dorsal and ventral spaces and the origin of one of the pennate bristles. × 13. p. 423.

Fig. 19. Transverse section through the intestine at the anal orifice. On each side of the gut is a pair of large transversely striped muscles; on the ventral aspect a pair of non-striped muscles which form a sphincter arrangement around the intestine. × 50. p. 422.

Fig. 20. Transverse section through the upper part of the genito-abdominal segment, showing the two ovaries placed dorsally, with a cement and an oviduct on each side of the ventral space, separated from each other by an interval; also the alimentary canal with its foldings or diverticula. × 15. p. 423.

Fig. 21. Transverse section through the same region, showing the same parts, but with a less complicated alimentary canal. × 13.

Fig. 22. Transverse section through the upper part of the genito-abdominal segment, showing ovaries and cement glands in the same plane, also oviducts and cement ducts. × 15. p. 425.

Fig. 23. Transverse section through the genito-abdominal segment, showing the pair of cement glands at the sides of the dorsal space; the cement ducts and oviducts are near the ventral space: the walls of the ducts on each side are connected by an intermediate band. The pigmented lining membrane has shrunk away from the wall of chitin. The alimentary canal is reniform in section. × 13. p. 425.

Plate IV.

Fig. 24. Section through an ovary, showing the contained ova. p. 424.

Fig. 25. Transverse section through the lower part of genito-abdominal segment, no ovaries or cement glands, but the walls of the oviduct and cement duct on each side are in contact: bundles of striped muscles, *s.m.*, are also seen. × 13. p. 425.

Fig. 26. A similar transverse section, showing fusion of the oviduct with the cement duct. × 13.

Fig. 27. A similar transverse section, where the two ducts on each side are blended, and form a receptaculum. A loop-like arrangement across the mesial plane connects the two receptacula. × 13.

Fig. 28. A transverse section a little above the genital openings. Striped muscles are attached by distinct tendons to the outer wall of each receptaculum. × 9. p. 426.

Figs. 29, 30, 31. Three transverse sections in succession from above downward in proximity to the genital openings. They show the receptacula, each containing a plug of cement. In 29 is also seen the short canal of the receptaculum which leads to the genital opening; and in 30 the opening itself is visible with a plug of cement protruding through it. × 13.

Fig. 32. Transverse section through the upper part of the caudate abdominal segment. The origin of a pair of bristles is seen and the pair of ova strings lie ventrally to the segment. × 13.

Lettering of Plates II., III., IV., the figures in which are reproductions by Messrs M. & T. Scott of photographs of sections through Pennella.

A. Alimentary canal.
ar. Areolated tissue.
C. Cement gland.
c.d. ,, duct.
Ch. Chitin.
D. Dorsal space.
g. Œsophageal nerve ganglion.
g.o. Genital opening.
m. Mesentery.
n.m. Non-striped sphincter muscle.
n. Nerve cells.
O. Ovary.
oa. Ova.
od. Oviduct.
Œ. Œsophagus.
or. Oral cleft.
o.s. Ova string.
p. Pinnate bristles.
pg. Pigmented lining membrane.
R. Receptaculum.
s.m. Striped muscles.
t. Tubercles.
V. Ventral space.

XIX.—On the Histology of the Blood of the Larva of *Lepidosiren paradoxa*. Part II. Hæmatogenesis.

By Thomas H. Bryce, M.A., M.D. (With Four Plates.)

(Read July 18, 1904; MS. received October 15, 1904. Issued separately May 6, 1905.)

Introduction.

In the first part of this memoir I described the structure of the corpuscles at a stage of larval development when the red cells were actively dividing and the blood contained several varieties of white cells. During the course of these more strictly cytological observations, it was impressed upon me that the great size of the elements and their very marked histological characters, combined with the simple character of the organisation of the animal, made *Lepidosiren* a very favourable case for the study of the first principles of Hæmatogenesis. I was specially interested in what may be termed a middle phase in the history of the blood. I refer to a period after the primitive corpuscles have acquired hæmoglobin and there are leucocytes present, but before the blood-forming organs are unfolded. This stage lasts a relatively long time in *Lepidosiren* up to the differentiation of the spleen, as the liver takes no part in blood-formation at any period.

Since Bizzozero first discovered that hæmoglobin-containing cells divide by mitosis, and emitted the hypothesis that the red cells are a stirp kept up only by division, it has been largely held that all forms of the coloured corpuscles are descendants of those first laid down.

The admission of a non-hæmoglobin-containing element into the erythrocyte series (Lowit and Denys) only pushed back the argument from the 'hæmatoblast' of Neumann to the 'erythroblast' of Lowit. If the erythrocyte series constitutes a 'tissue' *sui generis*, when does it cease to be laid down?

We know from observations on the characters of the corpuscles at various stages of development that in all vertebrates the elements are at first identical, but that their characters change, and the adult corpuscles are different in character from those which first appear. It is sometimes assumed that the adult nucleated erythrocytes of the lower vertebrates correspond to the nucleated discs of mammals, but in the *Lepidosiren* larva the erythrocyte series shows all the stages seen in the higher forms. There are 'erythroblasts' without hæmoglobin; 'hæmatoblasts' with hæmoglobin; young erythrocytes with reticular nucleus; and mature or old forms with vesicular nuclei. All forms save the last divide by mitosis, and the last is probably only a hæmoglobin carrier like the mammalian red blood corpuscles.* Thus, though the outward form of

* *Cf.* Laguesse, *Journal de l'Anat.*, T. 26, 1890.

the elements is rather different, the *essential* problem is the same in both lower and higher animals. The heterogeneity of the erythrocyte series through the whole larval life, and the fact that all the elements except the mature erythrocyte multiply by mitosis and after their kind, seems a presumption in favour of continued new formation. If this were so the erythroblasts would necessarily bear a relation to some less specialised cells. Analogous facts in other forms have led to the theory, in all its varieties, of the origin of the red cells from the white elements.

From the theory that the vascular and lymphatic systems are formed in the mesenchyme (ZIEGLER), whatever view may be taken of the actual channels and spaces, it follows that the cells, fixed and free, red and white, are all homologous, having the same parentage. Further, the theory gives room for the acceptance of continuous formation of free cellular elements, at any rate up to the completion of histological differentiation, and of the genetic relation in some sort between the red and white corpuscles.

On the other hand, if the vessels arise *in situ* but the blood has a restricted local origin in one part of the embryo from which the corpuscles are dispersed—as pointed to by the works of RÜCKERT, RABL, ZWINCK, SWAEN and BRACHET—and the sites of origin are not also those of the germs of the connective tissue, the blood and the lymph may, from the embryological and morphological point of view, require to be distinguished (FELIX, SWAEN and BRACHET). The two classes of corpuscles might then belong to two separate stirps, multiplying by division and having no mutual relationship.

The origin of the first leucocytes has not yet been demonstrated beyond doubt, but the almost universal opinion has referred them to the mesoderm (mesenchyme). This is ZIEGLER's * view, and in this country it has been specially maintained by GULLAND. They are 'wandering cells' formed outside the blood stream; and as they appear at a later stage of ontogeny than the primitive blood corpuscles, they belong to a different category.

In recent years, however, there has been a tendency to derive the lymphoid cells direct from the endoderm. KÖLLIKER first described the epithelial cells of the thymus as becoming converted into lymphoid cells; PRENANT, SCHULTZE, MAURER, NUSSBAUM and PRYMAK, and BEARD † have come to a similar conclusion, and the last named claims for the gland that it is the sole source of the leucocytes.

A similar conversion of the epithelial cells of the gut into the lymphoid cells of the intestinal glands has been described by RETTERER, RUDINGER, DAVIDOFF, and KLAATSCH, but it has been denied by STOHR and KOLLMANN.‡

In the matter of the spleen there has also been a question of the endoderm providing the cells of its rudiment. This view has been put forward specially by MAURER and KUPPFER, but the older view that it is formed from the mesoderm as a mass of

* *Ber. der Naturforsch. Gesellsch. zu Freiburg*, Bd. iv., 1889. *Verhandl. d. deutschen Zool. Gesellsch.*, 1892.

† *Zool. Jahrbucher Abt. f. Anat.*, vol. 17, 1903. For a historical account of histogenesis of thymus and references to literature this work may be consulted.

‡ For critical review and references, see KOLLMANN, *Archiv f. Anat.*, 1900.

mesenchymatous cells, maintained by LAGUESSE, has been upheld by most of the recent workers, especially by TONKOFF, KOLLMANN, KEIBEL, and PIPER.*

My observations on the blood of *Lepidosiren* have not included the first origin of the blood and vessels, but have been directed specially on the later phases, for the study of which *Lepidosiren* presents very favourable opportunities. I have, however, made an exhaustive examination of the blood corpuscles of all the early stages, so that my observations extend over the whole history of the blood, from the time the first corpuscles appear in the heart onwards. I have necessarily studied the histogenesis of the spleen, and the development of the lymphoid tissue, so called, in the kidney, and have made incidental observations on the thymus gland. On this last head I shall have little to say, as I hope soon to study the development of the gland in detail.

For the purpose of the research, I have had the opportunity, through the kindness of my friend Professor GRAHAM KERR, of going through the whole of the larval stages. This has involved the exhaustive examination of more than twenty series—up to stage No. 38 of the sequence, when the larva is already practically a small adult *Lepidosiren*. I have also examined sections of the modified filamentous hind limb of the male during the degeneration which ensues after the breeding season, when the tissues are crowded with leucocytes.

As the result of these more extensive observations, I have not much to add in regard to the structure of the fully evolved elements.

The erythrocytes in the adult and later larval stages have almost invariably a vesicular nucleus, and the few corpuscles that have a nucleus with the coarse reticulum of the earlier phases are to be regarded as young erythrocytes. The equatorial band is less distinct in the adult corpuscles, and the reticular structure of the disc is more doubtful. The adult material is unfortunately not sufficiently well fixed to enable me to determine whether the reticular appearance of the adult corpuscle is wholly an artifact or not; but as the granular, irregularly reticular appearance of the disc closely resembles that of the reticulum of the early corpuscles in a badly fixed condition, I think it probable that the larval and adult elements resemble one another in this respect also. I make, however, the same reservations in regard to this structural feature as I did in my previous communication. I find that the leucocytes of the adult and late larval stages present no essential differences from those of the earlier stages described in Part I. I must add to that account, however, that I find forms with basophile granules. The cells containing these do not differ in general character from those with eosinophile granules, and, as in them, the granulation is either fine or coarse. I shall reserve what I have to say on the general morphology of the leucocytes until I have described their development.

Before proceeding to the record of my observations, I must refer to certain general points.

* Complete historical accounts are given by CHORONSCHITZKY, *Anat. Hefte*, Bd. 13, 1900; and PIPER, *Diss. Med.*, Freiburg, 1902. See also specially KOLLMANN, *Archiv f. Anat.*, 1900.

A description of the external characters of the embryo and larva of *Lepidosiren* and a general account of the affinities of the animal will be found in Professor GRAHAM KERR's original memoir in the *Transactions* of the Royal Society, B., vol. cxcii., and an account of the early stages of development in his paper in the *Quarterly Journal of Microscopical Science*, vol. xlv. part i.

The egg is markedly telolecithal, but the segmentation is holoblastic. The excessive lading of the cells with yolk, and the consequent large dimensions of the primitive yolk cells, impresses characters on the development remarkable in certain respects. Among the secondary features due to this cause, the one that chiefly affects the present study is the long postponement of the formation of the alimentary canal. Even at the stage numbered 31, the pharynx, gill clefts, and alimentary tract are solid, showing in no part an epithelial disposition of the cells. As a further consequence of this, the splanchnic mesenchyme is long in being differentiated. The splanchnic mesepithelium rests on a layer of smaller yolk-laden cells, in which blood-vessels form, but which is not separate from, or to be distinguished from, the general mass of the yolk cells. It is only when the alimentary canal commences to be differentiated that a layer of mesenchyme and a layer of definitive epithelial hypoblast cells can be distinguished.

The general mesenchyme is differentiated much earlier. It arises, GRAHAM KERR states, for the most part by a proliferation from the mesoderm, at about the level of the nephric rudiment, very much as in the Selachians, partly directly from the sub-notochordal region of the hypoblast.

The history of the blood corpuscles may be divided into three phases. The *first* phase extends over the period from their origin up to stage 30, when the alimentary canal commences to be cut off. It is coincident with the laying down of the heart and main vessels. While the phase lasts, the corpuscles, like the rest of the tissue cells, are laden with yolk; and though they vary much in character according to the amount of yolk borne, are at first all of one type, though very soon two kinds are to be distinguished.

The *second* phase is coincident with the formation of the alimentary canal and the differentiation of the splanchnic mesenchyme. The blood corpuscles, now free of yolk, like all the tissue cells save the hypoblast, are heterogeneous. There are erythroblasts, young erythrocytes in active division, and mature red cells, while, further, there are white elements belonging to different categories. The rudiment of the spleen appears at the beginning of this phase, and during its persistence is undergoing its histogenetic changes.

As the phase advances the erythrocytes become almost all of the mature variety, and the *third* phase is initiated, during which the permanent conditions are established. The great mass of the red cells are now mature, young corpuscles are sparsely distributed, while the erythroblasts seen here and there in the vessels are densely crowded in the spleen pulp, and to a lesser degree in the venous sinuses of the kidney

(mesonephros). The leucocytes, in their several varieties, are coursing in the blood stream or wandering among the tissues, especially in the walls of the alimentary tract; they are also crowded in the spleen and, as will afterwards be shown, along a tract around the kidney.

Though I have thus divided the history of the blood corpuscles into three phases, it is not to be understood that they are sharply marked off from one another, but that they overlap and merge into one another imperceptibly.

Phase I.

The Primitive Blood Corpuscles.

With the actual primary origin of the blood corpuscles, and the development of the heart and vessels, this paper will not concern itself, nor will it deal with the general question regarding the morphological relations of the blood in terms of the germ layers. This will be dealt with by Mr Graham Kerr himself in a future memoir.

I begin at a stage, No. 26, when the rudiment of the heart is laid down. In it there are free cells, not to be distinguished by any of their characters from the fixed cells forming its wall. In the middle line, below the notochord, the aorta appears like a cord of cells, identical in general characters with the cells of the developing general mesenchymatous tissue. The same is true for the cardinal vein.

On the surface of the mass of yolk cells, immediately beneath the fused somatic and splanchnic layers of mesepithelium, there are groups of free rounded cells, in spaces which later become definite vessels (fig. 1, Pl. I.).

As I have said, I have not actually studied the origin of these various free rounded cells, but from certain incidental observations, and from later phases of the development of the corpuscles, I may express my belief, subject to the revision of Mr Graham Kerr's future researches, that the corpuscles have a multiple origin *in situ* from the general mesenchyme in connection with the developing blood-vessels, and from an irregular layer of smaller yolk-laden cells lying beneath the splanchnic mesepithelium, on the exact provenance of which I do not wish to express an opinion. I make this statement with every reserve, and only make it at all because of its bearing on later stages.

The characters of the primitive blood corpuscles (figs. 1 and 2, Pl. I.) are determined by the size and number of the yolk masses in the protoplasm. In fig. 2 is represented at one pole of the nucleus an area free of yolk, including a spot I have taken for the centrosome, and from it, radiating among the yolk grains, there are delicate threads of protoplasm reaching the periphery of the cell. The nucleus is rounded or slightly oval, and sometimes shows a notch. The chromatin is collected into rounded karyosomes, from which delicate processes ramify to join those of other karyosomes to complete the reticulum.

In a stage older, No. 27, the appearance of the corpuscles is considerably altered

This is due in some measure to different methods of treatment, the earlier stages being celloidin sections stained with safranin, while these are paraffin preparations stained with iron hæmatoxylin.

Two kinds of corpuscles are to be recognised. The first, evidently the direct derivatives of the corpuscles of the previous stage, have a large round or slightly oval body, measuring 50 μ to 60 μ, and a round nucleus which has the same general features as described for that of the earlier elements.

The protoplasm round the periphery of the cells has undergone a transformation into a broad band of delicate concentric fibrils. Within this the protoplasm has a delicate alveolar structure and contains large yolk grains, and in some corpuscles clear vacuoles. In the particular cell drawn (fig. 3, Pl. I.) there was a homogeneous area at one pole of the nucleus which I take to represent the centrosomal area of the earlier corpuscles, but there is no distinct centrosome, nor is there a radial sphere. Most of these cells are round, but a few are oval in shape (fig. 4, Pl. I.).* The cell of this class drawn has only a few yolk grains, the alveolar disposition of the protoplasm is distinct, and at each end of the elongated body there is a group of minute darker-staining granules, which I interpret as the cross sections of the peripheral fibrillæ of the other cells, and which, as will appear later, in more advanced corpuscles come to occupy this situation in profile sections.

The second kind of corpuscles are smaller cells, about 39 μ in diameter (figs. 5 and 6, Pl. I.). The nucleus is notched or lobed, but has otherwise the same general characters as that of the larger corpuscles. The protoplasm is either uniformly alveolar or vacuolated. There is a distinct sphere and centrosome, and the peripheral band is absent.

The vacuolation of the corpuscles at this and later stages is evidently connected with the using up of the yolk. It is seen in the general tissue cells also. The fibrillar band is seen only in the blood corpuscles, and is clearly the first stage in the conversion of the primitive blood corpuscles into the passive hæmoglobin carriers of later phases.

It seems to be a fibrillar transformation of the peripheral layers of the protoplasm, and not a mere disposition of an alveolar meshwork; and as none of the corpuscles have yet the biconvex disc shape, it is not the mere consequence of the shape of the cell.

While the mass of the corpuscles are thus assuming a passive rôle, certain remain as free mobile elements, constituting the second type of cell described, with its sphere and centrosome, and the lobing of its nucleus somehow associated with the activities of the cell.

While one cannot at this stage, when there are yet no hæmoglobin-bearing cells, call these elements leucocytes, it is to be noticed that in general morphological

* It is probable that these are profile sections of the circular disc-shaped forms.

characters they are identical with the later leucocytes. I may also remark that the general mesenchyme cells show in certain instances all the characters of these free cells in the blood, and that some even have definitely polymorphic nuclei.

The next three stages may be described together, as there is no marked difference in the general characters of the blood. They are numbered 28, 29, and 30.

The large ringed corpuscles have increased greatly in number, partly by active division, partly by new formation. That new formation is proceeding, I conclude, because both in the earlier stages and in those now in review there are appearances in the developing vessels under the mesepithelium over the mass of undifferentiated yolk cells which indicate the budding off of free elements from the superficial layer of smaller yolk-laden cells, and it seems to me probable that the opening out of the cardinal veins in the mesenchyme round the nephric ducts is also associated with the setting free of elements into the blood stream. The enormous amount of yolk in all the cells at this stage, and the consequent large size of the cells (each cut through, in 10 μ sections, at least three times), make it very difficult, however, for one to satisfy one's mind regarding this.

Some of the yolk-laden blood corpuscles are at this stage of enormous dimensions. These giant corpuscles (fig. 6, Pl. VII.) are either round or oval. The round cells are about 60 μ in diameter; the ring is very broad and distinct, enclosing the yolk grains and vacuolated protoplasm, from which it is sharply marked off. The large oval cells (fig. 7, Pl. I.) attain a length of 80 μ to 90 μ; I have met with individuals even more than 100 μ in length. In the smaller cells of earlier stages, as far as I can make out, the layer of superficial fibrillar protoplasm surrounds the greater part of the corpuscle, but in the elongated corpuscles it is massed at the ends, *i.e.* at the equator of the oval disc. The nucleus of all the corpuscles is of the 'leucoblast' type. It is spherical in the great majority of these large cells, but numbers have lobed or even multiple nuclei, and these often of very unequal size (fig. 11, Pl. I.). In the earlier stages of these three series under consideration mitotic figures are numerous; in the later they also occur, but less frequently, and the question arises whether the multiple nuclei arise by direct or indirect division.

I have little doubt in referring to direct division certain cases in which the nucleus is a regularly shaped dumb-bell, and in explaining them as antecedent stages of cells with two equal nuclei lying side by side. In other cells of a smaller variety, to be mentioned immediately, the nucleus is irregularly lobed, and there are sometimes detached free portions, which must be produced by direct fragmentation. On the other hand, certain instances may be the result of indirect division of the nucleus without division of the yolk-laden body. The mitotic figures are not numerous enough to enable me to obtain a complete series without much labour, so that I have not followed this point out to a judgment. For the same reason I have been unable to determine the relation, if any, to the multiple and unequal condition of the nuclei, of a number of instances observed of irregular mitoses, some of which were multipolar.

Up to this stage none of the corpuscles contain hæmoglobin, as I have been able to ascertain, at least for the youngest of the series, by staining with methylene blue and eosin; but in the next following stage, between which and stage 30, however, there is a small hiatus, the corpuscles contain hæmoglobin, and the megalocytes have disappeared.

The giantism of the corpuscles can hardly, however, be related to the hæmoglobin formation, because there are other hæmoglobin-free cells in the blood, very different in character. These are much smaller cells, and belong to several different categories. *First*, there are elements having all the characters of the later red corpuscles, except that they have round nuclei (figs. 8 and 9, Pl. I.), which have precisely the same characters as those of the large yolk-laden corpuscles. As all intermediate stages are observed between these corpuscles and the large ones, there is reason for believing that they are simply the large cells in which all the yolk is used up and the coarsely vacuolated protoplasm has been reduced to a fine alveolar condition. To the same category, cells with lobed nuclei like that in fig. 15, Pl. I. probably belong. On the other hand, there are, *second*, still smaller elements (fig. 12, Pl. I.), with smaller rounded nuclei. These must arise in some other way. They are found in the heart, but perhaps more frequently in the cardinal veins and in the vitelline vessels.

The appearance represented in fig. 16, Pl. I. seems to point to one mode of formation. A large yolk-laden cell has apparently divided unequally, the larger moiety remaining among the yolk cells, and forming part of the layer of smaller cells still incompletely separated from the larger yolk cells, while the smaller portion, consisting merely of a nucleus and a small zone of yolk-free protoplasm, is budded off to become a free cell.

I have also observed one instance suggesting that a multinucleated cell is being resolved into a number of elements; but as I have been unable to discover another, I cannot speak definitely regarding this possible source of these small corpuscles. That these small elements become young erythrocytes is indicated by the occurrence of such cells as figured in figs. 13 and 14, Pl. I.

The first change in the conversion is the appearance of a delicate layer of fibrillæ in the protoplasm, forming an ill-defined ring when the corpuscle is seen on the flat, and showing as the group of apparent granules in profile view (fig. 14, Pl. I.).

Third, still another type of free cell also occurs (fig. 17, Pl. I.) in which the nucleus is polymorphic. The protoplasm is exceedingly homogeneous, and shows only very indistinctly any reticular or alveolar disposition, while there is an exquisite radial sphere and large single centrosome. These cells occur very rarely. They are in general aspect the same as the elements of an earlier stage (fig. 10, Pl. I.). I have said above that these latter possibly represented the earliest mobile stage of the blood cells, or that they were the earliest representatives of the free mobile white corpuscles. I have little hesitation in naming such a corpuscle at this stage an early leucocyte, both from its characters and because it is arising from the district of the potential mesenchyme, in which, in the next phase, the leucocytes begin to appear in large numbers.

Phase II.

As already stated, the alimentary tract up to stage 30 is represented by a solid mass of heavily yolk-laden cells, which can only be distinguished into central large and peripheral smaller cells. In no part is there any epithelial disposition of these primitive hypoblast cells. Immediately after this stage the gut begins to be cut off from the mass of yolk cells, at first as a solid cord with a palisade grouping of its cells. This cutting off of the alimentary tract is associated with the differentiation of a layer of splanchnic mesenchyme round it. I am not here concerned with the details of the process, nor with its general significance, but only with its special significance regarding the history of the blood. The only point I desire to state is that the splanchnic mesenchyme of that part of the alimentary tract from the pharynx to the region in which, later, the pancreas rudiment lies, is laid down as a direct derivative of the layer of smaller yolk-laden cells which seem simultaneously converted into the definitive hypoblast and the mesenchyme. The two layers are at first absolutely continuous, the line of demarcation being indicated only by the retention in the hypoblast for a longer period of the yolk grains. Later, however, they are sharply marked off by the rounded hypoblast elements assuming the form of epithelial cells.

Up to this point I think the evidence is fairly clear that the blood corpuscles increase in number both by direct division and also by new formation. This new formation can only be from two sources—from the somatic mesenchyme as the vessels are formed in it, or from the layer of yolk cells lying beneath the splanchnic mesepithelium. Both probably before, both almost certainly now, share in contributing to the blood; but as the definitive hypoblast and the splanchnic mesenchyme are now differentiated, a new phase is inaugurated.

I shall first enumerate the different types of free cells met with in the blood, and then refer to their seats of origin.

Four stages (-31, $31+$, 32, and $32+$) may be taken together for the classification of the types, though in referring to their origin I shall have to discriminate between 31 and 32. As stage 32 was the one selected for the study of the cytological characters of the corpuscles, I must refer to the plates published with Part I. of this memoir for most of the illustrations.

1st. *Erythrocytes.*

All the giant corpuscles have now disappeared from the blood. The corpuscles have assumed their definitive disc shape (figs. 1 and 2, Pl. I., Part I.). They now contain hæmoglobin. Three varieties occur, differing in the characters of the nuclei. The great majority have oval nuclei with a very coarse homogeneous chromatin reticulum, which takes the orange dye from the mixture of Ehrlich; a certain proportion have round nuclei and a relatively small cell body, while many have vesicular nuclei. As the

vesicular condition of the nucleus is characteristic of the great mass of the erythrocytes in all the later stages, we may take it that the corpuscles with this character are the mature elements. I may here remark that after stage 32 the number of dividing erythrocytes gradually diminishes, until in later phases they are very rare except in the spleen, where it is only the young erythrocytes which are dividing. I believe that the mature erythrocyte is probably incapable of division. Those with the oval nuclei showing a reticulum are therefore young corpuscles, and at the stages under consideration they are in very active division. Those with round nuclei are probably transitional forms between the erythroblast and the young erythrocyte; they are also in active division.

2nd. *Erythroblasts.*

These are small cells with a round or oval nucleus and small cell body, which, though basophile, has a delicate concentric fibrillation. Such cells are represented in figs. 18, 19, Pl. I., and fig. 24, Pl. III.). There are two stages in the fibrillation of the protoplasm. The cell drawn in fig. 24, Pl. III. has only a narrow zone of protoplasm and the fibrillation is extremely faint. In fig. 18 there is a distinct marginal band, with a zone of apparently alveolar protoplasm round the nucleus; when seen in profile section (fig. 19, Pl. I.) the band shows at the extremities of the oval body as a series of fine dots.

The characters of the nucleus are very important. Compared with the mononuclear cells of this stage and with the young red cells of the blood at stage 30, it is seen that the chromatin nucleoli are larger, massed closer together, and the intervening reticulum is coarser, so that the nucleus stains more deeply. These cells apparently correspond to what GIGLIO-TOS * calls 'thrombocytes,' after DEKHUYZEN. He regards them, not as stages of the red corpuscles, but as special elements derived from leucoblasts, and distinguished from the erythroblasts by several characters, one of which is the concentric disposition of the fibrillæ of the protoplasm, while in the erythroblasts the filaments are radially arranged. It is just the concentric fibrillation of the protoplasm, leading up to the fibrillar band, which determines in this case the nature of these cells.

3rd. *Large Mononuclear Cells.*

I have given this name to these elements because of the large simple nucleus, which is either round or notched (figs. 24, 25, Pl. III., Part I.). It is distinguished from the erythroblast by the more purely basophile reaction of the protoplasm, which is never fibrillar, showing only an extr nely delicate reticulum. The nucleus has its chromatin nucleoli rather widely scattered; they are relatively small, and the intervening reticulum is very delicate and filamentous. It is to be specially noticed that all the corpuscles up to stage 30, when the hæmoglobin appears, have nuclei of this type. The mono-

* *Arch. Ital. de Biol.*, T. xxix., 1898, and *Mem. a. R. Accad. delle Sc. di Torino*, s. ii. t. xlvii.

nuclear cells vary in size, some being smaller and also showing sometimes a deepish cleft (fig. 25, Pl. III.).

The contrast between the nuclei of the mononuclear cells and the erythroblasts is important. It corresponds, in general terms, exactly with the contrast between the erythroblast and leucoblast in higher forms, as established by LOWIT and then DENYS, and more recently, still further elucidated by PAPPENHEIM.*

4th. *Small Mononuclear Cells.*

These have a small round or notched nucleus, with a relatively dense disposition of the chromatin and a narrow zone of delicate hyaline basophile protoplasm.

5th. *Leucocytes proper.*

These are found sparingly in the blood, but crowded in the spaces of the mesenchyme. Outside the blood stream they are simply free wandering mesenchyme cells. They vary greatly in size; the nucleus shows all degrees of polymorphism; there is always a large centrosome and sphere; the protoplasm is either hyaline and basophile, or granular in various degrees, from a few scattered extremely small granules to larger bodies filling the whole cytoplasm, when the cells have all the characters of eosinophile leucocytes. In a few instances the granules have a basophile reaction, taking the blue instead of the eosin from a methylene blue and eosin stain.

In the succeeding section the origin and inter-relation of these various elements will be considered.

The four stages under consideration present a gradual unfolding of the conditions which characterise the second phase in the development of the blood.

The disposition of the splanchnic mesenchyme at stage 31, may be gathered from a section, passing through the liver just behind the pharynx and heart. The solid cord of cells which represents the future stomach is seen passing down on the left of the liver to be continuous below with the still undifferentiated yolk cells. Surrounding it, and passing below over the surface of the yolk, is a very cellular tissue, which also here surrounds the solid rudiment of the bile duct, and passes along it into the liver.

This mesenchyme is a loosely arranged tissue, everywhere permeated by irregular spaces containing red blood corpuscles, but without definite endothelial walls.

The component cells (fig. 2, Pl. II.) are both free and fixed. The latter seem to be elongating to form spindle-shaped elements, while the former appear to lie free in the intercellular spaces. These free cells are already of more than one variety. The majority are large mononuclear cells, but there are also cells with metamorphosed nuclei, and some even are distinctly granular. The layer is directly continuous with the underlying hypoblast, in which mitotic figures are observed.

* *Archiv f. path. Anat.*, vol. 151, 1898. PAPPENHEIM gives a very extensive bibliography. His theses, summarised at the close of his paper, agree in respect of these blood cells closely with those of this paper.

In the vessels in the liver there are numerous cells identical with the free mesenchymatous elements, and it is to be noticed that the spaces in the mesenchyme are continuous with the venous spaces in the liver.

Turning to the somatic mesenchyme *—there is round the pronephric duct and mesonephric tubules a specially cellular tract (fig. 21, Pl. II.), which is becoming canalised, as it were, from before backwards, to form irregular venous spaces round the tubules, as I shall show immediately. The section drawn (fig. 21) is far back in the

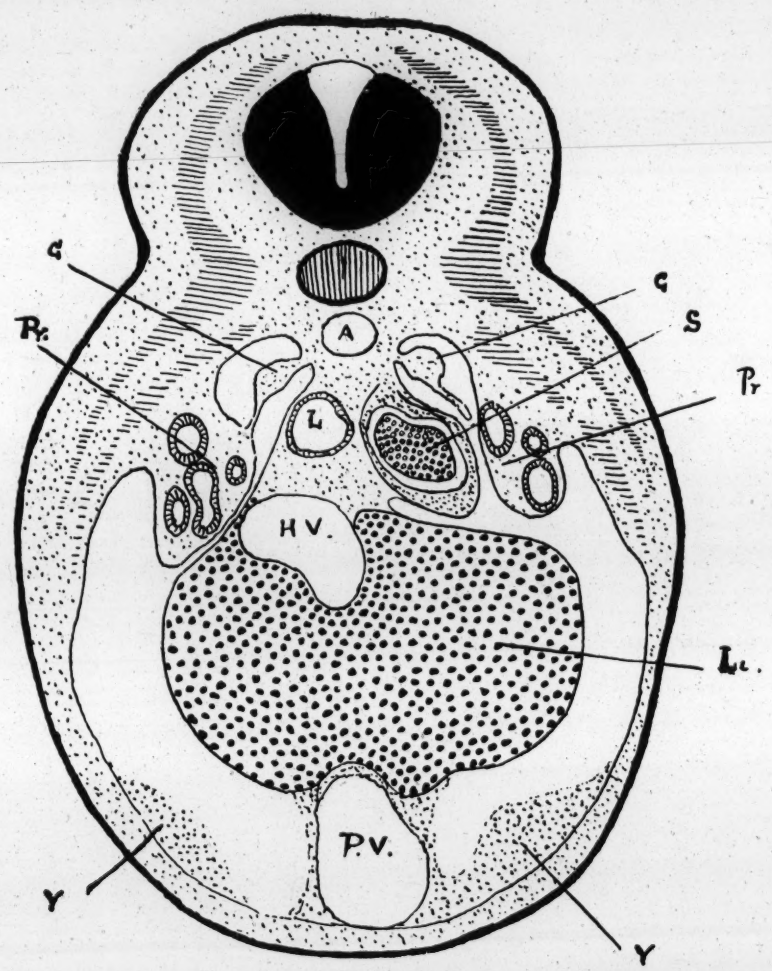


FIG. 1.—Section through larva, stage 32+, behind heart, but in front of undifferentiated yolk cells. ×33 d.
A, aorta; L, lung; Li., liver; Pr., pronephros; G, glomerulus: S, gullet; P.V., vena advehens of liver; H.V., vena revehens of liver; Y, mesenchymatous tissue covering anterior surface of mass of undifferentiated yolk cells.

series, and the tissue is not yet here penetrated by spaces containing blood corpuscles, but some of the cells seem to occupy spaces in the protoplasmic meshwork, and a few have polymorphic nuclei. The venous spaces naturally communicate with the cardinal vein, and these vessels contain cells, having again all the characters of the free elements in the mesenchyme. This tract of mesenchyme is the rudiment of the lymphoid tissue, so called, of the kidney. This has long been well known as a seat of blood formation in

* While it is quite legitimate to call the cellular tissue round the gut and on the mass of yolk cells *mesenchyme*, it is perhaps not strictly correct to use the term as applied to this tissue. I use it in quite a general sense as a convenient word to indicate the young connective tissue. Through the whole larval stages the so-called lymphoid tissue is mesenchyme in this sense, with free cells in its meshes.

the Teleosteans (BIZZOZERO, ZIEGLER, LAGUESSE, FELIX, and others). In the *Lepidosiren* larva the tissue is not true lymphoid tissue, as we understand the term in higher forms, and there are no glands. It consists simply of a tract of branched connective-tissue cells, with wide intervening spaces in which there are free lymph cells. Throughout the whole of larval life it has much the appearance of the mesenchyme at its first

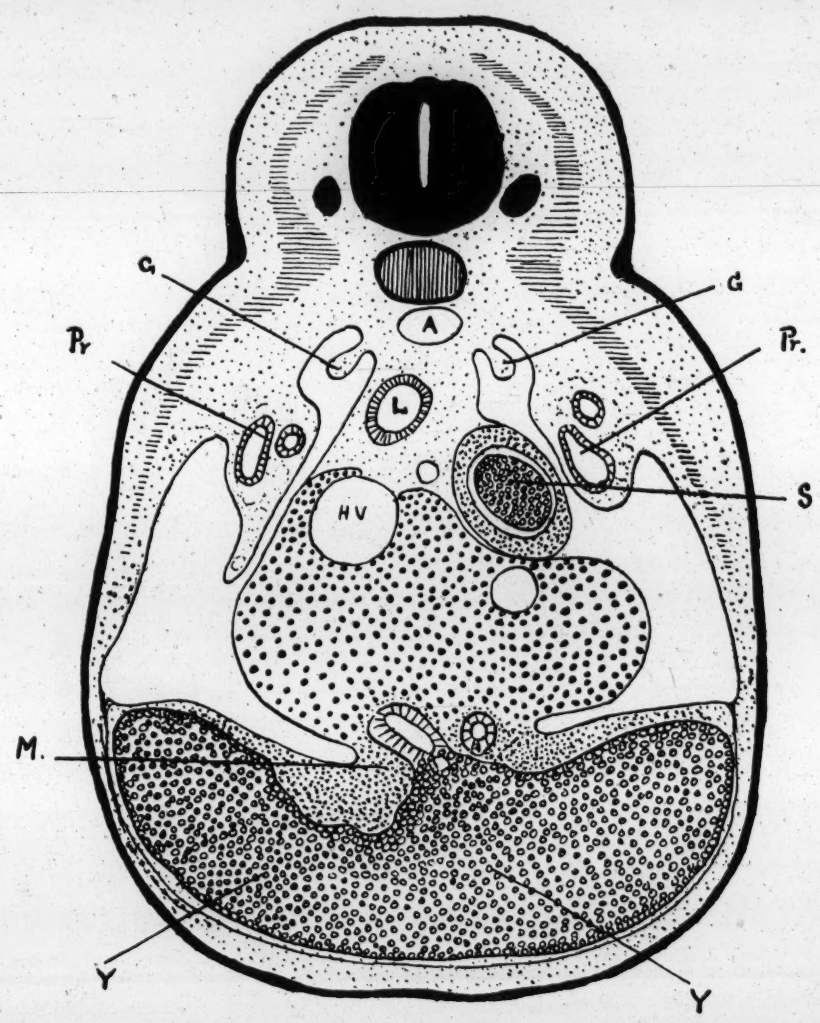


FIG. 2.—Section through larva, stage 32+, further back than figure 1, through liver ducts. ×33 d.
A, aorta; L, lung; Pr., pronephros; G, glomerulus; S, stomach; H.V., vena revehens of liver; Y, mass of undifferentiated yolk cells; M, mesenchyme covering mass of yolk cells.

appearance, and is, as it were, a tract of that tissue which has retained its undifferentiated characters.

FELIX * shows that in Salmonidæ it is not true lymphoid tissue, and he uses the term 'pseudo-lymphoid' tissue. The tissue in the Teleosts, as described by him, differs both in appearance and development from the corresponding tract in *Lepidosiren*. To this I shall return later.

At stage 32 the general disposition of parts is indicated in the diagrams in the text. In figs. 1 and 2 the still solid gut is seen lying dorsal to the liver; further back

* *Anatomische Hefte*, Bd. 8.

it descends on the left of that organ, and, as in stage 31, the cord passes into the general mass of the yolk cells.

Round the cord of cells the mesenchyme forms a layer, differing from the general somatic mesenchyme in being composed of rounded cells closely packed together. Traced back, the layer sweeps over the yolk, surrounds the vessels and ducts entering the liver, passing with them to join the general connective tissue of that organ.

The very anterior edge of the ventral mass of yolk cells is indicated in fig. 1. As we shall see later, this tissue is composed entirely of mesenchyme, and it passes into the liver round the portal vein.

Behind the point where the definitive hypoblast of the solid gut joins the general mass of yolk cells, a mass of the mesenchyme projects free (text fig. 3) into the body

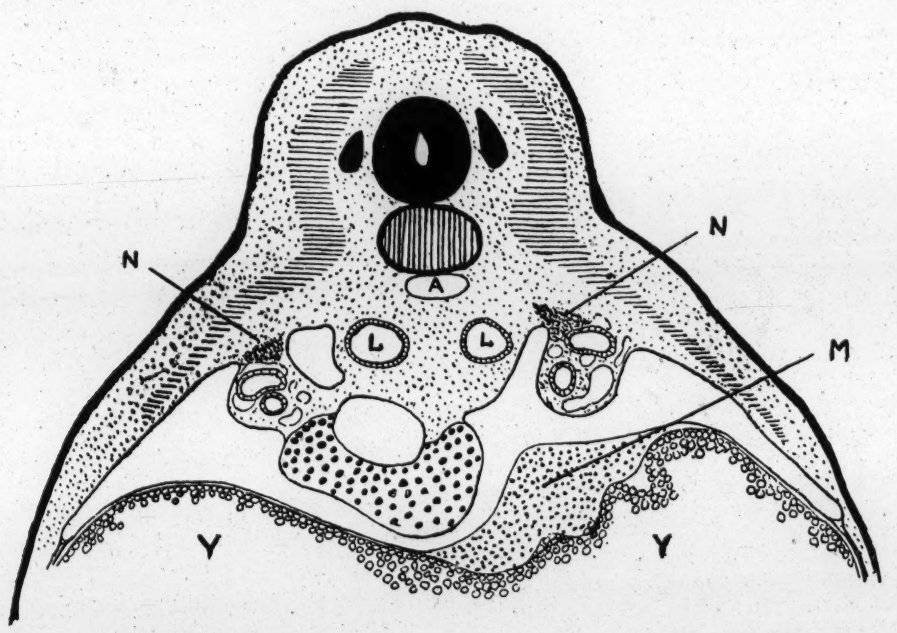


FIG. 3.—Section of larva, stage 32+, through posterior edge of liver, behind point where the solid gut rudiment becomes continuous with undifferentiated yolk cells. × 33d.

A, aorta; L L, lungs; Y, mass of undifferentiated yolk cells: M, mesenchyme on surface of mass of yolk cells; N, specialised tract of mesenchyme along pronephric duct.

cavity to the left of the hinder end of the liver. On each side this fades away into the general mesendodermic layer of the yolk, while behind this point the splanchnic mesenchyme, as such, is absent (text fig. 4), until again posteriorly it surrounds the solid hind gut which is being differentiated from behind forwards (text fig. 5). The general somatic mesenchyme is at this stage formed of widely separated stellate or fusiform cells, but the tract adjoining and surrounding the pronephric duct, the pronephric funnels, and mesonephric tubules has quite special histological characters. This tract can be followed all along the body of the embryo.

Before proceeding to describe the minute characters of the splanchnic mesenchyme and of this specialised tract of the somatic mesenchyme,* it may be observed, as a fact

* See note, p. 464.

of possible significance, that this specialised tract corresponds in situation to the original seat of origin of the 'sclerotome' from the mesoblast;* while the splanchnic mesenchyme is the differentiated representative of a previously undifferentiated mass of primitive hypoblast cells which had certainly an important share in the contribution of the early blood corpuscles.

I must also state that I have searched more than once every section through the pharynx and gill clefts at this stage for thymus 'placodes' giving origin to leucocytes, as described by BEARD.† I cannot identify any thymus rudiment, nor have I seen any appearances like those figured by BEARD. Moreover, though leucocytes occur here

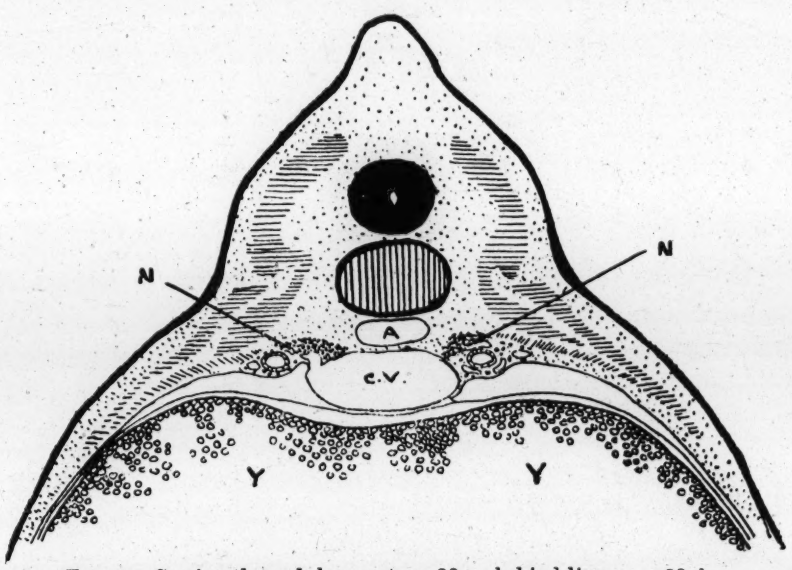


FIG. 4.—Section through larva, stage 32+, behind liver. ×33 d.
A, aorta; C.V., cardinal vein; N, specialised tract of mesenchyme in region of mesonephros.

and there in the general mesenchyme in that region, and also in the aortic arches, I do not happen to have seen a single one in the anterior cardinal vein.

Splanchnic Mesenchyme.

It has already been observed that this tissue surrounding the isolated portion of the gut is composed of rounded closely packed cells. There are numerous spaces in it containing red blood corpuscles, and here and there free elements with characters similar to those of the cells immediately to be described.

Dorsal to the gut, a tract is showing the first stages of differentiation from behind forwards, which will convert it into the spleen. The layer on the yolk which is in direct continuity with the undifferentiated hypoblast has, however, very special characters. If a section be taken at Y in text fig. 1 it will be seen that the tissue is composed of loosely arranged fusiform or stellate cells, with many free elements in the

* I find a remark almost in the same terms as this was made by ZEIGLER (*Arch. mikr. Anat.*, xxx., 1887), referring to the similar tract in *Teleosts*. See also LAGUESSE, *loc. cit.*, p. 364.

† *Loc. cit.*

spaces between them. These are either of the mononuclear type or, more frequently, show various degrees of polymorphism, and a large number have their protoplasm filled with granules. Further back, the tissue over the surface of the yolk has the same characters, and in certain situations, as at the point marked M in text fig. 2, it has the appearance represented in fig. 22, Pl. II. Above is the line of the mesepithelium, below and to the left is the yolk-laden hypoblast. Directly continuous with this the mesenchymatous tissue is composed of a network of protoplasmic threads, in the meshes of which are cells showing all degrees of polymorphism in their nuclei, while in the deeper layers near the hypoblast there are numerous cells with the protoplasm laden

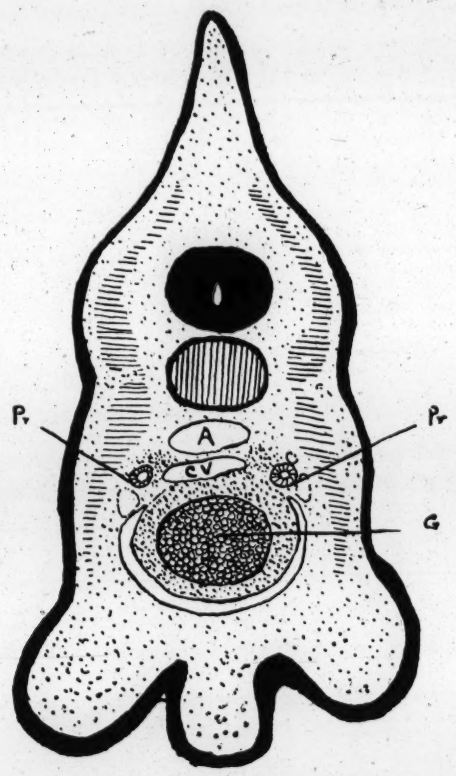


FIG. 5.—Section through larva, stage 32+, at level of the hind-limb buds.

A, aorta ; C.V., cardinal vein ; Pr., nephric duct between Pr. and A, the tract of mesenchyme lettered N in preceding figures ; G, solid hind gut.

with fine granules. The nuclei of the cells forming the framework are widely separated, and are thus apparently fewer in number than the free cells. They are elongated like the nucleus above and to the left (bounding here a space containing two erythrocytes), or rounded like those below, between the two horseshoe-shaped nuclei.

In other portions of the tissue the general characters are the same, but the polymorphic metamorphosis of the nuclei is not everywhere so pronounced, and large mononuclear free elements are more numerous. Throughout this tract of mesenchyme mitotic figures are of frequent occurrence.

Somatic Mesenchyme.

In the tract above mentioned the appearances differ according to the degree to which the tissue round the nephric ducts is opened out into venous spaces communicating with the cardinal vein.

In the region of the pronephros (fig. 25, Pl. III.) the duct is surrounded by an irregular sinus, interrupted here and there by delicate protoplasmic strands, belonging to cells of an endothelial character, with elongated nuclei. The sinus is crowded with dividing erythrocytes and with large mononuclear cells, with round, pitted, or notched nuclei. Here and there, there is a free polymorphonuclear corpuscle, either with hyaline protoplasm or with various degrees of granulation. Within the spaces of the trabecular framework are many polymorphonuclear cells, in some cases having all the characters of eosinophile leucocytes.

Further back (fig. 23, Pl. II.), in the angle between the cardinal vein, aorta, and duct (text fig. 5), the tissue is not so much opened out, the framework is closer, and in its meshes are massed free elements with round, notched, or polymorphic nuclei and various degrees of granulation. In the actual blood spaces the great majority of the cells are of the large mononuclear variety.

In the region of the mesonephros the general characters of the tissue round the tubules is similar, as will be gathered at once by reference to fig. 26, Pl. III.

The individual free cells outside the blood stream vary greatly in size, without any further indication of discriminating characters. Mitoses are frequent, but there is nothing to indicate that one type of cells is dividing to give origin to another. They seem to divide at various stages in the metamorphosis of the nucleus, and all after their kind. In fig. 24, Pl. III. there is drawn a cell in one of the venous spaces, which from its more crowded nuclear reticulum and the indication of a faint fibrillation of its protoplasm is to be regarded as an erythroblast, while outside the blood stream is a mononuclear cell of the largest variety, which in general dimensions is equal to the erythroblast.

The cardinal vein, compared with the aorta, contains a disproportionate number of white to red elements, and a preponderating number of mononuclear cells.

Thus in 112 sections the white elements were distributed between the two vessels thus :—

	Small Mononuclear.		Large Mononuclear.		Polymorphonuclear.
Aorta,	8	...	8	...	6
Cardinal vein,	14	...	100	...	26

The cardinal vein, however, being much larger than the aorta, it was necessary to arrive at the proportion of the white to the red elements in each. This was arrived at

by counting the erythrocytes and white cells in every third of the 112 sections. The figures were as follows :—

	Erythrocytes.		White Cells.
Aorta,	1003	...	22
Cardinal vein,	2634	...	140

The proportion of red cells in the aorta compared with the cardinal vein was thus less than 1 to 3, while the proportion of white cells was nearly 1 to 7.

I have used the term 'white cells' rather than 'leucocytes' here because the counts include the large mononuclear cells, which are not necessarily all true leucocytes. Those in the spaces of the mesenchyme outside the blood sinuses are certainly in great part 'leucoblasts,' in the sense that all the polymorphic and granular cells are derived from them by metamorphosis of the nucleus and by the deposition of granules in their protoplasm; but what is the relation of those within the blood stream to the non-hæmoglobin-containing erythroblasts?

As one has, unfortunately, no opportunity of actually seeing one form of cell change into another, this question can only be answered in terms of probability.

A careful scrutiny of these intravascular mononuclear cells in the cardinal vein and the spaces communicating with them shows that certain of them have rounder and larger nuclei than others, though identical in general characters.

Several considerations point to the probability that these cells are progenitors of the erythroblasts.

1*st*. The polymorphism of the erythrocyte series is in favour of the view that the blood is at this stage receiving new formed elements. There seems no reason why certain corpuscles in the general blood stream should retain, under the same conditions, their primitive generalised characters if all the corpuscles are survivors from earlier stages.

2*nd*. If new elements are being added, the fact that erythroblasts are never found outside the blood stream indicates that they must be derived from less specialised cells in the blood stream, which in turn should have the characters of cells outside the blood stream. The large mononuclear cells fulfil the conditions and complete a logical chain.

But the question arises as to the relations of the cells without and within the blood stream. Numerous cases prove that cells are passing from the spaces in the mesenchyme into the blood stream, or *vice versa*, but it is not possible absolutely to say in which direction they are moving.

The fact that these mononuclear cells occur in disproportionately large numbers in a vessel,* the current in which would carry them away from this locality, indicates that in all probability they do arise in this tract of mesenchyme, and are there set free

* The sections in which the above count was made were behind the liver, a long distance posterior to the point where the hepatic vein joins the cardinal.

in the blood stream. From the evidence available, it seems to me justifiable to conclude for this stage—

1*st*. That new elements are being added to the blood stream. These belong either to the erythrocyte or leucocyte series.

2*nd*. In regard to the leucocyte series, all the cells free in the blood which can be called leucocytes are derivatives of the mesenchyme; certainly of the splanchnic, almost certainly of the nephric tract of somatic mesenchyme.

3*rd*. In regard to the erythrocyte series—the erythroblast without hæmoglobin is derived from an intravascular 'mononuclear element,' which is in turn derived from the mesenchyme, probably from both splanchnic and somatic mesenchyme.

4*th*. The two series arise from a common mother cell—the mesenchyme cell. The changes which convert the mother cell into an erythrocyte take place in the blood stream, while its metamorphosis into a leucocyte is brought about *in situ*, in the spaces of the mesenchyme outside the blood stream.

Before leaving this stage I must refer to the character of the granulation in the granular leucocytes. In some instances, as I have already stated, these cells have all the characters of eosinophile cells. It would take me out of my direct way to go into the general question of the meaning of the granulation of leucocytes; and I would here merely remark, in regard to these particular cells, that the granulation may be merely yolk material in fine division, for I have observed precisely similar eosinophile granules in the yolk cells, and the yolk itself has strong affinity for eosin. At this stage I find the granular cells crowded together only in the splanchnic mesenchyme and in the tract of the somatic mesenchyme so frequently alluded to. It at once suggests itself, in the case of the cells near the yolk, that this may be the source of the material which constitutes the granulation; and in regard to those in the neighbourhood of the nephric duct, it is worth mentioning that the cells forming the wall of that duct are filled with yolk grains long after they have disappeared in the neighbouring tissues. It seems to me not impossible that these early leucocytes may be concerned in the distribution of yolk food. Were this the case, the granules would necessarily differ from those of the leucocytes of later stages, but I see no inherent improbability in the suggestion, which, if well founded, has a bearing on the nature of granulation in leucocytes generally.

Phase III.

After the second phase, as I have ventured to define it, is fully established at stage 32 or 33, the general conditions are maintained for a time, during which the histogenesis of the spleen is gradually accomplished and the atrophy of the pronephros completed. It is with the complete differentiation of the spleen as a hæmopoietic organ that what I call the 3rd phase in the history of the blood is established, and the renewal of the blood corpuscles is confined to that organ, and to the pseudo-lymphoid tissue round the kidneys, and possibly also round the gut. It is presumably the adult state of things

that is thus established. At no stage does the liver take any part in blood formation. The part taken by the spleen and the lymphoid tissue of the mesonephros and gut in the process will now be considered.

Histogenesis of the Spleen.

As already mentioned, the rudiment of the spleen appears in the splanchnic mesenchyme dorsal to the gut. It is simply a differentiated tract of that tissue. The hypoblast takes no direct share in its formation, but it must be observed that it takes form in the mesenchyme very shortly after that tissue is itself differentiated from the primitive undifferentiated mass of yolk cells.

The first sign of its formation is the appearance of a column of large rounded or oval cells, round which the remaining cells of the mesenchyme group themselves concentrically (fig. 28, Pl. IV.). The nuclei of the investing cells are oval or elongated, and their long axes are arranged in a general way parallel to one another, and concentrically to the central tract. As elsewhere, this tract of mesenchyme is permeated with spaces having no definite endothelial walls, but containing red blood corpuscles.

These spaces are at first irregular, but in the second stage (fig. 29, Pl. IV.) they run together so as to form a peripheral sinus surrounding the central tract, or island as it appears in sections, isolating it in great measure from the peripheral layers of investing cells. These latter become the investing connective-tissue coat of the spleen. The central tract is permeated by cleft-like spaces which communicate with the peripheral sinus. Both spaces and sinus contain red cells, and in the sinus are seen a number of leucocytes, evidently wandering in from the general mesenchyme. In the central mass, here and there are seen cells with simple nuclei surrounded by a layer of free protoplasm. Numerous mitotic figures occur among the cells of the central tract, which by multiplication of its constituent cells increases in size, while at the same time its spaces are opened out until (fig. 30, Pl. IV.) it is converted into a system of cellular columns or trabeculæ. The peripheral sinus is now less definite in its arrangement, because the spaces between the outer ends of the columns have enlarged and become continuous with the lumen of the sinus. The peripheral investing cells are now all flattened connective-tissue elements, and the capsule of the organ is complete.

The sinuses at this stage are full of red cells, nearly all of the mature variety. There are few erythroblasts, the phase contrasting in this respect markedly with the next, in which the sinuses are filled with young erythroblasts.

The cellular columns are composed of cells of various dimensions (fig. 3, Pl. IV.). Some have smaller and irregular nuclei, but the great majority have round or oval nuclei of large size. In the spaces within the trabeculæ there are numerous white elements (fig. 31, Pl. IV.). These have either simple nuclei and hyaline protoplasm, or polymorphic nuclei and hyaline or granular protoplasm. The polymorphic nuclei have a closer arrangement of the chromatin nucleoli than the cells with the simple

nuclei, so that they take a deeper coloration. The shape of the nucleus varies, but is most frequently horseshoe-shaped, and the protoplasm is either hyaline or granular in different degrees. Though it is true that at the earlier stages leucocytes are wandering in from without, I think there is little doubt that these cells are being formed at the expense of the cells of the columns or trabeculæ, which, after all, have exactly the same origin as the remainder of the mesenchyme, in which we have seen the cells being transformed into similar elements.

The mononuclear cells have certainly a local origin. They are seen crowding the splenic vein (fig. 27, Pl. III.). They are of varying dimensions, and are indistinguishable from the cells with simple nuclei first observed in the blood at stages 31 and 32. The problems regarding them are exactly the same as presented themselves at the earlier stages, but I shall postpone discussion of the point until I consider the next stage, as I have had the opportunity of staining a series of sections freshly cut from that stage with stains specially suited for discriminating between the different types of cells.

At stage 37 the spleen is an elongated organ, overlapping the pancreas behind and extending forwards along the whole length of the liver to a level close behind the point where the lung comes off from the pharynx. It is broad behind, but tapers in front, and it is in this anterior part that the general structure can be most easily determined (fig. 32, Pl. IV.).

Surrounding the central artery there is an axial mass of cells with small irregular nuclei. From this to the periphery extend radiating trabeculæ, with small oval or elongated nuclei. Between the trabeculæ are large sinuses packed with cells belonging to different categories, and in the meshes of the framework immediately round the central area are large and round nuclei, which sometimes, it is quite clear, have a zone of free protoplasm round them, while in other cases they seem imbedded in the general protoplasmic framework. In the apex of the organ the structure is simple because this zone is much reduced, but posteriorly it is more extensive and more loosely arranged, so that the picture is more complicated. It is a matter of great difficulty to say sometimes whether the large round nuclei with which this zone is studded belong to the trabecular framework or to cells in its meshes.

In the peripheral sinus, and the larger sinuses between the outer ends of the trabeculæ which are continuous with it, there are great numbers of hæmoglobin-containing erythroblasts. These and the erythrocytes give to the outer zone of the spleen characters which would justify the application to it of the name of 'pulpe rouge' used by LAGUESSE,* while the central portion is the 'pulpe blanche.'

The free cells in the spleen pulp belong to several different categories, and the discrimination between them is a matter of difficulty, as all the nuclei stain blue with hæmatoxylin, and the only differences are those of intensity of staining, associated with a difference in the disposition of the chromatin. I was unable, except in a

* *Jour. de l'Anatomie et de la Physiologie*, T. xxvi., 1890.

general way, to identify the elements in a series stained with this dye and counter-stained with eosin.

On a new series I tried many stains, but owing to some factor in the fixative, presumably the acid, I found the tissue would not stain with any basic dye. I had recourse, therefore, to Mann's double acid mixture of methyl blue and eosin, but found that for this particular material I got much more vivid differentiation by staining with the two dyes successively, first for three minutes in a saturated watery solution of eosin, and then, after rinsing in water, in a saturated watery solution of methyl blue for one or two minutes. The results varied according to the proportion in which the two dyes were held; and although there was some variation in the colorisation, the conclusions were not vitiated, because the picture was always a relative one.

The general effect depended partly on a difference in the reaction of the protoplasm, but chiefly on the relative proportions in the nuclei of bodies with different affinities to the two dyes.

In my most successful stainings I obtained the following results:—

A. *Fixed Cells.*

The protoplasmic trabeculæ stain pure blue, and the nuclei have a general blue tint owing to the general reticulum of the nucleus selecting the blue dye, but the nucleoli (chromatin) stain a violet-red colour. The central parts of the karyosomes stain yellowish-red. The violet colour is given by an outer covering or coating, as it were, of the blue staining general linin reticulum.

B. *Free Cells.*

1. In the meshes of the trabeculæ there are large numbers of cells with a large nucleus and a small amount of blue-staining protoplasm. The nuclei are round and vary in size, but roughly they may be divided into two classes—those with a diameter about 24 μ and those with a diameter of about 18 μ. Many have smaller nuclei, but as there are frequent mitoses they may be considered young cells. The nuclei are characterised (fig. 33, Pl. IV.) by the large amount of blue they select, the general fine reticulum taking up the methyl blue, while the chromatin nucleoli, which are relatively few in number, stain reddish-violet. The characters are thus exactly the same as the nuclei of the reticulum.

2. Distinguished from these cells are others which have nuclei of the same dimensions as the largest of the cells of the last category, but which react differently to the stain (fig. 33, Pl. IV.), their general tint being reddish-violet. This is due to a difference in the disposition of the chromatin. The nucleoli are larger and are more closely opposed, and the blue-staining reticulum between them is reduced, and has a violet, not a pure blue tint. These cells are very frequently seen dividing, and are distinguished during division by the red-violet colour of their chromosomes. The daughter cells have

naturally smaller nuclei, but are distinguished by the reaction of the nuclei. The protoplasm has an open reticulum, but is relatively small in amount. The outer layers are frequently seen to be fibrillar, the fibrillæ being arranged circumferentially. There are also fine reddish granules in many of the cells, which, I take it, would correspond to GIGLIO-TOS's hæmoglobigenic granules.

3. The next class of cells is found only in the large sinuses. The nucleus is of the same dimensions as that of the cell just described; it is round or oval, and selects only the eosin; the colorisation is reddish-yellow, the nucleoli are closely packed, and joined into a coarse reticulum by bars staining yellow like the rounded masses of the nucleoli. The mitotic figures, which are very numerous, belonging to this class of cell, are distinguished by their yellow chromosomes. The protoplasm varies in amount; it has an indefinite warm tint, and many of the cells have yellow granules. The peripheral layer is fibrillar, and round the nucleus the protoplasm shows a wide-meshed reticulum.

Looking at these three varieties of cell, it is clear that the third is an erythroblast containing hæmoglobin. It is almost equally clear that the second is a stage of the third—that, in fact, it is a primary erythroblast. The spleen, therefore, is a seat of origin of the erythroblasts, but do they arise by multiplication of erythroblasts which have entered the organ from without, or by new formation from the spleen cells? I conclude for the latter alternative, for the following reasons.

While it is possible that the erythroblasts having nuclei characteristic of hæmoglobin-containing corpuscles might be derived by division from the red blood corpuscles in the sinuses of the previous stage, it is not possible that cells with nuclei which do not show that reaction should be derived from hæmoglobin-containing corpuscles. They might arise, however, by division from non-hæmoglobin-containing erythroblasts derived from without, but their numbers are far out of proportion to the number of mitotic figures, so that one is driven to believe them to be a further phase of the larger cells of the first category. As a matter of fact, in cells with all the characters of these, here and there one occurs with a definite concentric fibrillar condition of the protoplasm (fig. 33, Pl. IV.), which I take to be the first stage in the conversion of the cell into an erythroblast.

I think, again, that it is reasonable to derive these cells of the first category from the mesenchyme of the original rudiment. It is very hard to say at this stage whether their protoplasm is actually free, or part of the general protoplasmic framework; and as this is the last stage of the series available, I am unable to say whether any part of the original cellular columns is retained in its primitive form, to give rise continuously to new budded-off elements; or whether the cells are all set free in the meshes of the reticulum, and give rise by continuous division to new elements.

It seems justifiable to conclude that the original spleen cells, by a series of changes in the protoplasm and nucleus, become converted first into non-hæmoglobin-containing erythroblasts, and that these, in turn, acquire hæmoglobin and become the young red cells, and that the cells I have described represent the stages in the process.

Besides the erythrocyte series, there is a large leucocyte contingent in the spleen. The fully unfolded leucocytes are of two main classes. 1*st*, Small cells, with horse-shoe or lobed nuclei, with rather closely packed chromatin nucleoli, and a small cell body composed of hyaline blue-staining protoplasm, though sometimes it contains granules. 2*nd*, Large cells, with polymorphic nuclei and a large cell body, in which the centrosome is always surrounded by a well-developed radial sphere and aster. The protoplasm is either hyaline and blue-staining or granular, and the granulation is either fine or coarse. The fine granules stain in some cells blue, in others red, and the coarse granulation of the eosinophile cells is not always of the same size.

Compared to the erythroblasts, the leucocytes are in relatively small numbers, but in the spaces and sinuses there are many of the cells which have all the characteristics of leucoblasts (fig. 33, Pl. IV.). The nuclei vary in size, but generally speaking are smaller than those of the largest spleen cells (18 μ against about 24 μ), but have otherwise the same character and reaction.

The protoplasm is reduced to a very narrow zone, is hyaline, and stains pure blue. Many nuclei are deeply notched (fig. 33, Pl. IV.), as if beginning to undergo polymorphic metamorphosis.

As leucocytes in all varieties are abundant in every tissue of the body, and especially round the kidney tubules and gut, as I shall presently describe, it is much less certain whether they are actually formed in the spleen at this stage than that the erythroblasts are rising there.

The great difficulty is, that it is impossible to distinguish a leucoblast from a primitive spleen cell. We have here repeated the same problem dealt with before in connection with the origin of the two classes of corpuscles from the mesenchyme cells; and considering that the spleen is merely a tract of mesenchyme, which may be supposed to retain its primitive potential characters, the same general scheme may not unreasonably be considered to apply, which would derive from the primitive cells of the rudiment both classes of corpuscles by specialisation along different lines. My observations on the spleen are thus in strict accord with those of Lagu esse * in his classical work. My large mononuclear cell takes the place of his *noyau d'origine*, a term borrowed from Pouchet, † who first formulated the general scheme here adopted from his work on the blood of Triton.

The leucocytes, in their several varieties, are found in the blood-vessels and scattered in every tissue, but are specially crowded in the wall of the gut in the mesentery and in the tissue surrounding the tubules of the kidney.

Pseudo-lymphoid Tissue of Kidney.

The tract described at stage 32 has disappeared in front with the atrophy of the pronephros, but it can be identified in the region of the kidney, and the general

* *Loc. lit.*

† *Gaz. Méd. de Paris*, 1879.

appearances are much like those figured at stage 32 in the region of the mesonephros (fig. 26, Pl. III.).

The tubules are imbedded in what is practically a huge sinus; in the blood stream a great variety of elements occur. There are numerous erythroblasts, many mononuclear cells, and numbers of leucocytes, while in the pseudo-lymphoid tissue there are large quantities of leucocytes of various kinds, but the polymorphic and granular varieties preponderate. The general impression when a section is compared with a section of the spleen is, that while in the latter the primary erythroblasts and secondary erythroblasts are greatly in the majority, the reverse is the case in the kidney.

I have made some attempts to arrive at some estimation of the relative numbers of erythroblasts and leucocytes in the renal-portal and cardinal veins. As the vessels are, in the greater part of their course, unequal in size, any count, to be a reliable index of the part taken by this tissue round the kidney, in contributing new elements to the blood, would require to be one relative to the number of erythrocytes. This I found impracticable, on account of the corpuscles being unequally distributed. I therefore, to reduce the balance in a rough way, counted in every fourth section of a continuous series of 200 the erythroblasts and leucocytes in one renal-portal, and put them against those in the two cardinals. I have not sufficient confidence either in the method or the figures themselves to submit them in detail, or to found a definite judgment on them, but I may say that the general result was in favour of the renal-portal as regards both erythroblasts, large mononuclears, and leucocytes; and that while erythroblasts and large mononuclears of the type seen issuing from the spleen pulp were very common in the renal-portal along the whole length of the kidney, they were practically absent in the cardinal. The possible explanation is, that in the 'backwater' formed by the great kidney venous sinus, the erythroblasts and their mother cells undergo their further transformation into erythrocytes.

Thus, though I have shown that it is highly probable that both orders of corpuscles are produced along this tract at an earlier stage, it seems doubtful whether in the later phases the pseudo-lymphoid tissue of the kidney is concerned in the new formation of red cells. I have shown how the spleen, at first distinctly lymphoid, becomes later more specially concerned in the formation of the erythrocytes, and there is some reason for believing that the kidney tract is differentiated in the opposite sense.

The development of this pseudo-lymphoid tissue, according to my account, is very simple. It is nothing more than the mesenchymatous tissue round the nephric duct and tubules, canalised, as it were, by venous spaces which communicate with the cardinal vein. From the first the cells are fixed and free; the fixed cells form the general connective-tissue basis; the free cells are either derived from the primitive mesenchyme cells of the tract itself, maintained by constant division, and set free in the blood stream by their own amœboid movement, or they wander to this site from the splanchnic mesenchyme. I have shown reasons for a belief in the first alternative.

This account is different from that given by FELIX * of the development of the corresponding tissue in the Salmonidæ. He derives the tissue from proliferation from the wall of the vein, and he describes and figures trabeculæ of *epithelioid* cells which give rise to the tissue, the cells in places becoming converted into both white and red elements. I have not seen appearances like these in *Lepidosiren.* In the early stages there is no sharp separation of the tract laden with lymphoid cells from the general mass of the mesenchyme, and all the indications are in favour of the view that it is merely a tract of the general mesenchyme in the spaces of which, for physiological reasons, the lymph cells are congregated. §

This lymphoid tissue in the larval kidney is the rudiment of a very remarkable mass of lymphoid tissue in the cortical part of the adult kidney. It forms a thick cap, so densely filled with pigment that the structure is quite concealed, but it can be seen that it is thickly studded with leucocytes.†

Lymphoid Tissue in Gut Wall.

The fact that GIGLIO-TOS ‡ describes the spiral valve as the hæmopoietic organ in the lamprey directed my attention specially to that structure in the later larval stages. I find that the fold is occupied by a denser tissue than that investing the gut. The small elongated nuclei proper to the tissue are more closely packed, while the free lymphoid cells or leucocytes are much less numerous.

The tissue of the spiral valve does not then differ in kind from the general investing tissue of the gut wall, and so far as the larva is concerned there are no appearances suggesting that it has a special hæmopoietic function.

The number of leucocytes in the connective tissue investing the gut is very great. They lie in the meshes of a loose alveolar reticulum, and belong to all the different categories. There are large numbers of mononuclear cells, here in every probability, leucoblasts. Mitotic figures occur frequently. Cells with all degrees in the metamorphosis of the nucleus, and with granules of all the varieties already mentioned, are very plentiful. Just as in the case of the kidney, this tissue represents, doubtless, the early stage of the lymphoid tissue surrounding the gut and occupying the spiral valve of the adult animal. This tissue is probably purely lymphoid—or, rather, the free cells are probably all lymphoid elements. There are no vessels to be seen crowded with erythroblasts, such as one would expect to find if the tissue had any relation to the formation of red cells.

Before closing the descriptive part of my paper, I must refer to the condition of the thymus gland at this stage. I have not studied the early stages of its development, but I now find it as a small organ, cut up into lobes by the passage through it of muscular fibres and a nerve cord. The cells of the gland are now free from the

* *Loc. cit.* † See paper by GRAHAM KERR, *Proc. Zool. Soc.*, 1901–2.
‡ *Arch. Ital. de Biologie*, vol. xxvii., 1897. § *Cf.* ZIEGLER, *loc. cit.*, 1892, note, p. 21.

hypoblast, and though differing from the cells lining the gill clefts and pharynx, in respect of their limited protoplasmic envelope they have a definitely epithelioid appearance. The nuclei are closely packed, are round or oval, and in no single instance fissured, horseshoe-shaped or lobed. In the whole gland I have detected only one leucocyte—an eosinophile, and therefore a mature variety. The appearance of the cells is wholly different from that of any class of the leucocytes; and while these elements occur in their thousands round the gut in the spleen and kidneys, and scattered in every tissue, only one or two occur in the neighbourhood of the gland.

The so-called lymphoid transformation of the epithelial cells described in various forms by KÖLLIKER, PRENANT, MAURER, SCHULTZE, NUSSBAUM and PRYMAK, and BEARD cannot, so far, have taken place. Up to this stage, then, the thymus can have no share in contributing leucocytes to the blood, unless on the impossible assumption that the epithelial cells have become lymphoid, gone off as leucocytes, and become again replaced by epithelial cells.

I hold this observation to be ample proof that the leucocytes in *Lepidosiren* do not originate in the thymus in the larval stages, but I have in preparation a note on the Thymus which will bring the matter to a perfectly conclusive issue.

General Remarks.

The divergence of opinion as to the first origin of the blood is so great that it is difficult to reconcile the various accounts. My results are suggestive in this connection, for they show how both primitive hypoblast and mesenchyme may, under certain conditions of development, share in blood formation. As in the later phases there is, in the nature of things, a degree of uncertainty, I shall attempt to summarise my facts in the strictest terms of accuracy, and then draw together such conclusions as I think they may reasonably bear.

Summary of Fact.

(1) The primary corpuscles, the first origin of which I have not studied in detail, are all alike in characters.

(2) At a stage sometime before there is any suspicion of hæmoglobin being present, there are two classes of corpuscles, one with a distinct circumferential equatorial fibrillation of the superficial layers of the protoplasm and without attraction sphere, the other without such modification of the protoplasm, but with distinct centrosome and sphere.

(3) In all later stages these two kinds of corpuscles coexist, but at stage 32 there is a sudden great increase in the proportion of the active type of cell.

(4) At stage 30, which is critical, there are (a) corpuscles presenting the features of intermediate stages between the large ringed yolk-laden bodies and an oval, disc-shaped,

ringed yolk-free corpuscle with a round nucleus; (*b*) small cells with a small amount of protoplasm formed (?) by budding (unequal division) of larger yolk-laden cells, and (*c*) similar cells with slight concentric fibrillation of the protoplasm.

(5) In all stages up to 30 the nucleus in every variety of corpuscle is identical except in size. In the arrangement of the chromatin it agrees with that of the large mononuclear basophile elements of all later phases.

(6) The cells from which the smaller elements are derived are located *certainly* in the mass of yolk cells under the splanchnopleuric mesepithelium, *possibly* in the somatic mesenchyme in the neighbourhood of the developing cardinal veins.

(7) From stage 30 onwards the active cells appear in progressively greater numbers up to stage 32+, in which they are very plentiful.

(8) From stage 30 onwards there are always cells with nuclei of the erythroblast type, and fine concentric fibrillation of a basophile protoplasm. Side by side with these are hyaline cells with small amount of basophile protoplasm and simple nuclei of the leucocyte type, identical with the nuclei of the earlier young red cells.

(9) The mass of yolk cells immediately under the splanchnopleuric mesepithelium begins at stage 30 to become differentiated with the growth of the liver, and isolation of the gut rudiment, into definitive hypoblast and splanchnic mesenchyme.

(10) At stage 31 the splanchnic mesenchyme is a cellular tissue with spaces containing blood corpuscles and free elements of two kinds: (*a*) cells identical with the large mononuclear cells appearing in increasing numbers in the blood; and (*b*) polymorphonuclear cells, with either hyaline or granular protoplasm. The nephric tract of somatic mesenchyme also contains a few free mononuclear and polymorphonuclear cells, but in smaller numbers than occur in the very cellular splanchnic mesenchyme. Mitoses are frequent in both tissues.

(11) At stages 32 and 32+ the splanchnic mesenchyme holds in its spaces numbers of mononuclear basophile cells, and in parts is composed almost wholly of polymorphonuclear cells, with hyaline or granular protoplasm. The whole perinephritic tissue has, massed in the spaces between its stellate cells, very large numbers of mononuclear basophile cells, with nuclei becoming transformed into every grade of lobing, and their protoplasm acquiring every degree of granulation. The portal and hepatic veins, and the cardinal vein and spaces communicating with it, contain white cells of different categories, the mononuclear preponderating. The cardinal vein contains three times as many erythrocytes as the aorta, but seven times as many white elements. The mononuclear outnumber the polymorphonuclear by four to one in the cardinal vein, while in the aorta they occur practically in equal proportions.

(12) The splanchnic mesenchyme retains its primitive characters in part round the gut along its whole length, forming the so-called lymphoid tissue of the adult.

(13) The nephric tract of the somatic mesenchyme also retains its primitive characters and forms the so-called lymphoid tissue of the adult kidney.

(14) The spleen rudiment is a tract of, at first identical, closely packed mesenchyme

cells. The outer cells, later, form the capsule; the central cells give rise to cellular trabecular tissue, which in part becomes the connective tissue of the fully formed organ, in part gives rise to free cells.

(15) In the earlier stages the splenic vein is full of mononuclear cells, with nuclei of the leucocyte type, identical with those in the blood at stage 32, before the spleen has differentiated. There are also in the vein a few polymorphonuclear cells, and these are in large numbers in the spaces of the trabeculæ.

(16) In later stages the spleen pulp contains *first* basophile cells, with the leucocyte type of nucleus. In some parts the remains of the basophile cellular trabeculæ are seen containing nuclei of the same type; *second*, similar cells with concentric fibrillation of the basophile protoplasm; *third*, cells with nuclei of the erythroblast type and concentric fibrillæ in the protoplasm; *fourth*, cells with nuclei of the erythrocyte type, and yellow granules in the ringed cell-body; *fifth*, mature erythrocytes; *sixth*, mononuclear leucocytes with basophile protoplasm; *seventh*, polymorphonuclear leucocytes, small and large, in all their varieties. The members of the erythrocyte series seem greatly to outnumber those of the leucocyte series in the spleen.

(17) The leucocytes are in great abundance before the thymus rudiment appears, and at the end of the larval series examined, the gland is still a mass of epithelial cells, showing no resemblance to the lymphoid cells in other tissues. There are practically no leucocytes in its substance, and no special grouping of them, either in the surrounding mesenchyme or in the veins.

Summary of Interpretation.

The study of the early corpuscles shows that they are at first all alike—probably wandering cells, such as those first described in the living Teleost embryo by Wencke-bach.* Almost immediately a change sets in, which leads to the adoption by most of them of a passive rôle, while others remain free mobile elements, with centrosome and sphere.

The latter have all the morphological characters of leucocytes, and exactly similar cells occur in the somatic mesenchyme. As this is long before hæmoglobin is developed, the cells are never at any stage wholly hæmoglobin-containing corpuscles; and if these leucocyte-like cells are the successors of the early corpuscles, mobile elements are never absent from the blood. It is possible that we should call the early corpuscles themselves 'leucocytes.' This may seem an unwarrantable use of the term, but I believe it might bear examination. The leucocyte is generally admitted to be the phylogenetically older cell. In the Dipnoi the blood is extraordinarily rich in leucocytes, and they appear at a very early stage. The postponement of the appearance of the leucocytes in ontogeny, supposed to exist in all vertebrates, has, it might be considered, only

* *Journal of Anat. and Phys.*, vol. xix., 1885; *Archiv f. Mikr. Anat.*, Bd. 28, 1886.

partially taken place in *Lepidosiren*. PARKER * says that in Protopterus the leucocytes bear a larger proportion to the red cells than in any other vertebrate, except in pathological conditions. It is a possibility that in *Lepidosiren* we have a stage in which the blood vascular and the lymph vascular systems are not sharply marked off from one another. Perhaps we see in the blood the division of labour going on, which makes one primitive corpuscle a respiratory, another a lymph cell. The phenomena of phase 2 support such a supposition.

Up to stage 30, when the hæmoglobin appears and phase 2 sets in, all the corpuscles, active or passive, have nuclei of the leucocyte type; the chromatin is collected into karyosomes, rather widely scattered, with a fine filamentous intervening reticulum. The erythroblasts in the following stages have the erythrocyte type of nucleus, richer in chromatin, with large closely-set chromatin nucleoli, joined by a coarser and more distinctly reticular intervening substance. (Compare figs. 13, 14, 15, 19, Pl. I.) At stage 31, and better marked at stage 32, the place of the earlier cells with nuclei of the leucocyte type is taken by the large mononuclears. These have either a single or a double origin. It may be that they are all derived from the splanchnic mesenchyme, and that they wander from thence, especially into the nephric tract of the somatic mesenchyme. The distribution of cells in the cardinal veins is, however, against this, and I am disposed to believe that they arise in both situations. The point is not very material—they are mesenchymatous in origin in either case.†

Exactly similar cells to those mononuclears in the blood stream become outside the blood stream leucocytes of the several varieties. From the larger number of those mononuclears in the blood, compared to the number of polymorphonuclear cells, while the opposite proportion prevails outside the blood stream, and from their identical characters with the basophile cells of earlier stages, I believe it is a logical induction, not an assumption, that they become converted in the blood into the hæmoglobin-containing elements. Mitoses are frequent in those cells outside the blood stream, but they do not occur in the blood stream itself. This inference is confirmed by the observations on the spleen; and thus I conclude that from common mother cells added to the blood during phase 2 from the mesenchyme, and in phase 3 especially from the spleen (a derivative of the mesenchyme), are derived two families of cells, one undergoing their metamorphosis in the blood to form the respiratory erythrocytes, the other undergoing their metamorphosis outside the blood stream in the connective tissue, *i.e.* lymph spaces, to form the typical wandering polymorphonuclear leucocytes.‡

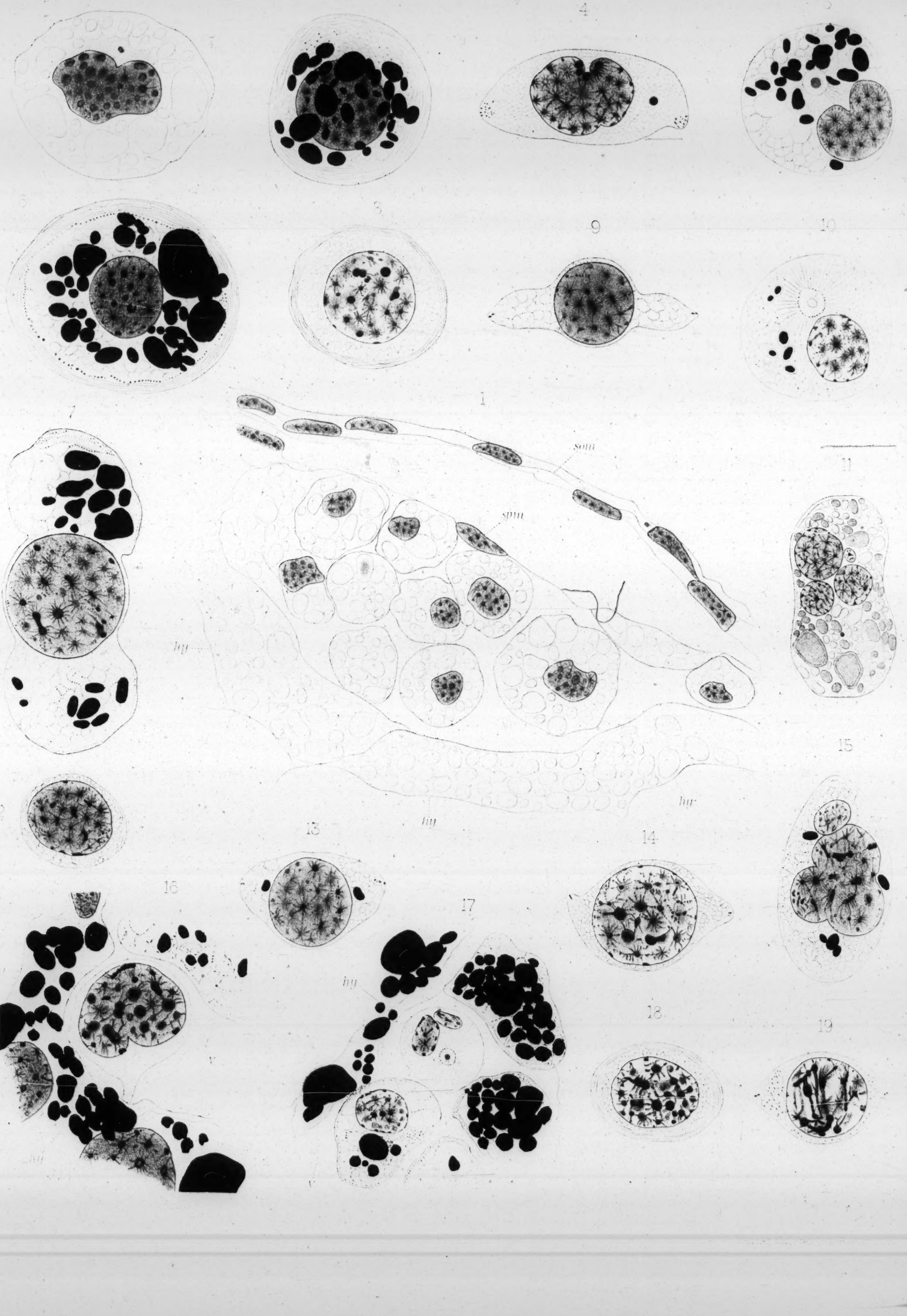
* *Trans. Roy. Irish Academy*, vol. xxx. part iii. p. 168.

† I am here assuming that the facts justify me in concluding for a continuous new formation, at least up to the end of phase 2. The statement in the text is perhaps too sweeping. If it were possible to establish that all the colourless cells were derived from the splanchnic mesenchyme, and that they wandered thence, it would be a fact of great significance, as it is derived directly from the primitive hypoblast, and it might be held that the free cells in it were directly derived from that layer. I think this would be a strained interpretation, and difficult to reconcile with the facts described regarding the nephric tract of the somatic mesenchyme, but it is a possibility which must not be overlooked.

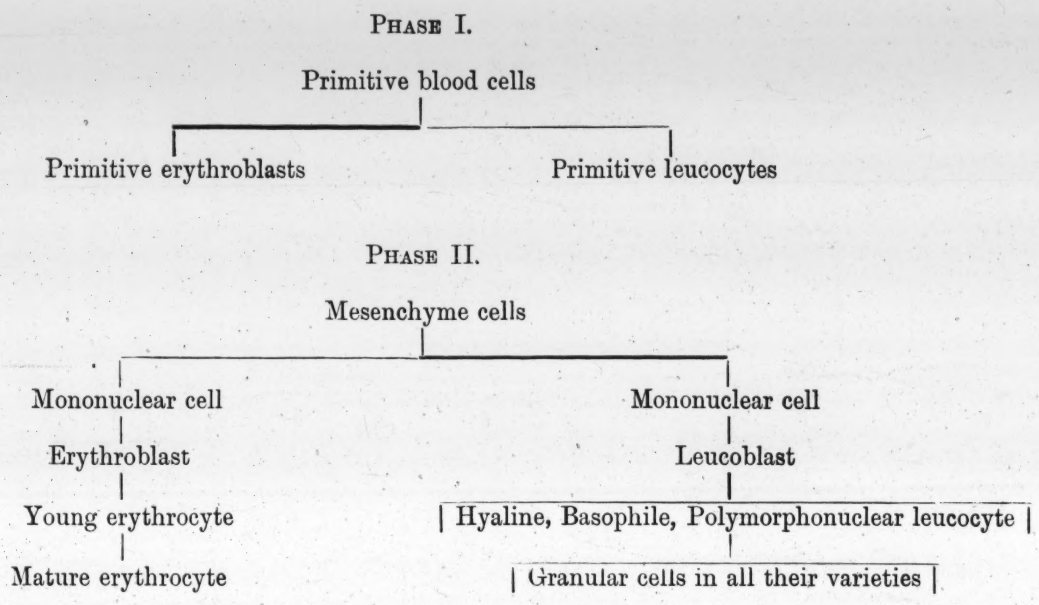
‡ An objection might here be raised that the nuclei of the polymorphic cells may regain their simple form in the blood, but as there are many polymorphs in the blood this objection would not have much force.

Bryce: Histology of the Blood of the Larva of Lepidosiren Paradoxa Part II. Plate I

Trans. Roy. Soc. Edin.



This may be expressed in the following scheme :—



As I have already said, LAGUESSE adopts a scheme very closely similar to this, and the origin of both red and white corpuscles from a common mother cell in the blood was first suggested by POUCHET in 1879. The same sort of scheme was set forth by GIGLIO-TOS in his account of hæmatogenesis in the adult lamprey. My large mononuclear mesenchyme cell would also represent the embryonic form of the parent white cell postulated by PAPPENHEIM as the forerunner of the erythroblast, while from an embryological point of view my results in a general way agree with those obtained by SAXER, according to whom the red cells in the mammalia are derived from polymorphic mesenchyme cells, while the leucocytes are derivatives of similar cells at a later stage of development.

DESCRIPTION OF PLATES.

All the drawings were carefully traced with Abbé's drawing apparatus (Zeiss). The lenses used were the 3 mm. 1·40 N.A. apochromatic lens or the 4 mm. 0·95 N.A. apochromatic lens of Zeiss with compensating oculars 4, 6 or 8. The substage condenser was the achromatic combination 1·0 N.A. of Zeiss.

The following lettering in the figures used to indicate the different catagories of free cells :—

s.m., small mononuclear ; *l.m.*, large mononuclear ; *b.p.*, basophile polymorph ; *g.p.*, granular polymorph ; *f.g.p.*, finely granular polymorph ; *c.g.p.*, coarsely granular polymorph ; *ery.*, erythroblast.

PLATE I.

Fig. 1. Group of free cells in a space between the mass of yolk cells (*hy*) and splanchnic mesepithelium (*spm*). × 500. Stage 26. *Som.*, somatopleure.

Fig. 2. Primitive yoke-laden blood corpuscle from heart tube ; safranin. Stage 26. × 800.

Fig. 3. Corpuscle from heart with fibrillar ring. Yolk stained black with iron hæmatoxylin. Stage 27, Series 84B. 2. × 800.

Fig. 4. Corpuscle from vessel on surface yolk in vertical section. Stage 27, Series 84B. 5. × 800.

Fig. 5. Corpuscle with centrosome and sphere from heart; fibrillæ of ring appear as equatorial groups of granules. Stage 27, Series 84B. 2. × 800.

Fig. 6. Large vacuolated ringed corpuscle from heart. Stage 30, Series 93C. 3. × 800.

Fig. 7. Giant corpuscle in vertical section from heart. Stage 30, Series 93C. 3. × 800.

Fig. 8. Small ringed yolk-free corpuscle with alveolar protoplasm and round nucleus. Stage 30, Series 93C. 3. × 800.

Fig. 9. Same in vertical section. Stage 30, Series 93C. 3. × 800.

Fig. 10. Corpuscle from heart, with centrosome sphere and lobed nucleus and without ring. Stage 30, Series 84B. 2. × 900.

Fig. 11. Giant apparently multinuclear cell from heart. Stage 30, Series 91C. 13. × 500.

Fig. 12. Small corpuscle with alveolar protoplasm faintly fibrillar. Stage 30, Series 93C. 9. × 800.

Figs. 13 and 14. Small corpuscles in vertical section, showing groups of granules at equatorial points. Stage 30, Series 93C. 5. × 800.

Fig. 15. Large cell with lobed nucleus. Stage 30, Series 93C. 5. × 800.

Fig. 16. Small cell apparently being budded off into vascular space (*v.*) on surface of yolk cells (*hy*). Stage 30, Series 93C. × 800.

Fig. 17. Cell with polymorphic nucleus centrosome and sphere in space on surface of yolk cells (*hy*). Stage 30, Series 93C. 4. × 500.

Fig. 18. Erythroblast with basophile protoplasm, and faint circumferential fibrillation. Nucleus with some closely-grouped, larger, and darker-staining karyosomes, and more distinct intervening reticulum. Stage 32+, Series 113C. 24. × 800.

Fig. 19. Same in vertical section. Stage 32, Series 113C. 35. × 800.

The character of the nucleus in both these cells is markedly different from that of the nucleus of the similar elements of earlier stages, figs. 13 and 14.

Plate II.

Fig. 20. Portion of splanchnic mesenchyme near its developing edge. Stage 31+, Series 106C. 11. × 400. Below is mass of yolk cells (*hy*) with nuclei and yolk grains, and very indistinct cell outlines; above, the mesenchyme consisting of fixed and free cells, and showing spaces filled with blood corpuscles.

Fig. 21. Section, through the mesonephros and adjoining parts. Stage 31+, Series 106C. 35. × 400. To right, the notochord (N.), aorta (A.), and cardinal vein (C.V.); to left, the inner outline of the muscle plate (*m.p.*).

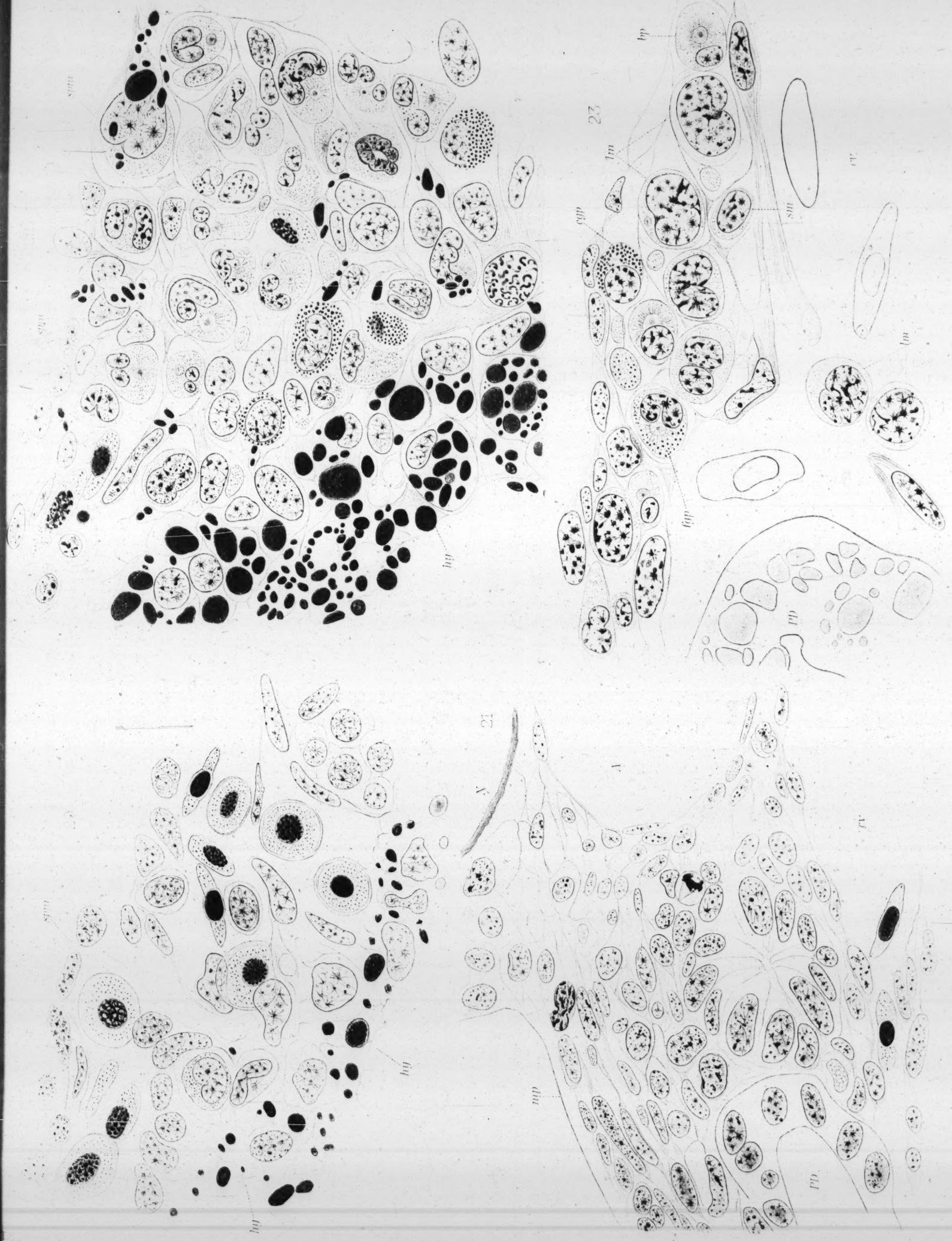
Fig. 22. Section of the splanchnic mesenchyme at point M, text figure 2. Stage 32+, Series 113C. 22. × 500. Below and to left is mass of yolk cells (*hy*), above is the line of the mesepithelium (*spm.*). The free cells of the mesenchyme have almost all become polymorphonuclear, and a number are coarsely granular.

Fig. 23. Section through the mesenchyme between aorta and pronephric duct dorsal to cardinal vein (*cf.* text figure 4). Stage 32+, Series 113C. 34. × 700. To left, outline of pronephric duct (P.D.); the section passes through an opening from the venous sinus round the duct into the cardinal vein (C.V.).

Plate III.

Fig. 24. Another portion of nephric tract of mesenchyme. Stage 32+, Series 113C. 34. × 600. Below, a typical erythroblast (*ery.*); above and to the right, a large mononuclear (*lm.*), contrasting in size with the smaller polymorphonuclear cells.

Fig. 25. Section through part of venous sinus round pronephric duct in region of pronephros. Stage 32+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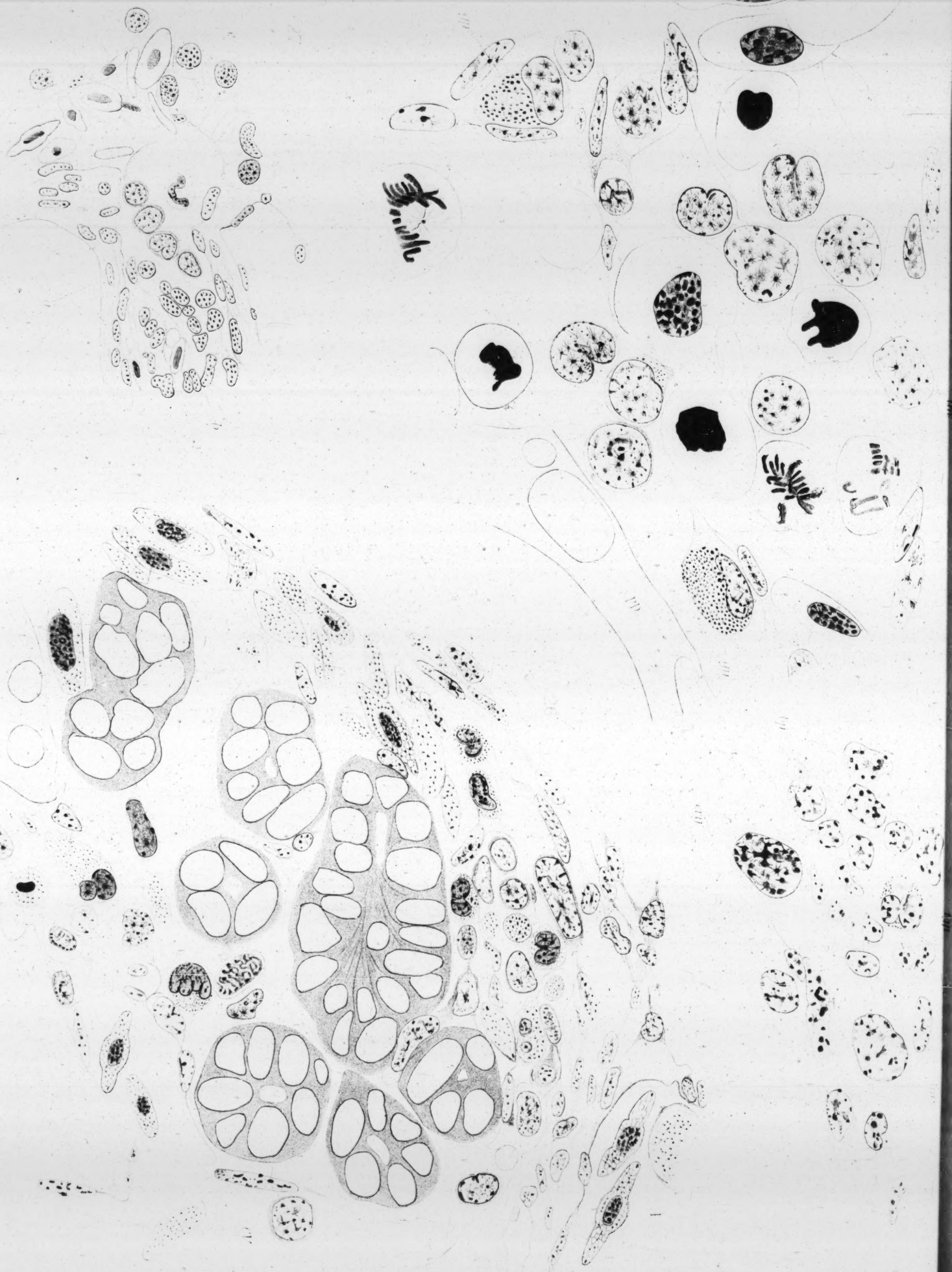

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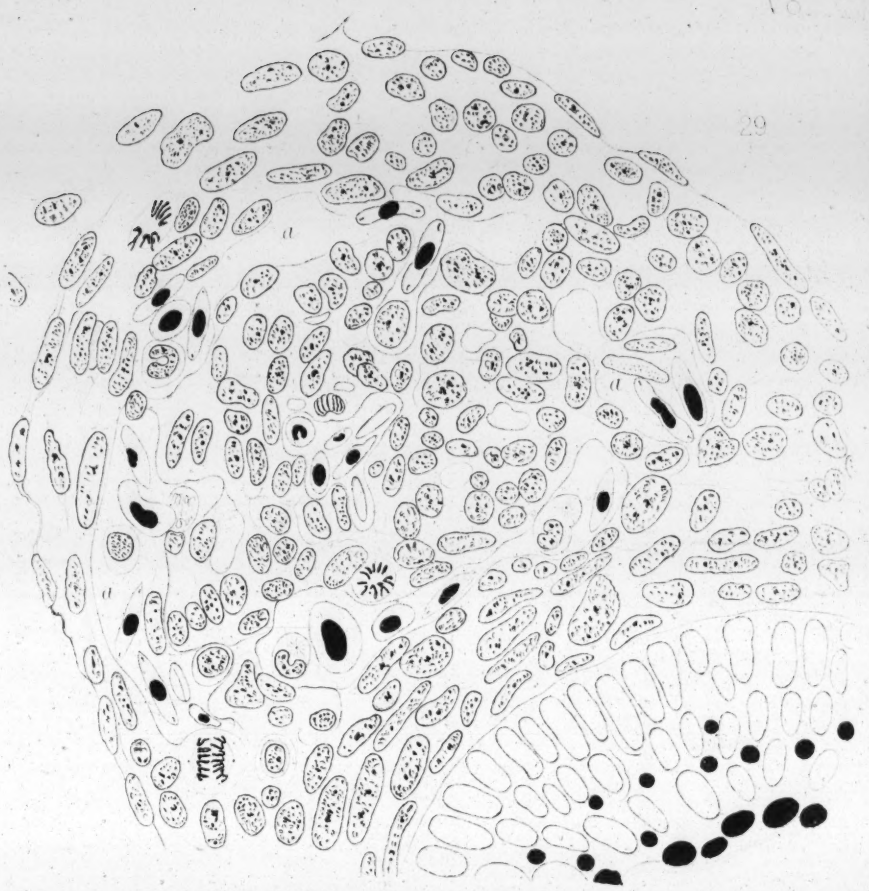

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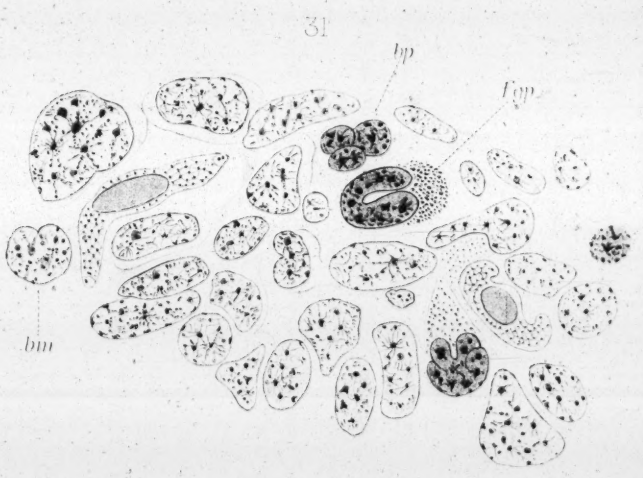

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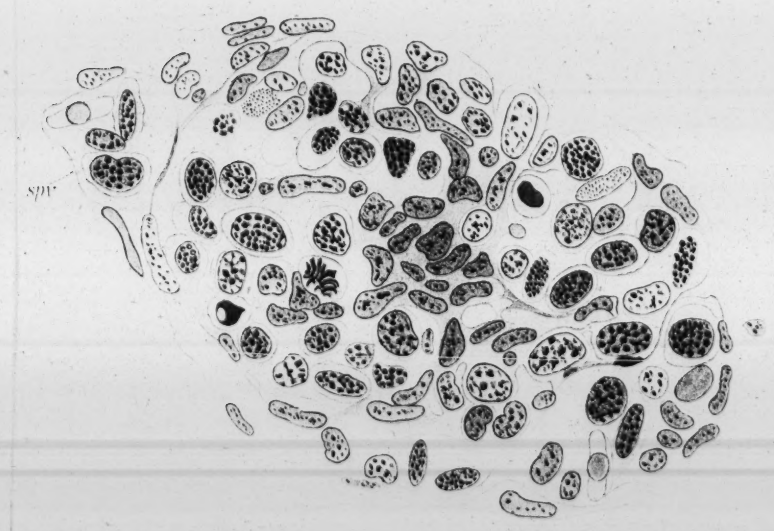

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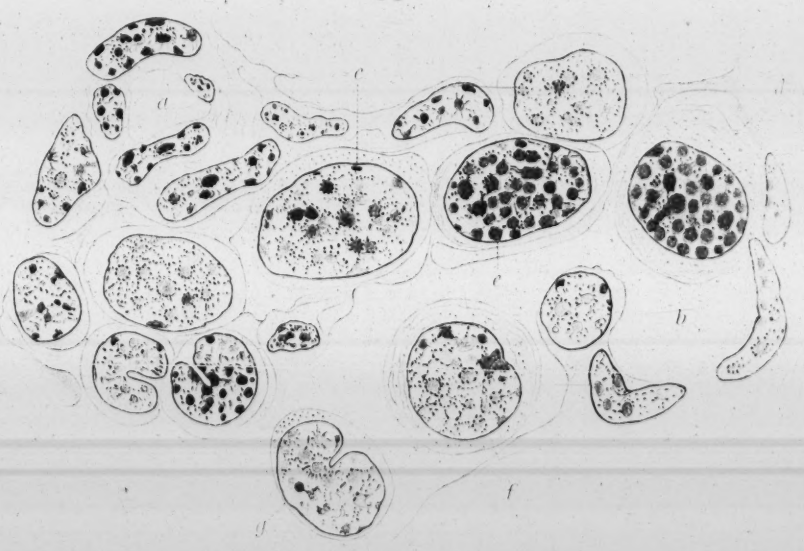

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Series 113C. 24. × 700. Below and to right is pronephric duct (P.D.) The sinus is crossed by delicate protoplasmic strands with elongated nuclei. In the meshes of this reticulum, polymorphonuclear granular leucocytes (*gp.*). In sinus space itself many large mononuclear cells, also erythrocytes drawn in outline, some of them in division.

Fig. 26. Section through mesonephros. Stage 32+, Series 113C. 40. × 500. A, aorta; C.V., cardinal vein; P.D, pronephric duct. The nuclei of the tubules have not been filled in. The meshes of the reticular tissue are filled with leucocytes of all varieties.

Fig. 27. Section through splenic vein. Stage 36, Series 139. 19. × 300. *a*, outline of stomach wall; *b*, spleen; *e*, liver. In vein, a polymorphonuclear leucocyte and four mononuclear cells.

Plate IV.

Fig. 28. Section through spleen rudiment. Stage 32+, Series 113C. 22. × 400. Below and to right, outline wall of stomach.

Fig. 29. Section through spleen rudiment. Stage 33, Series 108. 11. × 300. Below and to right, wall of stomach, with its nuclei represented in outline. *a*, *a*, *a*, peripheral blood sinus.

Fig. 30. Section of spleen; Stage 37; Series 139. 20. × 200. S, outline wall of stomach.

Fig. 31. High-power view of portion of same section, showing cellular trabeculæ and spaces containing red blood corpuscles and white cells of different varieties. × 700.

Fig. 32. Section of anterior end of spleen. Stage 39, Series 137B. 11. × 400. Above and to left, vein (*spv.*); in centre of spleen, artery (*a*).

Fig. 33. Portion of same. × 850. *a*, central artery; *b*, peripheral sinus. Four large cells in a row: to left, two large mononuclear spleen cells, *c*; to right, secondary erythroblast with hæmoglobin, *d*; in centre, primary erythroblast, *e*. Cells *f* and *g* were introduced from a neighbouring section; *f* has a nucleus like the spleen cells, *c*, but the basophile protoplasm shows delicate concentric fibrillation; *g* has a similar but deeply notched nucleus, and the protoplasm is very delicate and hyaline. It is probably a leucoblast.

XX.—Supplement to the Lower Devonian Fishes of Gemünden.

By R. H. Traquair, M.D., LL.D., F.R.S., Keeper of the Natural History Collections in the Royal Scottish Museum, Edinburgh. (With Three Plates.)

(Read December 19, 1904. Given in for publication April 14, 1905. Issued separately May 13, 1905.)

Since the publication, about a year ago, in the *Transactions* of this Society, of my paper on the Lower Devonian Fishes of Gemünden,* a review of it has appeared in the pages of *Science* by Professor Bashford Dean, of New York. In this review Professor Dean endeavoured to throw doubts on the correctness of my orientation of the elements of the dermal skeleton of *Drepanaspis Gemündenensis*, in the following words:†—

"Thus, his grounds seem inadequate for distinguishing dorsal and ventral sides. In no specimen figured is the relation of the dorsal lobe of the tail shown convincingly to be continuous with the so-called dorsal aspect; moreover, the eyes occur on the side which Traquair regards as ventral. Unless additional evidence is forthcoming, it would accordingly seem to me more probable that the 'labial'‡ of Traquair was the 'rostral' plate, a structure which appears constant in *Heterostracans*. This interpretation would permit the eyes to be seen at the sides of the dorsal armoring, as indeed they occur in *Pteraspis*, and would enable us at the same time to locate the greater number of the larger plates on the dorsal side. This conclusion is the more satisfactory on comparative grounds, since there is not an instance in the chordate phylum in which the eyes and the most complete part of the armoring appear on the (morphological) ventral side. And I doubt whether, on the present evidence, we can assume, with Professor Patten, that *Drepanaspis* might have evaded the law of vertebrate orientation by swimming on its back. Dr Traquair has attempted to solve this dorso-ventral difficulty by suggesting that either the orbits are 'sensory' pits, *i.e.* not orbits, or that, 'since the specimens are all crushed absolutely flat, it is by no means certain that in the original uncompressed condition the openings did not look out to the side.'"

The tail of *Drepanaspis* being heterocercal, the dorsal aspect of the caudal fin, in accordance with the universal condition of such tails among fishes, will be that along which the prolongation of the body-axis proceeds, as shown by its greater extent, by the squamation, and more especially by the larger size and (usually) greater number of the ridge scales or "fulcra." In the above-quoted criticism Professor Bashford Dean gave it as his opinion that I had failed to prove that this dorsal aspect of the tail was coincident with that aspect of the carapace which I described as dorsal, and which he,

* "The Lower Devonian Fishes of Gemünden," *Trans. Roy. Soc. Edin.*, vol. xl., Part iv., October 1903.
† *Science*, N.S., vol. xix., No. 471, January 8, 1904.
‡ The plate here meant is in my memoir termed *mental*, and not "labial."

from other reasons, thought much more likely to be ventral. In other words, I had turned the fish upside down.

It was, however, satisfactory to read in the same periodical, a few months later, the following counter-criticism by another eminent American palæichthyologist, Dr C. R. EASTMAN, of Cambridge, Massachusetts :—

"Whatever may be thought of TRAQUAIR's figures, though his Plate II. seems to us conclusive enough, there can be no question about the originals, and those who have examined them attentively are compelled to admit the correctness of the Scottish author's interpretations. The dorsal ridge-scales are larger than the ventral and form a more extended series, beginning further forward and continuing further back than the ventral fulcra. Several specimens in the Edinburgh Museum have been pointed out to the present writer by Dr TRAQUAIR in which this row of prominent ridge-scales can be traced continuously from a point shortly behind the median dorsal plate to the tip of the dorsal lobe of the tail. The extent to which the caudal lobes are covered with fulcra is well shown in Plate IV. and Plate I. fig. 1 of the memoir in question, and their connection with upper and lower systems of body-plates appears tolerably distinct." *

For my part, I certainly hold that the relations of the heterocercal tail to the two surfaces respectively of the carapace were quite satisfactorily proved in the plates *plus text* of my former paper. The great median plate of the surface on which the mouth and supposed orbits are placed, I described as being different from the one on the other side (see Plates I., II., and IV.) by being bilobate in front, and having behind a peculiar raised longitudinal fold continuing the direction of the posterior notch a little way forwards. It is true that I did not, among my plates, reproduce a photograph of an entire specimen of this plate, the one shown in Pl. III. being deficient posteriorly, but in the text-figure 3, p. 729, I "restored its contour from other specimens" (p. 728). And in the specimen, of which a good photograph is given, in Pl. V. fig. 1, the line of smaller fulcra, situated on the *presumably ventral margin* of the tail, is exhibited with absolute clearness running up to the posterior (cloacal) notch of that plate, which, as shown by its prominent median fold, is as undoubtedly the plate described by me as *ventral*.

But as Professor DEAN's remarks have been widely circulated in so well-known a periodical as *Science*, I shall in this "Supplement" go over the subject of the orientation of the exoskeleton of *Drepanaspis* once more, this time illustrating by specimens not figured in my previous memoir, though one of them (Pl. I.) was before me when it was written.

The depressed and flattened carapace of *Drepanaspis* shows in front a wide mouth-slit, which, though *nearly* terminal, is not quite so, and consequently is seen only on one aspect of the creature, which may meanwhile be called the *oral* one. It

* *Science*, N.S., vol. xix., No. 487, April 29, 1904, p. 704.

is this aspect, represented in Pl. III. of my former memoir, which I have denoted "ventral," and on which we find two median plates, the anterior and smaller of which bounds the mouth behind, while the larger one has posteriorly a conspicuous median notch (cloacal), the direction of which is continued forwards for some little distance in the middle line by the longitudinal fold to which I have already called attention. In this side of the creature are also seen the "orbital" or sensory plates, the anterior and posterior ventro-laterals, and a narrow external marginal portion of the postero-lateral on each side. In the specimen to which I have just referred the posterior extremity of the great median plate is deficient, but this deficiency I now remedy by figuring the one shown in Pl. I. of this Supplement, and I may say that this is the specimen which enabled me to complete the restoration of the plate in question, as seen in text-figure 3, p. 729 of my previous paper. Of this example an accurate photograph is given in Pl. I. of the present communication, and on comparing this figure with Pl. III. (former paper) it will be at once seen that the creature presents to us the same *oral* surface; for though the front part with the mouth is lost, there is no mistaking the "orbital" of one side (*x.*), the anterior ventro-laterals (*a.v.l.*), the postero-laterals (*p.l.*), and the posterior ventro-lateral of one side. In the centre of the specimen we see the great median plate (*m.v.*) of this surface, considered by me as *ventral*, in a state of nearly absolute completeness, though obliquely deformed like the rest of the specimen. With perfect clearness we see the bilobate front of this plate, the re-entering angle thereby formed being occupied by a portion of the mental plate (*m.*), while posteriorly the median fold, terminating on the notch behind, is shown with absolute distinctness. Then, below this, the hinder portion of the median plate (*m.d.*) of the opposite side is seen from its inner or visceral aspect, being brought into view by an oblique backward thrust.

The aspect of the carapace to which the last-mentioned plate (*m.d.*) belongs is seen in Plates I. and II. of my former memoir. On it the mouth-slit is never visible, and consequently the term *aboral* may be temporarily applied to it. It shows only one large median plate, which differs strikingly from the corresponding one on the oral aspect in its proportionally narrower shape, in not being emarginate or bilobate in front, in having its posterior notch smaller, pointed in front and filled up by a narrow plate, and finally in the total absence of the median fold which is so conspicuous on the posterior part of the great median plate of the oral surface. We also recognise the aboral surface by the much greater extent to which the postero-lateral plates are visible, by the greater number of small polyoral plates surrounding the median one, by the fusion of some of these little plates into what I have called the rostral plates in front, and by the two shallow pits, one on each side of the front of the head, caused by the compression of one of the small plates over the ring-like thickening round the margin of the orbital or sensory opening internally.

Having now made sure of the two surfaces of the carapace, the details of which are put together in my restored figures (pp. 726 and 729 of my previous memoir), it now

remains to settle accurately the relation to those surfaces of the two margins of the caudal fin.

In fig. 2, Pl. V. of my previous paper, a tail is represented which, though truncated behind, shows clearly that on one margin the fulcral scales are much larger than on the other; and an additional difference is, that those of the smaller series are, in front, peculiarly short and erect. Then, an inspection of fig. 1, Pl. II. of the present Supplement shows that the caudal fin, though not bilobate, is unsymmetrical above and below, and projects further back on that aspect on which the larger fulcra are placed; moreover, we observe that the lateral scales which clothe the fin are, under the line of larger fulcra, also larger. In other words we have, to all appearance, a normal piscine heterocercal tail, of which the longer margin, provided with the larger fulcra, is presumably the *dorsal* one, and it now remains to prove with which aspect of the carapace this margin of the tail coincides. This question, I maintain, was already settled by the specimen represented in Pl. V. fig. 1 of my previous memoir, in which the line of smaller fulcra is traceable to the apparent cloacal opening at the posterior extremity of the median fold of that plate, which is certainly the median plate of the *oral* aspect of the carapace.

But a still more complete demonstration of these relations is afforded in Pl. III. of the present Supplement. Here we have a specimen seen from the oral side, as shown by the form of median plate (*m.v.*) with its posterior fold, the presence of the sensory plate *x.* with its orbital (?) perforation, and of the plates *a.v.l.* and *p.v.l.*, designated by me *anterior* and *posterior ventro-lateral* respectively. It may also be noted that the median plate (*m.d.*) of the aboral side, seen from the internal surface, is shown displaced, and projecting from below the root of the tail. A considerable part of the caudal fin with the fulcra on both margins is shown, and, with absolute clearness, the line of smaller fulcra (*v.f.*) is seen to proceed forwards and end at the posterior extremity of the great median plate of the *oral* surface of the carapace. Compare this figure with the two in Pl. V. of my former memoir.

It is therefore proved, beyond all possibility of doubt, that that margin of the caudal fin which carries the row of smaller fulcra is coincident with the oral aspect of the carapace; and conversely, that the other margin, which projects further back, carries the large fulcra, and presumably contained the caudal body-prolongation, is coincident with the aboral one. If, then, the tail is constructed according to the normal piscine heterocercal type, the *aboral* surface of the carapace *is* the *dorsal*, and the *oral* one *is* the *ventral* surface.

But it may be asked whether in *Drepanaspis* the heterocercy might not have been reversed as in the reptilian *Ichthyosaurus*, by the caudal prolongation of the body axis having passed down along the ventral margin of the caudal fin, instead of along the dorsal one. In that case I should still be in the wrong as regards the orientation of the two surfaces of the carapace!

In the first place, we know of no such case among *fishes*. For, though the lower

lobe of the caudal fin in the Angel-fish (*Squatina*) is larger and projects further back than the dorsal one, there is no downward bend of the vertebral axis, which proceeds straight backwards. Nor has the greater size of the lower caudal fin lobe in the teleostean homocercal Flying-fish (*Exocœtus*) any bearing on the question.

Another circumstance which cannot be overlooked is the position of the mouth. This, as I have already shown, is not truly terminal, but is situated on that aspect of the carapace which is coincident with the shorter margin of the fin and the apparent cloacal opening. Now, although the mouth may appear to look upwards in such a peculiarly specialised bony fish as the recent Angler (*Lophius*), yet, judging from the analogy of other Ostracoderm types, such as *Pteraspis*, *Cephalaspis*, and *Asterolepis*, the *dorsal* side is *not* the one on which we would expect to find it in the case of *Drepanaspis*.

But of really crucial importance is the position of the cloacal opening or vent. *It*, at least, we cannot expect to find on the dorsal aspect of the tail of a fish or fish-like vertebrate, unless we should take upon ourselves to deny the presence of a notochordal vertebral axis in those creatures. Now if we look at Pl. II. of my former memoir, which represents the *aboral* surface of the fish, we find no trace of any such opening, although the scales of the middle line, between the posterior margin of the great median plate and the fulcra of the tail, are in complete order and well preserved. The aboral side of the fish is therefore *not ventral*; and if it be not ventral, then it is dorsal, and the oral side is the ventral one, in accordance with my original description.

Turning now to this oral side,—in the specimen represented in fig. 1, Pl. V. of my former memoir, the position of the cloacal opening seems to me to be distinctly marked just in the notch which follows the prominent posterior median fold of the great central plate, in front of the first fulcral scale in the middle line. Again, in the specimen represented in Pl. III. of the present Supplement, we have the great median plate (*m.v.*) of the aboral surface distinctly shown, as is also its median fold behind and the notch *c.*, though one margin of the latter is broken away. It is this notch *c.* which, in my opinion, marks the position of the cloacal opening, in perfect accordance with the appearances shown in the figure just referred to, and also represented in the restoration of the ventral surface in text-figure 3, p. 729 of my previous memoir. It is, however, to be noted that in this specimen (Pl. III.) the anterior extremity of the first (ventral) fulcral scale is slightly displaced, or shoved to one side, so that it no longer closes the cloacal notch (*c.*), which is consequently left open behind.

I submit, therefore, that I have amply shown—

First.—That the aboral aspect of the carapace of *Drepanaspis* is coincident with the apparent dorsal "lobe" or aspect of the caudal fin.

Second.—That the absence of a cloacal opening on the aboral aspect of the commencement of the tail, and its apparent presence on the oral one, is equivalent to a proof that the aboral margin of the tail, consequently of the entire creature, *is* the dorsal aspect. Conversely, the oral aspect is the ventral; and my previous orientation

of the creature is correct, no matter on which side of it the openings supposed to be orbits are placed.

No one will question the sensory nature of these openings, but that they really are eye-orbits, however possible or even probable that may be, is by no means certain. Their position is, however, analogous to that of the supposed orbits in *Pteraspis*; and I can only repeat that, situated as they are so near to the right and left edges of the vertically flattened carapace, they might well, in the living and uncompressed condition of the animal, have enjoyed a considerable amount of lateral outlook.

Scales of the tail-pedicle.—I have already in my previous memoir (p. 731) alluded to the fact that on that part of the tail which lies between the carapace and the caudal fin there is at least one longitudinal row of scales, which are considerably higher than broad, and which are seldom well seen, owing to that part being usually obscured by pyritous deposit. As the form of these scales can only be expressed in a direct lateral view, they could not be properly represented in my restored figures, in which the tail-pedicle is depicted as seen from above and from below.

However, in fig. 2, Pl. II. of this Supplement the tail-pedicle is seen free of pyrites, and here two rows of such vertically elongated scales are clearly visible. At the caudal fin they pass into smaller scales of a rhombic form, which become very small on the fin-membrane.

This specimen is also interesting in this respect, that, while lying on its ventral surface, that is to say, back upwards, the median dorsal plate has dropped out and the visceral aspect of the median ventral one has come into view, this plate being at once recognisable by the prominent emargination of its anterior border. This condition is the reverse of what more commonly occurs, for, as I have already stated, it is not at all rare to find in a specimen lying on its back that the median ventral plate has been lost, and the inner surface of the median dorsal one shown in consequence. See my previous memoir, Pl. IV., and explanation, p. 738.

In conclusion, I may remark that up to the present I have not been able to find in *Drepanaspis* any trace of a lateral sensory canal system.

Dr R. H. Traquair on Fossil Fishes of Gemuenden, Supplement—Plate II.

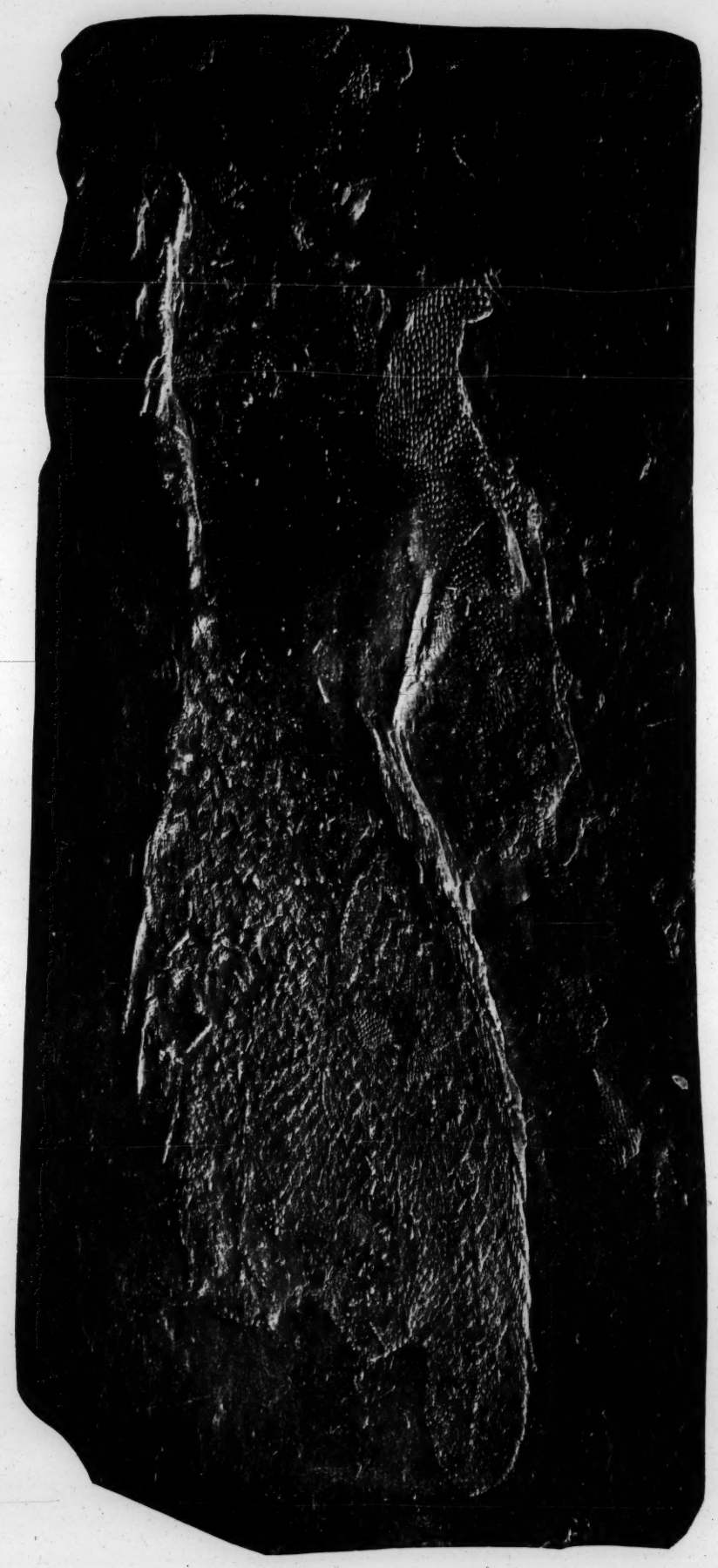

Fig. 1.
Natural Size.

Fig. 2.
One-half Natural Size.

M'Farlane & Erskine, Edinburgh.

DR R. H. TRAQUAIR ON FOSSIL FISHES OF GEMUENDEN, SUPPLEMENT—PLATE III.

Reduced by Two-sevenths.

M^cFarlane & Erskine, Edinburgh.

EXPLANATION OF THE PLATES.

All the figures in the following plates have been reproduced from photographs taken from specimens from the Lower Devonian Roofing Slate of Gemünden in the Royal Scottish Museum.

m.d., median dorsal plate.
m.v., median ventral.
m., mental.
x., sensory.
a.v.l., anterior ventro-lateral.
p.v.l., posterior ventro-lateral.
p.l., postero-lateral.
R.p.l., right postero-lateral.
L.p.l., left postero-lateral.
d.f., dorsal fulcral scales of tail.
v.f., ventral fulcral scales of tail.
c., position of cloacal opening.

Plate I.

Ventral surface of a carapace of *Drepanaspis Gemündenensis*, somewhat deficient in front and on the left side. The specimen is strongly obliquely deformed, so that the lateral plates on the right side are shoved in advance of those on the left. Here we have an exceedingly good view of the median ventral plate (*m.v.*) with its posterior median fold and notch, and the anterior emargination in which the hinder portion of the mental plate (*m.*) is seen to be lodged. Owing to oblique displacement of parts, the hinder portion of the internal surface of the median dorsal plate (*m.d.*) is also seen below and behind the median ventral. Compare this figure with that on Plate III. of my previous memoir.

Plate II.

Fig. 1. Tail and caudal fin of *Drepanaspis*, to show the oblique heterocercal configuration, and the greater size and strength of the fulcra on the upper or dorsal margin. The lateral scales of the tail-pedicle in front of the caudal fin are covered with pyritous deposit, and the commencement of the row of dorsal fulcra is also not exhibited. Natural size.

Fig. 2. Specimen of *Drepanaspis* somewhat deficient on the left side, lying on its ventral surface, but with the median dorsal plate wanting, so that the *inner* surface of the median ventral is exposed. At the antero-external corner of the carapace the small rounded pit formed by the compression of one of the external dorsal polygonal plates over the internal ring-like thickening of the opening in the sensory plate is well marked; behind this, the outer margin of the carapace and its postero-lateral angle are formed by the postero-lateral plate. The median row of dorsal fulcra is seen from its commencement, and the two rows of vertically elongated scales on the side of the tail-pedicle are unusually well shown. The ventral fulcra are not seen, having been cut off by the broken edge of the stone, but a great part of the expanse of the caudal fin, covered with small rhombic scales, is visible. One-half natural size.

Plate III.

Specimen of *Drepanaspis Gemündenensis* lying on its back, and showing the greater part of the outer surface of the median ventral plate (*m.v.*) with the posterior median fold, and the notch *c.* marking the position of the cloacal opening. This notch is followed by the line of median ventral fulcral scales (*v.f.*), the first of which is slightly displaced in front, so that its anterior margin no longer completes, as it ought, the notch into an opening. The dorsal fulcra (*d.f.*), of obviously larger size, are seen on the opposite side of the tail, the anterior ones being slightly confused. The scales of the beginning of the tail-pedicle are, as is commonly the case, obscured by pyritous deposit, but those further back and on the caudal fin are clearly exhibited. Portions of the left sensory plate (*x.*), of the left anterior ventro-lateral (*a.v.l.*), and of the left postero-lateral (*L.p.l.*) are seen at the top of the figure; a portion of the right postero-lateral (*R.p.l.*) is also seen below, removed from the rest of the fish, and turned right over so as to show its dorsal surface. Lastly, the median dorsal plate (*m.d.*) is so displaced from the rest of the body as to be seen almost in its entirety, —seen, of course, from its internal or visceral aspect.

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XXI.—A further Contribution to the Freshwater Plankton of the Scottish Lochs.

By W. West, F.L.S., and G. S. West, M.A., F.L.S. *Communicated by* Professor I. B. Balfour, F.R.S. (With Seven Plates.)

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I. Introduction.

Our first investigation of Scottish plankton in 1901–2, although only tentative and somewhat meagre, was sufficient to show that, as regards the phytoplankton, the lochs of the west and north-west of Scotland were probably richer than any lakes previously examined. Owing to the extraordinary richness of the few collections then examined, it was considered eminently desirable that the investigation should be further extended. This we have been enabled to do by means of a third successive grant from the Royal Society, and the present paper is one of the results of a visit to the north-west of Scotland in July, August, and September, 1903.

Material was collected from more than twenty of the lochs in Perth, Inverness, Ross, and the Outer Hebrides, and its examination has been most gratifying. The collections were made in the ordinary manner by silken tow-nets, about 9 inches diameter at the mouth, and the material was mostly preserved in 4 p.c. formalin. It is to be regretted that much of this material could not be examined in the living state, as there is every reason to believe from the preserved samples that some at least of the lochs were rich in Heliozoa.

In Perthshire, Loch Tay was investigated during the month of July along a great part of its length, and a number of collections were also made from the River Lochay.

In Inverness, six lochs were investigated during August, viz., L. Bairness, L. na Cloiche Sgoilt, L. na Criche, L. Gorma, L. Morar, and L. Shiel. Of these, the two latter yielded the best material.

In Ross, material was collected during September from L. Luichart and L. Rosque.

In Lewis, nine lochs were investigated during August. The plankton of most of these lochs was distinctly rich, especially that obtained from Loch Fadaghoda, which stands unrivalled for the abundance and diversity of its Desmid-flora.

In Harris, three lochs were examined, viz., L. Diracleet, L. a Mhorghain, and L. Laxadale.

In our previous contribution to Scottish freshwater plankton we recorded a number of algæ from Loch Tay, Perthshire, and from Loch Laxadale, Harris; and for the sake of better comparison these records have been included in the tabulated account of the plankton, along with additions obtained by subsequent collections from these lochs.

We have also included in this same table the results of an examination of some material forwarded to us by Mr J. Murray, of the Scottish Lake Survey (Pullar Trust), from Lochs nan Cuinne, Ghriama, and Ruar, in Sutherland, and from Loch Morar, in Inverness. We obtained permission from Sir John Murray to utilise these results to the best advantage, and we have therefore tabulated them alongside the other plankton records. The material from Loch Ruar was both rich and interesting, and consisted principally of Desmids and Diatoms. The Desmids were almost as abundant and conspicuous as those from Loch Fadaghoda, but were not nearly so diversified in character. The Diatoms, however, were especially noticeable, and constituted fully one-half of the plankton.

In preparing this paper we have largely made use of photomicrography. An inspection of photomicrographs, especially those taken under a magnification of 100 diameters, greatly facilitates the comparison of plankton from different lochs, and renders more manifest the distinguishing features of the material. In preparing the plankton-material for photography, it is absolutely essential to remove the larger animals, such as the Entomostraca, otherwise the thickness of the film of water is too great to allow of obtaining a reasonable focus of the majority of the floating objects. The photographs used for purposes of illustration are principally of material from Lochs Ruar and Fadaghoda, as they indicate very clearly the differences between the Scottish plankton and that of most of the lakes of continental Europe.

Mr Lemmermann, of Bremen, has kindly reported upon the *Peridinieæ* and some other flagellate organisms from certain of the lochs.

II. Detailed Account of Plankton of Lochs investigated.

Plankton has been examined from the following twenty-four lochs and from the River Lochay. The date refers to the date of collection.

1. *Loch Tay, Perthshire*, July 1903.—The loch is about 12 miles long and 290 feet above sea-level, and is situated in the vicinity of mountains which reach a height of 4000 feet. The collections were made from the western part of it during very fine weather. Desmids and Diatoms were abundant in the plankton, and there was also an abundance of *Cœlosphærium Kützingianum*.

2. *River Lochay, Perthshire*, July 1903.—The collections were made from the lower reaches of the river before its entrance to Loch Tay. Diatoms were very abundant.

3. *Loch Bairness, Inverness*, Aug. 1903.—This is a small loch situated among very rocky surroundings in Moidart. The plankton contained numerous Desmids and a great abundance of the Rotifer *Anuræa cochlearis*, Gosse.

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4, 5, and 6. *Lochs na Cloiche Sgoilt* (alt. 800 feet), *na Criche* (alt. 700 feet), and *Gorma* (alt. 800 feet), *Inverness*, Aug. 1903.—Small lochs in Moidart, with rocky surroundings, the largest one being barely a mile in length. A long-spined form of *Ceratium hirundinella* and *Crucigenia irregularis* were very abundant in Lochs na Cloiche Sgoilt and Gorma. The plankton of Loch na Criche was remarkable for the enormous quantity and large size of the colonies of *Kirchneriella lunaris* (Kirchn.), Möb.

7. *Loch Morar*, *Inverness*, Aug. 1903.—This loch is $11\frac{1}{2}$ miles in length, and at its widest part is only a mile and a half in breadth. It is a little more than 30 feet above the sea-level, and is one of the deepest lakes in Europe. It has a maximum depth of 1017 feet, and only seven of the lakes of continental Europe exceed this depth—four in Norway and three in Italy. The temperature of the surface-water in July 1902 was 55·2° F., and the temperature at the bottom 42·2° F.*

Three collections of plankton were examined from this loch. All were remarkable, owing to the quantity of *Stigonema minutum*, Hass., *St. mamillosum*, Ag., *Calothrix*, sp., and *Glæocapsa Ralfsiana* (Harv.), Kütz., they contained. These algæ are not plankton-forms, and had most probably entered the lake by mountain torrents during heavy rains.

8. *Loch Shiel*, *Inverness*, Aug. 1903.—This loch is about 18 miles in length, and one of the narrowest in Scotland, presenting a close resemblance to a Norwegian fjord. For much of its length it lies between high mountains, and its altitude is only 15 feet above sea-level. The plankton was mostly obtained from towards the western extremity, and species of *Mougeotia* and *Zygnema* were very common in it. Rhizopods were not uncommon, one species of *Nebela* being hitherto undescribed.

9. *Loch Luichart*, *Ross*, Sept. 1903.—This loch is above 4 miles in length, and is 280 feet above sea-level. It is situated in a valley between rocky hills, the lower slopes of which are thickly wooded. The plankton contained quantities of *Mesotænium macrococcum* (Kütz.), Roy & Biss., in small gelatinous colonies, and also an abundance of *Clathrulina elegans*, Cienk., and *Diplosigopsis frequentissima* (Zach.), Lemm.

10. *Loch Rosque*, *Ross*, Sept. 1903.—A loch about 3 miles in length, 508 feet above the sea-level, and situated between high mountains. The plankton contained quantities of *Clathrulina elegans*, Cienk., and was rather remarkable for the scarcity of *Peridinieæ*. It contained numerous specimens of a *Rhaphidiophrys*, probably *R. viridis*, Arch., and *Diplosigopsis frequentissima* (Zach.), Lemm.

11, 12, and 13. *Lochs nan Cuinne, Ghriama, and Ruar, Sutherland.*—Loch nan Cuinne is about 3 miles in length, 390 feet above sea-level, and is situated among a number of low hills of north-east Sutherland. Lochs Ghriama and Ruar are much smaller lochs. The plankton was collected from these three lochs by Mr James Murray, of the Scottish Lake Survey (Pullar Trust), and samples were forwarded to us for the investi-

* Bathymetrical Survey of the Freshwater Lochs of Scotland, part v.—Lochs of the Morar Basin—by Dr T. N. Johnston, *The Scot. Geog. Mag.*, Sept. 1904, vol. xx., No. 9.

gation of the algæ. The material was in every case fairly rich, that from Loch Ruar being especially rich in Desmids and Diatoms.

14. *Loch east of Cearnabhal, Lewis*, Aug. 1903.—This was a small deep loch, about 200 feet above sea-level, and situated amongst rocky bog-land.

15. *Loch Cuthaig, Lewis*, Aug. 1903.—A small loch among rocky and boggy land, about 200 feet above sea-level. The plankton contained many Desmids, the most abundant of which were forms of *Arthrodesmus Incus* (Bréb.), Hass., and *Staurastrum lunatum*, Ralfs, var. *planctonicum*.

16. *Loch Fadaghoda, Lewis*, Aug. 1903.—This loch is about 2 miles long, with a very irregular contour. It contains several rocky islands, and its margins are both rocky and boggy. The plankton was well investigated, and the material collected was the richest we have ever examined. The Desmids were in great quantity, and also in great variety. Many species of great rarity and interest occurred in abundance in this plankton, and the investigation of this material has extended the geographical range of quite a number of Desmids. *Staurastrum Ophiura*, Lund, and *Sphærocystis Schroeterii*, Chod., were especially noticeable.

17. *Loch Langabhat, Lewis*, Aug. 1903.—A loch about 8 miles in length and about 200 feet above the sea-level. The plankton contained large numbers of *Mallomonas caudata*, Iwanoff.

18, 19, 20, 21, and 22. *Lochs Roinebhall, an Sgath, Shubhaill, Stranabhat, and an Tomain, Lewis*, Aug. 1903.—These lochs are all of small size and are similarly situated, none being at an altitude of more than 300 feet above sea-level. The plankton obtained from them was of a somewhat uniform character, that from Loch Stranabhat being undoubtedly richer than the others. Loch an Sgath was remarkable for the great abundance of Rotifers. L. Roinebhall contained *Mallomonas longiseta*, Lemm., and *M. producta* (Zach.), Iwanoff.

23. *Loch Diracleet, Harris*, Aug. 1903.—A small loch south of Tarbert, with rocky shores, and situated very little above sea-level. The plankton was characterised by the great abundance of *Staurastrum jaculiferum*, West.

24. *Loch a Mhorghain, Harris*, Aug. 1903.—A small loch among the hills 5 miles north of Tarbert; its altitude is 480 feet. The plankton had no distinguishing features, although *Asterionella gracillima*, Heib., occurred in considerable quantity.

25. *Loch Laxadale, Harris*, Aug. 1903.—This loch is about $1\frac{1}{2}$ miles in length, a quarter of a mile in breadth, and is 40 feet above sea-level. It is situated near Tarbert, in the midst of high rocky mountains. The plankton was somewhat varied in character, and contained an abundance of *Staurastrum paradoxum*, Meyen, and *St. jaculiferum*, West. Seven species of Rhizopods were observed in this plankton. LEMMERMANN also records *Mallomonas caudata*, Iwanoff, and *M. longiseta*, Lemm.

Table of Phytoplankton.

The algæ observed from the various collections have been tabulated to facilitate comparison. The relative frequency of a species is indicated by the letters "ccc" = very abundant, "cc" = common, "c" = fairly common, "r" = infrequent, "rr" = rare, and "rrr" = very rare.

SPECIES.	Perth.		Inverness.						Ross.		Sutherland.			Lewis.									Harris.		
	Loch Tay.	River Lochay.	Loch Bairness.	Loch na Cloiche Sgoilt.	Loch na Criche.	Loch Gorma.	Loch Morar.	Loch Shiel.	Loch Luichart.	Loch Rosque.	Loch nan Cuinne.	Loch Ghriama.	Loch Ruar.	Loch near Cearnabhal.	Loch Cuthaig.	Loch Fadaghoda.	Loch Langabhat.	Loch Roineabhall.	Loch an Sgath.	Loch Shubhaill.	Loch Stranabhat.	Loch an Tomain.	Loch Diracleet.	Loch a Mhorghain	Loch Laxadale.
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
Phæophyceæ.																									
Stichoglœa olivacea, Chod.,	...	...	...	...	...	...	...	...	...	...	..	...	...	...	...	r									
Phæococcus planctonicus, sp. n.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	r	...	...	...	...	...	...	...	...	r
Dinobryon Sertularia, Ehrenb.,	c	...	...	...	...	r	cc	cc																	
„ *utriculus* (Ehrenb.), Klebs,	...	...	...	...	...	...	...	r																	
„ *protuberans*, Lemm.,	...	...	...	...	..	...	...	c																	
„ *cylindricum*, Imhof., var. *pediforme*, Lemm.,	...	...	...	...	...	c	...	cc	...	rr	...	...	...	cc	r	c	c	ccc	rr	c	r	c	rr	cc	
„ „ var. *palustre*, Lemm.,	...	...	...	...	...	...	...	r	...	...	...	...	...	...	c	...	...	c	...	...	...	...	r	rr	c
„ „ var. *divergens* (Imhof.), Lemm.,	...	...	...	...	...	..	cc	...	...	...	...	...	...	...	cc	...	...	ccc	...	rr	...	r			
„ „ var. *Schauinslandii*, Lemm.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	c	
„ *elongatum*, Imhof.,	...	...	...	...	...	c																			
„ „ var. *undulatum*, Lemm.,	...	...	...	...	...	...	...	...	...	...	...	...	...	..	...	...	...	...	ccc	...	...	...	cc	...	ccc
Halobryon Lauterbornii, Lemm.,	...	...	...	...	...	...	...	rr																	
Chlorophyceæ.																									
Œdogonium, spp. (sterile),	cc	c	c	c	r	...	...	cc	...	rr	...	...	...	cc	c	ccc	...	rr	c	rrr	c	c	...	r	r
„ *punctato-striatum*, De Bary,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	..	...	rr				
Ulothrix zonata (Web. & Mohr), Kütz.,	r	cc	...	...	...	...	...	...	...	r															
„ *subtilis*, Kütz., var. *variabilis* (Kütz.), Kirchn.,	...	...	r	...	r	...	...	...	...	..	...	...	...	...	c	...	...	r	...	...	...	...	r		

TABLE OF PHYTOPLANKTON—*continued.*

SPECIES.	PERTH.		INVERNESS.						ROSS.		SUTHERLAND.			LEWIS.								HARRIS.		
	Loch Tay.	River Lochay.	Loch Bairness.	Loch na Cloiche Sgoilt.	Loch na Criche.	Loch Gorma.	Loch Morar.	Loch Shiel.	Loch Luichart.	Loch Rosque.	Loch nan Cuinne.	Loch Ghriama.	Loch Ruar.	Loch near Cearnabhal.	Loch Cuthaig.	Loch Fadaghoda.	Loch Langabhat.	Loch Roinebhall.	Loch an Sgath.	Loch Shubhaill.	Loch Stranabhat.	Loch an Tomain.	Loch Diracleet.	Loch a Mhorghain	Loch Laxadale.
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
Geminella interrupta, Turp.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	c						
Microspora amœna (Kütz.), Lagerh.,	r	...	...	rr	...	r	...	r	...	c	...	...	...	...	...	r	...	...	...	c	r	...	c	...	c
Mougeotia, spp. (sterile),	r	c	c	r	c	...	c	...	cc	cc	c	c	cc	r	c	ccc	cc	r	cc	ccc	cc	ccc	r	...	cc
Zygnema, spp. (sterile),	...	...	...	rr	...	...	c	...	cc	ccc	...	...	...	...	...	c	r	...	c	r	...	r	c	r	c
Spirogyra, spp. (sterile),	...	...	...	...	...	...	...	r	...	r	...	...	r	...	rrr	c	...	...	rrr	rrr	...	c	c	...	r
Gonatozygon monotænium, De Bary,	...	...	...	...	cc	r	...	cc	...	...	...	...	rr	...	rrr	r	rr	...	r	...	r	r	r	...	rr
,, ,, var. *pilosellum*, Nordst.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rr									
,, *Kinahani* (Arch.), Rabenh.,	...	...	...	...	...	...	...	rr	...	...	...	...	c	...	...	rrr									
Genicularia elegans, W. & G. S. West,	...	..	...	...	...	...	c	rr	...	...	...	..	...	...	...	ccc	...	...	r	...	...	...	...	...	c
Mesotænium macrococcum (Kütz.), Roy & Biss.,	...	...	c	c	...	...	...	...	cc	...	...	...	...	...	...	...	...	...	...	c					
Cylindrocystis diplospora, Lund,	...	...	rrr																						
Netrium Digitus (Ehrenb.), Itzigs. & Rothe,	...	...	...	...	...	...	...	r	...	...	...	...	rr	...	r	r	...	...	rrr	...	rr	...	...	...	r
Penium Libellula (Focke), Nordst.,	...	...	...	...	...	...	...	r																	
,, *minutum* (Ralfs), Cleve,	rrr	...	...	...	...	...	...	...	...	...	...	...	rr	...	...	r									
Closterium macilentum, Bréb.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rrr									
,, *striolatum*, Ehrenb.,	...	...	...	...	...	...	...	rr	...	...	...	...	...	...	...	...	rr	rrr							
,, *Dianæ*, Ehrenb.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rrr									
,, *parvulum*, Näg.,	r	rr	...	...	...	...	...	rrr	...	...	...	...	...	...	rr	...	...	...	...	...	...	...	r		
,, *Jenneri*, Ralfs,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rr										
,, *Venus*, Kütz.,	rrr	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rr				
,, *moniliferum* (Bory), Ehrenb.,	...	rrr																							
,, *turgidum*, Ehrenb., forma *glabra*, Gutw.,	...	...	...	...	...	...	...	rrr																	
,, *acerosum* (Schrank), Ehrenb. forma *minor*,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rrr									
,, *pronum*, Bréb.,	...	...	...	...	...	...	rrr	rr																	
,, *aciculare*, T. West, var. *subpronum*, W. & G. S. West,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rrr									
,, *acutum*, Bréb.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	r									
,, *lineatum*, Ehrenb.,	r	r																							

Closterium Kützingii, Bréb.,	r	rr	...	...	...	...	...	c	c	...	...	...	r	...	...	rrr	...	...	...	...	r						
,, ,, var. *onychosporum*, var. n.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rrr						
,, *rostratum*, Ehrenb.,	c	r	...	...	...	...	...	...	...	...	...	...	r														
,, *setaceum*, Ehrenb.,	...	...	...	...	...	...	...	...	...	rr	...	...	c	...	...	r	...	...	c								
,, ,, var. *elongatum*, v. n.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	cc								
Pleurotænium coronatum, Bréb.,	...	...	...	...	...	...	rr	..	...	...	rr	...	r	...	...	rr	...	...	rr	...	...	rr					
,, ,, var. *fluctuatum*, West,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rr											
,, *Ehrenbergii* (Ralfs), De Bary,	...	...	...	rr	...	...	...	r	rr	...	...	...	r	...	rrr	c	rrr	...	rr	rrr	r	...		rr	rrr		
Tetmemorus Brébissonii (Menegh.), Ralfs,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	r	...	...	...	...	...	...	...	...	...	rrr		
,, *granulatus* (Bréb.), Ralfs,	rr	...	...	rr	...	...	rrr	c	...	...	...	...	...	...	c	rr	rrr	...	rrr	rr	rr	...	rrr				
Euastrum crassum (Bréb.), Kütz.,	...	...	...	...	...	...	r	...	...	...	...	...	...	...	rr	rrr											
,, *oblongum* (Grev.), Ralfs,	...	...	...	...	...	...	rr	...	...	...	...	...	...	...	...	...	...	rr	...	...	rrr						
,, *pinnatum*, Ralfs,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rrr		
,, *ansatum*, Ralfs,	...	...	...	rr	...	...	...	c	...	r	...	...	...	rr	c	r	...	...	rr	...	rr						
,, *bidentatum*, Näg.,	r	c	...	...	rrr	...	...	...	...	r	...	...	r	...	r	c	...	r	...	...	cc	...	r				
,, *elegans* (Bréb.), Kütz.,	...	c	...	r	rr	...	...	...	rr	...	...	...	rr	r	c	c	...	r	rrr	...	r	...	rrr				
,, *binale* (Turp.), Ehrenb.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rrr	...	...	...	...	r						
,, *denticulatum* (Kirchn.), Gay,	...	r	...	...	...	...	...	r	...	...	...	...	...	rrr	r	...	...	...	rr								
,, *pectinatum*, Breb., var. *inevolutum*, W. & G. S. West,	...	...	...	rrr	...	rr	...	r	...	...	...	...	r	...	rr	rr	...	...	...	...	...	...	...	...	r		
,, *verrucosum*, Ehrenb., var. *reductum*, Nordst.,	c	...	rr	c	...	r	r	c	...	...	r	...	cc	...	c	c	rr	...	...	...	...	rrr	r	...	c		
,, ,, var. *planctonicum*, W. & G. S. West,	...	...	...	...	...	...	...	...	...	...	r	...	cc														
Micrasterias pinnatifida (Kütz.), Ralfs,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	r	...	...	...	...	...	...	...	...	rr		
,, *Jenneri*, Ralfs,	...	...	...	...	...	...	...	...	rr																		
,, *truncata* (Corda), Bréb.,	...	...	r	rr	...	...	...	r	...	...	...	...	...	...	...	r	...	rr	...	...	rr	...	...	..	rr		
,, *conferta*, Lund,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rr	rr	...	...	...	...	...	...	...	...	rrr		
,, *papillifera*, Bréb.,	...	...	...	r	...	...	...	c	...	...	...	...	...	...	...	r	...	...	rr	...	rrr						
,, *Murrayi*, W. & G. S. West,	...	...	...	...	...	...	...	...	...	...	...	...	cc														
,, *radiosa*, Ralfs,	...	...	...	...	...	...	...	...	...	...	...	r	...	...	...	rr											
,, ,, var. *ornata*, Nordst.,	...	...	r	...	...	...	...	r	...	...	...	...	...	...	...	c											
,, *apiculata*, Menegh., var. *fimbriata*, Ralfs,	...	...	...	...	..	...	...	...	...	...	...	...	...	..	...	rrr											
,, ,, var. *brachyptera* (Lund),	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rr	...	...	...	...	...	rrr					
,, *rotata* (Grev.), Ralfs,	...	rr	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	r	...	...	...	...	...	rr		
,, *denticulata*, Bréb.,	...	...	...	...	...	...	...	r	...	...	..	...	...	...	rr	c	...	...	...	...	...	...	rr				
,, *furcata*, Ag.,	...	...	...	...	...	...	...	...	...	...	r	...	...	...	...	c											
,, *Americana*, Ehrenb.,	...	...	...	rr	...	...	...	r																			
,, *Wallichii*, Grun.,	...	...	...	...	...	...	...	...	...	...	...	...	rr														
Cosmarium connatum, Bréb.,	...	...	...	...	...	...	...	...	...	...	c	...	c	...	...	r											
,, *contractum*, Kirchn.,	...	...	c	rrr	...	r	...	...	...	...	...	...	...	...	rr	cc	...	rr	cc	c	...	rrr	rr				
,, ,, var. *ellipsoideum* (Elfv.), W. & G. S. West,	...	...	cc	...	...	...	...	...	...	...	...	...	...	c	...	...	rrr	...	...	...	r	rr	r	rrr			

TABLE OF PHYTOPLANKTON—*continued.*

SPECIES.	PERTH.		INVERNESS.						ROSS.		SUTHERLAND.			LEWIS.									HARRIS.		
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	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
Cosmarium subcontractum, W. & G. S. West,	...	...	...	c																					
,, *granatum*, Bréb.,	...	...	...	...	...	...	...	c	...	...	...	...	...	...	...	...	...	...	rr	r	rrr				
,, *depressum* (Näg.) Lund, var. *achondrum* (Boldt), W. & G. S. West,	c	...	c	c	...	c	...	c	...	...	...	...	c	...	rrr	cc	...	r	r	...	r	r	...	...	r
,, *moniliforme* (Turp.), Ralfs,	...	...	...	...	...	...	...	...	..	...	...	...	...	...	r	cc	...	...	c	rr	rr	...	r		
,, ,, forma *panduriformis*, Heimerl.,	...	...	cc	...	..	...	...	...	...	...	...	...	...	...	...	c									
,, *subarctoum* (Lagerh.), Racib.,	..	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rr	...	...	...	...	...	...	...	...	cc
,, *bioculatum*, Bréb.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	cc	rr	...	...	c	...	rr	rr			
,, *Klebsii*, Gutw.,	...	...	c	...	...	...	r	c	...	...	...	...	...	r	r	...	...	rrr	...	r	...	...	rrr		
,, *abbreviatum*, Racib.,	r																								
,, ,, var. *planctonicum*, var. n.,	ccc	...	...	...	...	...	c	...	r	ccc	...	...	...	r	cc	c	...	cc	r						
,, *capitulum*, Roy & Biss.. var. *grœnlandicum*, Börges.,	...	...	...	.	...	...	cc	...	..	...	...	...	...	...	cc	...	...	cc							
,, *difficile*, Lütkem., var. *sublæve*, Lütkem.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rrr	r									
,, *impressulum*, Elfv.,	...	...	...	...	...	...	...	rr																	
,, *Meneghinii*, Bréb., forma *octangularis*, Wille,	...	r	c																						
,, *humile*, Gay,	rr	...	rr	...	...	...	r	...	...	...	...	...	...	...	c	...	...	...	r	rr	rr	c	...	r	rr
,, *punctulatum*, Bréb.,	...	...	...	...	...	...	...	...	r	...	...	...	...	...	c	...	...	...	...	...	...	...	...	..	rrr
,, *Blyttii*, Wille,	...	...	...	rr	...	...	...	r	...	...	...	...	r	rr	...	...	...	...	rrr	...	...	rr			
,, *Kjellmanii*, Wille, var. *grande*, Wille,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rr
,, *formosulum*, Hoff.,	...	...	...	...	...	...	...	rrr	...	...	...	...	...	...	...	...	...	...	rr	cc					
,, *margaritiferum* (Turp.), Menegh.,	...	...	r	r	...	rrr	...	rr	...	...	...	...	...	...	r	cc	rrr	rr	r	rr	c	rr	...	...	r
,, *Botrytis* (Bory) Menegh.,	c	cc	c	...	..	r	...	rr	...	...	...	...	...	...	rr	r	...	...	r	rr	rr	c	...	...	cc
,, var. *tumidum*, Wolle,	...	...	...	...	...	...	...	r	...	...	...	...	...	...	rr										
,, var. *depressum*, var. n.,	...	...	...	...	...	...	...	...	...	...	...	...	c												
,, *reniforme* (Ralfs), Archer,	...	...	...	...	...	...	...	r	...	...	...	...	...	rr	rr	r	...	...	...	...	r	rr			
,, *ornatum*, Ralfs,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rr	...	...	...	...	...	...	rr		
,, *subcostatum*, Nordst.	r	c	...	...	...	...	...	...	...	...	...	...	...	...	..	...	...	...	...	...	rrr				

Cosmarium tetraophthalmum (Kütz.), Menegh.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	r	r	...	..	rr	...	rrr							
„ *controversum*, West,	...	...	...	...	...	...	...	...	...	...	...	...	..	...	...	rrr												
„ *ovale*, Ralfs,	...	...	...	...	...	...	...	rrr	...	...	...	...	...	...	...	rr												
Xanthidium antilopæum (Bréb.), Kütz.,	cc	...	c	...	...	...	r	r	...	...	...	...	rr	c	cc	cc	...	rr	r	c	c	r	...	...	c			
„ „ var. *polymazum*, Nordst.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	r												
„ „ var. *depauperatum*, W. & G. S. West,	c	...	...	ccc	...	c	...	...	...	...	c	r																
„ „ var. *Hebridarum*, var. n.,	...	...	...	cc	...	...	...	c	r	r	r	rr	r	...	...	ccc	rr	c	r	r	c	...	c	r	c			
„ „ var. *læve*, Schmidle,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	c												
„ *subhastiferum*, West,	...	...	...	...	...	...	cc	cc	r	...	c	c	...	...	...	cc												
„ „ var. *Murrayi*, W. & G. S. West,	...	...	...	...	...	...	r	...	...	...	rr																	
„ *tetracentrotum*, Wolle forma,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rrr			
„ *controversum*, W. & G. S. West, var. *planctonicum*, W. & G. S. West,	...	...	...	...	...	...	...	...	...	...	...	c	...	...	...	...	r	...	rr	r	rrr	rr						
„ *cristatum*, Bréb.,	...	...	r	...	...	rr	...	...	...	...	...	...	...	...	...	c	...	...	rr	...	rr	rr						
„ *fasciculatum*, Ehrenb.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rr													
„ *armatum* (Bréb.), Rabenh.,	...	...	...	...	...	...	...	rrr	...	...	...	...	...	...	...	r	...	...	rr									
Arthrodesmus quiriferus, W. & G. S. West,	...	...	...	...	...	...	...	...	ccc	cc	...	r																
„ *Incus* (Bréb.), Hass.,	rr	c	...	...	...	...	...	rrr	...	...	...	...	...	c	ccc	rr	c	cc	c	...	r	c	c	...	cc			
„ „ var. *longispinum*, var. n.,	...	...	...	...	...	...	...	...	...	...	...	...	...	r														
„ „ var. *Ralfsii*, W. & G. S. West forma,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	r	...	rr	...	r								
„ *triangularis*, Lagerh.,	...	...	...	cc																								
„ „ var. *subtriangularis* (Borge), W. & G. S. West,	...	...	r	rr	...	r	...	...	r	rr	...	...	...	...	...	...	r	...	rr									
„ *crassus*, W. & G. S. West,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	r													
„ *convergens*, Ehrenb.,	r	...	...	...	...	rr	...	...	...	...	...	...	...	...	...	c												
„ *octocornis*, Ehrenb.,	...	..	...	...	...	...	...	rr																				
Staurastrum dejectum, Bréb.,	r	...	c	...	...	...	...	...	...	...	...	...	...	c	c	c	...	r	r	...	c	r	c	...	r			
„ „ var. *inflatum*, West,	...	...	...	...	...	...	...	...	...	...	...	...	r	...	rr	...	...	...	...	...	...	...	...	...	rr			
„ *Dickiei*, Ralfs,	...	r	...	...	...	...	...	rr	...	...	...	...	...	...	...	r												
„ *curvatum*, West,	rrr	...	...	r	...	...	rr	cc	c	cc	c	...	...	...	rrr	c	r	...	...	r rr	r	c						
„ *jaculiferum*, West (end-view biradiate),	...	...	r	...	...	...	rrr	ccc	r	...	...	...	cc	...	...	ccc	ccc	...	...	rr	c	cc	r	c				
„ „ „ (end-view triradiate),	ccc	...	...	...	...	...	ccc	...	..	...	...	...	...	...	...	ccc	...	...	rr	r	c	r	ccc	r	cc			
„ „ var. *excavatum*, W. & G. S. West,	...	...	...	...	...	...	...	...	...	...	...	cc	...	c	...	...	c	...	...	...	...	...	c					
var. *subexcavatum*, W. & G. S. West,	...	...	...	...	...	...	rr	...	...	...	...	r	r															
„ *megacanthum*, Lund,	...	...	...	...	...	...	...	rr	...	...	...	rr	...	...	r	c	...	rr	r	rr	...	r						
„ „ var. *Scoticum*, W. & G. S. West,	...	...	...	...	...	...	...	...	...	...	c	r	...	...	...	...	r	...	...	...	...	...	...	...	c			

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	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
Staurastrum cuspidatum, Bréb., var. *maximum*, W. & G. S. West,	cc	...	...	...	...	r	ccc	...	c	ccc	...	...	...	c	cc	ccc	...	c	c	ccc	...	c	c	c	
,, ,, var. *divergens*, Nordst.,	...	...	cc																						
,, *inelegans*, sp. n., ...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	r									
,, *longispinum* (Bail.), Arch., . .	...	...	...	...	...	...	...	c	...	...	...	...	...	...	...	c	...	...	...	...	r				
,, ,, var. *bidentatum* (Wittr.), W. & G. S. West,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	r	...	...	c						
,, *Brasiliense*, Nordst., var. *Lundellii*, W. & G. S. West,	...	...	...	...	...	...	...	rr	...	...	...	...	...	...	...	c	...	...	r						
,, *grande*, Buln., ...	...	...	...	...	...	...	...	...	...	...	r	...	...	...	r	cc	...	...	...	...	...	...	...	...	r
,, *conspicuum*, W. & G. S. West, .	..	...	...	...	...	...	...	...	...	...	...	rr													
,, *tumidum*, Bréb., ...	..	...	...	...	...	...	...	...	r	...	...	...	...	...	...	rr	...	...	rr						
,, *subpygmæum*, West, ...	...	...	...	...	...	...	...	...	r	...	...	...	...	...	...	rr									
,, *brevispinum*, Bréb., ...	...	...	...	...	...	...	...	rr	...	...	...	...	...	...	...	...	...	...	...	r					
,, ,, var. *altum*, var. n., .	...	...	...	...	...	...	...	...	...	...	r	...	rr	...	...	cc	...	...	rr	...	...	rr	r		
,, ,, var. *obversum*, var. n., .	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	cc									
,, *sibericum*, Borge, ...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	r									
,, *aversum*, Lund, ...	...	...	...	...	...	...	...	r	...	...	rr	...	...	...	...	c	rr	...	c	...	...	rr	...	rrr	
,, *lunatum*, Ralfs., var. *planctonicum*, W. & G. S. West,	cc	rrr	c	r	...	...	r	r	c	c	c	r	...	...	ccc	ccc	...	...	...	ccc	c	c	r	rr	r
,, *Avicula*, Bréb. var. *subarcuatum* (Wolle), West,	...	...	rr	rr	rr	...	...	...	...	...	...	...	...	...	rr	rrr	...	rr	...	...	rr	...	...	r	
,, *orbiculare*, Ralfs, ...	...	...	r	...	r	...	...	rr	...	...	...	...	...	...	r	...	...	...	rrr	...	rr				
,, *Bieneanum*, Rabenh., ...	...	...	r	...	...	...	...	...	...	...	...	...	...	...	rrr	r	...	rrr	...	...	...	rrr			
,, *sublævispinum*, W. & G. S. West, .	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rrr									
,, *subnudibrachiatum*, sp. n., . .	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rr									
,, *brachiatum*, Ralfs, ...	...	...	...	...	...	...	...	...	...	...	r	...	...	rr	...	c	...	...	cc	...	r				

Staurastrum dilatatum, Ehrenb., var. *obtusilobum*, De Not.,	...	...	...	...	...	...	rr																				
,, *erasum*, Bréb.,	...	...	.	...	...	rrr	...	...	...	...	...	...	ccc														
,, *teliferum*, Ralfs,	...	...	r	rr	...	rrr	...	c	rr	r	...	...	...	rr	r	rr	...	...	rr	...	r	r	...	...	rr		
,, *asperum*, Bréb.,	...	...	...	...	...	...	...	rrr																			
,, *polymorphum*, Bréb.,	...	rr	...	...	...	...	...	...	...	...	...	...	...	...	...	rr	...	...	...	c	...	rr					
,, *pseudopelagicum*, W. & G. S. West,	cc	...	...	...	...	...	ccc	r	...	...	...	...	rr	r	c	r	...	ccc	rr	c	c	r					
,, *paradoxum*, Meyen,	cc	...	...	rr	...	r	c	c	c	ccc	r	..	cc	rr	...	...	...	cc	r	c	...	rr	cc	rr	ccc		
,, ,, var. *longipes*, Nordst.,	ccc	...	r	...	..	c	...	...	cc	c	c	r	...	r	...	ccc	c	...	c	rr	r	c	...	...	r		
,, ,, var. *cingulum*, W. & G. S. West,	...	...	...	...	..	...	c	...	...	cc																	
,, *gracile*, Ralfs,	...	...	...	r	...	r	...	r	...	r	c	r	rr	...	c	...	rr	rrr	c	c	c	c	...	...	r		
,, ,, var. *nanum*, Wille,	...	...	r	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rr						
,, ,, var. *cyathiforme*, W. & G. S. W. forma,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	ccc	...	...	c								
,, *cyrtocerum*, Bréb.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rrr	...	...	rr								
,, *tetracerum*, Ralfs,	...	...	rrr	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rrr						
,, *subgracillimum*, W. & G. S. West,																											
,, *anatinum*, Cooke & Wills,	...	...	ccc	...	...	...	...	...	rr	...	c	r	rr	rr	c	c	c	c	cc	rr	...	r	c	rr			
,, ,, var. *grande*, W. & G. S. West,	...	...	...	...	...	...	...	r	..	...	...	...	...	...	...	cc	...	...	r	...	c	...	rr	...	r		
,, ,, var. *pelagicum*, W. & G. S. West,	...	...	...	...	rr	...	...	ccc																			
,, ,, var. *truncatum*, West,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	r	c	...	...	...	...	rr	c	r	...	r		
,, ,, var. *longibrachiatum*, var. n.	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	cc										
,, *Sebaldi*, Reinsch, var. *productum*, var. n.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	r											
,, *Ophiura*, Lund,	r	...	...	...	...	...	...	c	...	...	r	rr	...	...	r	ccc	...	...	ccc	...	cc	rr	rr	..	c		
,, *Arachne*, Ralfs,	...	...	rr	...	...	...	...	...	...	...	...	...	...	...	...	r	...	...	...	...	rrr						
,, ,, var. *curvatum*, W. & G. S. West,	...	...	...	...	...	...	...	...	...	...	rr	...	r														
,, *monticulosum*, Bréb., var. *bifarium*, Nordst.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	..	...	rr				
,, *forficulatum*, Lund,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rr											
,, *Tohopekaligense*, Wolle, var. *trifurcatum*, W. & G. S. West,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rrr											
,, *Clevei* (Wittr.), Roy & Biss.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rrr											
,, *furcigerum*, Bréb.,	c	...	...	...	...	...	r	...	...	...	...	...	...	...	...	c	...	...	...	...	...	rrr					
,, ,, forma *armigera* (Bréb.),	cc	...	...	...	...	...	...	r	r	...	rr																
,, *sexangulare* (Buln.), Rabenh.,	...	...	...	...	...	...	..	...	...	...	r	...	...	rr	...	c	r	...	rr	...	c	rr					
,, *Arctiscon* (Ehrenb.), Lund,	rrr	...	...	...	...	...	...	c	...	...	rr	r	...	...	...	cc	...	...	...	...	...	...	...	...	r		
Spondylosium pulchrum (Bail.), Arch. var. *planum*, Wolle,	cc	...	...	...	...	...	...	...	c	cc	r	...	...	...	r												

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	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
Sphærozosma granulatum, Roy & Biss.,	...	...	...	r	rr	...	...	...	...	...	...	...	...	rrr	r	r	rr	r	...	...	r				
,, *vertebratum*, Ralfs,	...	...	...	...	...	rrr	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	r		
,, *Aubertianum*, West, var. *Archerii* (Gutw.), W. & G. S. West,	...	...	r	...	...	...	...	rr	...	...	...	...	...	...	...	cc									
Desmidium Swartzii, Ag.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	..	...	...	...	rr	...	r
,, *aptogonum*, Bréb.,	...	...	...	...	...	...	...	...	...	...	..	...	rr	...	rrr	r	...	...	...	...	...	...	...	...	c
,, *occidentale*, sp. n.,	...	...	...	...	...	..	...	...	...	...	...	...	...	...	...	cc									
,, *graciliceps* (Nordst.), Lagerh.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rrr									
,, *coarctatum* Nordst., var. *cambricum*, West,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	cc									
Gymnozyga moniliformis, Ehrenb.	...	...	...	...	...	...	...	c	rr	...	rr	...	...	...	...	rr	rr	...	rr	...	...	...	...	...	c
,, ,, var. *gracilescens*, Nordst.,	...	...	..	...	...	.	...	...	...	...	...	...	...	...	...	rrr									
Hyalotheca dissiliens (Sm.), Bréb.,	rr	...	...	...	...	...	...	c	...	...	...	...	...	...	c	rr	...	r	...	...	rrr	...	rrr	r	
,, ,, forma *tridentula*, Nordst.,	...	...	...	...	...	...	...	...	...	...	...	...	...	rr											
,, *mucosa*, Ehrenb.,	r	rrr	...	...	...	...	r	c	rr	...	...	rr	...	rr	c	c	...	...	r	...	rrr	...	r	rr	
,, *neglecta*, Racib.,	...	...	...	...	...	...	...	rr	...	...	...	...	r	...	...	ccc	...	...	r	...	rrr	...	c	...	cc
,, *Indica*, Turn.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	c									
Pleodorina californica, Shaw,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rrr									
Eudorina elegans, Ehrenb.,	r	...	r	...	...	...	...	r	...	...	...	...	...	...	...	cc									
Gonium pectorale, Müll.,	...	...	cc																						
Pandorina morum (Müll.), Bory.,	...	...	...	...	...	...	...	rrr																	
Pediastrum Boryanum (Turp.), Menegh.,	rr	...	c	r	r	rr	...	r	...	rr	...	...	r	rrr	r	r	...	r	rr	...	c	rr	rr	...	r
,, *duplex*, Meyen,	...	...	...	rr	...	...	...	c	...	...	...	...	r	...	...	rr	...	...	...	...	...	r	rrr		
,, *Tetras* (Ehrenb.), Ralfs,	...	r	...	...	...	...	...	...	...	...	...	...	...	...	r										
Sorastrum spinulosum, Näg.,	...	...	rr	...	...	...	...	...	...	...	...	...	...	...	...	rrr									
,, *Americanum* (Bohlin), Schmidle,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rrr
Cœlastrum cambricum, Arch.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	r									
,, *sphæricum*, Näg.,	...	..	...	rr																					
,, *Morus*, W. & G. S. West,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rrr									

Crucigenia rectangularis (Näg.), Gay,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rr	rr	c	...	...	...	rr							
,, *irregularis*, Wille,	...	...	...	cc	r	c	r	...	...	...	...	...	...	...	...	...	...	...	ccc										
Scenedesmus quadricauda (Turp.), Bréb.,	cc	r	r	...	...	...	rr	...	...	rrr																			
,, *bijugatus* (Turp.), Kütz.,	rr	c	...	...	...	...	...	...	...	...	...	...	...	...	r	rrr	...	...	...	r									
,, *denticulatus*, Lagerh., var. *linearis*, Hansg.,	...	...	...	...	...	...	...	rr	...	...	...	...	...	...	c	...	...	...	...	rr									
Dimorphococcus lunatus, A. Br.,	...	...	rr																										
Ankistrodesmus falcatus (Corda) Ralfs,	r	rr	cc	...	r	...	c	cc	...	...	...	...	...	...	r	ccc	...	rr	rrr	...	rr	r	...	...	r				
,, ,, var. *acicularis* (A. Br.), G. S. West,	r	...	r	...	...	...	r	...	...	...	...	...	...	...	...	...	...	...	r	rr									
,, ,, var. *spirilliformis*, G. S. West,	...	...	...	...	...	...	...	..	r	...	...	...	...	...	...	rrr													
,, *Pfitzeri* (Schröd.), G. S. West,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	c	rr	...	rr										
Kirchneriella lunaris (Kirchn.), Möb.,	...	...	...	...	ccc	...	...	...	...	...	...	...	...	...	...	rrr													
Oocystis crassa, Wittr.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rr										
,, *parva*, W. & G. S. West,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	r														
Nephrocytium Agardhianum, Näg.,	...	...	...	r	...	rrr	r																						
,, *lunatum*, West,	rrr	...	...	...	...	...	c	...	...	r	...	...	...	...	...	rr	...	rrr	c										
Tetraëdron limneticum, Borge,	...	...	...	...	...	...	...	...	...	...	...	...	cc																
Dictyosphærium pulchellum, Wood,	...	...	r	...	...	...	...	c	rr	c	...	...	...	...	r	...	c	cc	...	...	...	c							
Actinobotrys confertus, gen. et sp. n.,	...	...	r	...	c	r	r	...	...	..	...	...	...	...	...	c	...	...	ccc	...	...	...	rr	...	r				
Botryococcus Braunii, Kütz.,	...	...	r	r	rr	c	...	...	c	cc	r	...	rr	. .	rr	...	rrr	rr	c	...	rr	...	...	...	r				
,, *protuberans*, sp. n.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...		c													
Ineffigiata neglecta, W. & G. S. West,	...	...	c	r	...	cc	...	...	...	c	r	rr	cc	...	...	ccc	...	rrr	cc	rrr									
Sphærocystis Schröterii, Chodat,	r	...	...	c	c	...	c	c	r	r	...	...	...	c	...	c	c	c	...	...	rr	c							
Glœocystis gigas (Kütz.), Lagerh.,	...	r	r	r	...	c	...	cc	...	...	...	...	rr	.	c	...	...	...	...	rr	r	r							
,, *vesiculosa*, Näg.,	...	...	c	...	...	...	...	...	r	...	...	...	...	...	...	rrr													
,, *infusionum* (Schrank), W. & G. S. West,	...	c	...	r	rr	...	...	cc	...	...	...	...	...	rrr	...	...	...	r	c	...	r								
HETEROKONTÆ.																													
Ophiocytium parvulum (Perty), A. Br.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	r											
Tribonema affine (Kütz.), G. S. West,	...	...	...	...	...	...	...	...	...	...	...	...	c																
BACILLARIEÆ.																													
Melosira granulata (Ehrenb.), Ralfs,	...	...	rr																										
Cyclotella operculata, Kütz.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	c				
,, *Meneghiniana*, Kütz.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rr	...	...	...	r	...	...	...	c					
Rhizosolenia longiseta, Zach.,	...	...	...	...	...	...	...	...	...	cc																			
,, *eriensis*, H. L. Sm., var. *morsa*, var. n.,	...	...	...	...	...	...	...	ccc																					
Tetracyclus lacustris, Ralfs,	...	...	...	...	...	...	rr																						
Tabellaria fenestrata (Lyngb.), Kütz.,	c	c	r	cc	r	cc	c	ccc	r	c	...	cc	c	cc	ccc	c	cc	rrr	r	ccc	ccc	ccc	c	c	ccc				
,, ,, var. *asterionelloides*, Grun.,	...	...	...	...	...	...	...	r	...	...	...	...	ccc	...	...	...	...	...	rrr	...	...	rrr							
,, *flocculosa* (Roth.), Kütz.,	ccc	ccc	cc	r	rr	r	rr	c	c	r	c	r	rrr	rr	c	...	r	r	rrr	r	c	c	r	c	ccc				

TABLE OF PHYTOPLANKTON—*continued.*

SPECIES.	PERTH.		INVERNESS.						ROSS.		SUTHERLAND.			LEWIS.									HARRIS.		
	Loch Tay.	River Lochay.	Loch Bairness.	Loch na Cloiche Sgoilt.	Loch na Criche.	Loch Gorma.	Loch Morar.	Loch Shiel.	Loch Luichart.	Loch Rosque.	Lochnan Cuinne.	Loch Ghriama.	Loch Ruar.	Loch near Gearnabhal.	Loch Cuthaig.	Loch Fadaghoda.	Loch Langabhat.	Loch Roinebhall.	Loch an Sgath.	Loch Shubhaill.	Loch Stranabhat.	Loch an Tomain.	Loch Diracleet.	Loch a Mhorghain	Loch Laxadale.
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
Denticula tenuis, Kütz.,	...	...	...	...	...	...	...	...	r																
Diatoma elongatum, Ag.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rr	rr			
Fragilaria mutabilis (W. Sm.), Grun.,	...	...	...	...	...	...	...	rr	...	...	...	...	...	...	c	...	...	...	...	...	...	...	...	...	ccc
,, *Crotonensis* (A. M. Edw.), Kitton,	...	c	rr	...	...	rrr	rr	r	r	...	...	...	...	rr	...	...	...	...	...	cc	r	cc			
,, ,, var. *contorta*, var. n.,	...	...	...	...	...	...	...	...	...	...	...	...	cc												
Synedra Ulna (Nitzsch), Ehrenb.,	c	r	r	...	r	...	...	...	...	...	...	...	...	...	rr	...	...	...	...	r	...	...	...	...	r
,, *pulchella*, Kütz.,	...	...	...	r	...	...	...	...	...	ccc	r	...	...	r											
,, *Acus* (Kütz.), Grun.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	cc
Asterionella formosa, Hass.,	ccc	...	cc	cc	...	c	c	c	ccc	ccc	...	...	...	...	...	...	...	...	...	...	rrr	...	c	...	cc
,, *gracillima*, Heib.,	...	...	...	...	...	...	...	...	...	...	cc	c	ccc	...	...	ccc	c	...	r	...	...	cc	...	cc	
Ceratoneis Arcus (Ehrenb.), Kütz., var. *Amphioxys* (Rabenh.), De Toni,	r	c																							
Eunotia lunaris (Ehrenb.), Grun,	c	cc	r	...	...	...	c	c	...	...	...	...	...	rr	...	...	...	...	...	...	c	...	...	...	c
,, *pectinalis* (Kütz.), Rabenh.,	cc	c	...	r	c	r	rr	cc	...	...	...	...	...	c	c	c	...	r	r	...	...	c	r	...	cc
,, *gracilis* (Ehrenb.), Rabenh.,	...	...	...	...	...	...	...	...		...	...	...	...	...	...	rr	...	...	...	...	...	...	...	...	r
,, *major* (W. Sm.), Rabenh.,	...	...	...	...	...	...	...	r																	
,, *biceps* (W. Sm.), G. S. West,	...	...	...	...	..	...	...	c	...	...	...	...	...	...	...	rr	...	...	...	...	...	...	...	...	ccc
,, *Diadema*, Ehrenb.,	...	...	...	rr	r	...	...	rr																	
Achnanthes flexella (Kütz.), Bréb.,	r	...	...	...	...	...	...	rr	...	...	...	...	...	...	r	...	...	...	...	...	...	...	...	...	c
Navicula major, Kütz.,	...	r	c	r	...	...	rr	cc	...	...	...	...	...	c	...	rr	...	...	...	...	r	...	rr	...	rr
,, *nobilis* (Ehrenb.), Kütz.,																									
,, ,, var. *Dactylus* (Ehrenb.), Van Heurck,	r	...	...	...	...	rr	...	...	...	...	...	...	...	...	...	rrr									
,, *viridis*, Kütz.,	r	...	...	...	r	...	r	c	...	rr	...	...	...	...	rr	rr	rr	...	r						
,, *alpina* (W. Sm.), Ralfs,	..	...	...	...	...	...	...	rr	...	r															
,, *divergens* (W. Sm.) Ralfs,	...	...	..	...	...	...	...	r	...	...	...	...	...	...	...	rrr	...	...	...	...	rr	...	...	...	rrr
,, *gibba* (Ehrenb.), Kütz.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	cc				
,, *radiosa*, Kütz.,	c	c	...	c	...	r	...	r	rr	...	...	...	...	...	c	r									
,, *elliptica*, Kütz.,	r	...	...	...	...	...	...	rr																	

Navicula limosa, Kütz.,	...	...	...	...	...	...	...	r																			
Stauroneis Phœnicenteron (Nitzsch), Ehrenb.,	...	r	...	...	rr	...	...	rr																			
„ *anceps*, Ehrenb.,	r	rr	...	..	...	r	...	...	...	...	...	...	...	...	r												
Vanheurckia rhomboides (Ehrenb.), Bréb.,																											
„ var. *Saxonica* (Rabenh.), G. S. West,	r	r	...	c	r	c	c	c	r	r	...	...	...	cc	c	rr	c	...	r	r	cc	rr	...	r	cc		
Amphipleura pellucida, Kütz.,	r	c	...	...	r	r	...	...	r	...	...	...	...	...	...	rrr	c	...	rr	rr	...	...	...	rr			
Gyrosigma acuminatum (Kütz.),	...	...	...	...	...	...	...	rrr																			
Gomphonema acuminatum, Ehrenb.,	...	c	r	...	...	...	...	rrr	...	...	...	...	...	...	r	...	...	...	rrr	...	...	...	...	...	cc		
„ *constrictum*, Ehrenb.,	r																										
„ *geminatum*, Ag.,	c																										
„ *intricatum*, Kütz.,																											
„ „ var. *Vibrio*, (Ehren.), Van Heurck,	...	c																									
Cocconema Ehrenbergii, (Kütz.), G. S. West,	...	...	...	...	...	...	...	rrr																			
„ *cuspidatum*, (Kütz.), G. S. West,	r	c																									
„ *lanceolatum*, Ehrenb.,	...	r																									
„ *cymbiforme*, Ehrenb.,	c	c	...	rr	...	...	...	c	...	...	...	...	...	...	r	...	...	...	...	rr							
„ *Cistula*, Ehrenb., var. *maculata* (Kütz.),	cc																										
„ *gastroides* (Kütz.),	...	...	...	...	...	...	...	..	...	...	...	...	c														
Amphora ovalis, Kütz.,	r	...	r	...	...	rr	...	rrr																			
Epithemia turgida (Ehrenb.), Kütz.,	...	...	...	rr																							
„ *gibba*, Kütz.,	...	...	c	...	...	...	...	...	rr																		
Nitzschia sigmoidea (Ehrenb.), W. Sm.,	...	...	...	...	...	rr	...	rr																			
„ *linearis* (Ag.), W. Sm.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rr	...	...	...	...	r						
„ *Palea* (Kütz.), W. Sm.,	...	rr	...	...	...	c	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rr						
Surirella biseriata, Breb.,	...	...	...	rr	...	rr	...	c	...	...	...	...	...	...	...	rr	...	...	...	rrr	r						
„ *linearis*, W. Sm.,	...	r	...	...	...	rr	...	...	...	...	...	...	...	...	c	rrr	...	rrr	rr	...	rr	...	...	...	c		
„ *robusta*, Ehrenb.,	...	...	...	rr	...	...	r	c	...	...	...	...	c	...	...	rr	...	...	...	rrr	r						
„ „ var. *splendida* (Ehrenb.), V. Heurck,	...	...	c	r	..	r	...	ccc	...	r	c	.	ccc	...	..,	c	...	...	...	rr	rr	...	...	...	r		
„ *ovalis*, Bréb., var. *angusta* (Kütz.), V. Heurck.,	...	...	...	...	...	...	...	r	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	r r		
„ *spiralis*, Kütz.,	...	rr	...	r	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rr	...	rrr			
Campylodiscus Hibernicus, Ehrenb.,	...	...	...	...	...	...	...	rr																			
MYXOPHYCEÆ.																											
Stigonema minutum, Hass.,	...	...	...	...	...	...	...	...	c	...	...	...	...	...	...	r	r	c	c	...	...	rr	...	c			
Nostoc microscopicum, Carm.,	...	...	c	...	...	...	...	...	c	...	...	...	...	...	...	r	...	...	...	...	...	c					
Anabæna circinalis (Kütz.), Rabenh.,	...	...	...	...	...	...	...	c	...	...	...	...	...	...	c	rr	cc	rr	cc	cc	c	rr	...	r	ccc		
Lyngbya contorta, Lemm.,	...	..	...	...	...	...	rr	...	...	...	...	...	...	...	...	...	...	...	...	...	r						
Oscillatoria tenuis, Ag.,	...	...	...	...	...	...	rr	cc	c	r	...	...	...	...	rrr	r	...	c	...	r	...	rr					
„ *rubescens*, D. C.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	r											

TABLE OF PHYTOPLANKTON—*continued.*

SPECIES.	PERTH.		INVERNESS.						ROSS.		SUTHERLAND.			LEWIS.									HARRIS.		
	Loch Tay.	River Lochay.	Loch Bairness.	Loch na Cloiche Sgoilt.	Loch na Criche.	Loch Gorma.	Loch Morar.	Loch Shiel.	Loch Luichart.	Loch Rosque.	Loch nan Cuinne.	Loch Ghriama.	Loch Ruar.	Loch near Cearnabhal.	Loch Cuthaig.	Loch Fadaghoda.	Loch Langabhat.	Loch Roinebhall.	Loch an Sgath.	Loch Shubhaill.	Loch Stranabhat.	Loch an Tomain.	Loch Diracleet.	Loch a Mhorghain	Loch Laxadale
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
Glœotrichia echinulata (Eng. Bot.), Richter,	...	...	...	...	...	...	...	...	...	...	..	...	...	...	...	...	...	...	...	...	rrr				
Glœothece linearis, Näg.,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	rr						
Merismopedia glauca (Ehrenb.), Näg.,	...	...	...	...	...	...	r	c	...	...	c	...	...	...	c	cc	...	cc	ccc	...	r	...	...	...	c
,, *æruginea*, Bréb.,	...	...	...	...	...	...	...	rr	...	...	...	...	...	...	r	ccc	...	r	...	...	c	rr			
,, *punctata*, Meyen,	...	...	...	...	...	...	...	...	...	...	...	...	...	rr	rr	r	...	...	r	...	...	...	r		
Cœlosphærium Kützingianum, Näg.,	ccc	...	...	..	...	...	c	...	...	..	c	...	cc	...	...	...	...	...	...	rr	r				
,, *minutissimum*, Lemm.,	...	...	...	...	...	...	...	c	...	...	...	...	...	ccc	...	r	...	...	...	...	...	...	...	...	c
Microcystis Flos-aquæ (Wittr.), Kirchner,	...	...	...	...	...	...	r	...	r	...	...	...	...	...	cc	c	...	...	...	...	rr	...	...	...	r
,, *elabens* (Bréb.), Kütz.,	...	...	c	cc	r	...	...	rr	...	...	r	c	...	...	c	cc	...	...	c	rr	...	...	...	...	c
,, *stagnalis*, Lemm.,	...	...	...	..	...	...	...	rr	...	...	...	...	...	...	...	rr	c								
,, *marginata*, Menegh.,	...	r	c	c	c	...	c	c	...	...	...	...	...	...	c	...	r	cc	c	...	c	r			
Chroococcus turgidus (Kütz.), Näg.,	...	...	..	...	...	...	...	...	...	...	...	...	...	...	...	r	c	...	...	...	c				
,, *limneticus*, Lemm.,	...	...	...	...	...	...	rr	...	...	...	...	...	...	...	...	...	cc								
,, *pallidus*, Näg.,	...	...	...	...	...	...	...	...	r	...	...	...	...	...	c	rr									
,, *minor* (Kütz.), Näg.,	...	...	...	...	...	...	...	c																	

Peridinieæ.

In this account of the Peridinieæ we have to acknowledge our indebtedness to Mr E. Lemmermann, of Bremen, for most of the determinations. We forwarded him material from ten of the Scottish lochs, and he very kindly acceded to our request that he would report upon the Peridinieæ therein. He has decided that one of these organisms is a new species, to which he has given the name *Peridinium Westii*.

The most abundant and conspicuous of the Scottish Peridinieæ is undoubtedly *Ceratium hirundinella*, which occurs very abundantly, and is subject to considerable variation, especially with regard to the length of the horns and the amount of their divergence. This is partially illustrated by the photomicrographs, and the form of this species from Loch Ruar should be compared with the form from Loch Fadaghoda (*vide* fig. xylogr. 1, A and C).

The following is a tabulated account of the species observed :—

SPECIES.	Perth.	Inverness.						Ross.		Lewis.								Harris.		
	Loch Tay.	Loch Bairness.	Loch na Cloiche Sgoilt.	Loch na Criche.	Loch Gorma.	Loch Morar.	Loch Shiel.	Loch Luichart.	Loch Rosque.	Loch near Cearnabhal.	Loch Cuthaig.	Loch Fadaghoda.	Loch Roinebhall.	Loch an Sgath.	Loch Langabhat.	Loch Shubhaill.	Loch Stranabhat.	Loch an Tomain.	Loch Dìracleet.	Loch a Mhorghain	Loch Laxadale.
Gymnodinium paradoxum, Schilling,	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	.. r	...	r
Glenodinium pulvisculus (Ehrenb.), Stein,	...	r	..	...	...	...	...	...	...	...	...	...	r	...	...	...	...	...	r	...	
Ceratium cornutum (Ehrenb.), Clap. et Lachm.,	...	...	...	...	...	rr	c	..	...	...	...	rr	...	...	...	...	...	...	r	...	rr
,, *curvirostre*, Huitf.-Kaas,	..	...	...	...	...	..	rr														
,, *hirundinella*, O. F. Müller,	r	...	ccc	c	c	rrr	c	rr	rrr	c	c	ccc	cc	c	cc	c	cc	cc	cc	cc	cc
Peridinium alatum, Garbini,	...	r	...	...	...	...	...	r	...	...	r	...	r	...	...	...	...	...	r		
,, *inconspicuum*, Lemm.,	...	r	..	...	...	...	r	...	...	...	...	rr	r	...	...	...	...	...	r		
,, *pusillum*, Schilling,	..	r																			
,, *umbonatum*, Stein,	...	r	...	...	...	...	...	...	...	...	...	...	...	...	...	r					
,, *Willei*, Huitf.-Kaas,	...	r	r	...	...	...	...	...	..	...	...	...	...	r	c						
,, *Westii*, Lemm., n. sp.,	...	cc	...	...	...	...	c	...	r	...	...	...	...	r	...	r					
,, *tabulatum*, (Ehrenb.), Clap. et Lachm.,	...	...	...	...	...	...	...	...	...	...	...	...	rr								

Ceratium hirundinella, O. F. Müller.

The abundance of this organism is one of the most conspicuous features of the Scottish plankton, but sufficient observations have not been made to decide definitely the time of its maximum period.

The length and divergence of the antapical horns are exceedingly variable.

LEMMERMANN gives the following measurements :—

Locality.	Length of Apical Horn.	Length of 1st Antapical Horn.	Length of 2nd Antapical Horn.	Length of 3rd Antapical Horn.	Total Length of Body.
Loch Cuthaig, } Lewis,	103μ	68·5μ	44μ	15μ	234·5μ
Loch Roinebhall, } Lewis,	96–109·6μ	72–85μ	33–42μ	7μ	212–245·6μ
Loch Shubhaill, } Lewis,	105–124μ	60–93μ	37·5–45μ	4–15μ	210·5–281μ
Loch Diracleet, } Harris,	103–111μ	82–104μ	34–49μ	15–23μ	238–291μ
Loch a Mhorghain, } Harris,	101–153μ	86–120μ	45–60μ	15–30μ	251–357μ
Loch Laxadale, } Harris,	105–120·5μ	73–103μ	40–45μ	15–31·5μ	237–304μ

To these we add two others—

Loch Ruar, Sutherland,	86μ	46μ	21μ	0	198μ
Loch Fadaghoda, Lewis,	118–129μ	67–82μ	60–67μ	31–38μ	235–260μ

In the specimens from Loch Ruar there was a complete absence of the third (short) antapical process, and in those from Loch Fadaghoda the three antapical processes were nearer a uniform development than in any other specimens examined.

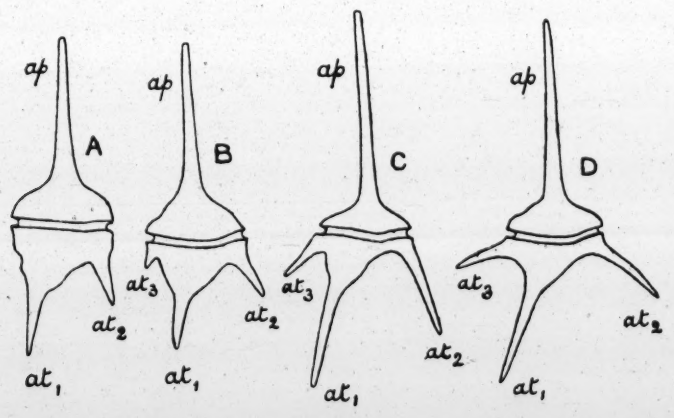


FIG. 1.

Ceratium hirundinella, O. F. Müller. A, from Loch Ruar, Sutherland; B, from Loch Asta, Shetlands; C, from Loch Fadaghoda, Lewis; D, from Loch Beosetter, Bressay, Shetlands. (All × 200.) *ap*, apical horn; at_1, at_2, at_3, the three antapical horns.

In fig. xylogr. 1, A, the third antapical horn is absent, but in fig. 1, D, it has reached its maximum development (at_3). LEMMERMANN has figured a large number of different

forms of this species from the plankton of Sweden (*vide* Lemm. in *Archiv f. Bot.*, Bd. ii., No. 2, t. 2, f. 1–49). We have found all these forms in the Scottish plankton, and cysts occur plentifully. Figs. xylogr. 1, B and D, are from specimens observed in plankton from the Shetland Islands. Fig. 1, B, shows the first appearance of the third antapical horn (at_3), and fig. 1, D, shows the maximum development and divergence of the three antapical horns.

Peridinium Westii, Lemm., sp. n.

Body nearly globular, at the posterior margin slightly sinuate, 44–45μ long and 42·5–59μ broad, in the girdle-view (fig. xylogr., 2, C and D) nearly reniform, divided by the transverse furrow into two nearly equal halves. Transverse furrow disposed in a conspicuously spiral manner (*vide* fig. 2, A). Longitudinal furrow reaching a short distance into the apical half and downwards as far as the posterior margin of the body, in the posterior part much dilated, and in the middle of the left margin with a blunt papilla. Membrane ornamented with irregular, undulating, and sometimes ramified ridges.

Plates irregularly developed, in the lower part of the body much larger than in the upper part. *Epivalve* (fig. 2, C) with a transversely elongated rhomboidal plate; æquatorial plates seven, the dorsal one much larger than the others; first apical plate quadrangular, dilated outwards; second apical plate pentagonal, nearly as long as broad; third apical plate on the dorsal side of the first dorsal plate, forming part of an apical circle of plates; fourth apical plate irregularly quadrangular; first dorsal plate nearly triangular, situated at the apex of the upper part of the body; second dorsal plate trapezoid; apex wanting. *Hypovalve* (fig. 2, D) with two antapical plates, of which the right one is the larger; suture between these plates running from the middle of the ventral margin obliquely towards the left dorsal side.

Intercalary strips of the carapace very variable, in old cells sometimes very broad.

Hab.—Lochs Bairness and Shiel, Inverness; Loch Rosque, Ross; Loch Shubhaill, Lewis, Outer Hebrides.

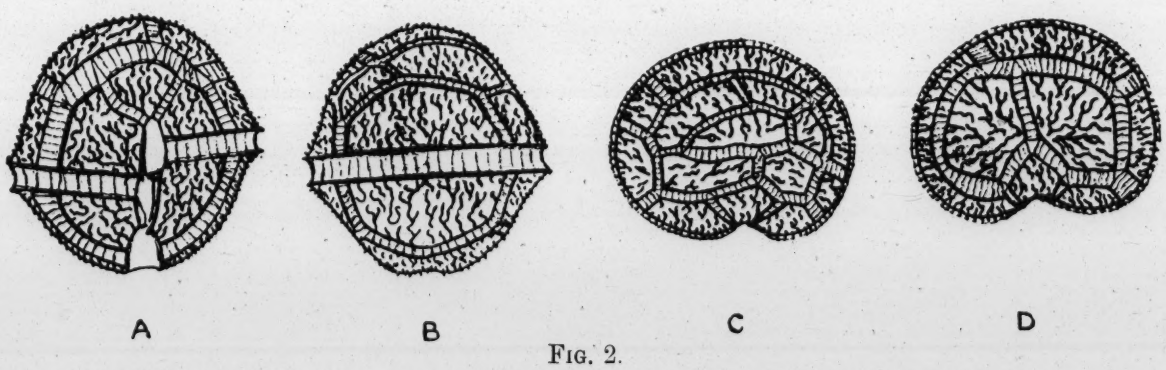


FIG. 2.

Peridinium Westii, Lemm. × 610. A, ventral view; B, dorsal view; C, Epivalve; D, Hypovalve. (After drawings by LEMMERMANN.)

P. Westii is distinguished from any other species of *Peridinium* by the disposition of the plates and the curious ridges of the membrane.

III. Systematic Account of the More Important Algæ of the Plankton.

This part of the paper deals specifically with the most interesting of the Algæ found in the plankton. Many of these belong to the family Desmidiaceæ, in which the Scottish plankton is remarkably rich. Three species, viz., *Staurastrum inelegans*, *St. subnudibrachiatum*, and *Desmidium occidentale*, and a number of varieties, are here described (and figured) for the first time, most of them having been found in Loch Fadaghoda, Lewis. The following are rare Desmids which have not previously been recorded from the British Islands:—*Gonatozygon aculeatum*, Hast.; *Hyalotheca Indica*, Turn.; *Micrasterias Wallichii*, Grun.; *Staurastrum sublævispinum*, W. & G. S. West; and *St. Tohopekaligense*, Wolle, var. *trifurcatum*, W. & G. S. West.

A new brown alga—*Phæococcus planctonicus*—occurred in several of the lakes examined, and the occurrence of *Cœlastrum Morus*, W. & G. S. West, is very interesting. An undescribed genus of the Protococcoideæ—*Actinobotrys*—occurred in quantity in some of the lakes, and a few specimens of *Pleodorina californica*, Shaw, were observed from Loch Fadaghoda. *Botryococcus protuberans* is also a well-marked new species.

Of the Diatoms, *Rhizosolenia eriensis*, H. L. Sm., var. *morsa*, and *Fragilaria Crotonensis* (A. M. Edw.), Kitton, var. *contorta*, are worthy of special mention.

Class PHÆOPHYCEÆ.

Order Syngeneticæ.

Family Phæocapsaceæ.

Genus *Phæococcus*, Borzi, 1892.

1. *P. planctonicus*, sp. n. (Pl. 6, figs. 15, 16).

P. cellulis parvis, globosis vel subglobosis, 16–64 in tegumento libere natante aggregatis; tegumentis gelatinosis, homogeneis, hyalinis et subellipsoideis, vix conspicuis; chromatophoris singulis vel binis, parietalibus, luteo-brunneis, cum marginibus irregulariter lobatis; membrana cellularum firma et tenue. Propagatio bipartitione cellularum in omnes directiones.

Diam. cell. 7·2–9·6 μ; diam. fam., 73–120 μ.

Hab.—Loch Mor Bharabhais and Loch Fadaghoda, Lewis; and Loch Laxadale, Harris, Outer Hebrides.

This alga is distinguished from the other species of the genus by its free-floating colonies of small size and its globose cells. Each cell contains either one large, lobed chromatophore, occupying the greater part of the interior of the cell-wall, or two smaller, lobed chromatophores.

The gelatinous investment is very inconspicuous, and towards its outer surface often exhibits a radiating fibrillar structure.

Zoogonidia were not observed.

The only other British species is *Phæococcus paludosus*, W. & G. S. West, a species with ellipsoidal cells which occurs in moorland ditches and pools.

Family DINOBRYACEÆ.

Genus *Dinobryon*, Ehrenb., 1833.

2. *D. elongatum*, Imhof., var. *undulatum*, Lemm., in *Berichte Deutsch. botan. Ges.*, 1900, p. 28; p. 516, t. 18, f. 21–22.

Hab.—Loch an Sgath, Lewis; Lochs Diracleet and Laxadale, Harris, Outer Hebrides.

3. *D. cylindricum*, Imhof., var. *pediforme*, Lemm., *l.c.*, p. 517, t. 19, f. 12–14.

Hab.—This variety occurred in great abundance in the plankton of most of the lochs examined.

Class CHLOROPHYCEÆ.

Order ŒDOGONIALES.

Family ŒDOGONIACEÆ.

Genus *Œdogonium*, Link., 1820.

Numerous sterile species of this genus occur abundantly in the plankton of almost all the Scottish lochs, most of them being relatively small species of the genus. The filaments are mostly short, and they most probably accumulate in the plankton by breaking away from the margins of the loch.

Order MICROSPORALES.

Family MICROSPORACEÆ.

Genus *Microspora*, Thur., 1850; em. Lagerh., 1888.

4. *Microspora amœna* (Kütz.), Lagerh. Quantities of this alga occurred in the plankton of most of the Scottish lochs. The filaments were mostly in a fragmentary condition, and the cell-walls were greatly thickened, some of the plants being very near *M. crassior* (Hansg.), Hazen.

Order CONJUGATÆ.

Family ZYGNEMACEÆ.

Genus *Mougeotia*, Ag., 1824.

Species of this genus are very abundant in the Scottish plankton, especially in the lakes at high altitudes. They are mostly species with long and exceedingly narrow cells, and they very rarely produce spores. Some of the thicker species exhibit a remarkable contortion and coiling of the filaments which we have never observed except in the

plankton. We previously mentioned this curious feature as characteristic of the plankton of Loch a Bhursta, Benbecula (vide *Journ. Linn. Soc.*, 1903, bot., xxxv. p. 524).

Family Desmidiaceæ.

Genus *Gonatozygon*, De Bary, 1856.

5. *G. aculeatum*, Hastings, in *Amer. Month. Micr. Journ.*, 1892, p. 29 ; Johns, in *Bull. Torr. Bot. Club*, 1895, vol. 22, No. 7, p. 291, t. 239, f. 9. [*G. aculeatum*, f. *minor*, W. & G. S. West in *Trans. Linn. Soc.*, bot. ser. 2, v., 1896, p. 230, t. 12, figs. 1, 2.]

Long. 266 μ ; lat. in med. sine acul. 10·5 μ, ad apic. 12·5 μ ; long. acul. 7·5–7·7 μ (Pl. 6, fig. 2).

Hab.—Loch Fadaghoda, Lewis, Outer Hebrides.

This rare species has not been reported previously as being found in Europe. It seems distributed over certain districts of the eastern United States, and Turner has figured two forms of what he termed "*G. pilosum*, Wolle," from Central India, which undoubtedly belong to *G. aculeatum* (*vide* Turn. in *Kongl. sv. Vet.-Akad. Handl.*, Bd. 25, No. 5, 1893, p. 25, t. 20, figs. 1, 2). The Scottish specimens are relatively narrower than the American ones, but all the species of this genus exhibit great variation, not merely in size, but in relative proportions.

6. *G. monotænium*, De Bary, var. *pilosellum*, Nordst. in Wittr. and Nordst., *Alg. Exsic.*, 1886, No. 750, fasc. 21, p. 48 ; W. & G. S. West, *Brit. Desm.*, 1904, vol. i. p. 31.

Long. 278 μ ; lat. in med. 7·8 μ, ad apic, 9 μ (Pl. 6, fig. 1).

Hab.—Loch an Sgath, Lewis, Outer Hebrides.

This variety is of very rare occurrence, and is characterised by the short papillate projections on the cell-wall. It has not previously been recorded for Scotland.

Genus *Genicularia*, De Bary, 1858.

7. *G. elegans*, W. & G. S. West, in *Journ. Linn. Soc.*, bot., xxxv., 1903, p. 536, t. 14, figs. 1, 2 ; *Brit. Desm.*, i., 1904, p. 36, t. 5, figs. 1, 2.

Hab.—Lochs Morar and Shiel, Inverness ; Lochs Fadaghoda and an Sgath, Lewis ; and Loch Laxadale, Harris, Outer Hebrides.

This species appears to be confined to the plankton of the west of Scotland. It has only previously been recorded from Loch nan Eun, N. Uist, Outer Hebrides.

Genus *Closterium*, Nitzsch, 1817.

8. *Cl. acerosum* (Schrank), Ehrenb., forma *minor*.

Long. 242 μ ; lat. 20 μ (Pl. 6, fig. 20).

Hab.—Loch Fadaghoda, Lewis, Outer Hebrides.

9. *Cl. turgidum*, Ehrenb., forma *glabra*, Gutw., in *Rospraw. Wydz. matem.-przyr. Akad. Umiej. Krakow*, xxxiii., 1896, p. 37, t. 5, f. 10.

Long. 436 μ; lat. 54·5 μ; lat. apic., 16 μ.

Hab.—Loch Shiel, Inverness.

The Scottish examples were proportionately shorter than those described and figured by GUTWINSKI from Poland, but were otherwise similar. The apices of the cells are a little stouter than in typical *Cl. turgidum*.

10. *Cl. Kützingii*, Bréb., var. *onychosporum*, var. n. (Pl. 6, fig. 22).

Var. cum angulis zygosporæ incrassatis et productis in ungues acutos quattuor, ungues prope partem dorsalem semicellulæ intra marginem.

Lat. cell. 15–18 μ; long. zygosp. sine ung. 43–46, cum. ung. 45–50 μ; lat. zygosp. sine ung. 32 μ, cum ung. 40–43 μ; long. ung. 5–6 μ.

Hab.—Loch Stranabhat, Lewis, Outer Hebrides.

This variety is well characterised by its peculiar zygospore.

11. *Cl. setaceum*, Ehrenb., var. *elongatum*, var. n. (Pl. 6, fig. 21).

Var. cellulis elongatis et angustis, diametro 60–65-plo longioribus.

Long. 740–780 μ; lat. 12–12·5 μ.

Hab.—Loch an Sgath, Lewis, Outer Hebrides.

This variety is chiefly remarkable for the great prolongation of the attenuated extremities. The cell-wall was colourless and the striolations scarcely discernible.

Genus *Micrasterias*, Ag., 1827.

12. *M. Wallichii*, Grun., "Süssw. Diat. u. Desm. Ins. Banka," *Rabenh. Beiträge z. Kenntniss u. Verbreit. der Alg.*, H. 2, Leipzig, 1865, p. 14, t. 2, f. 21.

Long. 181–190 μ; lat. 152–160 μ; lat. isthm. 30 μ.

Hab.—Loch Ruar, Sutherland (J. Murray)!

This Desmid is a close relative of *M. Mahabuleshwarensis*, Hobson, but differs in the subdivision of the superior lateral lobes. It is a rare plant, and was previously unknown from the British Islands.*

Genus *Cosmarium*, Corda, 1834.

13. *C. subcontractum*, W. & G. S. West.

Hab.—Loch na Cloiche Sgoilt, Inverness.

This species has only previously been observed from the Shetland Islands.

14. *C. moniliforme* (Turp.), Ralfs, forma *panduriformis*, Heimerl, "Desm. Alp," *Verhand. k. k. zool.-bot. Ges. Wien*, 1891, p. 598, t. v., f. 11.

Two forms of this were observed :—

a. Cells small; cell-wall distinctly punctate. Long. 21–22 μ; lat. 11·5–13 μ; lat. isthm. 7·5 μ (Pl. 7, fig. 6).

Hab.—Loch Bairness, Inverness.

* Mr MURRAY has also collected this species from the following lochs:—Lochs Nan Eun and Bhaic, Perthshire; Loch nan Cuinne, Sutherland; Lochs Burraland and Littlester, Shetlands. It is entirely due to Mr MURRAY's investigations that this species is known from the British Islands.!

b. Cells large, slightly angular at the apices; cell-wall smooth. Long. 37–52 μ; lat. 19–27 μ; lat. isthm. 12–14·8 μ (Pl. 7, figs. 4, 5).

Hab.—Loch Fadaghoda, Lewis, Outer Hebrides.

15. *C. abbreviatum*, Racib. in *Pamietnik. Wydz. matem.-przy. Akad. Umiej. Krakow*, x., 1895, p. 83, t. 10, f. 13.

Var. *planctonicum*, var. n. [*C. abbreviatum*, var. W. & G. S. West, in *Journ. Linn. Soc.*, bot., xxxv., 1903, p. 541, t. 15, f. 6].

Var. major, angulis superioribus rotundatioribus.

Long. 19–29 μ; lat. 22–30 μ; lat. isthm. 5·5–8 μ; crass. 10·5–13·5 μ.

Hab.—Loch Tay, Perthshire; Loch Morar, Inverness; Lochs Luichart and Rosque, Ross; Lochs Cuthaig, Fadaghoda, Roinebhall, an Sgath, and Loch near Cearnabhal, Lewis, Outer Hebrides.

This variety appears to retain its characters very constantly, and we have only observed it from the plankton. It is fairly common in the Scottish plankton, and we have previously recorded it from Loch Shin, Sutherland, and from Loch Mor Bharabhais, Lewis.

16. *C. capitulum*, Roy & Biss., in *Journ. Bot.*, xxiv., 1886, p. 195, t. 268, f. 9.

Var. *grœnlandicum*, Börgesen, Ferskv. alg. Östgrönl., *Meddelsen om Grönl.*, 18, Kjöbenhavn, 1894, p. 16, t. 1, f. 5.

Long. 19–21 μ; lat. 16·4–19·5 μ; lat. isthm. 7·7–8·6 μ; crass. 11·3 μ (Pl. 7, figs. 2, 3).

Hab.—Loch Morar, Inverness; Lochs Cuthaig and Roinebhall, Lewis, Outer Hebrides.

The specimens observed were all slightly smaller than those which have already been recorded as occurring in the plankton of Loch Doon, Ayrshire.

17. *C. Botrytis* (Bory.), Menegh.

Var. *depressum*, var. n. (Pl. 7, fig. 1).

Var. cellulis circiter tam longis quam latis, angulis basalibus semicellularum latioribus et rotundatioribus, angulis superioribus obtusissimis.

Long. 68–72 μ; lat. 65–68 μ; lat. isthm. 16–17 μ.

Hab.—Loch Ruar, Sutherland (J. Murray)!

This variety occurred in quantity in the plankton of the above-mentioned loch. It seems well characterised by its depressed semicells, which easily distinguish it from any other variety of *C. Botrytis*.

Genus *Xanthidium*, Ehrenb., 1834.

18. *X. antilopœum* (Bréb.), Kütz.

Var. *Hebridarum*, var. n. (Pl. 7, fig. 21).

Var. cum spinis tribus supra marginem superiorem semicellulæ uniuscujusque utrobique, spina singula ad angulum superiorem et spinis duobus prope angulum lateralem

utrobique; semicellulæ a vertice visæ ellipticæ, tuberculo mediano obtuse mamillato utrobique, spinis plerumque in serie unica.

Long. sine spin. 46–50 μ, cum spin. 61–71 μ; lat. sine spin. 42–49 μ, cum spin. 69–82 μ; lat. isthm. 12–14 μ; crass. 31 μ; long., spin. 13·5–19 μ.

Hab.—Common in the plankton of many lochs in Inverness, Ross, Sutherland, and the Outer Hebrides.

Instead of the normal pair of spines at each upper angle of a semicell, there is in this variety only a single spine, and the pair of spines which should be attached side by side at the lateral angles are here separated some distance apart, and placed more or less vertically over each other. The semicells are therefore not so angular as in typical *X. antilopæum*. The length of the spines varies in different specimens of the var. *Hebridarum*, but otherwise the distinguishing features are retained very constantly.

Genus *Arthrodesmus*, Ehrenb., 1838.

19. *A. Incus* (Bréb.), Hass.

A number of very large forms of this species were observed from Loch Cuthaig, Lewis. The spines were in all cases widely divergent, and in some specimens the cell-wall possessed minute scrobiculations. Long. sine spin. 25–32·5 μ; lat. sine spin. 21–26 μ; long. spin. 21–32 μ; lat. isthm. 7·5–8·5 μ.

Var. *Ralfsii*, W. & G. S. West. [*A. Ralfsii*, West.]

Forma spinis brevissimis; long. 15·4 μ; lat. 15 μ; lat. isthm. 7·5 μ; long. spin. 3–3·8 μ (Pl. 7, fig. 10).

Hab.—Lochs Fadaghoda, Roinebhall, and Stranabhat, Lewis, Outer Hebrides.

Var. *longispinum*, var. n. [*A. Incus*, Wolle, *Desm. U.S.*, 1884, t. 24, f. 3.]

Var. spinis perlongis, multe sursum divergentibus.

Long. sine spin. 14 μ; lat. sine spin. 11·5 μ; lat. isthm. 5·7 μ; long. spin. 21–25 μ (Pl. 7, fig. 22).

Hab.—Small loch near Cearnabhall, Lewis, Outer Hebrides.

Genus *Staurastrum*, Meyen, 1829.

20. *St. inelegans*, sp. n. (Pl. 7, figs. 11, 12).

St. mediocre, paullo latius quam longum, profunde constrictum, sinu latissimo et obtuso, isthmo breviter cylindrico; semicellulæ late subtriangulares, marginibus inferioribus subrectis vel leviter biundulatis, apice in medio convexo, angulis productis acutis vel apiculatis (rare subobtusis); a vertice visæ triangulares, lateribus convexis in medio, angulis submamillatis et leviter apiculatis; membrana glabra.

Long. 53·5–60 μ; lat. 51–62 μ; lat. isthm. 9–10 μ.

Hab.—Loch Fadaghoda, Lewis, Outer Hebrides.

This remarkable *Staurastrum* is somewhat variable in its characters. The angles of the semicells are generally produced, and either acute or apiculate, but in some

specimens they are relatively obtuse. They may be horizontally disposed, but are more often slightly turned upwards. The isthmus is very shortly cylindrical.

The species can be compared with *St. Clepsydra*, Nordst. (and its var. *acuminatum*), but the differences are very considerable, especially in the vertical view.

21. *St. brevispinum*, Bréb.

Var. *obversum*, var. n. (Pl. 7, fig. 15).

Var. cellulis latioribus quam longis; semicellulis obverse semiellipticis cum mucrone prominentiori ad angulum superiorem unumquemque, sinu angustiori apicem ampliatum versus, dorso semicellularum leviter convexis, ventro multe convexis; a vertice visis cum mucronibus prominentioribus.

Long. 38–42 μ; lat. 45–49·5 μ; lat. isthm. 10·5–11 μ.

Hab.—Loch Fadaghoda, Lewis, Outer Hebrides.

Var. *altum*, var. n. (Pl. 7, fig. 16).

Var cellulis longioribus quam latis; semicellulis multe altioribus, dorso valde convexo; a vertica visis lateribus minus concavis, angulis crassioribus, mucronibus minutis.

Long. 57–66 μ; lat. 43–50 μ; lat. isthm. 12·5–15·3 μ.

Hab.—Lochs nan Cuinne and Ruar, Sutherland (J. Murray)! Lochs Fadaghoda, an Sgath, and an Tomain, Lewis; and Loch Diracleet, Harris, Outer Hebrides.

These two varieties differ from one another considerably, but are undoubtedly forms of *St. brevispinum*.

22. *St. sibiricum*, Borge, in *Bihang. till K. Sv. Vet.-Akad. Handl.*, Bd. 17, No. 2, 1891, p. 9, t. 1, f. 4.

Long. 17–18 μ; lat. 17–21 μ; lat. isthm. 7·5–8·5 μ; crass. 8–9 μ (Pl. 7, fig. 20).

Hab.—Loch Fadaghoda, Lewis, Outer Hebrides.

We have recorded a triangular form of this species from the south of England (*vide Journ. Roy. Micr. Soc.*, 1896, p. 157, t. 4, f. 39), but the typical form has not been observed before from the British Islands.

23. *St. sublævispinum*, W. & G. S. West, in *Journ. Linn. Soc.*, bot., xxxiii., 1898, p. 314, t. 18, f. 20–22.

Long. 33 μ; lat. 46 μ; lat. isthm. 8·5 μ.

Hab.—Loch Fadaghoda, Lewis, Outer Hebrides.

The specimens were slightly larger than the American ones, but otherwise precisely similar. The "body" of the *Staurastrum* is very small, and the processes are cylindrical, except for their obtusely attenuated apices. The specimen figured (Pl. 7, fig. 23) shows the first stage of cell-division, the elongation of the isthmus.

24. *St. subnudibrachiatum*, sp. n. (Pl. 7, figs. 18, 19).

St. mediocre, circiter $1\frac{1}{2}$-plo latius quam longum (cum processibus), leviter constrictum, sinu latissime aperto; semicellulæ suborbiculares, marginibus lateralibus in processus glabros longos rectos leviter subdivergentes productis, apicibus processuum

plerumque furcatis (dente unoquoque obtuso) vel interdum integris et obtusis; a vertice visæ 4-vel 5-radiatæ, processibus tam longis quam latitudo corporis; membrana glabra; pyrenoide singulo magno in chromatophora unaquaque; processibus alterius semicellulæ cum iis alterius alternantibus.

Long. sine proc. 31–37 μ, cum proc. 40–44 μ; lat. sine proc. circiter 20–22 μ, cum proc. 53–61 μ; lat. isthm. 15–15·5 μ.

Hab.—Loch Fadaghoda, Lewis, Outer Hebrides.

In the nature of its processes this species is precisely similar to certain forms of *St. brachiatum*, Ralfs, but it differs from that species in the shape of its body, its slight constriction, and in the radiate character of the vertical view. The processes are variable in character, even on the same specimen. They are usually bifurcate at the extremity, each lobe being rounded, but sometimes they are obtuse and entire.

It is unquestionably nearest to *St. nudibrachiatum*, Borge (*Archiv för Botanik af K. Sv. Vet.-Akad.*, 1903, p. 109, t. 4, f. 20), from which it is distinguished by the fewer number and different shape of the processes, which possess *rounded* (not acute) apical teeth, and by the smooth cell-wall. The processes of the Scottish species are gradually attenuated from the base towards the apex, whereas those of *St. nudibrachiatum* are distinctly constricted at the base and widest in the middle.

St. subnudibrachiatum should also be compared with *St. Clevei* (Wittr.), Roy & Biss., from which it differs in the processes of each semicell being all in one plane, in their rounded apical teeth, in the broader isthmus, and other features.

Most of the specimens were inclosed in a wide mucous investment.

25. *St. Clevei* (Wittr.), Roy & Biss., in *Ann. Scot. Nat. Hist.*, 1893, p. 179. [*St. læve*, Ralfs, var. *Clevei*, Wittr., in *Nova Acta Soc. Upsala*, ser. 3, vii. p. 18, t. 1, f. 9. *St. Kitchellii*, Wolle, 1882, *Desm. U.S.*, 1884, p. 150, t. 40, figs. 35, 36; W. & G. S. West, in *Journ. Linn. Soc.*, bot., xxxiii., 1898, p. 319, t. 18, figs. 10, 11.]

Long. sine proc. 32 μ; cum proc. 61 μ; lat. cum proc. 52–57 μ; lat. isthm. 11·5 μ.

Hab.—Loch Fadaghoda, Lewis, Outer Hebrides.

27. *St. Tohopekaligense*, Wolle, 1885; *Freshw. Alg. U.S.*, 1887, p. 45, t. 59, figs. 4, 5. [*St. nonanum*, Turn., in *Kongl. Sv. Vet.-Akad. Handl.*, xxv., 1893, No. 5, p. 119, t. 15, fig. 14.]

Var. *trifurcatum*, W. & G. S. West, in *Trans. Linn. Soc.*, bot. ser. 2, v., 1895, p. 80, t. 9, f. 8; vi., 1902, p. 181, t. 21, f. 27.

Long. sine proc. 38 μ, cum proc. 61 μ; lat. sine proc. 28, cum proc. 57 μ; lat. isthm. 14 μ (Pl. 7, fig. 7).

Hab.—Loch Fadaghoda, Lewis, Outer Hebrides.

This Desmid has not previously been recorded for the British Islands. The Scottish specimens exhibited variation in the apices of the processes. Those of the lower whorl were invariably trifurcate, but some of those of the upper whorl were often only bifurcate.

28. *St. pseudopelagicum*, W. & G. S. West, in *Journ. Linn. Soc.*, bot., xxxv., 1903, p. 547, t. 18, f. 1–3.

This species occurs in quantity in the plankton of many of the Scottish lochs. A specimen was observed from Loch Shubhaill, Lewis, in which one semicell was quite normal, whereas the processes of the other were each terminated by a single stout spine.

29. *St. paradoxum*, Meyen, 1829; Ralfs, *Brit. Desm.*, 1848, p. 138, t. 23, f. 8.

Var. *longipes*, Nordst., in *Acta Univ. Lund.*, 1873, ix., p. 35, t. 1, f. 17.

This is one of the commonest Desmids of the British plankton. We give a figure of a typical specimen from Lewis (Pl. 7, fig. 13); long. sine proc. 26 μ, cum proc. 77 μ; lat. sine proc. circ. 15 μ, cum proc. 84–87 μ; lat. isthm. 8 μ.

In the plankton of Loch Langabhat, Lewis, all the forms of this variety were very large, and the processes were relatively longer and more robust (Pl. 7, fig. 14); long. sine proc. 29 μ, cum proc. 84 μ; lat. sine proc. circ. 17 μ, cum proc. 134–139 μ; lat. isthm. 9·5 μ.

30. *St. gracile*, Ralfs, var. *cyathiforme*, W. & G. S. West, in *Trans. Linn. Soc.*, bot. ser. 2, v., 1895, p. 77, t. 9, f. 2.

Forma lateribus ad basin semicellularum integris (non undulatis).

Long. 58 μ; lat. cum proc. 80–83 μ; lat. isthm. 8 μ.

Hab.—Lochs Fadaghoda and an Sgath, Lewis, Outer Hebrides.

31. *St. subgracillimum*, W. & G. S. West, in *Trans. Linn. Soc.*, bot. ser. 2, v., 1896, p. 263, t. 17, figs. 3, 4.

Forma processibus paullo crassioribus, et denticulis minutis 3 ad apicem processuum.

Long. (sine proc.) 15·5 μ; lat. sine proc. 13 μ, cum proc. 55 μ; lat. isthm. 5·5 μ.

Hab.—Small loch near Cearnabhal, Lewis, Outer Hebrides.

The specimens seen were very like those observed from near Glenties, Ireland; *vide* W. & G. S. West, in *Trans. Roy. Irish Acad.*, 1902, xxxii. p. 56, t. 1, figs. 21, 22. They differ in the smaller teeth at the extremities of the processes.

32. *St. anatinum*, Cooke & Wills.

This is one of the most characteristic Desmids of both the Scottish and Irish plankton. It is a *Staurastrum* which is more or less covered with emarginate warts, and the var. *grande*, W. & G. S. West, is one of the handsomest Desmids of the genus.

Var. *longibrachiatum*, var. n. (Pl. 7, figs. 8, 9).

Var. corpore semicellularum infra apicem subglabra, processibus multe longioribus et gradatim attenuatis apicem versus.

Long. sine proc. 37 μ; lat. sine proc. circ. 29 μ, cum proc. 110–131 μ; lat. isthm. 11·5 μ.

Hab.—Loch Langabhat, Lewis, Outer Hebrides.

33. *St. Sebaldi*, Reinsch, in *Abhandl. d. Naturhist. Gesellsch. zu Nürnberg*, iii., Heft 2, 1867, p. 175, t. 11, f. 1.

Var. *productum*, var. n. (Pl. 7, fig. 24).

Var. apicibus semicellularum levissime convexis et glabris, cum serie verrucarum emarginatarum circ. 6 intra marginem, angulis in processus verrucosos longos productis; a vertice visæ triangulares, lateribus subrectis et glabris, cum serie verrucarum emarginatarum circ. 6 intra marginem unumquemque, angulis in processus longos verrucosos productis.

Long. 83 μ; lat. cum proc. 108–115 μ; lat. isthm. 20 μ.

Hab.—Loch Fadaghoda, Lewis, Outer Hebrides.

34. *St. forficulatum*, Lund.

Long. 55–57 μ; lat. 54–69 μ; lat. isthm. 9·6–13·5 μ (Pl. 7, fig. 17).

Hab.—Loch Fadaghoda, Lewis, Outer Hebrides.

The specimens observed of this rare Desmid were exactly typical, but all were tetragonal in vertical view. The species has been recorded from Perthshire, but as no British specimen has ever been figured we give a figure of one (Pl. 7, fig. 17).

Genus *Sphærozosma*, Corda, 1835.

35. *S. Aubertianum*, West, in *Journ. Bot.*, xxvii., 1889, p. 206, t. 291, f. 17; in *Journ. Linn. Soc.*, bot., xxix., 1892, p. 115, t. 19, f. 1.

The specimens showed considerable variation, more particularly with regard to the relative width of the cells and the form of the semicells. The semicells were mostly narrowly oblong or oblong-elliptical in form, and each lateral margin clearly showed two granules. In some specimens granules were observed across the front of the semicell, the forms thus passing into the var. *Archerii* (Gutw.), W. & G. S. West.

Long. 17·5–19 μ; lat. 25–27·5 μ; lat. isthm. 5·5–6 μ; crass. 11·5 μ (Pl. 6, fig. 7).

Genus *Hyalotheca*, Ehrenb., 1841.

36. *H. Indica*, Turn., in *Kongl. Sv. Vet.-Akad. Handl.*, xxv., 1893, No. 5, p. 152, t. 22, fig. 17 (*not* t. 19, fig. 18).

Forma, long. 13–15·5 μ; lat. 11–12 μ (Pl. 6, fig. 6).

Hab.—Loch Fadaghoda, Lewis, Outer Hebrides.

Genus *Desmidium*, Ag., 1824.

37. *D. coarctatum*, Nordst., in *Botaniska Notiser*, 1887, p. 155; in *Kongl. Sv. Vet.-Akad. Handl.*, xxii., 1887, No. 8, p. 25, t. 2, f. 3.

Var. *cambricum*, West, in *Journ. Roy. Micr. Soc.*, 1890, p. 283, t. 5, f. 2.

Long. 35–40 μ; lat. 40–42·5 μ; lat. apic. 17·5–20 μ; crass. 32–34 μ (Pl. 6, fig. 5).

Hab.—Loch Fadaghoda, Lewis, Outer Hebrides.

The cells were considerably longer than in the specimens observed from Wales, and the plant has much of the habit of *Desmidium graciliceps*.

38. *D. occidentale*, sp. n. (Pl. 6, figs. 3, 4).

D. mediocre, filis tortis, sine vagina mucosa; cellulæ $1\frac{1}{4}$-plo latiores quam longæ,

leviter constrictæ, sinu subangusto-lineari; semicellulis oblongo-semiellipticis, lateribus levissime biundulatis, angulis rotundatis, apicibus convexo-truncatis cum processibus connexis brevissimis; inter cellulas lacunis nullis; a vertice visæ triangulatæ, lateribus subrectis vel levissime convexis, angulis leviter subproductis et rotundatis.

Long. 26–27·5 μ; lat. 32·5–34 μ; lat. isthm. 25–27 μ; lat. apic. 21–22 μ.

Hab.—Loch Fadaghoda, Lewis, Outer Hebrides.

This *Desmidium* occurred abundantly along with *D. coarctatum*, var. *cambricum*, in the plankton of the above-mentioned lake. It is distinguished from *D. Swartzii*, Ag., by the proportionately greater length of the cells, and by the shortened and more rounded lateral angles of the semicells. The vertical view is likewise more robust, and the sides are slightly convex (not concave as in *D. Swartzii*).

Although *D. occidentale* occurred in quantity in Loch Fadaghoda, we have never seen it from any other locality. In none of the specimens is there any trace of a space between the apices of contiguous cells.

It may also be compared with *D. Bengalicum*, Turn.

Genus *Gymnozyga*, Ehrenb., 1840.

39. *G. moniliformis*, Ehrenb., var. *gracilescens*, Nordst., in *Botaniska Notiser*, 1880, p. 119; in Wittr. & Nordst., *Alg. Exsic.*, No. 367.

Hab.—Loch Fadaghoda, Lewis, Outer Hebrides.

This variety has not previously been recorded for the British Islands.

Order Protococcoideæ.

Family Volvocaceæ.

Genus *Pleodorina*, Shaw, 1894.

40. *P. californica*, Shaw, in *Botan. Gazette*, xix., 1894, pp. 279–283, t. 27; Kofoid, in *Bull. Illinois State Lab.*, v., 1898, p. 291.

Diam. colon. 90–113 μ; diam. veg. cell. 6·5–7·5 μ; diam. gonid. cell. 13·4–16·5 μ (Pl. 6, fig. 14).

Hab.—Loch Fadaghoda, Lewis, Outer Hebrides.

Among a quantity of *Eudorina elegans* which occurred in Loch Fadaghoda, several colonies were observed which we think correctly placed as *Pleodorina californica*, Shaw.

They differed from *Eudorina* in the outward form of the colony, which was ellipsoid with slight posterior lobes, whereas all the colonies of *Eudorina* were globular. There was also a well-marked distinction into vegetative aud gonidial cells, the latter being more than twice the diameter of the former.

The specimens observed did not strictly agree with the characters of either of the two known species of *Pleodorina*. The number of cells in the colony was 32 in every

case, which is the number usually present in *P. illinoisensis*, Kofoid, but the relatively large size of the gonidial cells, and the fact that the vegetative cells constitute about one-half of the colony, are characters more in agreement with *P. californica*, Shaw. The gonidial cells contained a large chloroplast, in which was imbedded a single pyrenoid, often of considerable size.

Whether these colonies should or should not be regarded as belonging to a genus closely allied to *Eudorina*, but distinguished by a differentiation of reproductive from vegetative cells, or merely as "*Pleodorina*-stages" of *Eudorina elegans*, is a question for further investigation.

Family Protococcaceæ (or Autosporaceæ).

Genus *Cœlastrum*, Näg., 1849.

41. *C. Morus*, W. & G. S. West, in *Journ. Bot.*, Sept. 1896, p. 5 (sep.), t. 361, f. 4.
Diam. cell. 17–19 μ; diam. colon. 50–53 μ, (Pl. 6, fig. 12).
Hab.—Loch Fadaghoda, Lewis, Outer Hebrides.

Genus *Tetraëdron*, Kütz., 1845.

42. *T. limneticum*, Borge, "Schwedisches Süssw. Plankton," *Botaniska Notiser*, 1900, p. 5, t. 1, f. 2.
Diam. cell. 53–62 μ (Pl. 6, figs. 10, 11).
Hab.—Loch Ruar, Sutherland (J. Murray)!
The specimens were slightly smaller than Borge's Swedish ones, and the bifurcations of the extremities more marked.

Genus *Botryococcus*, Kütz., 1849.

43. *B. protuberans*, sp. n. (Pl. 4, No. 4, pl. 6, figs. 8, 9).
Cellulæ in coloniis irregularibus, familiarum associatarum compositis; familiæ e 4, 8, 16, plusve cellularum formatæ; cellulæ anguste cuneatæ vel ovatæ, cum mucilagine adhærentes, cellulæ periphericæ subdistantes et exertæ ex massa mucilaginis, solum ad partem angustiorem adhærentes, parte latiori libera extrorsum directa.
Diam. colon. circ. 100–220 μ; long. cell, 16·5–20 μ; lat. max. cell. 9·5–11·5 μ.
Hab.—Loch Fadaghoda, Lewis, Outer Hebrides.
This species is readily distinguished by the few cells which form the smaller aggregations, and by the prominent way in which they project from the mucus in which they are imbedded. The colonies reach a large size and frequently become dark in colour. Living specimens were not seen, and it was quite impossible to arrive at any definite conclusions as to the nature of the chloroplast from the specimens preserved in formalin.
The cells are ovate or cuneate in shape, and are imbedded by their narrower ends in a mass of mucus. Each of these small aggregations or families consists of 4, 8, or 16

cells (rarely of more), and the families are joined into large colonies by irregular mucous strands. Two of the families of cells with the mucous strand for attachment to the rest of the colony are shown on Pl. 6, figs 8 and 9. A characteristic feature of the colony, and one by which this species can always be recognised, is the projection of the cuneate cells all round the margin. This is well shown in the photograph of a small colony on Pl. 4, No. 4.

Actinobotrys, gen. n.

Cellulæ in coloniis mucosis parvis associatæ, subcylindricæ, oblongæ, oblongo-ellipticæ vel subglobosæ, radiatim vel subradiatim (demum subirregulariter) dispositæ; seriebus radiosis cellularum irregulariter et spurie subramosis; chromatophoris parietalibus singulis, cum marginibus irregulariter lobatis, pyrenoidibus nullis.

44. *A. confertus*, sp. unica (Pl. 6, figs. 17–19).

Cellulæ confertim radiantes, plerumque oblongo-ellipticæ, cum axibus longioribus radiatim dispositæ; coloniæ e cellulis 20–110 (circiter) formatæ.

Long. cell. 6–8·7 μ; lat. cell. 4·5–5·8 μ; diam. colon. 43–65 μ.

Hab.—Lochs Bairness, na Criche, Gorma, and Morar, Inverness; Lochs Fadaghoda and an Sgath, Lewis; and Lochs Diracleet and Laxadale, Harris, Outer Hebrides.

This alga occurred in quantity in some of the lakes examined, and was also observed from some of the bogs of Sutherland.

The colonies are somewhat small, and they are enveloped in a more or less globular mass of mucilage sufficient to inclose the entire colony, but which is only rendered conspicuous by staining. The cells are small, oblong-cylindrical or oblong-elliptic in shape, and are arranged in series (of from 4 to 7 cells each) radiating from the centre of the colony. These series are often irregular, and commonly exhibit a trace of branching. The long axes of the cells are disposed radially, and there is generally a diminution in the size of the cells from within outwards, the outermost cell being not merely the smallest, but also more globular in shape.

The chloroplast is solitary and parietal, generally with lobed margins, and is quite destitute of a pyrenoid.

The genus stands near to *Dictyocystis*, Lagerh., but differs in the more crowded cells, which are rather smaller towards the exterior of the colony, and in the parietal chloroplasts. In *Dictyocystis* the chloroplasts are axile and substellate, with a central pyrenoid, but no pyrenoid is present in the parietal chloroplast of *Actinobotrys*.

Class BACILLARIEÆ.

Order CENTRICÆ.

Family RHIZOSOLENIACEÆ.

Genus *Rhizosolenia*, Ehrenb., 1858; em. Peragallo, 1892.

45. *Rhizosolenia eriensis*, H. L. Smith, var. *morsa*, var. n. (Pl. 6, fig. 23).

Cellulæ a aspectu cincto visæ apicibus inæqualiter emarginatis ad angulos duobus oppositos.

Long. sine setis 77–230 μ; lat. 7–21 μ; long. set. 21–39 μ.

Hab.—Loch Shiel, Inverness.

This Diatom was very abundant in the plankton of Loch Shiel, and varied greatly in size. It is well characterised by the excavation at two of the opposite angles of the girdle-view. Mr E. LEMMERMANN, of Bremen, very kindly verified this determination.

Order PENNATÆ.

Family FRAGILARIACEÆ.

Genus *Fragilaria*, Lyngb., 1819.

46. *F. Crotonensis* (A. M. Edw.), Kitton, var. *contorta*, var. n.

Cellulæ in fasciis brevissimis junctæ, 20–40 in fascia unaquaque, fasciis multe sed gracillime subspiraliter contortis.

Long. cell. 98–115 μ.

Hab.—Loch Ruar, Sutherland (J. Murray)!

We have never before seen filaments twisted in the manner exhibited by this variety. Photographs of it will be found on Pl. 2, No. 5, and also on a smaller scale of magnification on Pl. 1, Nos. 2 and 5.

IV. GENERAL REMARKS ON SCOTTISH PHYTOPLANKTON.

These remarks apply to the phytoplankton only, as, with the exception of the Rhizopods and some of the Rotifers, the animal life of the plankton has not been noted.

We have now examined the plankton of thirty-six of the lochs of Scotland, mostly those of the north-west of the country and of the Outer Hebrides. In our first paper * we gave a detailed account of the plankton of fourteen of these lakes, and some general conclusions with regard to the main features of the plankton. In many of the lochs examined the phytoplankton is relatively abundant, and some of the most salient features of the plankton are to be noted among the Algæ.

A large part of the phytoplankton—we are inclined to think considerably more

* W. West and G. S. West, "Scottish Freshwater Plankton, No. 1," *Journ. Linn. Soc.*, bot., xxxv., 1903, pp. 519–556, pls. 14–18.

than half—consists of Chlorophyceæ, and most of these green Algæ belong to the Conjugatæ. Slender species of *Spirogyra*, *Zygnema*, and *Mougeotia* occur in abundance in the plankton of many of the lochs, and the filaments of *Mougeotia* are particularly conspicuous. These species appear to rarely conjugate in the plankton, and they are most likely replenished from the shores of the lochs and by the streams draining more elevated boggy areas. Some of the filaments of *Mougeotia* become greatly coiled and contorted, presenting an appearance which we have never observed in specimens from any other situations. *Zygnema ericetorum* sometimes occurs in sufficient quantity to attract attention.

As we have previously pointed out, the Scottish plankton is unique in the abundance of its Desmids. The richness of its Desmid-flora is at once the most conspicuous feature of the plankton.

Some of the Desmids which are characteristic of this western area of Scotland are more abundant in the plankton of the lakes than in any other situations. To these must be added a number of Desmids which have hitherto only been found in the plankton. The following species can be regarded, so far as Scotland is concerned, as *true plankton Desmids*, being more abundant in the pelagic region of the lochs than in any other situations :—

Gonatozygon monotœnium.
Genicularia elegans.
Closterium Kützingii.
Euastrum verrucosum, vars. *reductum* and *planctonicum.*
Micrasterias Wallichii.
Cosmarium contractum.
,, *depressum*, var. *achondrum.*
,, *abbreviatum*, var. *planctonicum.*
Xanthidium antilopœum and a number of varieties.
,, *subhastiferum.*
,, *controversum*, var. *planctonicum.*
Arthrodesmus Incus, and several varieties.
,, *quiriferus.*
,, *crassus.*
,, *triangularis*, var. *subtriangularis.*
Staurastrum dejectum, var. *inflatum.*
,, *curvatum.*
,, *jaculiferum* and varieties.
,, *megacanthum* and var. *Scoticum.*
,, *cuspidatum*, var. *maximum.*
,, *inelegans.*
,, *longispinum* and var. *bidentatum.*
,, *Brasiliense*, var. *Lundellii.*
Staurastrum grande.
,, *brevispinum* and several varieties.
,, *lunatum*, var. *planctonicum.*
,, *aversum.*
,, *subnudibrachiatum.*
,, *teliferum.*
,, *erasum.*
,, *pseudopelagicum.*
,, *paradoxum* and varieties (especially var. *longipes*).
,, *gracile* and varieties.
,, *anatinum* and a number of varieties (especially var. *grande*).
,, *Ophiura.*
,, *furcigerum.*
,, *sexangulare.*
,, *Arctiscon.*
Spondylosium pulchrum, var. *planum.*
Desmidium graciliceps.
,, *coarctatum*, var. *cambricum.*
,, *occidentale.*
Hyalotheca mucosa.
,, *neglecta.*

Some of these Desmids develop very large mucous investments, and others are remarkable for the length and strength of their spines.

We call attention here to the abundance of *Mesotænium macrococcum* in certain

of the lochs. This Desmid occurred in small subspherical or ellipsoidal colonies containing upwards of 500 individuals imbedded in a copious mucus. These colonies were freely floating in the water, although the natural habitat of the plant is on wet rocks. It would be of interest to determine whether this species is or is not a constant feature of the plankton of these lochs. It would thus furnish a case of a species normally confined to wet rocks having become adapted to a pelagic existence. It normally occurs in irregular mucous colonies, and the only modification has been the reduction in size of the colonies, and their assumption of a more regular shape.

The remarkable abundance of Desmids is one of the most salient features of the Scottish plankton, and requires further comment. At the same time, it must be remembered that the plankton of the lakes in the extreme west of Ireland and North Wales, and that of some of the lakes in the English Lake District,* is almost equally rich in Desmids; and therefore this feature is not an isolated phenomenon confined to the lakes of the Scottish Highlands. *We attribute the abundance and variety of these Conjugates entirely to the geological character of the districts in which the lakes are situated.* Most of them are in rocky hollows in the old formations, either Precambrian or Older Palæozoic, and this fact always increases very considerably the richness of the Desmid-flora of any area, whether it be the flora of the bogs or of the lakes. In conjunction with this it must be remembered that these areas are suitable for the existence of peat-bogs and peaty pools, and that the water which drains into the lakes is mostly distinctly peaty and slightly acid. This is a condition eminently favourable for the multiplication of Desmids; but although it accounts for the abundance of certain of these plants, it does not account for their variety. Many of the Carboniferous areas are equally suitable for the formation of peat, and are likewise studded with peat-bogs, peaty pools and tarns; but neither the bogs nor the plankton of the tarns and lakes, although containing numerous Desmids, contain the great diversity of species met with on the older formations. In districts in which the geological formations are of the Secondary or Tertiary Age there is a still further reduction in the quality of the Desmid-flora. There are slight exceptions to this on the Lower Greensand of Surrey and on the Middle Eocene of Hampshire, but as a general rule the Desmid-flora of Secondary and Tertiary areas is relatively very poor, except where there is a considerable amount of intrusive igneous material.

Many of the most handsome of the Scottish plankton-Desmids, such as *Staurastrum Arctiscon*, *St. longispinum*, *St. Ophiura*, *St. sexangulare*, *St. aversum*, *St. Brasiliense*, *Micrasterias furcata* and others, are exclusively confined to Older Palæozoic or Precambrian areas. As an instance, *Micrasterias furcata* is confined in the British Islands to North Wales, Sutherland, the Outer Hebrides, Donegal, and the extreme west of Galway.

Probably most of the Desmids of the plankton were originally derived from the

* We hope in a short time to give a full account of the phytoplankton of the loughs of the west and south-west of Ireland, and also of the lakes of the English Lake District.

peaty moors and bogs situated at a slightly higher altitude, and gained the pelagic region of the lakes by the streams which drain the peat-bogs above. It must not be supposed, however, that all the Desmids of the plankton are continually being replenished from the peaty areas of higher altitudes. This may be partially true of some species, but evidence does not support it in the case of others. Most of the species found in the Scottish plankton have also been found in the rocky pools of the peat areas, but some of them are so very scarce and so rarely met with in such situations, and at the same time are so very abundant in the plankton, that one can only conclude that the percentage of recruits arriving in the pelagic region of the lochs is infinitesimal. A good instance of this is furnished by *Staurastrum Ophiura*. Only once have we found this Desmid in the peat-bogs of the west of Scotland, and we have examined many of these situations much more minutely than the pelagic region of the lakes. Also, the late Dr J. Roy, although investigating Scottish Desmids for more than twenty-five years, never observed this species; and yet *Staurastrum Ophiura* is one of the most abundant and characteristic Desmids of the Scottish Plankton. One can only infer from these facts that, so far as this species is concerned, recruiting for the plankton from surrounding bogs must go on very slowly, and be more or less insignificant.

Some of the plankton-Desmids have become slightly and others considerably modified by the assumption of this pelagic life, and the modifications are mostly in the development of long spines and processes which give them both a greater floating capacity and a greater protection against the depredations of Desmid-eating animals. We have previously stated * that this increase in the length of spines and processes is chiefly exhibited by species of *Staurastrum* and *Arthrodesmus*. Other species of the genera *Cosmarium*, *Micrasterias*, and *Euastrum* do not appear to have acquired any special characters either for further protection or for augmenting their floating capacity.

Dr C. Wesenberg-Lund † has recently remarked upon the abundance of Desmids in the Scottish plankton, and suggests that many of these Conjugates are adopting a pelagic life as they arrive in the lakes from the peat-moors. We think, however, that *most of the Desmids of the Scottish plankton have existed under these pelagic conditions for a very long time*, as there is every indication of this in the modifications some of them have undergone, and in the species and varieties which at present are only known to occur in the plankton.

A point which should not be overlooked is the almost complete absence from the plankton of those Desmids which are most conspicuous and abundant in the surrounding peat-bogs, and which are therefore the most likely species to be carried into the pelagic region of the lakes by the streams and rivers. It would appear from this that

* W. West and G. S. West, *l.c.*, p. 554.

† C. Wesenberg-Lund, "A Comparative Study of the Lakes of Scotland and Denmark," *Proc. Roy. Soc. Edin.*, xxv., 1905.

the greater quantity of the Desmids brought into these new conditions of pelagic life find them most unsuitable for their further existence, and rapidly die. Thus a specific selection has been gradually going on, certain species only having been able to maintain their existence in the plankton. In some cases these species have succeeded not only in maintaining their existence as plankton-organisms, but in prodigiously increasing in numbers, and in further developing those characters which have proved of most use to them under their new conditions of life.

The Family Œdogoniaceæ is represented in the plankton of almost all the lakes by more or less fragmentary filaments of various species of *Œdogonium*. These are mostly of very small size, and they never appear to produce oospores. They are most probably replenished from the margins of the lochs.

In the Protococcoideæ the Scottish plankton is not very rich, and in this fact there is a marked contrast between it and the plankton of some of the areas of continental Europe. There is a striking absence of *Richteriella*, *Golenkinia*, *Lagerheimia*, and many other genera which the investigations of CHODAT, LEMMERMANN, SCHRÖDER, SCHMIDLE, and others have shown to occur abundantly in the lakes of Germany and Switzerland. *Pediastrum Boryanum* is general in almost all the lochs, but is never abundant, and the same is true of *Ankistrodesmus falcatus*. *Crucigenia rectangularis* and *C. irregularis* were each noticed from several lochs, but they were not general, and no other species of the genus was observed. Most of the green Algæ belonging to the Protococcoideæ have a decided preference for shallower and warmer water than is met with in the Scottish lochs, so that their comparative scarcity is not very remarkable. The four most abundant were *Botryococcus Braunii*, *Ineffigiata neglecta*, *Sphærocystis Schroeterii*, and *Glœocystis gigas*.

Diatoms are very abundant in the Scottish plankton, and *they do not disappear* from it in May and June, but are very numerous even after that date. This abundance of Diatoms is largely due to the presence in quantity of a few species, most of those observed being relatively scarce, and principally littoral species, which have found their way to the pelagic region. The true plankton Diatoms which thrive in the surface waters of the Scottish lakes are chiefly—

Tabellaria fenestrata and var. *asterionelloides*.
„ *flocculosa*.
Fragilaria Crotonensis.
Asterionella formosa.
„ *gracillima*.
Eunotia pectinalis.
Navicula major.
Vanheurckia rhomboides, var. *Saxonica*.
Surirella biseriata.
„ *robusta* and var. *splendida*.

To these may be added *Synedra Ulna*, *Melosira granulata*, several species of *Cyclotella*, *Amphipleura pellucida*, *Rhizosolenia longiseta*, and *R. eriensis*, which occur widely distributed, but in smaller quantity.

We have remarked the curious absence of *Cymatopleura* from the plankton of most of the Scottish lakes, as this genus is commonly present in the plankton of shallower lakes.

It is doubtful if the two forms of *Asterionella* are specifically distinct. *A. gracillima* possesses longer frustules than *A. formosa*, but it is questionable if there are any other distinctions between them. They occur as stellate groups, the frustules being arranged like the spokes of a wheel. In the centre of the colony they are joined by minute mucous cushions in a spiral manner, so that there may be up to two and a half turns; that is to say, there is often more than two complete circles of frustules. The commonest number of frustules in a circle is eight, but seven and nine occur. Some continental authors have observed only three, four, or five in a circle, but the Scottish ones contain more than that. Intermediate forms occur.

Tabellaria fenestrata is very abundant and very variable, especially in the relative length and breadth of the frustules as seen in girdle-view. The commonest form is in zigzag chains, but the star dispositions (var. *asterionelloides*) are abundant in some lakes. This variety '*asterionelloides*' is very variable in itself, as can be seen from the photographs on Plates 1 and 2. Sometimes the frustules are short and broad, whereas in other cases they are long and narrow; Nos. 1 and 2, Plate 2, are examples of these extremes. The frustules are disposed in a spiral after the manner of those of *Asterionella*, and they may complete two circles. The commonest Scottish form is the 6-rayed one, but the number of frustules completing a circle varies from four to eight.

Tabellaria flocculosa is also very common in zigzag chains. No stars of this species were seen, although they have been recorded by Holmboe from Norway and by Lund from Denmark.

There are relatively few Myxophyceæ in the Scottish plankton. *Anabæna circinalis* occurs in most of the lochs, but other species are not frequent. Two sterile species of *Anabæna* were observed from Loch Bairness, Inverness. Species of *Oscillatoria* and *Lyngbya* are quite scarce, and *Cœlosphærium* and *Microcystis* are both uncommon.

The quantity of the plankton in the Scottish lochs is never very great and scarcely affects the colour of the water. As a rule it has no marked periodicity such as occurs in many of the shallower lakes of continental Europe. The slight seasonal variations of the plankton is principally accounted for by the more uniform temperature of the surface water than is met with in the continental lakes. There is sometimes a preponderance of Diatoms at a time when the Desmids are not so numerous, but this is not very conspicuous. The only known plankton elsewhere than in the British Islands which approaches the Scottish plankton in the number of its Desmids is the Swedish plankton. The Danish plankton is relatively much poorer in Chlorophyceæ, especially Conjugates. This is to be attributed principally to the fact that the geological formations of Denmark are mostly of Tertiary Age, and therefore the conditions of environment are unsuitable for the existence of a rich Desmid-flora.

The following is a summary of our knowledge of the phytoplankton of the inland waters of the west of Scotland:—

1. *The quantity of plankton is relatively small at any time and scarcely affects the colour of the water. It exhibits little periodicity, the seasonal variations being*

slight. This absence of any marked variation in character is to be attributed to the relatively slight changes in the temperature of the surface waters at different seasons.

2. *The greater part of the phytoplankton consists of Chlorophyceæ, and the major portion of these belong to the Conjugatæ. Most of the filamentous Algæ of the plankton are Conjugates.*

3. *The plankton contains a rich Desmid-flora.*

4. *This Desmid-flora owes its existence to the Older Palæozoic and Precambrian formations of the areas in which the lakes are situated. This rich Desmid-flora is not an isolated phenomenon peculiar to the plankton of the Scottish lochs, but it is also found in the plankton of those lakes of the English Lake District, N. Wales, and W. Ireland which are similarly situated on the Older Palæozoic formations. The* abundance *of Desmids is due to the absence of lime and the presence of humic acid in the water, these conditions being rendered possible by the geological nature of the areas in question, which are suitable for the formation of peat-bogs and peaty pools. The* diversity *of the Desmid-flora and the presence of some of the handsomest species of the family appear to be directly connected with the antiquity of the geological formations.*

5. *The Desmid-flora differs essentially from that of the small peaty pools and bogs of the same area, especially in the relative abundance of the species common to both the bogs and the plankton.*

6. *The Desmids were doubtless originally derived from the pools and bogs of the mountains, and only those species have flourished which found the conditions most suitable for their existence as pelagic organisms, some of them having in course of time produced distinct and characteristic plankton-varieties.*

7. *There is no very obvious maximum development of Diatoms, and some of the larger species of the Naviculoideæ and Surirelloideæ have firmly established themselves.*

8. *The proportion of Myxophyceæ is relatively small, and species of* Oscillatoria, Lyngbya, *and other genera are somewhat scarce.*

DESCRIPTION OF PLATES.

Plate 1.

Photomicrographs of plankton from Loch Ruar, Sutherland; all × 100.

No. 1. 1, *Mougeotia* sp.; 2, *Cosmarium contractum*; 3, *Staurastrum gracile*; 4, *St. paradoxum*; 5, *St. jaculiferum*, f. *biradiata*; 6, *St. cuspidatum*, var. *maximum*; 7, *Asterionella gracillima*; 8, *Tabellaria fenestrata*, var. *asterionelloides*; 9, *Surirella robusta*, var. *splendida*; 10, *Cœlosphærium Kützingianum*.

No. 2. 1, *Euastrum verrucosum*, var. *planctonicum*; 2, *Staurastrum jaculiferum*; 3, *St. cuspidatum*, var. *maximum*; 4, *St. paradoxum*, var. *longipes*; 5, *St. Arachne*, var. *curvatum*; 6, *St. erasum*; 7, *Tabellaria fenestrata* and var. *asterionelloides*; 8, *Fragilaria Crotonensis*, var. *contorta*; 9, *Surirella robusta*, var. *splendida*. 10, *Ceratium hirundinella*.

No. 3. 1, *Euastrum verrucosum*, var. *reductum*; 2, *Cosmarium Botrytis*, var. *depressum*; 3, *St. jaculiferum*, var. *subexcavatum*; 4, *St. dejectum*, var. *inflatum*; 5, *St. paradoxum*; 6, *St. erasum*; 7, *Tabellaria fenestrata*, var. *asterionelloides*; 8, *Asterionella gracillima*; 9, *Surirella robusta*, var. *splendida*; 10, *Anuræa cochlearis*.

No. 4. 1, *Pleurotænium Ehrenbergii*; 2, *Euastrum verrucosum*, var. *reductum*; 3, *St. jaculiferum*, f. *biradiata*; 4, *St. dejectum*, var. *inflatum*; 5, *St. gracile*; 6, *St. paradoxum*; 7, *St. erasum*; 8, *Sphærocystis Schroeterii*; 9, *Tabellaria fenestrata*, var. *asterionelloides*; 10, *Surirella robusta*, var. *splendida*; 11, *Cocconema gastroides*.

No. 5. 1, *Mougeotia*, sp.; 2, *Closterium rostratum*; 3, *Euastrum verrucosum*, var. *reductum*; 4, *E. verrucosum*, var. *planctonicum*; 5, *Cosmarium Botrytis*, var. *depressum*; 6, *Staurastrum paradoxum*; 7, *St. jaculiferum*; 8, *Asterionella gracillima*; 9, *Fragilaria Crotensis*, var. *contorta*; 10, *Tabellaria fenestrata*, var. *asterionelloides*; 11, *Surirella robusta*, var. *splendida*; 12. *Ceratium hirundinella*; 13, *Anuræa cochlearis*.

No. 6. 1, *Micrasterias Murrayi*; 2, *Staurastrum jaculiferum*, forma *biradiata*; 3, *St. paradoxum*; 4, *St. paradoxum*, var. *longipes*; 5, *St. erasum*; 6, *Pediastrum duplex*; 7, *Tetraëdron limneticum*; 8, *Asterionella gracillima*; 9, *Tabellaria fenestrata*, var. *asterionelloides*; 10, *Surirella robusta*, var. *splendida*; 11, *Anuræa cochlearis*.

Plate 2.

Photomicrographs of plankton from Loch Ruar, Sutherland. Nos. 1 and 2 × 500; Nos. 3–6 × 200.

No. 1. *Tabellaria fenestrata*, var. *asterionelloides* This shows the most frequent form of this variety.

No. 2. *Tabellaria fenestrata*, var. *asterionelloides*. A more slender form, with fewer cells in the colony. This form is much more rarely met with than No. 1.

No. 3. 1, *Staurastrum jaculiferum*, forma *biradiata*; 2, *Asterionella gracillima*; 3, *Tabellaria fenestrata*, var. *asterionelloides*.

No. 4. 1, *Euastrum verrucosum*, var. *planctonicum*; 2, *St. jaculiferum*, forma *biradiata*; 3, *Tabellaria fenestrata*, var. *asterionelloides*; 4, *Asterionella gracillima*; 5, *Surirella robusta*, var. *splendida*; 6, *Dinobryon*, sp.

No. 5. 1, *Staurastrum jaculiferum*, forma *biradiata*; 2, *Sphærocystis Schroeterii*; 3, *Pediastrum Boryanum*; 4, *Fragilaria Crotonensis*, var. *contorta*. 5, *Anuræa cochlearis*.

No. 6. 1, *Euastrum verrucosum*, var. *reductum*; 2, *Staurastrum erasum*; 3, *Sphærocystis Schroeterii*; 4, *Surirella robusta*, var. *splendida*.

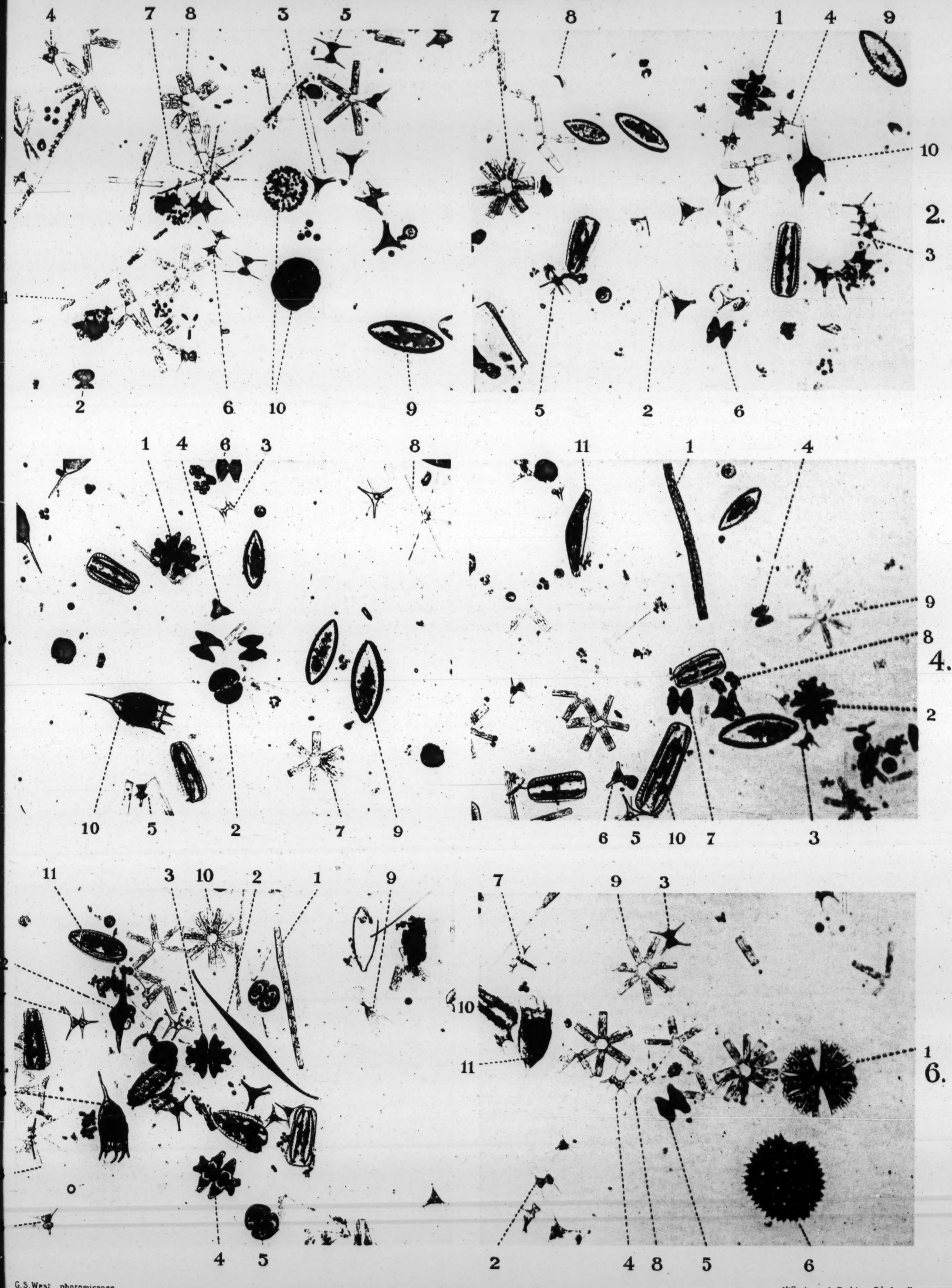
Plate 3.

Photomicrographs of plankton from Loch Fadaghoda, Lewis, Outer Hebrides; all × 100.

No. 1. 1, *Staurastrum Ophiura*; 2, *St. longispinum* (the wide gelatinous investment of this species is just visible in the original photograph); 3, *St. Sebaldi*, var. *productum*; 4, *St. jaculiferum* (forms); 5, *St. gracile*; 6, *Xanthidium antilopœum*; 7, *Mougeotia*, sp.; 8, *Tabellaria fenestrata*.

No. 2. 1, *Staurastrum Ophiura*; 2, *St. grande*; 3, *St. anatinum*, var. *grande*; 4, *St. anatinum*, var. *truncatum*; 5, *St. jaculiferum*; 6, *St. dejectum*; 7, *Spirogyra*, sp.; 8, *Sphærocystis Schroeterii*; 9, *Tabellaria fenestrata*; 10, *Asterionella gracillima*; 11, *Ceratium hirundinella*; 12, *Anuræa cochlearis*; 13, *Polyarthra platyptera*.

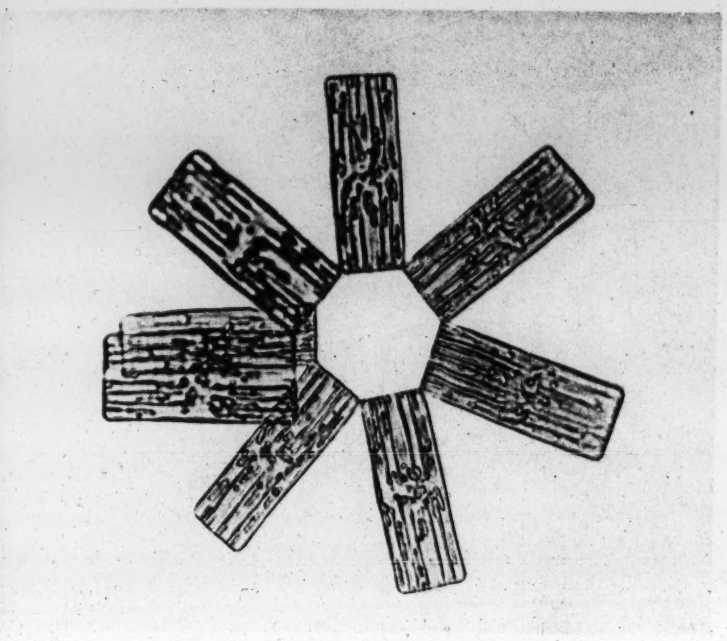
W. & G. S. WEST: A FURTHER CONTRIBUTION TO THE FRESHWATER PLANKTON OF THE SCOTTISH LOCHS — PLATE I

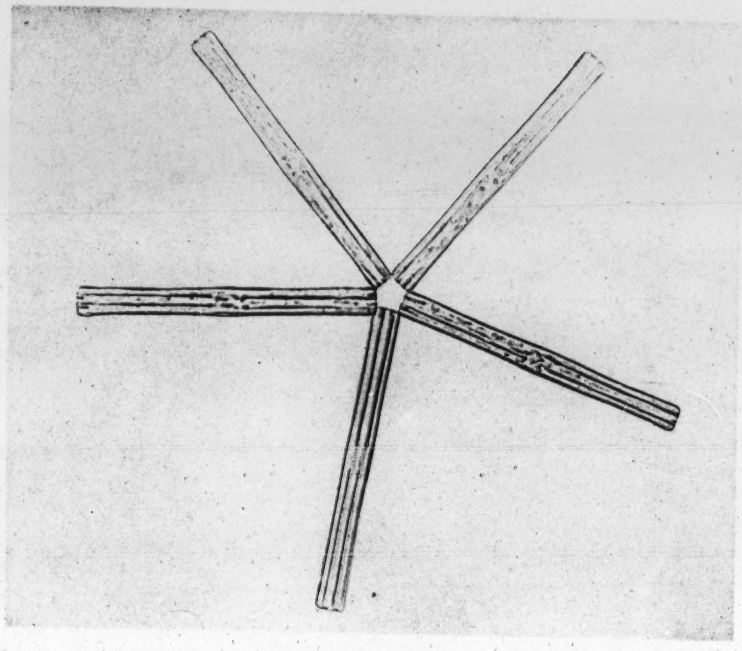


G.S.West, photomicrogr. M'Farlane & Erskine, Edinburgh.

PLANKTON FROM LOCH RUAR.

W. & G. S. WEST: A FURTHER CONTRIBUTION TO THE FRESHWATER PLANKTON OF THE SCOTTISH LOCHS — PLATE II.





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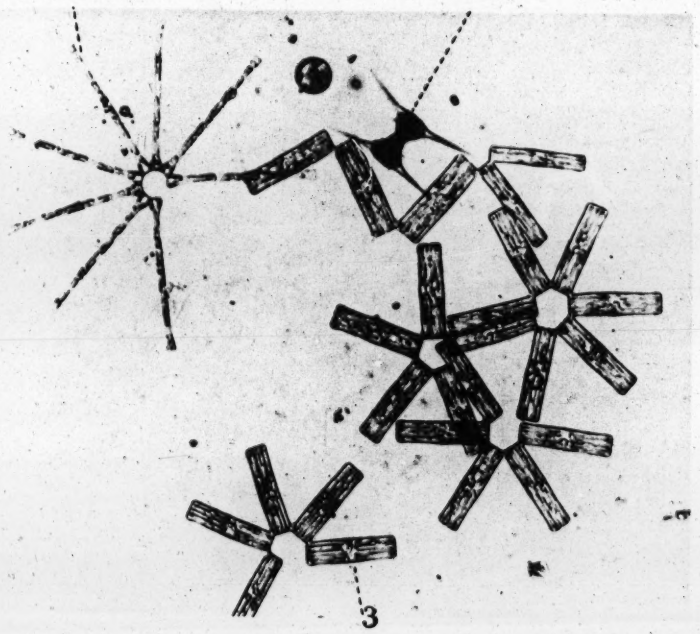

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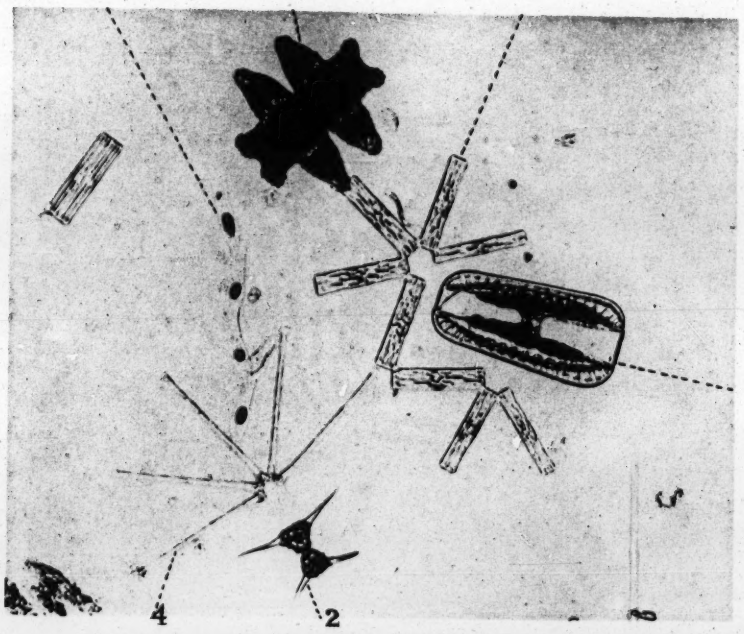

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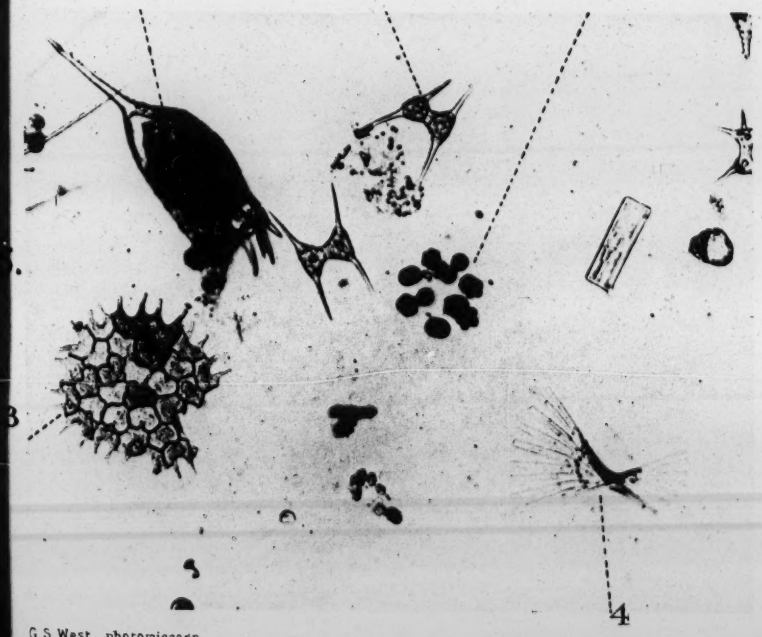

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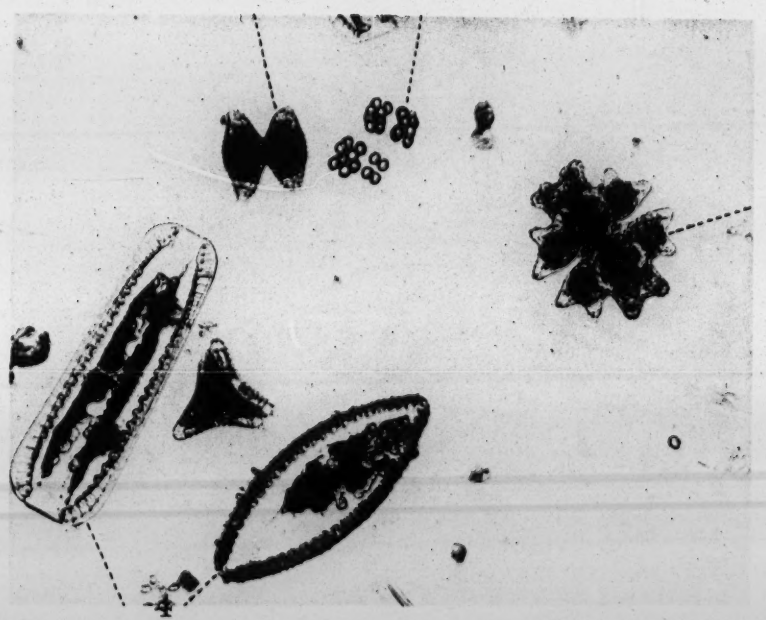

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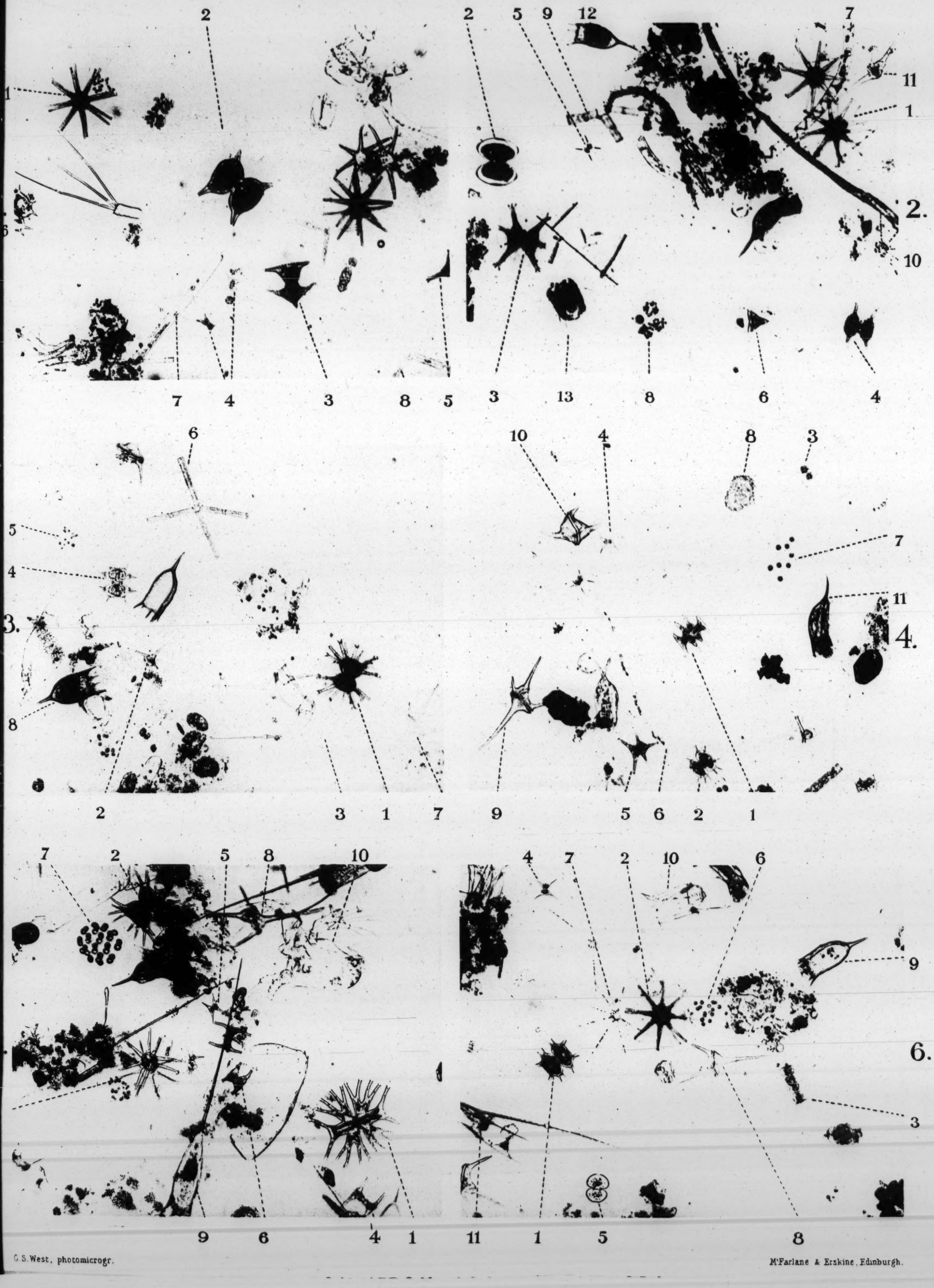
G. S. West. photomicrogr. M'Farlane & Erskine. Edinburgh

PLANKTON FROM LOCH RUAR.

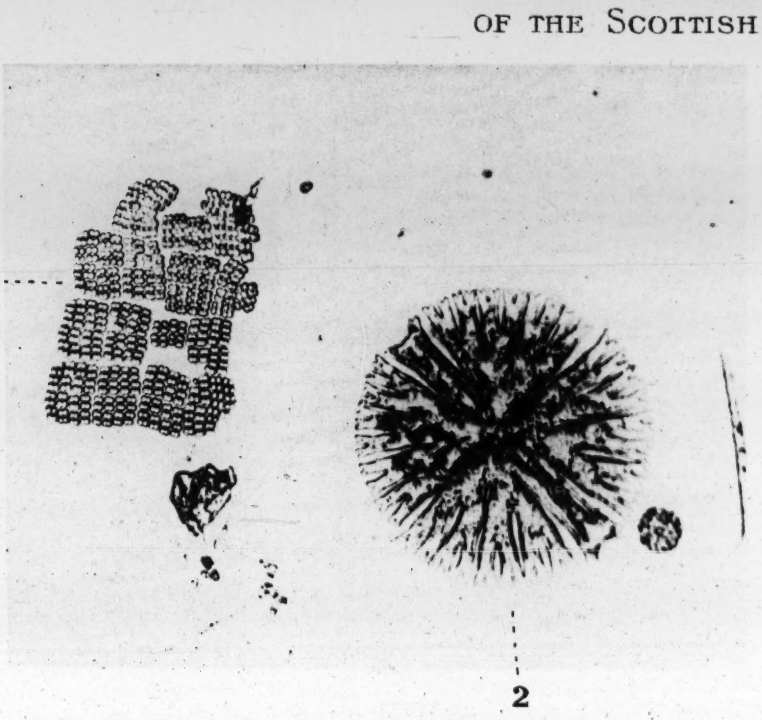
W. & G. S. WEST: A FURTHER CONTRIBUTION TO THE FRESHWATER PLANKTON OF THE SCOTTISH LOCHS — PLATE III.

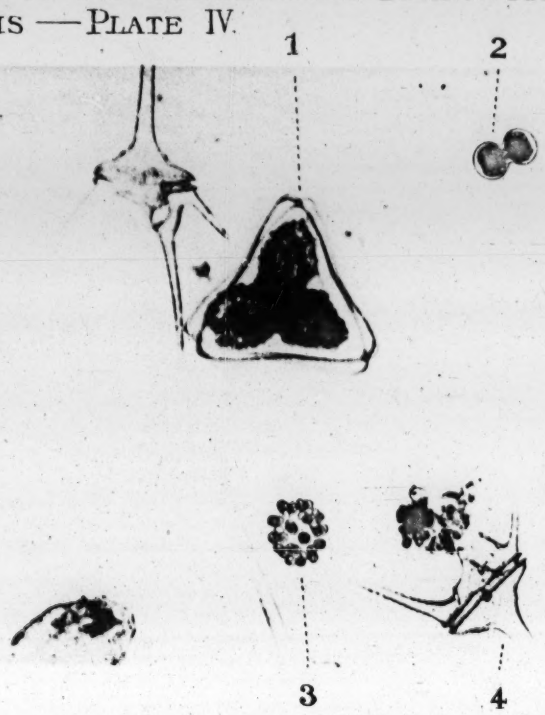


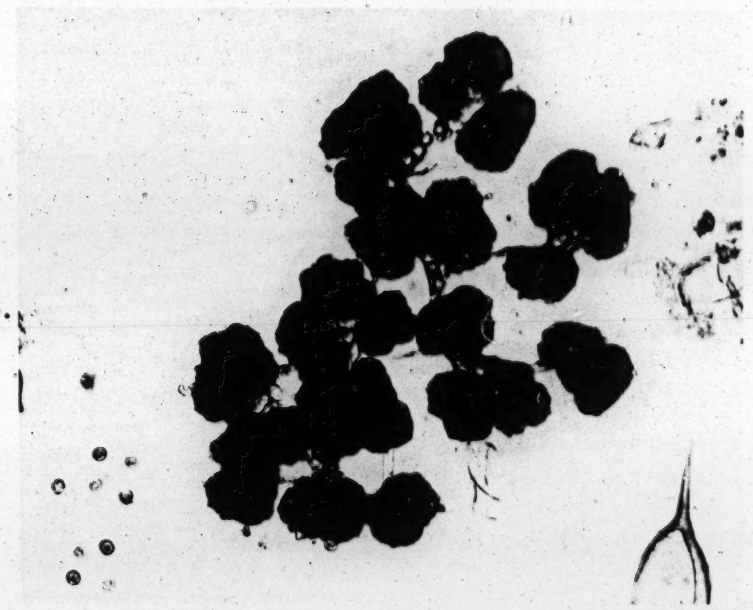
G. S. West, photomicrogr.

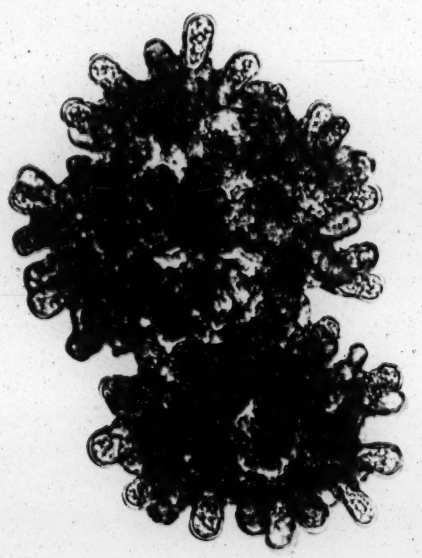
M'Farlane & Erskine, Edinburgh.

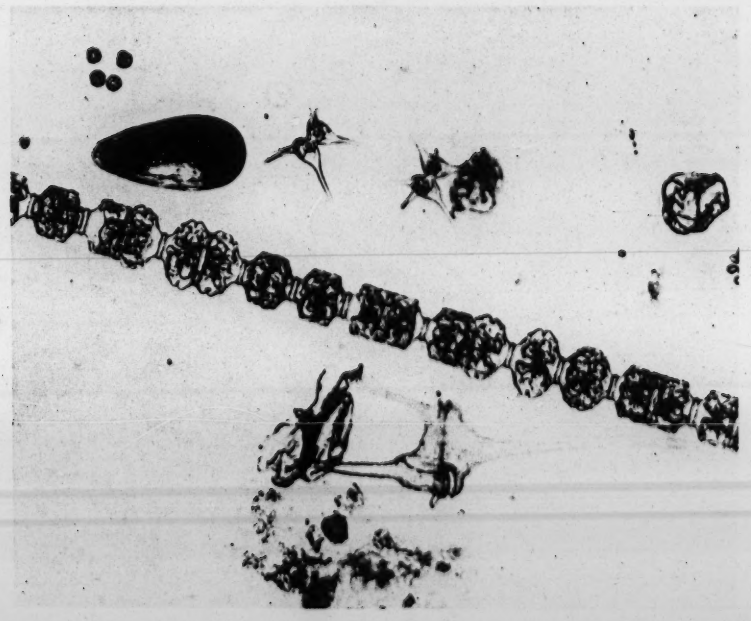
W. & G. S. WEST: A FURTHER CONTRIBUTION TO THE FRESHWATER PLANKTON OF THE SCOTTISH LOCHS — PLATE IV.

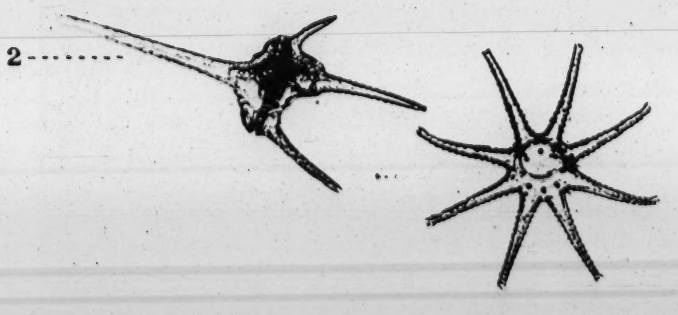








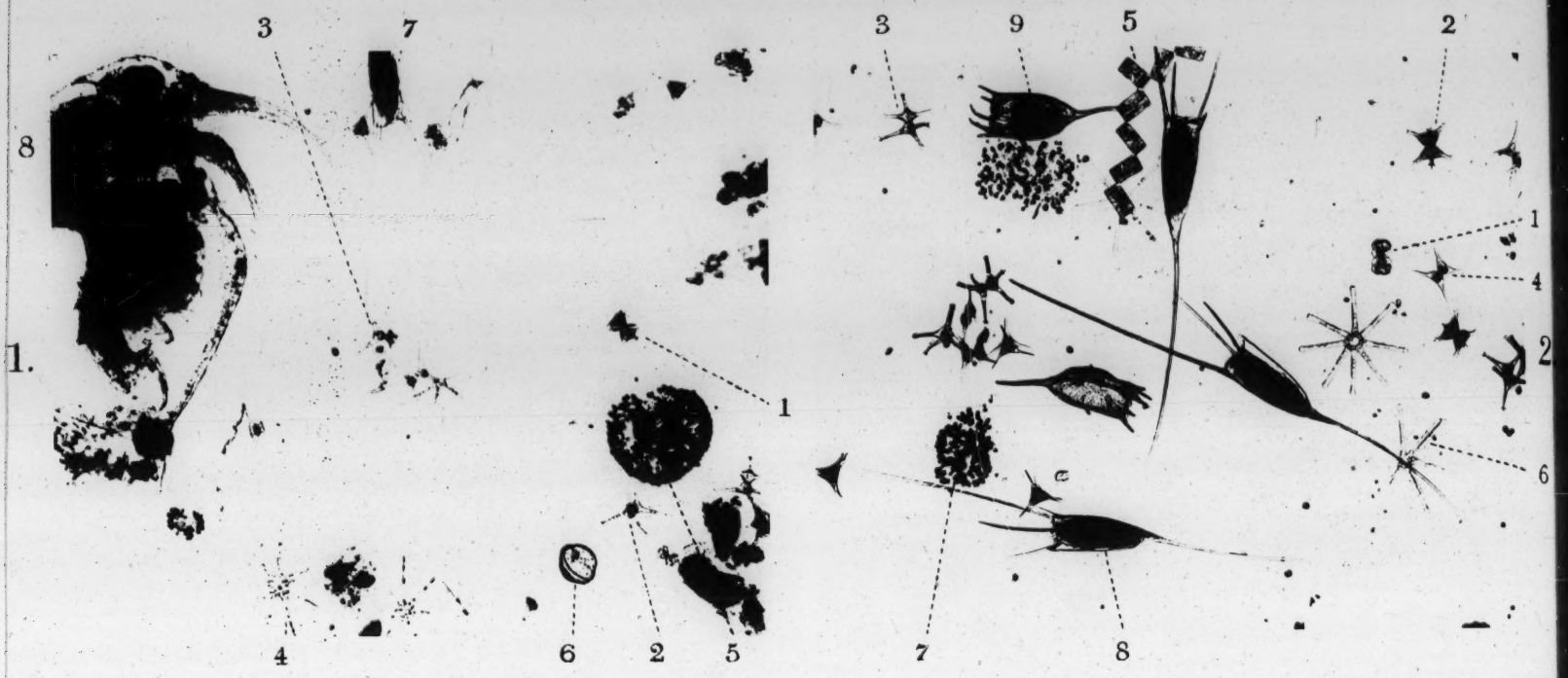


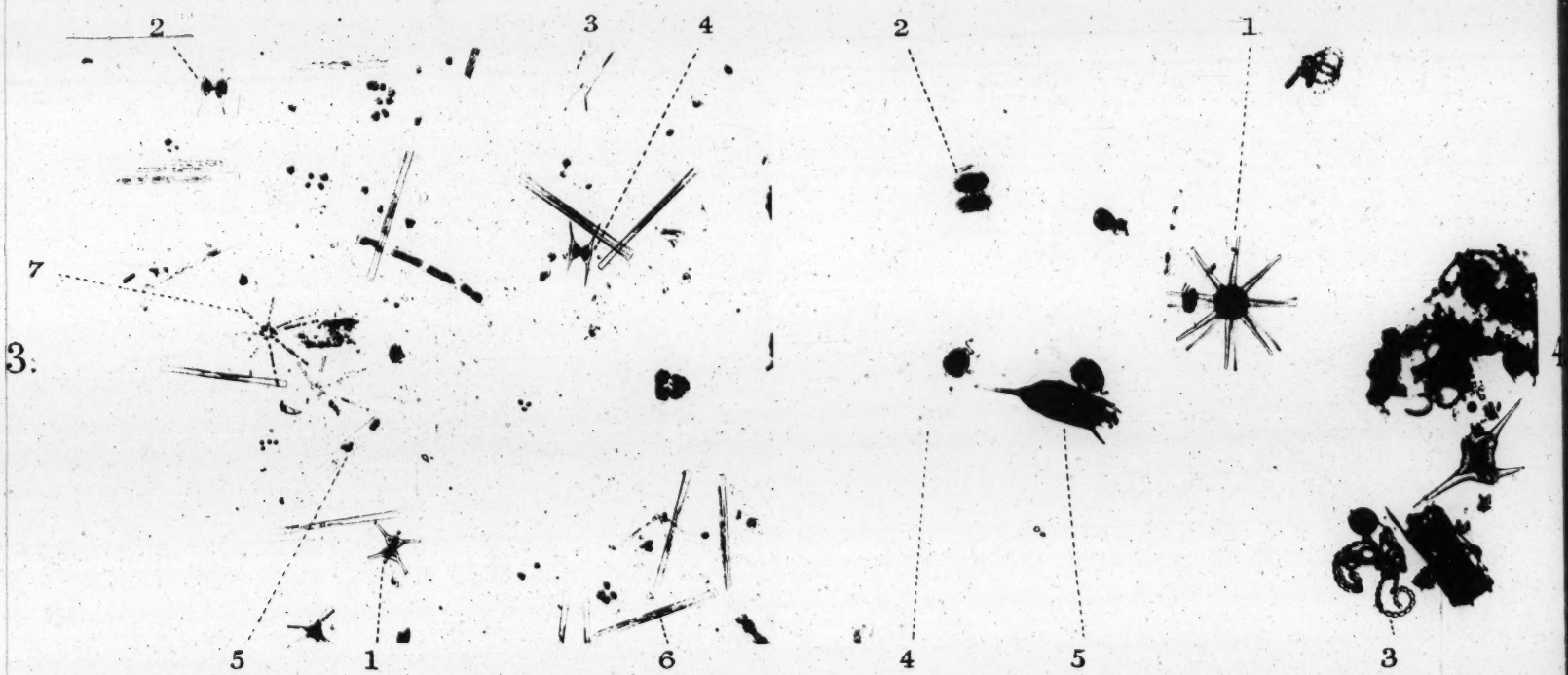


G. S. West, photomicrogr. M'Farlane & Erskine, Edinburgh.

PLANKTON FROM LOCH FADAGHODA.

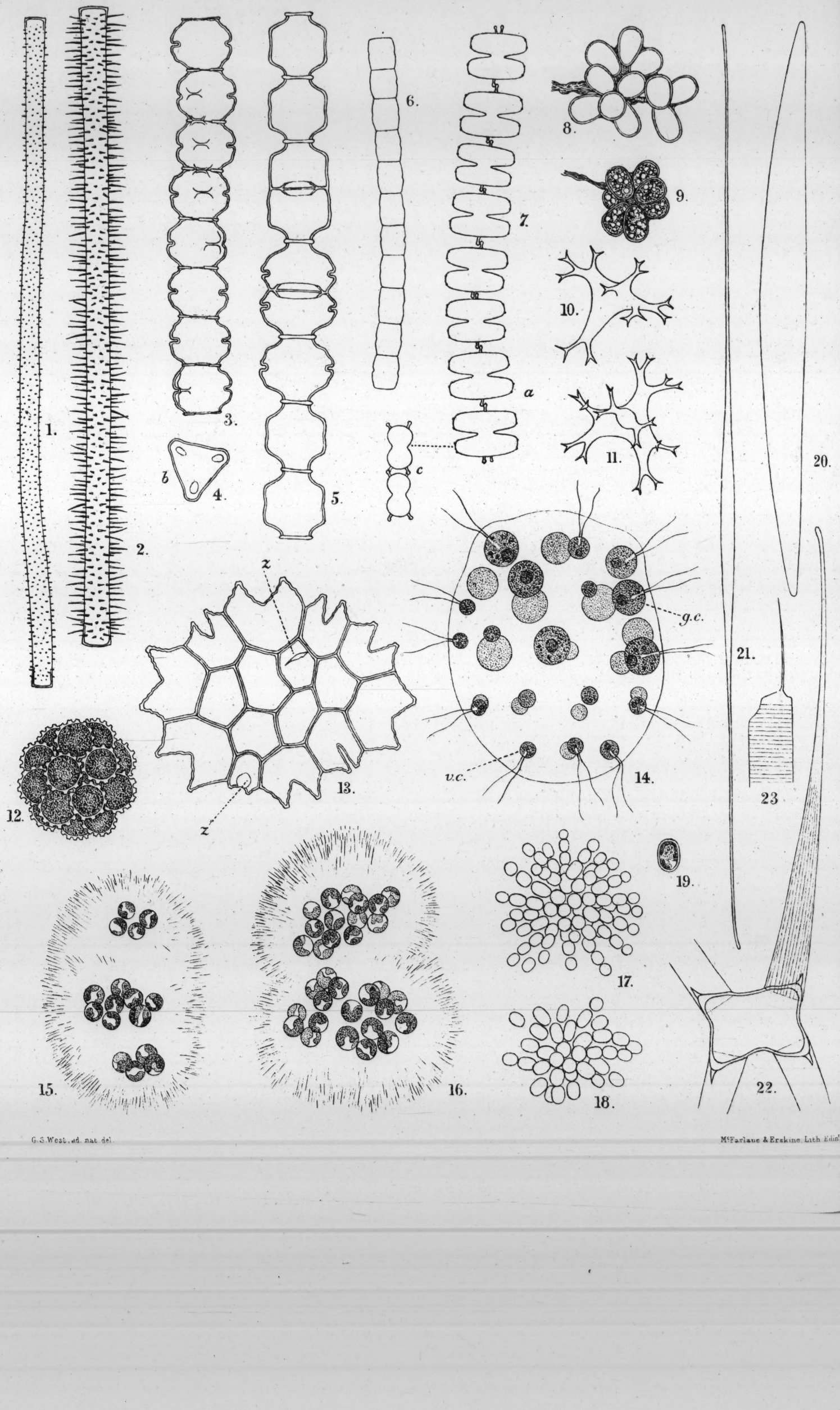
W. & G. S. WEST: A FURTHER CONTRIBUTION TO THE FRESHWATER PLANKTON OF THE SCOTTISH LOCHS — PLATE V.





G.S.West. photomicrogr.

M'Farlane & Erskine. Edinburgh.



G. S. West, ad. nat. del.

McFarlane & Erskine, Lith. Edin^r

No. 3. 1, *Staurastrum Arctiscon*; 2, *St. gracile*; 3, *St. jaculiferum*; 4, *Xanthidium antilopœum*; 5, *Sphærocystis Schroeterii*; 6, *Tabellaria fenestrata*, var. *asterionelloides*; 7, *Ceratium hirundinella*; 8, *Anuræa cochlearis*.

No. 4. 1, *Micrasterias pinnatifida*; 2, *Xanthidium antilopœum*; 3, *Staurastrum dejectum*; 4, *St. jaculiferum*; 5, *St. gracile*; 6, *Mougeotia*, sp.; 7, *Sphærocystis Schroeterii*; 8, *Microcystis*, sp.; 9, *Ceratium hirundinella*; 10, *C. cornutum*; 11, *Anuræa cochlearis*.

No. 5. 1, *Micrasterias furcata*; 2, *Staurastrum Brasiliense*, var. *Lundellii*; 3, *St. Ophiura*; 4, *St. gracile*, var. *cyathiforme*; 5, *St. jaculiferum*; 6, *St. anatinum*; 7, *Eudorina elegans*; 8, *Ceratium hirundinella*; 9, *Notholca longispina*; 10, Nauplius larva.

No. 6. 1, *Xanthidium antilopœum*; 2, *Staurastrum Ophiura*; 3, *St. anatinum*; 4, *St. jaculiferum*; 5, *St. brevispinum*, var. *altum*; 6, *Sphærocystis Schroeterii*; 7, *Asterionella gracillima*; 8, *Ceratium hirundinella*; 9, *Anuræa cochlearis*; 10, *Polyarthra platyptera*; 11, *Notholca longispina*.

Plate 4.

Photomicrographs of special objects in the plankton of Loch Fadaghoda, Lewis, Outer Hebrides. Nos. 1–3, 5 and 6 × 200; No. 4 × 400.

No. 1. 1, *Merismopedia æruginea*; 2, *Micrasterias radiosa*.

No. 2. 1, *Staurastrum grande*; 2, *Cosmarium moniliforme*; 3, *Eudorina elegans* (young colony); 4, *Ceratium hirundinella*.

No. 3. *Ineffigiata neglecta*; a typical colony from the plankton.

No. 4. *Botryococcus protuberans*, a small colony.

No. 5. 1, *Desmidium coarctatum*, var. *cambricum*; 2, *Staurastrum jaculiferum*; 3, *Ceratium hirundinella*.

No. 6. 1, *Staurastrum Ophiura* (semicell in vertical view); 2, *Ceratium hirundinella*.

Plate 5.

Photomicrographs of plankton from various lochs. × 100.

No. 1. Material from Loch Tay, Perthshire. 1, *Staurastrum paradoxum*; 2, *St. paradoxum*, var. *longipes*; 3, *St. cuspidatum*, var. *maximum*; 4, *Asterionella formosa*; 5, *Cœlosphærium Kützingianum*; 6, *Peridinium*, sp.; 7, *Notholca longispina*; 8, *Bosmina*, sp.

No. 2. Material from Loch Tay, Perthshire. 1, *Cosmarium abbreviatum*, var. *planctonicum*; 2, *Staurastrum paradoxum*; 3, *St. paradoxum*, var. *longipes*; 4, *St. jaculiferum*; 5, *Tabellaria fenestrata*; 6, *Asterionella formosa*; 7, *Cœlosphærium Kützingianum*; 8, *Notholca logispina*; 9, *Anuræa cochlearis*.

No. 3. Material from Loch Rosque, Ross. 1, *Staurastrum paradoxum*; 2, *St. curvatum*; 3, *St. jaculiferum*, forma *biradiata*; 4, *Arthrodesmus triangularis*, var. *subtriangularis*; 5, *Mougeotia*, 2 sp.; 6, *Synedra pulchella*; 7, *Asterionella formosa*.

No. 4. Material from Loch Laxadale, Harris, Outer Hebrides. 1, *Staurastrum Ophiura*; 2, *Xanthidium antilopœum*, var. *Hebridarum*; 3, *Anabæna circinalis*; 4, *Dinobryon elongatum*; 5, *Anuræa cochlearis*.

No. 5. Material from Loch Diracleet, Harris, Outer Hebrides. 1, *Staurastrum jaculiferum*; 2, *St. gracile*; 3, *St. brevispinum*, var. *altum*; 4, *Tabellaria fenestrata*; 5, *Notholca longispina*; 6, *Polyarthra platyptera*; 7, *Anuræa cochlearis*.

No. 6. Material from Loch Laxadale, Harris, Outer Hebrides. 1, *Hyalotheca neglecta*; 2, *Xanthidium antilopœum*, var. *Hebridarum*; 3, *Anabæna circinalis*; 4, *Tabellaria flocculosa*; 5, *Ceratium hirundinella*; 6, *Anuræa cochlearis*.

Plate 6.

In this Plate and in Plate 7 the following letters are used:—

a, *a'* = front view.
b = vertical view.
c = side (or lateral) view.

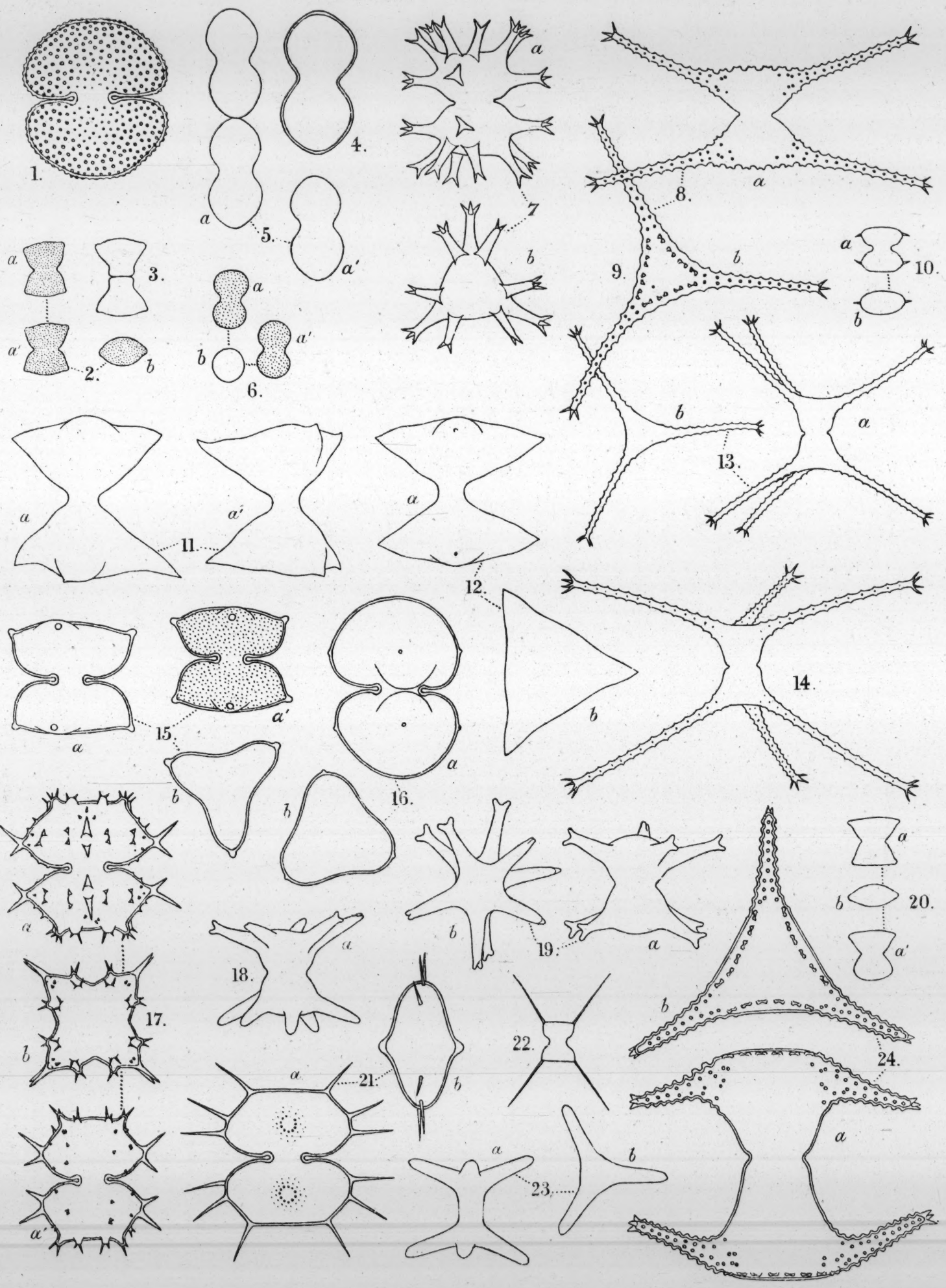
Fig. 1. *Gonatozygon monotænium*, De Bary, var. *pilosellum*, Nordst. × 520.
„ 2. „ *aculeatum*, Hastings. × 520.

Fig. 3, 4. *Desmidium occidentale*, sp. n. × 400.
„ 5. „ *coarctatum*, Nordst., var. *cambricum*, West. × 400.
„ 6. *Hyalotheca Indica*, Turn. × 520.
„ 7. *Sphærozosma Aubertianum*, West. × 520.
„ 8, 9. *Botryococcus protuberans*, sp. n. Small portions of two colonies. × 520.
„ 10, 11. *Tetraëdron limneticum*, Borge, forma. × 520.
„ 12. *Cœlastrum Morus*, W. & G. S. West. × 520.
„ 13. *Pediastrum Boryanum* (Turp.), Menegh. Large cœnobium, × 520. *z*, empty zoogonidangia, showing slit through which the zoogonidia escaped.
„ 14. *Pleodorina californica*, Shaw. *v.c.*, vegetative cells; *g.c.*, gonidial cells.
„ 15, 16. *Phæococcus planctonicus*, sp. n. × 520.
„ 17–19. *Actinobotrys confertus*, gen. et sp. n. 17 and 18, two colonies, the mucous investment not being represented, × 520; 19, single cell to show parietal chloroplast, × 1000.
„ 20. *Closterium acerosum* (Schrank), Ehrenb., forma *minor*. × 520.
„ 21. „ *setaceum*, Ehrenb., var. *elongatum*, var. n. × 250.
„ 22. „ *Kützingii*, Bréb., var. *onychosporum*, var. n. × 520.
„ 23. *Rhizosolenia eriensis*, H. L. Sm., var. *morsa*, var. n. Extremity of large specimen. × 520.

Plate 7.

Fig. 1. *Cosmarium Botrytis* (Bory), Menegh., var. *depressum*, var. n. × 430.
„ 2, 3. „ *capitulum*, Roy & Biss., var. *grœnlandicum*, Börges. × 520.
„ 4, 5. „ *moniliforme* (Turp.) Ralfs, forma *panduriformis*, Heimerl. × 520.
„ 6. „ „ Another smaller form of f. *panduriformis*. × 520.
„ 7. *Staurastrum Tohopekaligense*, Wolle, var. *trifurcatum*, W. & G. S. West forma. × 520.
„ 8, 9. „ *anatinum*, Cooke & Wills, var. *longibrachiatum*, var. n. × 520.
„ 10. *Arthrodesmus Incus* (Bréb.), Hass., var. *Ralfsii*, W. & G. S. West forma. × 520.
„ 11, 12. *Staurastrum inelegans*, sp. n. × 520.
„ 13. „ *paradoxum*, Meyen, var. *longipes*, Nordst. × 520.
„ 14. „ „ „ „ forma permagna. × 520.
„ 15. „ *brevispinum*, Bréb., var. *obversum*, var. n. × 520.
„ 16. „ „ var. *altum*, var. n. × 520.
„ 17. „ *forficulatum*, Lund. × 520.
„ 18, 19. „ *subnudibrachiatum*, sp. n. × 520.
„ 20. „ *sibiricum*, Borge. × 520.
„ 21. *Xanthidium antilopœum* (Bréb.), Kütz., var. *Hebridarum*, var. n. × 520.
„ 22. *Arthrodesmus Incus* (Bréb.), Hass., var. *longispinum*, var. n. × 520.
„ 23. *Staurastrum sublævispinum*, W. & G. S. West. × 520. Individual showing the initial stage of division.
„ 24. „ *Sebaldi*, Reinsch, var. *productum*, var. n. × 520.

WEST:— FRESH WATER PLANKTON OF THE SCOTTISH LOCHS.— PLATE VII.



G. S. West, ad. nat. del.

McFarlane & Erskine, Lith. Edin.r

XXII.—The Nudibranchiata of the Scottish National Antarctic Expedition.

By **Sir Charles Eliot**, K.C.M.G. *Communicated by* Sir John Murray, K.C.B.

(MS. received March 24, 1905. Read May 15, 1905. Issued separately June 9, 1905.)

The nudibranchs collected by the Scottish National Antarctic Expedition comprise only six species, but these include two new and interesting genera.

The species are—

1. *Notaeolidia gigas*, gen. et spec. nov.
2. *N. purpurea*, sp. nov.
3. *Tritonia appendiculata*, sp. nov.
4. *T. pallida*, Stimpson.
5. *Tritoniopsis brucei*, gen. et sp. nov.
6. *Scyllaea pelagica*, L.

The Scyllaea was caught on the return voyage, in the Atlantic, 32° N., 33° W.; *Tritonia pallida*, off Dassen Island, forty miles north of Cape Town; *Tritoniopsis brucei*, off Gough Island, 40° 20′ S., 9° 56′ W.; and the three remaining species in Antarctic waters.

The most remarkable point about the collection is the entire absence of Dorids. The collection made by the *Discovery*, which has also been entrusted to me for examination, shows the same character, and contains only one Bathydoris,* but several Aeolids and Tritonids (not the same as in this collection), one Doto and one Notaeolidia. The results of Northern Arctic expeditions are similar. The Dutch "Willem Barent" Expedition obtained five Aeolids and two Dendronotus; the Danish "Ingolf" Expedition two Tritonids, two Dendronotus, seven Aeolids, one Bathydoris, one Doridoxa, and only three normal Dorids out of a total of sixteen species. In the Tropics the proportion is reversed. Semper's collection from the Philippines contained only five kinds of Aeolids and fifty-four of Dorids.

None of the animals appear to have been brightly coloured. Most were, as far as one can judge, white or pinkish. *N. purpurea* is, as preserved, of a dull purple. In all, eyes are either absent or minute.

With these nudibranchs were two small holothurians (identified by Mr F. Jeffrey Bell as young specimens of Psolus), which closely resemble Dorids superficially. It is not clear that this resemblance is of any advantage to the animals, and it is probably due to mechanical reasons. Both Psolus and Dorids are slug-like animals, of moderately tough consistency, and possessing a clearly differentiated ventral creeping surface, and

* A second Bathydoris has been sent to me since.

branchial or tentacular appendages at one extremity of the body. Similarly, Elysiadae and Planarians are much alike in form, and constantly mistaken for one another; both are thin sheets of living matter, which progress by creeping and swimming.

Notaeolidia, gen. nov.

This new genus seems worthy to form the type of a new family intermediate between the Aeolididae and such forms as Dendronotus and Lomanotus. Externally the animals resemble the Aeolids, and are chiefly remarkable for their great size. They have no frontal veil, but large oral tentacles, perfoliated rhinophores without sheaths, and numerous cerata, arranged somewhat as in Gonieolis. The internal digestive organs, however, deviate from the type of Aeolis. The portion of the hepatic system within the body cavity consists of a folliculate mass as well as of tubes, and lies under the large hermaphrodite gland; the radula consists of a few rows (generally eighteen), each containing nine or sometimes eleven teeth. On the one hand, it is practically the radula of Coryphella and Gonieolis enlarged and extended; on the other, it presents resemblances to that of Dendronotus on a small scale.

Two or three species are known. *N. gigas* and *N. purpurea* described below are perhaps only varieties of one form, and differ chiefly in size and colour. *N. depressa*, obtained by the *Discovery*, which will be shortly described by me, is flatter, and superficially resembles Lomanotus, as it has only a single row of cerata on the mantle margin. It resembles *N. gigas*, however, in all essential points, and the radula is similar, though specifically distinguishable, the laterals being narrower and the denticles larger.

The characters of the genus, which are at present those of the family, are as follows:—

Large animals of Aeolidiform appearance. Oral tentacles large: rhinophores perfoliate without sheaths. Foot rounded and grooved in front. Dorsal margin undulated, and bearing one or more rows of close set cerata. Jaws not denticulate. Radula consisting of a central tooth and four (rarely five) laterals on each side. Central tooth with strong median cusp and side denticles; laterals denticulate on inner side. The liver forms a lobed flocculent mass within the body cavity, and in the body walls a thick spongy layer, from which rise the diverticula which enters the cerata. The hermaphrodite gland lies above the liver.

Notaeolidia gigas, gen. et spec. nov.

The collection contains six specimens, which, though differing considerably in size and somewhat in appearance, all seem referable to the same species. The largest (to which the details given in the description below refer unless otherwise stated) is no less than 122·5 mm. long. The measurements of the others are as follows:—

	Length.	Breadth, including Cerata.	
	mm.	mm.	
(2)	108·5	33·5	
(3)	50	33	This specimen is exceptionally broad.
(4)	59	19	Long and slender.
(5)	43·5	20	
(6)	broken in two, but about 54,	19	

All the specimens except the last were captured at Station 325, Scotia Bay, South Orkneys, in 9–10 fathoms. In two cases it is recorded that the temperature was 29° Fahr. The last specimen was found in a shore pool, Scotia Bay, South Orkneys; temperature, 30° Fahr. It is much damaged, nearly all the cerata being broken though not detached, but it probably belongs to the same species as the others, the anatomy being identical.

The body, rhinophores, and tentacles are of a dirty white; the cerata are of a faded pink, but this tint is stronger in the large specimens that in the smaller ones. The skin appears to be naturally smooth, but in some specimens is covered with wrinkles and blisters, apparently caused by the preserving fluid. When the integuments are held up to the light (but not otherwise), a network formed by intersecting bundles of fibres can be seen within the skin. This pattern is much more developed in some specimens than in others.

The largest specimen is 122·5 mm. long, 35 high, and 39 across the broadest part of the back, including the cerata. The shape and proportions of the body somewhat recall Tritonia, but the external characters are those of an aeolid. The foot is broad, with expanded margins, and measures 30 mm. at its widest part. It is rounded and grooved in front, with no trace of tentacular prolongations at the sides.

There is no trace of a frontal veil, but the oral tentacles which rise on each side of the mouth are unusually large and stout, being about 20 mm. long and 6 wide at the base. They are curved upwards and inwards, so as to present the appearance of a crescent when seen from the front. Between them is a slight prominence. The rhinophores are about 6 mm. behind the oral tentacles and close to one another, the interval between them being only 4·5 mm. They are 10 mm. high, and bear respectively fifteen and seventeen rings, most of which run completely round the stalk, though the last few are less regular, and interrupted here and there.

The cerata amount to about 800 on each side, that is, 1600 in all. They are set upon the dorsal margin, which is sinuous as in Lomanotus, with five undulations outwards and as many inwards. They begin slightly before the rhinophores, but on the lateral, not the anterior margin of the body, and are continued until its posterior termination, the tail projecting only 5 mm. They are of varying size, the tallest being about 18 mm. high, 4 mm. broad at the base, and 2 at the tip. The colour is faded pink. The largest are inside; the smallest, which are mere tubercles, less than 1 mm.

high, outside. They are not at all caducous, or even easy to detach. They are not set in groups, though the undulation of the dorsal margin produces a superficial appearance of such an arrangement, but are all crowded close together, except a few large ones which stand further inside, 2–3 mm. from the rest. The bare space in the middle of the back measures 15–20 mm., and the row of cerata which forms the border, though very irregular, is generally four or five deep, and consists of two large and two or three small ones. The shape of the cerata varies greatly. Some are symmetrically tapering, some cylindrical with blunt tips, and some, particularly the larger ones, are swollen at the bases, and taper somewhat suddenly in the upper half. The outline is irregular, and often presents knots and projections. The hepatic diverticula within the cerata are, like the cerata themselves, of irregular outline, and covered with knots, but are not ramified. They are similar in substance to the liver in the body, and in colour vary from brown to dull pink. In many cerata, at any rate, the liver cavity communicates with a small cavity above, which, in its turn, communicates with the exterior by a pore, which is sometimes visible externally. This cavity contains nematocysts of two shapes, spherical and elliptical. Mr G. H. Grosvenor, who has made a special study of these organs and kindly examined for me some from *Notaeolidia gigas*, informs me that the spherical nematocysts contain a convoluted cord inside and are of a type found in actiniae. In the elliptical nematocysts the cord is hard to see, but, as far as it can be followed, is straight.

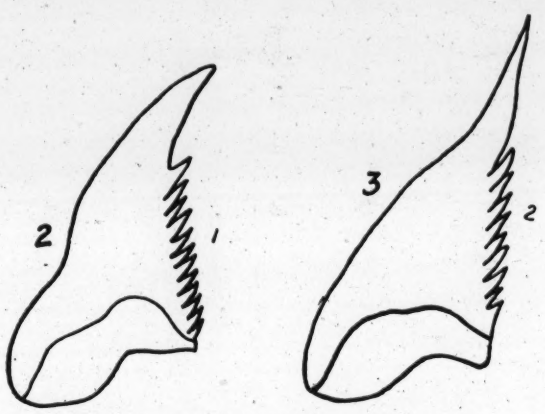
The large pericardial prominence lies a little to the right of the centre of the back, and is 18 mm. long by 14 broad. The genital orifices are about 38 and the anal papilla 52 mm. from the anterior end. This papilla lies just under the cerata, and its margin bears five crenulations, which are perhaps not natural.

On opening the body, the large heart is seen. It appears to be as usual. On the auricle are two lumps, possibly glandular. Considering the size of the animal, the central nervous system is small, the eyes in particular being minute specks. The ganglia are yellowish-white, and arranged as usual in the Aeolididae. The cerebro-pleural ganglia are elliptical, the pedal rather rounder. On the buccal commissure are situated the two elliptical buccal ganglia, separated from one another by a considerable interval, and connected with the gastro-œsophageal ganglia. The other commissures appear to be united in a common sheath.

The buccal mass measures 11 mm. by 14. The front part of it is formed by two moderately elongate jaws, which do not enclose the sides. They are brownish-yellow, and not very strong. The length is 8 mm., the breadth 6 mm. at the top, 3 at the bottom. The masticatory process is 3·2 mm. long; the edge is not denticulate, though it bears a few lumps and irregularities here and there.

The radula consists of eighteen rows of yellow teeth, and this number was constant in the five specimens dissected. Each row consists of a central tooth and normally four laterals. In some rows the outermost lateral is lost, and in one or two there appears to be a fifth rudimentary plate. The radula is brittle, and the central tooth liable to

split in two. This central tooth (fig. 1) is broad, and bears a strong pointed median cusp and eight lateral denticles on each side. The divisions between the denticles are continued as striæ on the body of the tooth. The first lateral (fig. 2) tooth is roughly triangular in shape, and bears about twelve fine distinct denticles on the inner side, facing the central tooth. The second lateral (fig. 3) is similar, but the top part is somewhat straighter and more pointed, and the denticles rather fewer (eight to nine). The third lateral (fig. 4) is still narrower and more pointed, but has about the same number (eight to nine) of quite distinct denticles. The fourth lateral (fig. 5)



FIGS. 2, 3.—First and second lateral teeth.

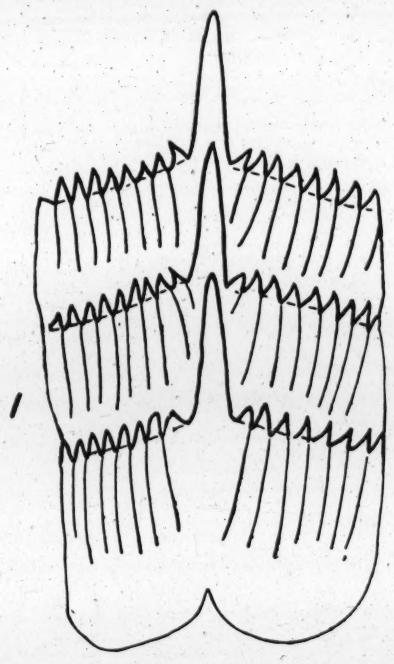
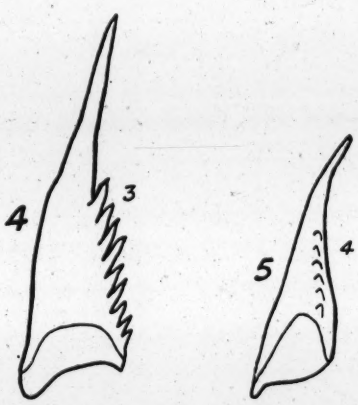


FIG. 1.—Three median teeth, from above.



FIGS. 4, 5.—Third and fourth lateral teeth.

FIGS. 1 to 5.—*Notaeolidia gigas.*

is smaller, and bears about seven indistinct and inconspicuous denticles. The fifth, when it exists, is a minute elongate smooth plate.

The salivary glands are two long white flocculent bands, 23 m. long and 4·5 wide. The digestive organs are almost empty in the larger specimens, and so crushed and compressed by the various parts of the reproductive system as to suggest that the animals do not take much nourishment during the breeding season. The rather short œsophagus leads straight into the stomach, which lies on the top of the much swollen mucous gland. The interior of the stomach is laminated. From it proceed two tubes, which enter the body wall right and left. Just below them issues the intestine, which goes first to the right and then turns backwards. At its commencement it bears

several ridges on the outside, and there is a glandular mass at the point where it leaves the stomach. Internally, both the intestine and the adjacent parts of the stomach bear very strong laminæ, which resemble plates but are not detachable. Posteriorly, the stomach is produced into a prolongation which extends to the end of the body cavity, and gives off on each side, at points not exactly opposite to one another, six branches, which enter the body wall. In front the stomach adheres pretty closely to this wall, and the liver is almost entirely within the sides of the body, but posteriorly it lies also within the cavity of the body, and covers the arrangement of tubes described with a dull purplish-brown mass of irregular shape, consisting of many lobes formed of minute convoluted tubes. The branches of the digestive system are not subdivided in the body cavity, but as soon as they enter the sides they are extensively ramified and form a thick spongy layer of tubes covered with liver cells, from which arise the diverticula which enter the cerata. This feature seems similar to the arrangement found in *Gonieolis typica* by Bergh (R. Bergh, *die Nudibr. gesammelt während d. Fahrten d. Willem Barents in das nördl. Eismeer*, 1885, p. 17).

The hermaphrodite gland lies on the top of the liver, and the posterior two-thirds of the body are almost entirely filled by a large mass of genital products, falling roughly into two halves, lying on the right and left. These halves are divided into numerous lobes of irregular shape, about 10 mm. long, 7 broad, and 4 thick. The lobes are composed of packets (about 2 mm. × 1 mm.), consisting of a number of yellowish bodies set in colourless jelly. They contain two different kinds of elements, which are presumably ova and spermatozoa, the first round, the second more or less elongate but of varying shape. The anterior portion of the body is filled chiefly by the huge mucous gland (about 33 mm. × 22 mm.) which lies under the stomach. It is white, rather slimy, and formed of innumerable windings. Inside it is the much smaller yellow albumen gland. The spermatotheca is of moderate size and roundish; the vas deferens much convoluted. The verge is cylindrical, not tapering, and unarmed. Within it is seen a twisted channel. The external orifices of these organs are protected by ample folds, one of which lies in front of them and the other behind, with a continuation below.

The species is distinguished by its great size, high shape, and numerous cerata set in several irregular rows.

Notaeolidia purpurea, sp. nov.

One specimen captured in Scotia Bay, 10 fathoms. It is of a uniform dull purplish-brown, and the preserving fluid has also become purplish. The form is rather elongate, the measurements being, length 41 mm., breadth across cerata 17, height 14. The left oral tentacle is missing, having apparently been bitten off; the right one is very large, 19 mm. long and 5 broad at the base, but tapering. It curves straight backwards, and not at all outwards. The other external characters are as in *N. gigas*.

The rhinophores bear about fifteen perfoliations, which become less distinct at the top. The dorsal margin makes five distinct undulations, so that the cerata appear to be set in groups. But this is not really the case, the arrangement being as in *N. gigas*, though perhaps the line of cerata is thinner and the bare space in the middle of the back larger. There are a few large cerata inside the line. The longest measure 16 mm., but most are short.

The internal anatomy is as in *N. gigas*. It is hard to formulate any real difference for the radula (18 × 4.1.4), but perhaps the denticulation of both the lateral and rhachidian teeth is more distinct, and the innermost denticles of the latter are set higher up on the side of the central cusp.

I am doubtful if this form should be regarded as a separate species, or merely a variety of *N. gigas*, but its general appearance and colour are different; it is much smaller, though apparently sexually mature, and the oral tentacles are proportionately longer.

Tritoniadae.

This family comprises the genera Tritonia and Marionia, the former without stomach plates, the latter with them. To them, I think, should be added Atthila, which Bergh makes the type of a separate family. It appears to me difficult to maintain the distinction between the genera Tritonia, Cuv., and Candiella, Gray. According to Bergh, the "margo veli frontalis" is in the former "papilligerus" and in the latter "digitatus": the former has many and the latter few lateral teeth. Yet *Tritonia exsulans*, Bergh, has 8–9 "einfache Finger" on each side of the frontal veil and a radula with a formula of about 41 × 61.1.1.1.61, whereas *Candiella ingolfiana*, Bergh, has "six fingers" on each side of the veil and a radula of 67 × 83.1.1.1.83. I am, however, inclined to add to the family two new genera, Tritoniopsis and Tritoniella. The former, described below, has a divergent radula. Tritoniella, which I propose to describe among the nudibranchs found by the *Discovery*, resembles Tritonia in most points, but has dorsal ridges, and instead of ramose branchiæ, simple projections or crenulations. Some of the specimens are exceptionally well preserved, and it seems clear that no appendages are lost.

Bergh's list of Tritonia and Candiella in the *System der Nudibranchiaten Gasteropoden* contains sixteen species, to which the following have since been added :—

17. *T. diomedea*, Bergh.
18. *T. exsulans*, Bergh.
19. *T. incerta*, Bergh.
20. *T. gigantea*, Bergh.
21. *T.* (*candiella*) *australis*, Bergh.
22. *T.* (*candiella*) *ingolfiana*, Bergh.
23. *T.* (*candiella*) *villafranca*, Vayssière.
24. *T. appendiculata*, sp. nov.

Tritonia appendiculata, sp. n.

One specimen marked "9 Fathoms. April 1903. Harbour S Orkneys." (Station 325. Scotia Bay.)

The animal has the usual shape of Tritonia: on the left side is a large blister, probably accidental. The length is 51·5 mm., the maximum height and breadth 12 and 16 respectively. The colour is a uniform dirty greenish-yellow. The back is thickly covered with small round flat warts. The oral veil is 12·7 mm. wide. It does not project much from the head, and bears twelve simple digitate processes, most of which are about 2 mm. long, but two are very small tubercles. At the ends of the veil and below the outermost process on each side is a large grooved tentacle, of the shape usual in the genus. The lips project on each side of the mouth as distinct ridges, prolonged at the top into free cylindrical processes 2·5 mm. long, resembling tentacles.

Fig. 6.—Branchia.

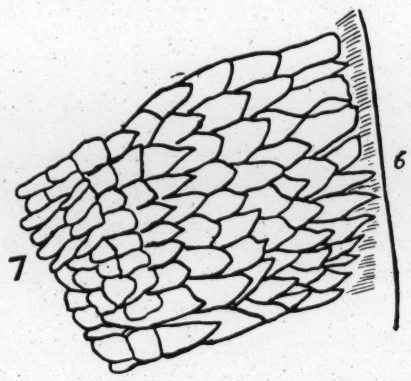


Fig. 7.—Portion of edge of jaw.

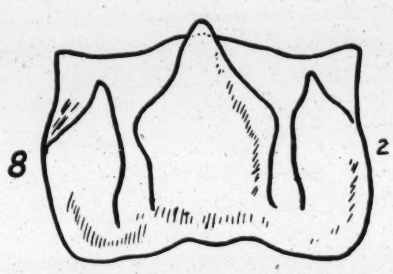


Fig. 8.—Median tooth.

Figs. 6 to 8.—*Tritonia appendiculata.*

The sheaths of the rhinophores are 3 mm. high and 3·5 mm. broad; the margins are jagged. The rhinophores are thick clubs, surrounded by about ten simply pinnate or bipinnate plumes, white, with greenish tips, and hard to separate from one another. The dorsal margin is distinct, 3 mm. broad, and starts from the back of the rhinophore sheaths, to which it is attached, giving them a somewhat elongated appearance behind.

On each side are nineteen branchiæ (fig. 6) of various sizes, but those on the left are, on the whole, rather larger than those on the right. They are scanty, and not foliaceous. The smaller are simply bifid; the larger consist of three processes set on a common prominence; each process is twice bifurcate. The anus is 22 and the genital orifice 15 mm. from the anterior end of the body. The former is just under the dorsal margin, the latter half-way up the side of the body and surrounded with ample folds.

There is no tail separate from the body. The foot is rounded and grooved in front, where it is thickened by a layer of what appear to be glands.

The pericardium and heart are as usual. The central nervous system is large, but no eyes were found. The ganglia are yellow and smooth, showing no signs of granula-

tion. The pedal ganglia are round, the cerebro-pleural elliptical, with traces of a division into two parts.

The jaws are strong, horny, elongate, yellow, with black edges. They are 10 mm. long and 4·3 wide at the broadest part. The region of the hinges is straight and flat. The rest of the jaw curves outwards and is convex. The edge (fig. 7) itself is smooth, but behind it are about six rows of stout denticles, which are blackish in the jaw and yellowish on the short (1·5 mm.) masticatory process. Behind them are about fifteen rows of flatter, roundish prominences, not amounting to denticles. The radula is of the type usual in Tritonia, with a formula of 29 × about 50.1.1.1.50. The median (fig. 8) tooth is tricuspid, the central cusp, pointed, those at the side blunt. The first lateral (fig. 9) is of the usual clumsy shape; the rest (fig. 10) are hamate, moderately stout

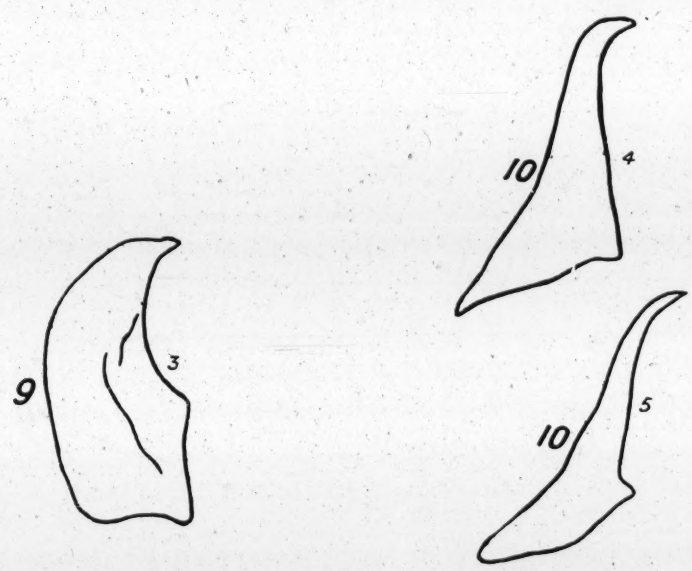


Fig. 9.—1st lateral. Fig. 10.—Laterals.

Figs. 9 and 10.—*Tritonia appendiculata.*

and moderately curved. The tips are often broken off, particularly near the middle of the radula.

The salivary glands are 8 mm. long, thin, ribbon-shaped above, slightly flocculent below. The œsophagus is at first narrow, but rapidly broadens out and enters the thin membranous stomach. About half of the stomach is surrounded by the brownish-yellow liver, which is itself surrounded by the hermaphrodite gland. The intestine leaves the stomach at the point where it emerges from this covering of liver and gland. It is strong and thick, and turns to the right after a slight bend forwards. Both the stomach and the intestine were filled with blackish matter, with which were mixed some bright red spiculous animal fragments.

The hemaphrodite gland consists of bright pale-yellow bodies set in colourless jelly. The mucous and albumen glands are large, both greyish-yellow. The spermatotheca is elongate, with a short duct. The vas deferens is convoluted. The verge is broadly conical at the base, with a thin pointed top.

This species offers many points of resemblance to *T. challengeriana* (Bergh,

Challenger Reports, Nudibranchiata, p. 45), but the veil is different, and the grooved tentacles are, as preserved, below it; there are more tubercles on the back, and the branchiæ are fewer in number. The tentacular prolongations of the lips are also remarkable. Rudiments of such formations may be seen in other species, but here they are unusually distinct.

Though I hesitate to refer the specimen to *T. challengeriana*, it is quite possible that the two species may not really be distinct.*

Tritonia pallida, Stimpson. Stimpson, *Proc. Acad. Philadelphia* (1854), p. 388.

One specimen, with the label "8 m. N. of Dassen Island in 35 fath." (Cape Colony.)

The animal is perfectly smooth and white; the yellowish viscera can be seen through the semitransparent integuments. It is somewhat bent and measures 35 mm. in length, equivalent to at least 40 if it were straightened out. The breadth is 14 and the height 12·5 mm., the foot is 12 mm. broad.

There appears to be no tail distinct from the body. The back is bordered by a distinct dorsal margin, projecting about 2 mm., and bearing seventeen branchial plumes on each side. The foot also has an expanded lateral margin and is rounded in front. The middle of the anterior margin is drawn up towards the mouth, but not notched.

The branchial plumes are of various sizes. The largest are the third, fifth, ninth and thirteenth on the right, and the fourth, seventh, eighth, ninth and eleventh on the left. The two or three foremost and hindmost are quite small. The largest plumes stand out from the back about 4 mm., and measure 6 mm. across. The primary axis is bifurcate; each bifurcation bears two to four secondary branches, and these branches in their turn bear irregular, simply pinnate projections. The smaller branchiæ are from a quarter to half the size of the larger ones and simpler, generally consisting of a short bifurcate stem, bearing on each side two or three simply pinnate plumes. The genital orifices are not conspicuous, and are situated under the fourth plume on the right side, rather high up. The vent lies just under the dorsal margin, between the sixth and seventh plumes.

The frontal veil (fig. 11) is of moderate size, about 8 mm. wide and projecting 3 mm. from the head, not counting the appendages. There are four of these on each side, digitate, and about 3 mm. long. The veil is divided into two halves by a central curve inwards, in the middle of which is a very small papilla. There are only slight and uncertain traces of a tentacular groove on the outermost process.

The rhinophore sheaths are wide and open, 2 mm. high and 3 wide, with irregularly crenulate edges. The club of the rhinophores is quite simple and surrounded by about ten plumes, united at their bases and of various sizes, simply pinnate or bipinnate, and occasionally imperfectly tripinnate.

* Since writing the above I have examined the type specimens of *T. challengeriana* in the British Museum. They are almost smooth, whitish, and, in addition to other differences, the branchiæ are more numerous, finer, and more elaborately ramified than in *T. appendiculata*.

The central nervous system is much as in *Candiella lineata*. The ganglia are smooth and yellowish; the nerves white. There is a large common commissure. The cerebro-pleural ganglia are pear-shaped, and show signs of a division into two halves. The pedal ganglia are round, and separated from the cerebro-pleural more clearly than in *C. lineata*. The eyes are black and very small. The pericardium is white, and as usual in the genus.

The buccal mass is rather elongate, measuring 12 mm. by 5·5, and strongly muscular. The inner parts and the radula have a faint yellowish tinge. The jaws are yellow, about 7 mm. long and 4 broad in the widest part, somewhat curved outwards. The edge of the jaw and the masticatory process bear five rows of very distinct denticles of somewhat varying shape. The radula consists of forty-one rows. Those in front are much worn and incomplete. The longer rows contain forty teeth or slightly more on each side of the rhachis, so that the formula is about 41 × 40.1.1.1.40. The central tooth (fig. 12) is tricuspid; the first lateral (fig. 13) of the usual clumsy shape; the

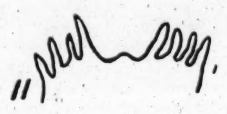


Fig. 11.—Frontal veil.

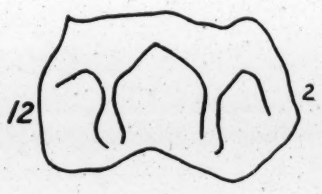


Fig. 12.—Median tooth.

Fig. 13.—First lateral.

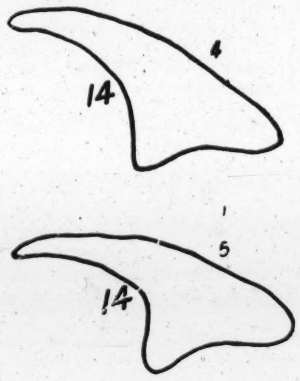


Fig. 14.—Other laterals.

Figs. 11 to 14.—*Tritonia antarctica*.

remaining laterals hamate, and slightly curved at the tip. None of the teeth bear any denticles, and the bases are not large.

The salivary glands are 5 mm. long, white and flocculent. The œsophagus is rather broad, 12 mm. long by 3·5 wide, with rather thin walls, irregularly laminated internally. It dilates into a stomach of moderate size, the greater part of which is enclosed by the liver. The liver is greyish, and surrounded below as well as above by a thick layer of the hermaphrodite gland, which consists of pale yellow bodies set in a colourless jelly. There is no trace of stomach plates. The stomach is filled with a yellowish mass, containing numerous black particles.

The spermatotheca measures 5 mm. by 3, and is yellowish, slightly striated, and apparently empty. Its duct is 5 mm. long. The albumen and mucous glands are moderately large and both white. The vas deferens is longish, not much convoluted; the verge conical, sharply pointed, unarmed, with a coiled duct inside.

I think that this specimen may be identified with *Tritonia pallida*, Stimpson, from Table Bay, Cape of Good Hope. Differences are not wanting: the white line mentioned

by STIMPSON is not visible, and the arrangement of branchiæ is not quite the same. But though STIMPSON's description is very slight and superficial, the similarity in colour and in the structure of the frontal veil seems sufficiently great to warrant identification in specimens from the same coast.

This form offers resemblances to *Tritonia* (*candiella*) *australis* and *ingolfiana*, but both of these have the first lateral tooth denticulate, and differ in other details.

Tritoniopsis, gen. nov.

The teeth of this form seem to differ from those of Tritonia too decidedly to allow of its being included in the same genus. Whereas in Tritonia the median tooth is broad, and the first lateral lower and of a more clumsy form than the others, in Tritoniopsis the median tooth is narrow and pointed, without wings or accessory cusps. The first lateral does not differ markedly from the others, but the outer laterals are very long and almost filamentous in appearance.

In the only known species there is one longitudinal and several transverse ridges on the back; the rhinophore sheaths bear appendages resembling branchiæ.

I have dedicated the species to Mr BRUCE, leader of the Expedition.

Tritoniopsis brucei, gen. et spec. nov.

Three specimens. The label says "April 22, 1904. Fathoms 10. Temperature 55° F. Gough Island." 40° 20′ S., 9° 56′ W.

The animals are of a transparent white (in one specimen with a slightly bluish tinge), allowing the yellow viscera to be seen through the integuments.

FIG. 15.—Branchia.

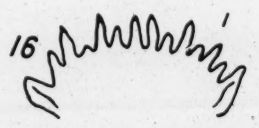


FIG. 16.—Frontal veil from below.

FIGS. 15 and 16.—*Tritoniopsis brucei*.

The largest specimen is much bent, but would measure about 22 mm. in length if stretched out. The breadth is 8 mm., the height 8·5. The others are slightly smaller. In all the shape is high and rather narrow, rising up from the head to the centre of the back, and then sloping down to the tail.

On the dorsal margin, which does not project, are twelve to fourteen branchial tufts (fig. 15), of which the alternate ones are larger and set more inward, whereas the smaller are directed outwards. The longest do not project more than 2 mm. from the body and are stout, but not at all arborescent or foliaceous. They consist of two or three stems, arising from a prominence which hardly amounts to a common stalk. Each of these

stems is bifurcate, more rarely trifurcate, and each of these secondary divisions ends in three (sometimes only two) small blunt points.

Down the middle of the back runs a low distinct ridge, sending off side ridges to the large branchiæ but not to the small ones. There are two branchiæ on each side before the first of these ridges. There are four of the transverse ridges in the anterior part of the body, but in the posterior portion both the longitudinal and transverse ridges become obliterated.

The veil (fig. 16) is ample, not bilobed, 9 mm. wide and projecting 2·5 from the head without the processes. It bears at each end a grooved tentacle of the shape usual in Tritonia, and twelve to fourteen digitate appendages, large and small, alternating with fair but not absolute regularity. The larger measure 2 mm., the smaller are about half the size.

The rhinophore sheaths are rather low (2 mm.), fairly wide, with a wavy margin.

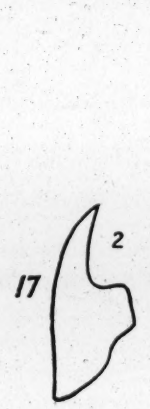
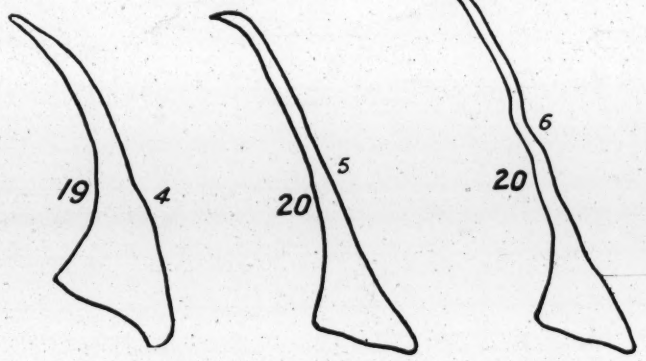


FIG. 17.—Central tooth, from the side.

FIG. 18.—Central tooth, from above.



FIGS. 19, 20.—Lateral teeth. 4 is nearer the rhachis than 5 and 6.

FIGS. 17 to 20.—*Tritoniopsis brucei.*

In front they carry two or three appendages, each bearing three points, and suggesting that a branchia is fused with the sheath. The club of the rhinophore is smooth and is surrounded by about twelve appendages, many of which are quite simple, while others bear a few pinnæ.

The orifices are not at all conspicuous. In the specimen in which they can be seen best the genital orifices lie below and between the fifth and sixth plumes, and the anus between the seventh and eighth, rather higher up but some distance from the dorsal margin.

The central nervous system resembles BERGH's figure of this organ in *Atthila ingolfiana* (*Nud. Gasteropoda of the Ingolf. Exp.*, pl. v. fig. 12). The four ganglia are all of much the same size and round. They are mottled and apparently granulate. The cerebro-pleural ganglia are not pear-shaped or larger than the pedal, and show no signs of a division into two halves. The buccal ganglia are rather large. No eyes are visible.

The jaws are yellowish, fairly hard and strong, rounded, not elongate, very convex.

The edge is smooth, and there seems to be no masticatory process. The radula is colourless and transparent. Seen from above, the median teeth (fig. 18) appear as simple, straight, pyramidal spines, rising from broadish bases; seen from the (fig. 17) side they are slightly bent downwards towards the tip, and somewhat resemble the teeth of Favorinus. They are quite smooth. The first laterals (fig. 19) are rather stouter than the others, but not of a different shape, as in Tritonia. The remaining laterals (fig. 20) are very long and thin, sometimes almost like filaments. They vary somewhat in shape: those nearer the rhachis are more distinctly hamate, those in the outer half of the row have a wavy or almost straight outline. They are difficult to count, as they seem to lie in sheaves, but the number on each side does not much exceed thirty at most.

The short and broad œsophagus leads straight into a rather small membranous and fragile stomach, almost entirely covered by the liver, and with no trace of plates. The liver is of a pale yellowish colour, covered with a thick layer of the hermaphrodite gland, which is of much the same hue, but still lighter. The albumen and mucous glands are greyish and of moderate size. The spermatotheca is yellow, roundish, small, with a long duct. The vas deferens not much convoluted. The verge is long, pointed, not armed; as preserved, it is curved at the end.

Scyllaea pelagica, L.

Ten specimens, captured on 1st July 1904, off floating gulf weed, 33° 53′ N., 32° 27′ W. They vary in length from 7·5 mm. to 13·5 mm. The colour is semitransparent white, with occasional minute spots of opaque white and a certain amount of yellowish-brown pigment, found chiefly on the margins and bases of the appendages, and differing in intensity and quality in the different individuals.

In some specimens there is nothing that can be called a caudal crest, the tail being merely rudder-like, and not raised above the level of the dorsal surface; but this peculiarity is not accompanied by any anatomical variation from the type, and passes into the normal form through intermediate stages.

XXIII.—On the Internal Structure of *Sigillaria elegans* of Brongniart's "Histoire des végétaux fossiles." By Robert Kidston, F.R.S.L. & E., F.G.S. (With Three Plates.)

(MS. received February 23, 1905. Read May 1, 1905. Issued separately June 30, 1905.)

Introduction.

Before giving a description of the specimen which forms the subject of this communication, it seems desirable that a brief summary of the literature dealing with the internal structure of *Sigillaria*, and some general remarks on the classification of the genus, should be given.

More for the purpose of convenience than on scientific grounds, the genus *Sigillaria* is usually divided into four groups. These groups were originally supposed by their founders to be of generic value, but experience has shown that the characters on which they were founded are more or less common to all four divisions, and in some cases the distinctive divisional characters even occur on the same specimen.

Group I. RHYTIDOLEPIS, Sternberg, 1823.

Stem ribbed, furrows distinct, straight or slightly flexuous. Leaf scars more or less distant, as wide as, or narrower than, the rib.

Group II. FAVULARIA, Sternberg, 1823.

Stem ribbed, furrows flexuous. Leaf scars with prominent lateral angles, and occupying the whole width of the rib. The lateral angles project slightly, and, alternating with those of the neighbouring leaf scars, impart a zigzag course to the furrows.

Group III. CLATHRARIA, Brongniart, 1822.

Stem without ribs. Leaf scars placed on contiguous rhomboidal, slightly elevated cushions, which are separated by deep oblique furrows.

Group IV. LEIODERMARIA, Goldenberg, 1857.

Stem without ribs. Leaf scars distant and unprovided with cushions.

Groups I. and II. pass into each other; and though in a few cases one can refer certain species to the *Favularia* section, such as *Sigillaria elegans*, there are others which so combine the characters of groups I. and II. that in practice it is impossible to treat them as members of distinct groups.

In the same manner, groups III. and IV. run into each other; and though some species appear to possess only the characters of the *Clathrariæ*, as *Sigillaria discophora*, König, sp. (= *Ulodendron minus*, L. & H.), and others those of the *Leiodermariæ*, as *Sigillaria camptotænia*, Wood, sp., still typical specimens of the groups *Clathraria* (*Sigillaria Brardi*, Brongt.) and *Leiodermaria* (*Sigillaria spinulosa*, Rost, sp.) have been found more than once organically united on the same example.*

A transition from the *Clathraria* to the *Favularia* can also be seen in *Sigillaria semipulvinata*, Kidston.†

It is therefore evident that, although the outer surface of the stems of *Sigillaria* differs in being ribbed or smooth, and in the leaf scars being distant or approximate, all these forms are closely connected by intermediate links; and though the larger division of ribbed and non-ribbed stems is generally very distinctive, still a case is not wanting to show how closely even these two groups stand to each other.

There are, however, differences in the structure of the vascular system of those stems of *Sigillaria* whose internal organisation is known, and though these differences are only variations of a single type of structure, they may hold a definite relation to the group of which the species is a member; so it is not without interest to consider this point in connection with the *group* to which the species belongs. Stems of *Sigillaria* showing their internal structure and the outer surface of the bark, and thus permitting of a specific determination, are, however, very rare.

The earliest description of the internal structure of *Sigillaria* is that given by Brongniart in his well-known memoir, "Observations sur la structure intérieur du *Sigillaria elegans* comparée à celle des *Lepidodendron* et des *Stigmaria* et à celle du végétaux vivants."‡ It is rather remarkable, however, that the plant Brongniart identified as *Sigillaria elegans* in this memoir is, as Zeiller§ has pointed out, the *Sigillaria Menardi*, Brongt.,‖ and therefore a member of the Clathrarian section.

In 1872 Williamson described some Sigillarian remains which he referred to *Favularia*, but his specimens were very imperfect, and threw little additional light on the subject.¶

It was not till the publication of Renault and Grand' Eury's memoir, "Étude sur

* Weiss, *Zeitsch. d. deut. geol. Gesell.*, 1888, p. 566. Zeiller, *Bull. Soc. géol. d. France*, 3e sér., vol. xvii. p. 603, pl. xiv., 1889. Zeiller, *Flore fossile, Bassin houiller et permien de Brive*, p. 83, pl. xiv. fig. 1, 1892. Kidston, *Proc. Roy. Phys. Soc. Edin.*, vol. xiii. p. 233, pl. vii. fig. 1, 1896.

† *Trans. Roy. Soc. Edin.*, vol. xxxix. p. 57, pl. iii. figs. 1–5, especially figs. 1, 2, 1897.

‡ *Archives du Muséum*, vol. i. p. 405, pls. xxv.–xxviii., 1839, Paris. See also Renault, "Structure comparée de quelques tiges de la flore carbonifère," *Nouvelles Archives du Muséum*, ii., 2e sér., 1879, p. 262, pl. xi. fig 13. Renault, *Flore fossile, Deux. part, Bassin houiller et permien d'Autun et d'Épinac*, fasc. iv. p. 200, pl. xxxvi. figs. 8–11, pl. xxxvii. figs. 3–7, 1896.

§ *Ann. d. Scienc. nat.*, 6e sér., *Bot.*, vol. xix. p. 259, 1884. Weiss, *Sitz. Bericht. d. Gessell. naturforsch. Freunde zu Berlin*, 1886, No. 5, p. 70.

‖ *Hist. d. végét. foss.*, pl. clviii. fig. 6 (? non fig. 5).

¶ "On the Organisation of the Fossil Plants of the Coal Measures." Part II. Lycopodiaceæ: Lepidodendra and Sigillariæ. *Phil. Trans.*, 1872, p. 197.

le *Sigillaria spinulosa*" * that any real advance was made in our knowledge of the structure of Sigillarian stems. This important memoir is a worthy companion to BRONGNIART's "Observations," and is one of the most valuable contributions which have been made to the subject.

Sigillaria spinulosa, Rost, sp., is the Leiodermarian condition of *Sigillaria Brardi*, Brongt., and is the type of the section *Leiodermaria*. Probably there are species of *Sigillaria* which possess a Leiodermarian type of cortex in all stages of their growth, but unfortunately the internal structure of none of these is known.

RENAULT gives some additional details of the structure of *Sigillaria* (*Clathraria*) *Menardi*, Brongt., and *Sigillaria* (*Leiodermaria*) *spinulosa*, Rost, sp., in the *Bassin houiller et permien d'Autun et d'Épinac* † and also in the same work describes a Sigillarian axis under the name of *Sigillaria xylina*. ‡

All the Sigillarian specimens whose structure was known previous to 1899 belonged to the *non-ribbed* members of the genus, but in that year Professor BERTRAND read an account of a ribbed *Sigillaria* (*S. elongata*, Brongt.) before the Botanical Section of the British Association at Dover. An abstract of this communication appears in the *Annals of Botany*, vol. xiii., 1899, p. 607. This paper gives the first clear account of the structure of a *ribbed Sigillaria*, and embraces all we know of the structure of this group, with the exception of a short account given by Dr SCOTT of a transverse section of a Sigillaria (*Rhytidolepis*) type, § and a note by WILLIAMSON, ‖ with a few explanatory remarks, where he refers the *Diploxylen* of his Memoir II. to *Sigillaria reniformis*, Brongt. ¶ This identification by WILLIAMSON is improbable, as *Sigillaria reniformis* has never, as far as I am aware, been found in so low a horizon as that from which WILLIAMSON's specimen came—the Lower Coal Measures.

The above brief sketch contains a note of the papers and works dealing with original investigations on the structure of undoubted stems of *Sigillaria* as far as known to me,** but before passing from the literature of this subject I wish to refer to a stem which has been described by Professor F. E. WEISS as "a biseriate Halonial branch of *Lepidophloios fuliginosus*," †† and which I think more probably belongs to the Ulodendroid group of Clathrarian *Sigillaria* than to *Lepidophloios*. I have come to

* Mém. présentés par divers savants à l'Acad. des Sciences de l'institut national de France, vol. xxii., No. 9, 1875, Paris, pls. i.–iv., pl. vi. figs. 33, 34. See also RENAULT, "Structure comparée de quelques tiges de la flore carbonifère," p. 264, pl. xi. figs. 17–21, pl. xii. figs. 1, 2, 1879. "Notice sur les travaux scientifiques de M. BERNARD RENAULT," Autun, 1896, p. 132.

† Fasc. iv. part ii., 1896, pp. 200, 208, pl. xxxvi. figs. 8–11, pl. xxxvii. figs. 3–7 and fig. 40 (*Sig. Menardi*), pl. xxxvi. figs. 2–5, pl. xli. figs. 4–11, 18–21, 23–26 (*S. spinulosa*).

‡ *L.c.*, p. 237, pl. xxxviii. figs.1–3. § *Studies in Fossil Botany*, p. 207, fig. 80, 1900.

‖ General, Morphological, and Histological Index to the Author's Collective Memoirs on the Fossil Plants of the Coal Measures. Part II. *Mem. and Proc. Manchester Lit. and Phil. Soc.*, Session 1892–93, p. 35, 1893.

¶ Pl. xxviii. figs. 33, 34.

** See, in addition to references already given, RENAULT, *Cours d. botan. fossile*, vol. ii., 1881. POTONIÉ, *Lehrbuch der Pflanzenpalæontologie*, 1899. ZEILLER, *Éléments de Paléobotanique*, 1900. SOLMS-LAUBACH, *Fossil Botany* (English translation), Oxford, 1891. KIDSTON, "Carboniferous Lycopods and Sphenophylls," *Trans. Nat. Hist. Soc. Glasgow*, vol. vi., new series, p. 101, 1891.

†† WEISS, *Trans. Linn. Soc. London*, 2nd ser., Bot., vol. vi. part 4, pp. 217–236, pls. xxiii.–xxvi., 1903.

this conclusion after a very careful examination of the specimen which was collected by the late Mr G. WILD, and sent to me for identification by Mr JAMES LOMAX, into whose possession it had come.

The following are my reasons for adopting the opinion that the more probable systematic position of this fossil is with the Ulodendroid Sigillariæ :—

I. The only carboniferous genera possessing biseriate cone scars, which have been definitely identified from the presence of leaf scars on their bark, are *Lepidodendron*, *Sigillaria* (Ulodendroid section), and *Bothrodendron*.

II. The specimen under discussion differs from the biserial *Lepidodendra* in the closely placed cone scars, and from *Bothrodendron* (*B. punctatum*, L. & H.) in the cone scars possessing a *central* vascular cicatrice, which in Bothrodendron is *eccentric*.

III. That in the position of the cone scars and their vascular cicatrices, and the distance arrangement of the leaf vascular cicatrices, it agrees entirely with specimens of *Sigillaria discophora*, König, sp.* (= *Ulodendron minus*, L. & H., sp.), when partially decorticated.

IV. That every Halonial (fruiting) branch of *Lepidophloios* which has shown the leaf scars, and so admitted of an undoubted identification, has had more than two rows of cone scars on the fully developed fruiting portion, and these are spirally arranged.

V. Dr HOYLE, Director of the Manchester Museum, has very kindly sent me for examination the specimen figured by Professor WEISS on his pl. xxiii. figs. 2, 3. In comparing these figures with fig. 1 of the same plate, it should be remembered that fig. 1 is natural size, and that figures 2, 3 are only $\frac{2}{5}$ natural size.

With the exception of a few places, the outer bark is removed from the Manchester Museum specimen (WEISS, *l.c.*, figs. 2, 3), and what remains is converted into bright coal. These coaly patches have the appearance of downward imbricating scales or cushions, but what their real structure has been cannot be clearly made out. Their lower ends are terminated by a fracture, and no leaf scars are shown.

The upper part of the fossil shows the two stumps of a bifurcation. On fruiting branches of *Lepidophloios* (*Halonia*) the fructification is frequently borne on the two forks of the stem immediately above the bifurcation. Below the fork the fructification sometimes begins as a single or double row, but when it extends to the forks the true multispiral arrangement is developed. This was pointed out many years ago by Dr (now Professor) J. M. MACFARLANE.†

The part represented by the specimen in the Manchester Museum may very probably represent a similar condition to that described and figured by Professor MACFARLANE, but as the two arms of the dichotomy are broken over on the example figured by Professor WEISS, the true spiral series which would naturally have occurred on the two arms is wanting. Had only a similar portion of Dr MACFARLANE's specimen been preserved, there would have been here a so-called "biseriate" *Halonia*, but that term cannot be applied,

* KÖNIG, *Icones fossilium sectiles.*, London, pl. xvi. fig. 194.
† *Trans. Bot. Soc. Edin.*, vol. xiv. pp. 186, 190, pl. vii.

as the "biseriate" *Halonia*—in this case at least—is only the beginning of the usual and typical multiseriate *Halonia*. It is quite possible that the specimen figured by Professor WEISS is a fragment of a Halonial branch of *Lepidophloios*, though the state of preservation of the specimen makes it quite impossible to affirm this positively; but though this point cannot be settled, I am perfectly satisfied that the fossil which forms the subject of Professor WEISS' figures 2, 3, pl. xxiii., does not represent the same species as that of which he describes the structure, and of which a figure is given, natural size, at fig. 1 of the same plate. The disposition of the cone scars shows this.

The specimen also figured by Professor WEISS on his plate xxiv. fig. 5, which is contained in the Williamson Collection, British Museum, No. 1945B, has very kindly been sent me for examination by Dr A. SMITH WOODWARD, F.R.S., Keeper of the Geological Department. Of this specimen Professor WEISS says, "The leaf bases are perfectly distinct over the whole surface, and their broad and fimbriating nature mark them out as belonging to *Lepidophloios*, as indeed was recognised by Williamson."* The characters here given as distinctive of *Lepidophloios* are not those which distinguish the genus, and afford no data for a generic identification. They would apply equally to most of the other genera of the Carboniferous Arborescent Lycopods if the leaves were forcibly broken over at a point above their attachment to the leaf cushion, as they appear to have been on this specimen.

This example, as I interpret it, is given in Professor WEISS' figure in inverted position. The smaller end, which he places downwards, I think is the upper end of the fossil. The outer surface bears the broken-over portions of the lower part of the leaves—not the persistent portion, which forms the leaf cushion on which the leaf scar occurs, but parts of the leaf while still attached to the cushion; hence no leaf scars are seen on the specimen. The fossil is too imperfectly preserved for a satisfactory determination, but in all the characters it shows they agree perfectly with those of *Sigillaria discophora*; and if I am correct in thinking that Professor WEISS has inverted the specimen in his figure—a view which I think his figure seems to bear out—then the "fimbriating" leaf bases point upwards. As already stated, from such imperfect material as that under discussion, it is extremely difficult, if not impossible, to arrive at any certain conclusion as to the nature of the fossil, but the fossil does not appear to me to show any of the characters of *Lepidophloios*, Sternberg.

The reference to Professor WILLIAMSON's figs. 27 and 28 of Memoir II.† throws no light on the point in question. WILLIAMSON describes his specimen as a *Ulodendron*, and at that date that was the genus into which specimens of *Sigillaria discophora* (= *Ulodendron minus*, L. & H.) would be placed; and, as far as can be learned from WILLIAMSON's figures and description, the specimen might as well belong to *Sigillaria discophora* as to *Lepidophloios fuliginosus*, Williamson.

I am further indebted to Dr A. SMITH WOODWARD for specimen No. 1949A of the Williamson Collection, and a transverse section cut from it (slide No. 1949), to which

* *Loc. cit.*, pp. 220–221. † *Phil. Trans.*, 1872, p. 209.

Professor Weiss also makes reference. My thanks are also due to Professor Bower for the loan of another transverse section from the same block, which he received from the late Professor Williamson.

The remaining portion of the block from which these slides were made (No. 1949A) is about $1\frac{1}{2}$ inches long, and shows a *subepidermal* surface with a single tubercle. The leaf cushions have been removed, so it thus possesses no external characters for a generic determination.

Part of a section made from this specimen is figured by Williamson in his Memoir XIX., on pl. iv. fig. 30.*

When one examines the remains of the original specimen and the slides made from it, it is difficult to interpret the relation of the parts to each other. One difficulty in explaining the relationship of the parts is, that if the two sides of the specimen represent the subepidermal surfaces of the original stem, how does the portion figured by Williamson come to hold the position that it does? Again, the pressure to which the specimen, and especially the vascular axis, has been subjected, makes a critical examination of its structure very difficult. These circumstances have prevented me from arriving at any conclusion as to the true position of this fossil.†

My thanks are due to Dr A. Smith Woodward, F.R.S., Keeper of the Geological Department of the British Museum, for kindly giving me the opportunity of examining the specimens in the "Williamson Collection," and to Dr Hoyle for sending me the one contained in the Collection of the Manchester Museum.

VI. That the vascular axis of the specimen described by Professor Weiss belongs to the same *type* of structure as the vascular axis of *Lepidophloios fuliginosus* is beyond dispute. The stems of these two plants are closely related in anatomical structure, but I do not think they are *identical*. Professor Weiss in his paper points out some slight differences, but what I regard as three important differences seem to be passed over in his description as of too little value for a separation of the two plants. The first is the very prominent and continuous (except where broken by the presence of leaf traces) band of tissue described by him as *phloem*. This layer is very much more developed in the specimen under discussion than in typical *Lepidophloios fuliginosus*. The second distinguishing point is the greater number of leaf traces given off by the vascular cylinder of Professor Weiss' specimen, while the third distinctive character is the presence of a *well-defined* pericycle.

The leaf traces are generally composed of a larger number of elements than those of *Lepidophloios fuliginosus*, and therefore appear more prominent, and are frequently larger in transverse section. From these causes. and the fact of the leaf traces being more numerous, they form a much more distinct ring of leaf traces surrounding the axis than do those of what I regard as the true *Lepidophloios*

* *Phil. Trans.* vol. clxxxiv. p. 20, 1893.

† Another portion of this specimen (Williamson, 1949A) is in the Wild Collection, Manchester Museum.

fuliginosus of WILLIAMSON. If I am correct in my opinion that the specimen described by Professor WEISS belongs to *Sigillaria discophora*, the leaf traces should be more numerous than those given off by *Lepidophloios fuliginosus*, for in *Sigillaria discophora* the leaf scars are smaller, and consequently more numerous in a given space than those of *Lepidophloios*, and the close position of the leaf scars is very clearly seen in tangential sections of the specimen described by Professor WEISS, of which I possess a fine series of sections. The same character and their quadrangular arrangement are also seen on the outer surface of the specimen as figured by Professor WEISS on plate xxiii. fig. 1.

These differences are slight, and even if they did not exist within the two stems the arrangement of the cone scars is sufficient to separate the *Ulodendroid Sigillaria* from *Lepidophloios*; and in a group containing so many closely allied genera as are known to exist amongst the Carboniferous Arborescent Lycopods, one must expect to find in the internal structure of their stems a great similarity.

VII. In regard to the specimen under discussion, and which has been referred to *Lepidophloios fuliginosus* by Professor WEISS, it appears to me that the external characters of the fossil point more strongly to its belonging to *Sigillaria discophora* (= *U. minus*) than to *Lepidophloios*; and further, that its internal structure, though of the same type, is not identical with that of *Lepidophloios fuliginosus* of WILLIAMSON.

In concluding this criticism, I wish clearly to state that I do not say that a "biseriate *Halonia*"—that is, a Halonial condition of *Lepidophloios*, on which the cones are arranged in two rows, "Ulodendroid" fashion—does not exist; but what I do maintain is, that if there is a *Lepidophloios* which bears its cones in two opposite and vertical rows, and in this mode alone, proof of its existence has still to be given.

DESCRIPTION OF SPECIMEN.

Sigillaria elegans of Brongniart's *Histoire des végétaux fossiles*, vol. i. p. 438, pl. cxlvi. fig. 1, pl. clv., pl. clviii. fig. 1.

As some botanists doubt the identity of BRONGNIART'S *Sigillaria elegans* with the *Sigillaria* (*Favularia*) *elegans* of STERNBERG,* I have adopted BRONGNIART'S name as my authority for the species whose structure is about to be described, as my specimen agrees in all respects with the descriptions and figures given in the *Hist. d. végét. foss.* On the other hand, I believe that BRONGNIART was quite correct in identifying his specimen with *Favularia elegans*, Sternberg; for if STERNBERG'S figure be inverted and so brought into its natural position, and if a very slight allowance be made for the delinquencies of the artist, it is difficult to see how there can be any real difficulty in recognising the specific identity of BRONGNIART'S and STERNBERG'S specimens.†

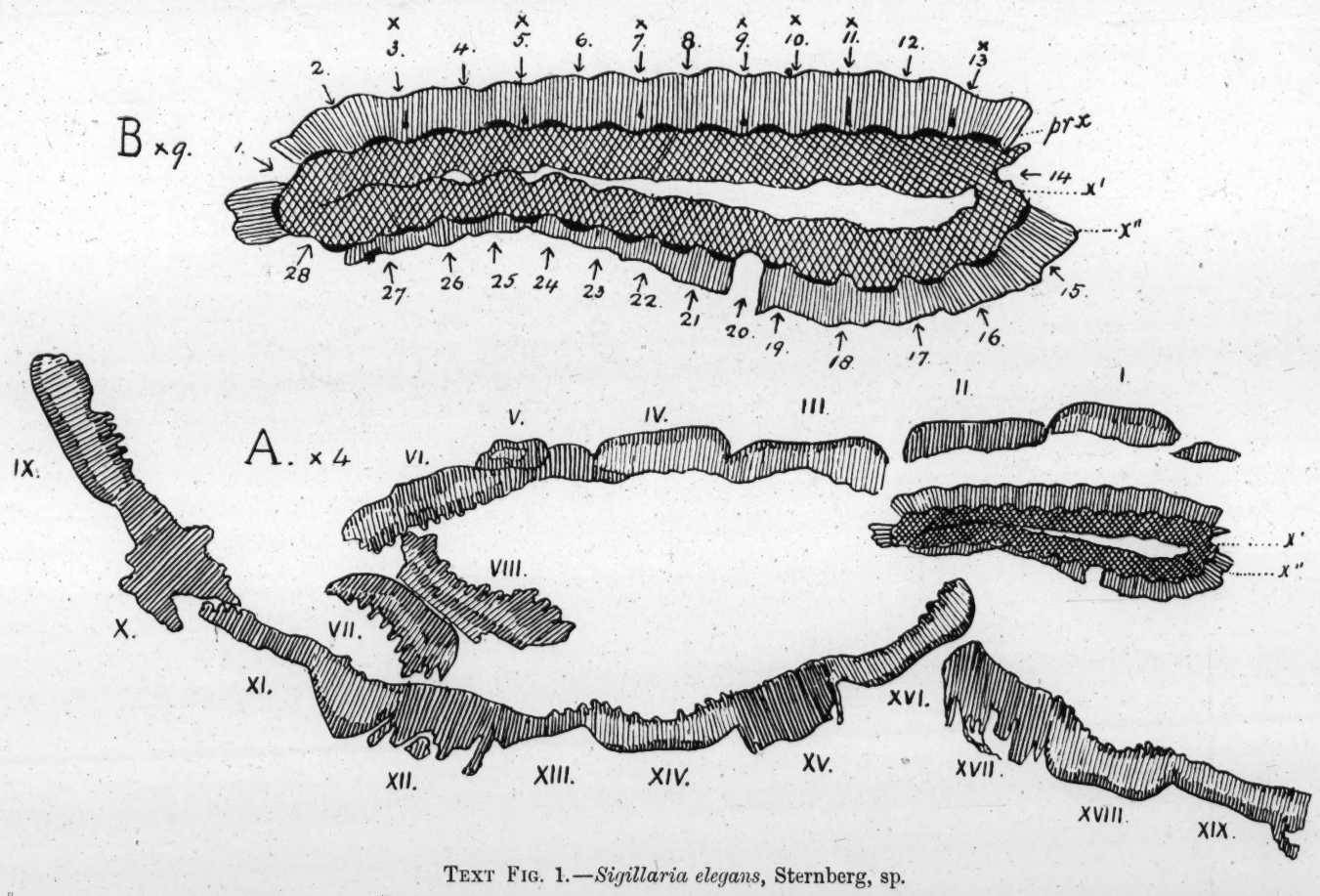
* STERNBERG, *Essai flore monde prim.*, vol. i. fasc. 4, pp. xiv, 48, pl. lii. fig. 4, 1826.

† See also ZEILLER, *Flore foss. Bassin houil. d. Valenciennes*, p. 582, pl. lxxxvii. figs. 1–4, Atlas 1886, Text 1888.

General Description of Specimen.—The specimen was contained in one of the well-known coal balls from the Halifax Hard bed near Huddersfield, Yorkshire (Lower Coal Measures), and was found by Mr W. Hemingway, by whom it was communicated to me, and to whom my thanks are due for the opportunity of describing the specimen.

Through a fortunate fracture in the stone, part of the outer surface of the specimen was exposed, showing a well-preserved row of leaf scars. This surface of the specimen is shown natural size at fig. 1, Pl. I., and enlarged two times at fig. 2.

With the exception of the row of leaf scars, the remainder of the surface of the specimen exposes the layer which lies immediately underneath that on which the leaf cushions sit. Its surface shows the leaf traces surrounded below by the parichnos,



Text Fig. 1.—*Sigillaria elegans*, Sternberg, sp.

A. Transverse section of the specimen showing the vascular cylinder and portion of the cortex. The ribs are numbered I. to XIX. × 4.

B. Transverse section of the vascular cylinder. The furrows on outer margin of primary wood are numbered 1 to 28. × 9. Protoxylem, *prx*. Primary xylem, *x'*. Secondary xylem, *x''*. Leaf traces are shown in the furrows marked with a ×. Slide No. 964.

which form a semicircle, Pl. I. fig. 2. At this point the parichnos do not seem to completely surround the leaf trace. Between the narrow ridges on which the leaf trace scars are seen, the exposed surface is striated with close, slightly flexuous longitudinal ridges.

An outline sketch of a transverse section of the specimen is shown at text fig. 1, A, enlarged four times. Towards the right is seen the vascular cylinder, the two sides of which are pressed together, and only a small part of the pith-cavity is seen.

The complete zone of cortex bore twenty-eight ribs, of which nine are not seen in the section, but the figure allows one to estimate the proportional size of the vascular cylinder to that of the circumference of the stem.

The stele, which measures slightly under 13 mm. in its longest compressed diameter, consists of a perfectly continuous ring of primary wood, x', which is surrounded by a zone of secondary wood, x''.

The pith and the tissue which originally composed the inner portion of the cortex and all remains of the phloem elements, have been entirely destroyed. The only other remaining portion of the stem in which the structure is preserved is the outer layer of the cortex on which the leaf cushions are situated. The structure of these parts may now be considered in detail.

The Primary or Centripetal Xylem.—At fig. 3, Pl. I., is shown a transverse section of the vascular cylinder, enlarged about $4\frac{1}{2}$ times. The ring of primary xylem $x.'$ is quite continuous, and is about 0·70 mm. thick. As seen in fig. 4, Pl. I., its outer margin is deeply and regularly undulate or crenate, so as to form a number of blunt ridges alternating with as many intervening furrows. The inner margin of the xylem ring is very uneven, sending irregular toothed projections into the now empty pith-cavity (Pl. I. fig. 4, $x.'$ and *p.c.*).

The main mass of the primary xylem consists of large tracheides, more or less hexagonal in transverse section, and without any intervening parenchymatous cells. They diminish slightly and gradually in size towards without, but just underneath the ridges a decrease in size takes place somewhat abruptly, and the ridges themselves are composed of much narrower elements, that are to be regarded as constituting the protoxylem (Pl. I. fig. 4, *prx.* Text fig. 1, B, *prx.*).

At the inner margin of the primary xylem a certain amount of thin-walled parenchymatous cells are to be found between the tracheides, and a few of the latter may even be surrounded by parenchyma, and separated from the rest of the mass. Some especially narrow tracheides also occur here and there along the inner margin.

Radial longitudinal sections, Pl. II. figs. 6 and 8, show that all the elements of the primary xylem, both protoxylem (Pl. II. fig. 7, *prx.*) and metaxylem (Pl. II. fig. 9, $x.'$), are elongated scalariform tracheides with pointed ends (fig. 7, *prx.*, fig. 9, $x.'$), except a few of the innermost tracheides bordering on the pith, which are quite short, blunt-ended, and irregular in shape, but still scalariform. So far as observed, no spiral or annular elements at all were found at any point in the primary xylem, not even in the protoxylem.

The protoxylem elements are seen in radial section at *prx.*, figs. 6 and 8, Pl. II. They are long and narrow, and terminate in an elongated conical point (Pl. II. fig. 7, *prx.*).

Secondary or Centrifugal Xylem.—This forms a zone of varying thickness surrounding the primary xylem, and attaining at its widest part a breadth of about 0·75 mm. (Pl. I. figs. 3, 4, and 14, $x.''$). Its inner margin follows the crenulate outline of the primary xylem. The outer margin exhibits the same crenulations, but to a slightly less degree.

The tracheides of the secondary xylem are arranged in radial rows (Pl. I. figs. 3, 4, and 14, *x.''*), interspersed at intervals by numerous medullary rays which run uninterruptedly through the whole thickness of the secondary wood.

The number of rows of tracheides at the inner margin of the secondary wood is greater than at the outer margin, so that many of the rows begun at the inner edge come to an end after running a certain distance, their loss being compensated for by the increased size of the tracheides in the rows which persist.

At figs. 6 and 8, Pl. II., are radial longitudinal sections of the stele, which show that the tracheides of the secondary xylem are elongated and scalariform, with pointed ends, similar to those of the main body of the primary xylem (metaxylem), only somewhat smaller in diameter.

The tracheides are of the same size throughout the secondary xylem, there being no difference between those opposite the ridges and those opposite the furrows.

Occasionally one or two thin-walled cells intervene between the primary xylem and the inner tracheides of the secondary xylem, but frequently the primary and secondary xylem are in direct contact.

The medullary rays (Pl. I. fig. 14, *m.r.*, Pl. III. fig 10, *m.r.*) are usually one cell wide, rarely two, and are formed of thin-walled cells, slightly elongated radially. Their walls sometimes bear delicate scalariform thickenings (Pl. I. fig. 14, *m.r.*). Similar cells have been figured by WILLIAMSON.*

In tangential section (Pl. I. fig. 28, *m.r.*) the medullary rays are seen to vary much in height, and may consist of from one to nine superposed cells. In many cases the cells have become decayed, and their position is only indicated by a larger or smaller lenticular space.

Leaf traces.—The leaf traces arise from the periphery of the primary xylem, and invariably at the base of the furrows, never from the tops or sides of a ridge (text fig. 1, B; Pl. I. fig. 4, *l.t.*; Pl. II. figs. 11, 12, 13, *l.t.*); and as the leaf traces appear to keep in the same vertical plane in their course through the cortex, the furrows on the primary xylem must correspond in position to the ribs on the surface of the stem.

In text fig. 1, B, the vascular cylinder is enlarged about nine times. This shows twenty-eight furrows, so there must have been twenty-eight vertical rows of leaves; and although text fig. 1, A, only shows nineteen, before the specimen was cut, the full number was actually present as already mentioned.

When the leaf trace is about to become free from the metaxylem at the bottom of the furrow it consists of a group of about a dozen small tracheides, arranged radially around the smallest of the group, which lies approximately in the centre (Pl. II. fig. 11, *l.t.*). Followed downwards, these small tracheides are seen to spread out over the surface of the furrow, and finally die out below (Pl. II. fig. 17, *l.t.*).

When the leaf trace becomes free from the primary xylem it first runs upwards and slightly outwards (Pl. II. figs. 12 and 13), then bending abruptly outwards, passes

* Memoir XIX., *Phil. Trans.*, vol. clxxxiv. (1893), p. 33, pl. iv. fig. 38.

out through a medullary ray in an almost horizontal direction. Once free of the secondary xylem it suddenly bends upwards again (Pl. II. fig. 15, *l.t.*).

By the time the leaf trace emerges from the secondary wood one or two additional rows have been added, mostly on the side next the axis, but a few may have been added to the opposite side, and the leaf trace now shows a distinct mesarch structure (Pl. II. fig. 15, *l.t.*). On another leaf trace, which is slightly further removed outward from the axis, a greater number of tracheides have been added, and the increase seems to have been more equally placed on all sides of the primary group (Pl. II. fig. 16, *l.t.*).

The course of the leaf traces through the xylem, and their behaviour on their emergence from it, seems to be similar to that so well represented in the figure of the radial section of *Sigillaria Menardi* given by RENAULT.*

It was impossible to discover any annular or spiral tracheides in the leaf trace, but their structure is very much effaced in their passage through the secondary xylem.

Sigillaria elegans does not show any secondary wood in connection with the leaf trace up to their entrance into the inner cortex and in the outer layer of the bark (the only portion of the cortex preserved); the leaf traces are too imperfect to permit of their structure being made out.

Cortex.—The only part of the cortex which has been preserved is that formed by the ribs, which must probably be regarded as composed of the confluent persistent leaf bases and a small portion of the underlying periderm.

Fig. 18, Pl. I., shows the outer surface of a specimen of *Sigillaria elegans*, collected by Mr W. HEMINGWAY from the Yorkshire Middle Coal Measures. In this condition the leaf cushions are usually compressed. They are arranged on the ribs in vertical rows, the leaf scars of one row alternating with those on the contiguous rows. The leaf scars are subhexagonal, with prominent lateral angles, which alternate with those of the neighbouring leaf scars, and thus impart a zigzag course to the furrows between the ribs. The cones are borne in a verticil formed of a single row, and some of the cone scars are seen in the figure at *c.s.*

In the uncompressed condition the leaf cushions slope outwards, and their lower margin becomes considerably elevated. This is shown on Pl. III. fig. 19, which is a radial section through a rib of *Sigillaria elegans*. The leaves are attached to the sloping surfaces, *s.a.* This figure is from a specimen received from Mr JAMES LOMAX, and was derived from the Halifax Hard bed.

The surface of the leaf scar is formed of a stratum of dark broken-down parenchymatous tissue. The main body of the leaf cushion below this consists of fairly thick-walled parenchyma, the cells of which are more or less oval, with their longest diameter directed outwards. Towards the upper adaxial side of the cushion these cells become smaller, and seem to contain a dark-coloured material, which causes the tissue in this region to become opaque (Pl. III. fig. 5). The lower margin is also formed by similar small opaque cells, but not developed to the same extent.

* *Flore fossile, Bassin houiller et permien d'Autun et d'Épinac*, Deux. part, Atlas, pl. xxxvii. fig. 6, 1893.

At *l.t.*, fig. 5, Pl. III., is seen the opening through which the leaf trace has passed to the leaf. The leaf cushions are attached to a continuous zone of underlying periderm (Pl. III. fig. 19, *pd.*), formed of very dense elongated fibres of small diameter, with fairly thick walls. In longitudinal section this tissue is usually very opaque, and seldom shows its structure distinctly. In transverse section (Pl. III. fig. 21) the fibres are seen to be arranged in close radial rows.

If a transverse section of cortex with several attached leaf bases be examined, it will be seen that some of the leaf bases have only a narrow band of dark dense tissue, as at Pl. I. fig. 3, *l.c.*, whereas others seem to be entirely composed of it, as at fig. 3 *l.c.'* This is explained by reference to fig. 5 and fig. 19, Pl. III. If the section passes through the lower part of the cushion the greater part of the section will show transparent parenchyma, whereas if the section passes through the upper part of the cushion it will be composed of dense opaque tissue.

A leaf cushion in transverse section is given on Pl. III. fig. 20. The dark broken-down margin is reduced here in thickness (*p.p.*), while the more transparent parenchyma occupies at this position the greater width of the cushion. At *pd.* are seen some fragments of the periderm.

The cause of the flexuous striations on the surface of the stem (Pl. I. fig. 2) where the leaf cushions have been removed is not very clear, but they seem to be formed at the point of union of cortex and periderm, and are probably caused by the projecting teeth of the latter (see Pl. III. fig. 20, *pd.*).

A section of a cushion approximately parallel with and a short distance below the outer surface is given on Pl. III. fig. 22. The two parichnos (*par.*) are seen, but the delicate tissue with which they were filled has mostly decayed. The leaf trace lying between them is too imperfectly preserved to admit of a detailed description. The parenchyma of the groundmass is here well preserved (Pl. II. fig. 24).

Cone Scars.—On the specimen showing the outer surface of the bark given on Pl. I. fig. 18 part of a verticil of cone scars is seen at *c.s.*

In one of the tangential sections of the cortex of the specimen shown on Pl. I. fig. 1 two vascular strands belonging to cones are cut through. This specimen is shown enlarged eight times at fig. 23, Pl. III., where the branches going to the cones are lettered *c.s.* and *c.s.'*

The cone axis *c.s.'*, fig. 23, Pl. III., is seen enlarged on Pl. III. fig. 25.

The peripheral zone of cortex, *b*, which has accidentally separated off from the rest, is too much disorganised to show the structure, but probably corresponds to the periderm. The space lettered *d* is a mechanically produced rupture.

The rest of the cortex consists of fairly thick-walled and apparently parenchymatous cells, which increase in diameter as they are traced inwards (Pl. III. fig. 26). This tissue, which undoubtedly belongs to the cortex, is separated from the solid axis of the cone by a clear empty zone (Pl. III. fig. 25, *g*, and Pl. I. fig. 27, *g*), which was probably occupied by some delicate tissue.

The cells of the innermost layer of cortical tissue lining this zone consist of smaller cells, roughly tabular in transverse section, and have very much the appearance of an endodermis (Pl. III. fig. 26, *e*; Pl. I. fig. 27, *e*).

Within the clear zone is a central mass of xylem (Pl. III. fig. 25, *k*; Pl. I. fig. 27, *k*), surrounded by a fairly stout zone of comparatively large thin-walled cells, separated from the xylem by a dark line of disorganised material (Pl. I. fig. 27, *h*). The xylem is too badly preserved for detailed description, but it is quite clear its development is centripetal. The smallest elements are at the periphery, and there is a gradual increase in size towards the centre. There is no trace of any secondary thickening.

The zone of thin-walled tissue surrounding the xylem possibly represents the so-called phloem of *Lepidophloios*, but the preservation is so imperfect that a definite decision as to its true nature is not permissible.

Remarks.—The structure of *Sigillaria elegans*, Brongt., is very similar to that of *Sigillaria elongata*, Brongt., the only other ribbed *Sigillaria* of which any detailed account of the structure has been given, but the short abstract of his paper given by Professor C. Eg. Bertrand scarcely enables one to make a critical comparison between the structure of the two species.* He has, however, very kindly favoured me with some photographs of his specimen, and the general agreement in structure between the two plants is very striking.

They both have a continuous circle of primary wood, with a distinct *corona*. In the Hardinghen specimen the projecting teeth of the protoxylem are pointed, while in the Yorkshire example they are rounded and contain a greater number of elements. The secondary xylem seems to offer no special point of difference.

On my specimen the leaf traces rise as a single strand directly from the base of the furrows of the primary xylem, and they may differ somewhat in their mode of origin from those described by Professor Bertrand;† but until the full description of the Hardinghen specimen is published, it is undesirable to institute any critical comparison between the anatomy of *Sigillaria elongata* and *Sigillaria elegans*.

In comparing the organisation of the various species of *Sigillaria* of which the structure has been described, one might be inclined to think that in the genus there were two types of structure; this, however, is not so. If one takes the *Sigillaria Menardi* described by Brongniart, with its isolated strands of primary xylem, and *Sigillaria elongata* or *Sigillaria elegans*, with their continuous circle of primary wood and prominent corona, they at first sight look very distinct. But in *Sigillaria spinulosa*, which normally is supposed to possess the same type of structure as *Sigillaria Menardi*, Solms-Laubach has pointed out that the separate bundles of primary wood sometimes coalesce, and in part form a continuous ring.‡ This character of the coalescence of the primary bundles of *Sigillaria* is also well seen in the figures of *Sigillaria spinulosa* given by Dr Scott.§ In *Sigillaria xylina*, Renault, the same

* *Annals of Botany*, vol. xiii. p. 607, 1899. † *Loc. cit.*, p. 608. ‡ *Fossil Botany*, English ed., p. 252, fig. 29, 1891.
§ *Studies in Fossil Botany*, p. 201, figs. 77–78, 1900.

character has been observed.* It is therefore seen that the isolated strand type of primary wood, and that even in the same specimen, passes into the continuous type, and that there is between the two forms an unbroken chain which connects them together.†

If the structure of *Sigillaria elongata* or *Sigillaria elegans* be compared with the structure of the large specimen of *Lepidophloios* described by SEWARD and HILL,‡ and which they believe to be the *Lepidophloios Harcourti*, Witham, sp., or with any of the Lepidodendroid stems whose primary wood is provided with a *corona*, the great similarity in structure is very apparent. The *corona* on the stems of *Lepidophloios*, though less prominent than in *Sigillaria*, is most distinctly present, and is also formed by the protoxylem elements. The leaf traces are also given off from the dividing bays, and the difference between the *corona* of many of the *Lepidodendreæ* and the *Sigillariæ* is only one of degree. If a series be arranged, beginning with *Lepidophloios Harcourti*, followed by *Sigillaria elongata*, and concluding with *Sigillaria elegans*, it will be seen that by a gradual increase of the size of the teeth of the corona you pass insensibly from the *Lepidophloios* structure to that of the *Sigillariæ* which possess the continuous ring of primary wood. The distinction at one time supposed to exist between the *Sigillariæ* and the *Lepidodendreæ*, of the former possessing secondary wood in connection with the leaf trace, is now found not to hold, for a considerable development of secondary wood takes place in the leaf traces of *Lepidophloios*.§ On the other hand, no development of secondary wood on the leaf traces of *Sigillaria elongata* or *Sigillaria elegans* has yet been observed.

Whether we are justified in classing all of the *Lepidodendreæ* with a *corona* on the primary xylem with *Lepidophloios* may be open to question, though it is certain that some *Lepidophloios* had primary wood so formed; and though on some stems of *Lepidodendron* the primary wood has an even contour, still in other *Lepidopendra* and in *Bothrodendron* ‖ the primary wood has a slightly undulating outline, so that in the Carboniferous *Lycopodiaceæ* there is a continuous chain of structure variation in the arrangement of the protoxylem elements which binds closely together all the genera of the Carboniferous Arborescent Lycopods. Between no two genera is there any outstanding character in the structure of the vascular cylinder which sharply separates them from each other. It seems, therefore, highly probable, as suggested by ZEILLER, that the Carboniferous Arborescent Lycopods have descended from a common stock.¶ In their fructification and certain other points, however, these ancient lycopods differed from each other in several important characters.

I am inclined to regard the Arborescent Lycopods as a group which has left no

* RENAULT, *Bassin houiller et permien d'Autun et d'Épinac, Flore fossile*, Deux. part, p. 238, 1896.

† It might be mentioned that RENAULT has described a *Lepidodendron* (*L. Jutieri*) in which the vascular system is formed of a circle of separate bundles. "Structure comparée de quelques tiges de la flore carbonifère" (in *Nouv. Archives du Mus.*, ii., 2e sér., 1879, p. 258; also RENAULT, *Cours d. bot. fos.*, vol. ii. p. 28, 1882.

‡ *Trans. Roy. Soc.* Edin., vol. xxxix. p. 907, pls. i.–iv., 1900. § SEWARD and HILL, *l.c.*, p. 914.

‖ From the discovery by Mr JAMES LOMAX of a specimen showing the outer surface of the bark, it has been shown that the *Lepidodendron mundum*, Williamson, is a *Bothrodendron*.

¶ ZEILLER, *Éléments paleobotanique*, p. 178, 1900.

descendants except in the case of *Sigillaria*, the structure of whose cone shows some similarity with the fructification of *Isætes*. Our other modern lycopods (*Lycopodium* and *Selaginella*) seem to have descended from the Carboniferous genus *Lycopodites*, with which they show much in common.

The geological distribution of the *Sigillaria* whose structure is known also brings out an interesting point; and though the evidence may not be sufficient for any definite conclusions, still it indicates changes which deserve recognition.

The following table gives the age of the rocks which have yielded *Sigillaria*, showing their internal structure, and also indicates a few of the more prominent characters of the species discovered.

TABLE.

Lower Permian	*Sigillaria Menardi*, Brongt.	Stem without ribs (Clathraria section)	Primary xylem a circle of separate bundles. Secondary xylem forming a centrifugal zone.
Do.	*Sigillaria spinulosa*, Rost, sp. (= *Sigillaria Brardi*, Brongt.)	Stem without ribs (Clathraria and Leiodermaria sections)	Primary xylem a circle of separate bundles, some of which frequently coalesce. Secondary xylem forming a centrifugal zone.
Upper Carboniferous Upper Coal Measures	No specimens showing structure from this horizon		
Middle Coal Measures (= "Westphalian Series")	*Sigillaria elongata*, Brongt.	Stem ribbed (Rhytidolepis section)	Primary xylem in form of a closed ring. Secondary xylem forming a centrifugal zone.
Lower Coal Measures	*Sigillaria elegans*, Brongt.	Stem ribbed (Favularia section)	Primary xylem in form of a closed ring. Secondary xylem forming a centrifugal zone.

The non-ribbed *Sigillariæ* are more characteristic of higher horizons and the ribbed *Sigillariæ* of the lower horizons, but neither group is restricted to either series.

The genus *Sigillaria* also extends into the Lower Carboniferous, where it is, however, very rare, but no specimens showing structure have yet been recognised from these rocks. In the following list I only mention the British species:—

Lower Carboniferous.

Carboniferous Limestone Series.

Sigillaria Youngiana, Kidston . . Stem ribbed.
Sigillaria Canobiana, Kidston . . Stem ribbed.
Sigillaria Taylori, Carr., sp. . . . Stem without ribs. (Ulodendroid Clathraria.)

Calciferous Sandstone Series.

Sigillaria Taylori, Carr., sp . . . Stem without ribs. (Ulodendroid Clathraria.)

The lowest horizon from which I have seen the *typical Clathrarian Sigillariæ* is the Lower Coal Measures, but from this horizon I have only seen a single example.

The non-ribbed *Sigillariæ* of the Ulodendron-Clathrarian group (*Sigillaria discophora*, König, sp. = *Ulodendron minus*, L. & H., and *Sigillaria Taylori*, Carr., sp.) extend into both divisions of the Lower Carboniferous, but the ribbed *Sigillariæ*, although they occur in the Lower Carboniferous, do not, as far as I know, extend to the base of the Carboniferous Limestone Series.

If I am correct in believing that the stem whose structure has been described by Professor Weiss as "a Biseriate Halonial Branch of *Lepidophloios fuliginosus*" is the *Sigillaria discophora*, König, sp., with which *Sigillaria Taylori*, Carr., sp., is very closely related, then the probability is that *Sigillaria Taylori* also possessed primary wood of the continuous ring type, and the same may be presumed for the two-ribbed *Sigillariæ* from the Carboniferous Limestone Series.

If it is permissible to assume these probabilities—and the assumption is not without some support from the known structure of the Middle and Lower Coal Measure species of *Sigillaria*—then it is probable that the continuous ring of primary xylem is the older type of Sigillarian stem structure, and that the circle of isolated strands which form the primary xylem of the Clathrarian Sigillariæ of the higher geological horizons has originated by a splitting up of the continuous ring type of bundle; and, as already mentioned, even in the few Clathrarian Sigillariæ from the higher horizon of which the structure is known, the actual transition from the one type to the other can be observed.

The *Lepidodendra* form, however, an older genus than *Sigillaria*, and extend to the base of the Carboniferous Formation. In beds not far above the base and low down in the Calciferous Sandstone Series specimens of *Lepidodendron* showing structure have been found; and of two of these occurring in the same bed, one species shows the continuous ring of primary wood, while the other possesses a solid cylinder of primary wood without any trace of pith; and although there occur here the two types of primary wood, side by side, still the solid cylinder type seems to be more common in the lower than in the upper horizons of Carboniferous Rocks, and the sequence of changes in the development of the primary xylem of the palæozoic Arborescent Lycopods seems to point to the *solid* vascular cylinder as the oldest type, from which has been derived the medullate cylinder with a *continuous ring* of primary wood, and this continuous ring of primary wood has, in turn, broken up to form the *isolated strands* of primary wood found in the Clathrarian *Sigillariæ*.*

I wish to express my thanks to Mr D. T. Gwynne-Vaughan, Glasgow University, for much kind criticism and advice while preparing this paper.

* See note on p. 546 (*Lepidodendron Jutieri*, Renault).

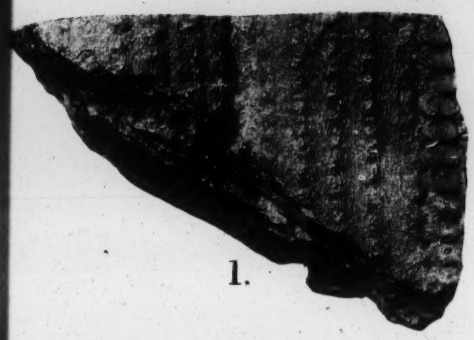
EXPLANATION OF PLATES I.–III.

Sigillaria elegans, Brongt. (Sternb., sp.).

Pl.	fig.	
I.	1.	Portion of outer surface of the specimen from near Huddersfield. Halifax Hard bed, Lower Coal Measures. Natural size. To the right is seen a row of leaf scars. Specimen No. 3497.
I.	2.	Part of the same specimen enlarged two times. To the right is seen the vertical row of leaf scars, and to the left the surface of the cortical layer which bears the leaf cushions.
I.	3.	Transverse section of the vascular cylinder and part of the cortex × 4½. Primary xylem, *x.'* Secondary xylem, *x.''* Leaf cushions, *l.c.* and *l.c.'* Slide No. 964.
I.	4.	Portion of vascular cylinder × 35. Protoxylem, *prx.* Primary xylem, *x.'* Secondary xylem, *x.''* Pith-cavity, *p.c.* Slide No. 964.
II.	5.	Longitudinal (radial) section of leaf cushions × 20. Spongy transparent parenchymatous tissue, *p.* Dense tissue of smaller cells, *p.p.* Leaf trace opening, *l.t.* Very dense tissue at upper left corner of figure, periderm. Slide No. 841.
II.	6.	Radial section passing through xylem × 95. Protoxylem, *prx.* Primary xylem, *x.'* Secondary xylem, *x.''* Slide No. 967.
II.	7.	Radial section passing through xylem × 170. At *prx.* is shown the termination of a tracheide belonging to the protoxylem. Slide No. 967.
II.	8.	Radial section passing through xylem × 95. Protoxylem, *prx.* Primary xylem, *x.'* Secondary xylem, *x.''* Slide No. 968.
II.	9.	Radial section passing through primary xylem × 95. At *x.'* is shown the termination of a tracheide. Slide No. 970.
III.	10.	Transverse section of secondary xylem × 95. Showing medullary rays, *mr.* Slide No. 964.
II.	11.	Transverse section showing leaf trace × 85. Protoxylem, *prx.* Primary xylem, *x.'* Leaf trace, *l.t.*, which is about to become free from the metaxylem. Slide No. 961.
II.	12.	Transverse section showing leaf trace × 85. Lettering as before. The leaf trace has just become free from the metaxylem. Slide No. 966.
II.	13.	Transverse section of leaf trace × 85. Lettering as before. The leaf trace has moved outwards and is becoming suddenly bent to pass out through a medullary ray. Slide No. 964.
I.	14.	Transverse section of secondary xylem, showing cells of medullary ray with spiral thickening *m.r.* × 115. Slide No. 962.
II.	15.	Transverse section showing leaf trace at outer margin of secondary xylem × 85. Medullary ray, *m.r.* Secondary xylem, *x''*. Leaf trace, *l.t.* The leaf trace has bent upwards after emerging from the secondary xylem, and is cut through at approximately right angle. Slide No. 962.
II.	16.	Transverse section of leaf trace quite free from the outer margin of the secondary xylem × 85. Slide No. 962.
II.	17.	Transverse section of xylem × 95. Primary xylem, *x.'* Secondary xylem, *x.''* At *l.t.* is seen the narrow spread-out base of the leaf trace shortly before it dies out. Slide No. 964.
I.	18.	Outer surface of a specimen showing the leaf scars and portion of a circle of cone scars, *c.s.* Natural size. From Wombwell Main Colliery, near Barnsley, Yorkshire. Horizon, Barnsley Thick Coal, Middle Coal Measures. Collected by Mr W. Hemingway. Specimen No. 989.
III.	19.	Radial (longitudinal) section passing through six leaf cushions × 4. Periderm, *p.d.* Surface to which leaf was attached, *s.a.* Opening through which leaf trace passed, *l.t.* The cushion marked *l.t.* is enlarged at fig. 5. Slide No. 841.
III.	20.	Transverse section of cushion × 14. Transparent parenchymatous tissue, *p.* Broken down margin of cushion, *pp.* Periderm, *pd.*, *pd.'* The part marked *pd.'* is enlarged at fig. 21. Slide No. 962.

Pl.	fig.	
III.	21.	Transverse section of periderm × 50. The part enlarged is seen on fig. 20, *pd.'* Slide No. 962.
III.	22.	Section of leaf cushion approximately parallel with the surface × 14. Position of leaf trace, *l.t.* Parichnos, *par.* Transparent parenchymatous tissue, *p.* Slide No. 973.
III.	23.	Section approximately parallel with outer surface of stem × 8. Free portions of leaf cushions, *l.c.* Cone pedicels, *cs.*, *cs.'* Slide No. 972.
II.	24.	Portion of transparent parenchymatous tissue of the leaf cushion shown at fig. 22, *p.*, × 60. Slide No. 973.
III.	25.	Transverse section of cone pedicel × 20. Slide No. 972. See description in text, page 544.
III.	26.	Portion of the cortex shown at fig. 25, *c.*, × 35. Slide No. 972.
I.	27.	Transverse section of vascular strand of cone pedicel × 70. Thin-walled disorganised tissue, *h.* Vascular strand, *k.* Endodermis-like structure, *e.* Slide No. 972.
I.	28.	Tangential section through secondary xylem × 95. Medullary rays, *m.r.*, cut through at right angles. Slide No. 969.

KIDSTON : STRUCTURE OF SIGILLARIA ELEGANS.—PLATE I.



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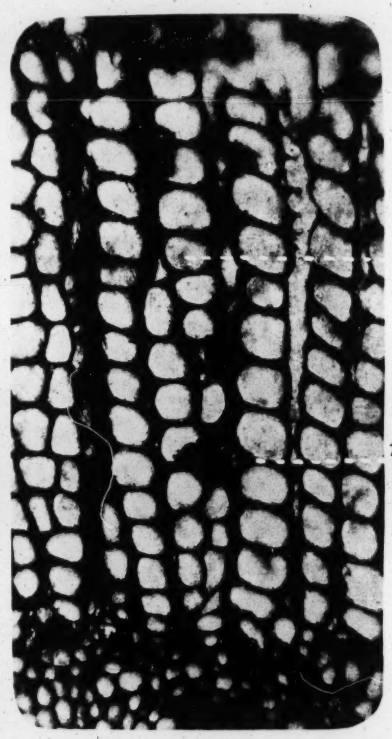

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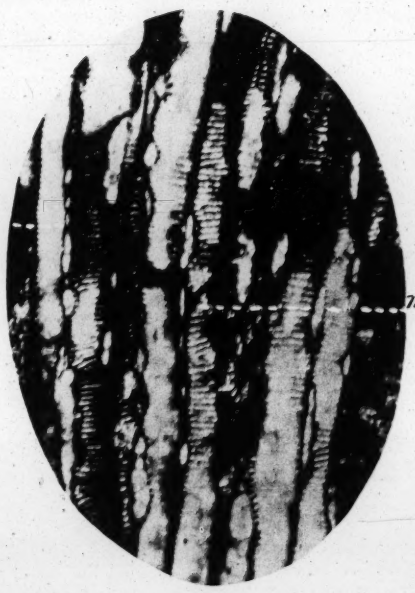

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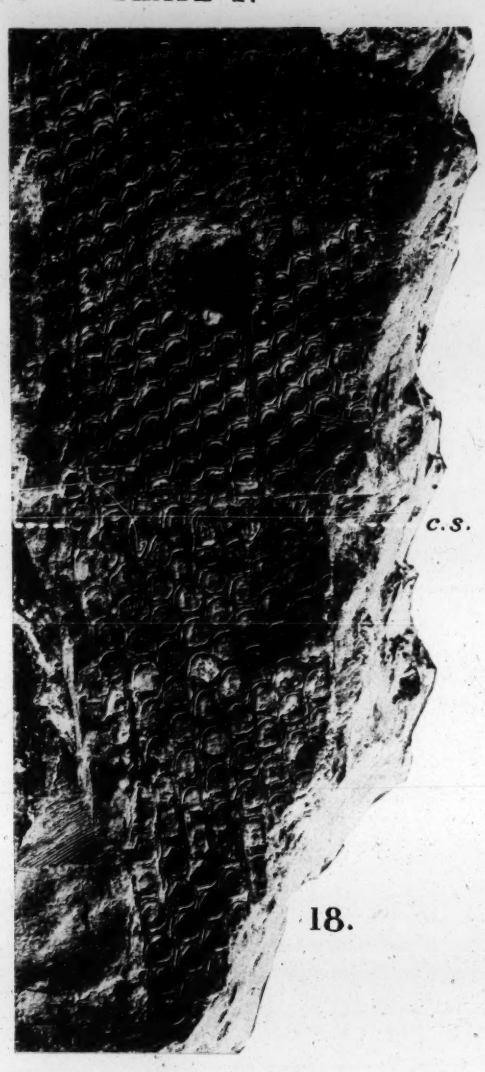

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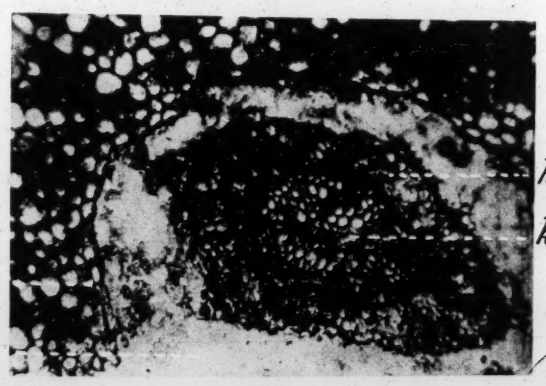

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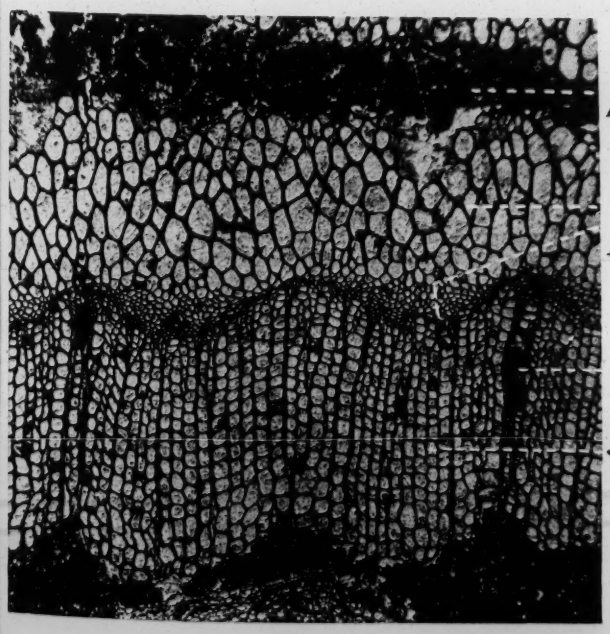

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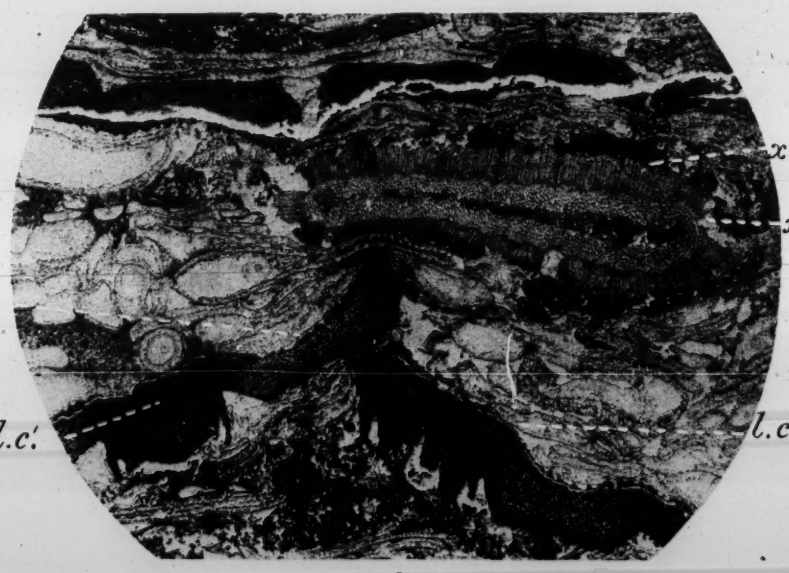

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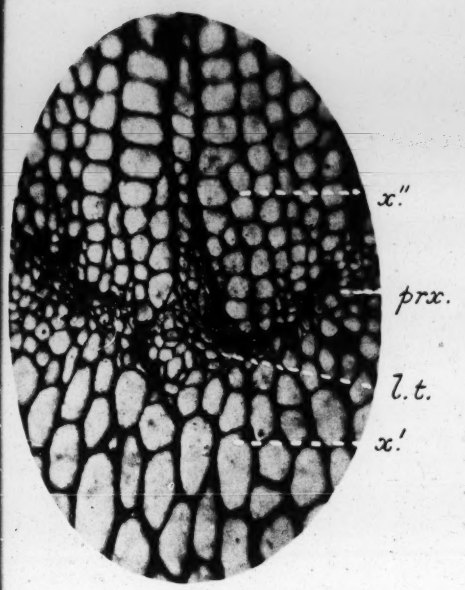
Kidston, Photo.

McFarlane & Erskine, Edinburgh.

KIDSTON : STRUCTURE OF SIGILLARIA ELEGANS.—PLATE II.



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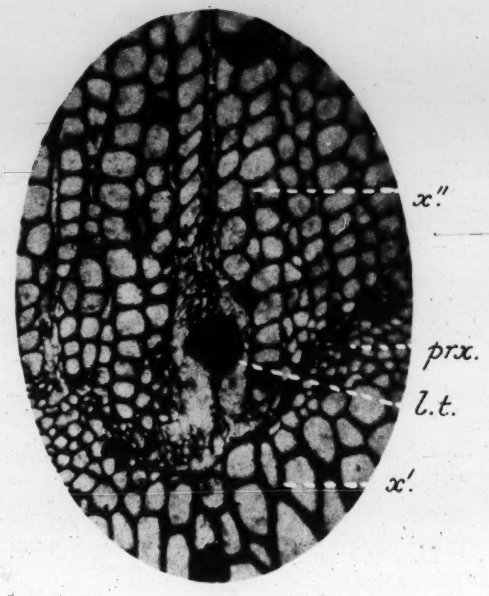

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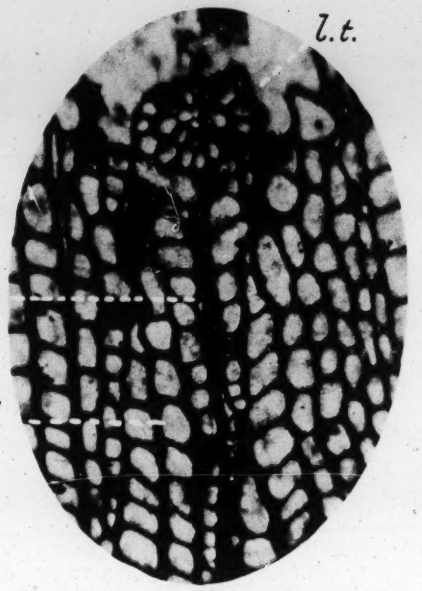

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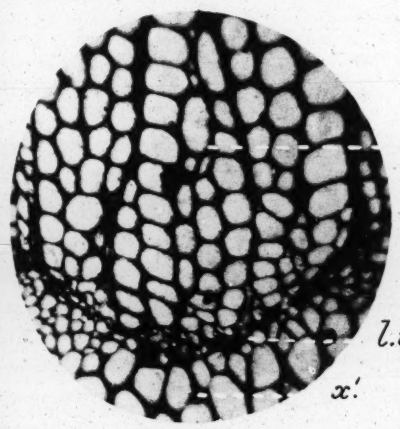

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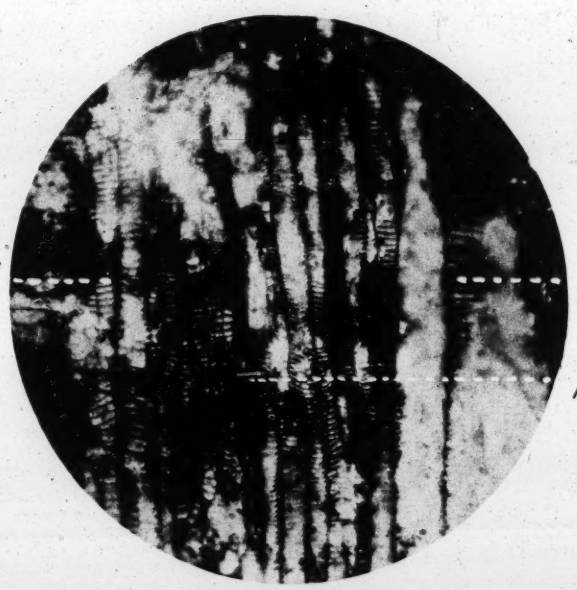

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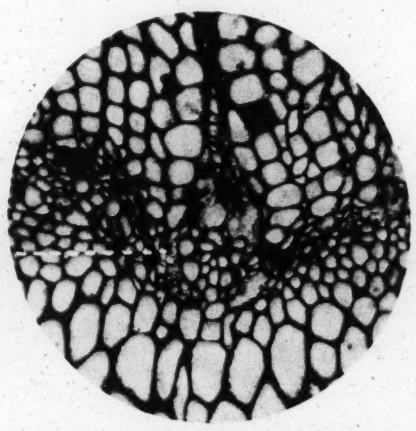

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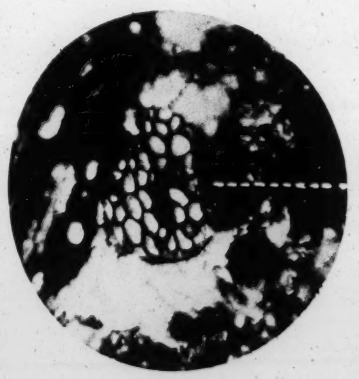

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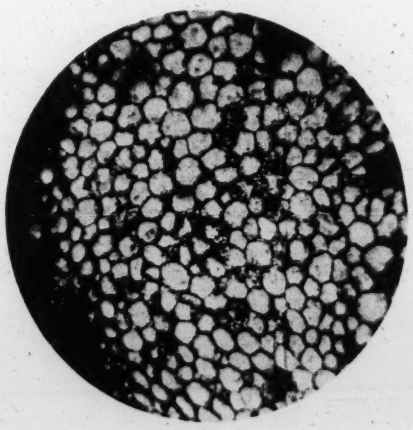
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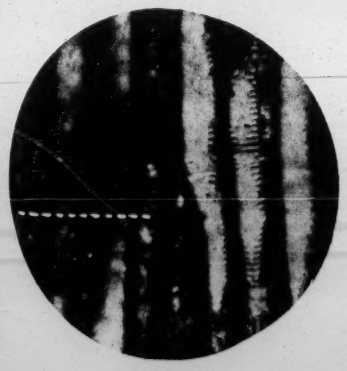
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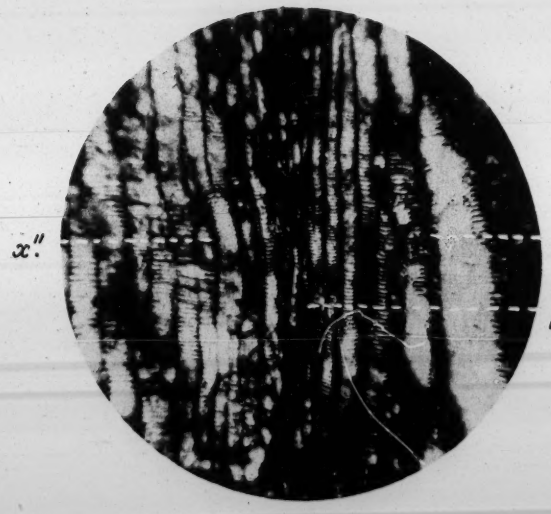

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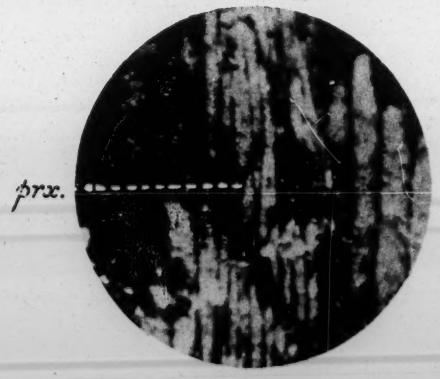

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J. Kidston, Photo.

McFarlane & Erskine, Edinburgh.

KIDSTON : STRUCTURE OF SIGILLARIA ELEGANS.—PLATE III.

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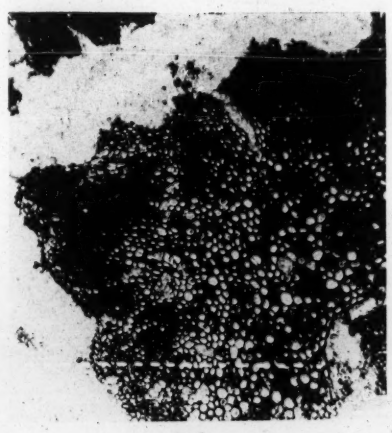

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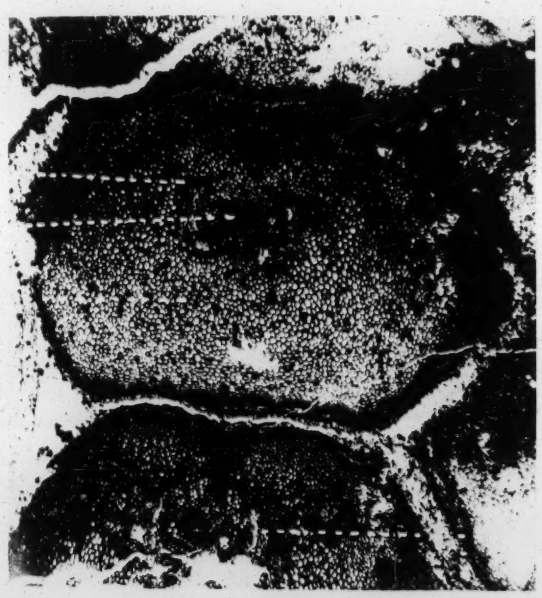

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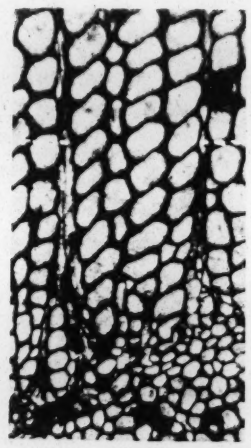

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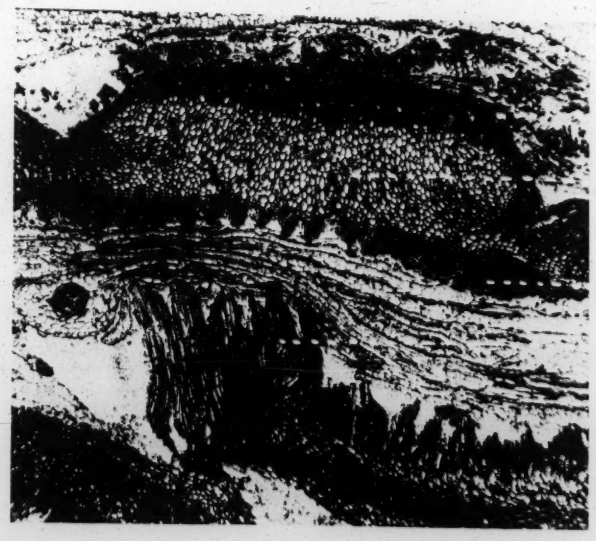

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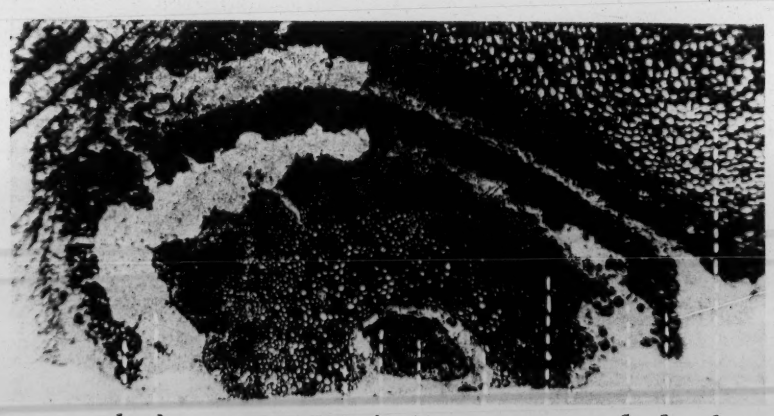

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R Kidston, Photo.

McFarlane & Erskine, Edinburgh.

XXIV.—On the Structure of the Series of Line- and Band-Spectra.

By J. Halm, Ph.D., Lecturer on Astronomy in the University of Edinburgh.

(Read July 4, 1904. MS. received October 14, 1904. Issued separately July 3, 1905.)

In a preliminary note read before the Society on July 4, 1904, I drew attention to the fact that a number of line-series, forming a group which includes the first series of Hydrogen, can be represented by an equation of the form

$$\frac{1}{\nu_\infty - \nu} = a_1 m^2 + b_1, \tag{1}$$

where ν denotes the wave-frequency of any line of the series, ν_∞ that of the so-called "tail" of the series ($m = \infty$), and a_1, b_1 constants depending on the nature of the emitting substance; the frequencies of successive lines being obtained by substituting successive integers for m. We see at once that this equation is a generalisation of BALMER's formula, into which it is transformed by equating b_1 to zero. In the same note I also pointed out the existence of another group represented by an equation of the same form, if $(m + \frac{1}{2})$ is substituted for m. As a special case ($b_1 = 0$) this group contains the second Hydrogen series discovered by Professor PICKERING in the spectrum of ζ Puppis. Subsequent investigations convinced me, however, that, although a considerable number of line-series may be classified into these two groups, there are numerous instances where the more general formula

$$\frac{1}{\nu_\infty - \nu} = a_1 (m + \mu)^2 + b_1 \tag{2}$$

must be employed, in which μ represents various fractional numbers. Also, in studying more thoroughly the literature on the subject, I found that the equation, in the last-mentioned form, is merely a modification of a mathematical expression already employed by Professor THIELE in his investigations on the band-series of the carbon spectrum and on the line-series of Helium. But, convinced as I was from my own computations of the accuracy and general importance of this equation, I was surprised to find it rejected by Professor THIELE, on the ground that it did not sufficiently represent the observed wave-lengths. This statement appeared to be so far from acceptable, that I resolved to demonstrate, by an exhaustive examination of all the known series, not only the general applicability of equation (2), but also its great superiority over any other formula hitherto proposed. The demonstration of this fact will be the first object of the present communication. It will be shown that the equation represents not only all the line-series hitherto known, but also the band-series, with an accuracy which leaves nothing to be desired. But far more important even than the demonstration of equation (2) as an empirical formula of practical usefulness are certain conclusions which may be drawn from the character of this equation and the numerical values of its constants. For instance, we shall see that equation (2) can be represented

in a geometrical form, by means of which we are enabled to collect all the line- and band-series into one single diagram revealing a community of properties between the two classes of spectral regularities and their individual members. This new geometrical connection between the series appears to be of theoretical importance, inasmuch as it shows a striking similarity between the vibrations of a radiating system of atoms and the nodal vibrations of elastic bodies. From this point of view an important relation has been discovered between the wave-frequencies of the "tails" of line-series and the atomic volumes of the emitting elements. In the course of this paper we shall have frequent opportunities of observing regularities in the constants of equation (2), and of drawing from them conclusions which cannot but be of some importance, however small, in connection with the theory of the phenomena of spectral regularities—a region into which the speculative mind has so far vainly attempted to penetrate. The outcome of the investigation must, I think, be to convey the impression that equation (2) is to be considered as more than a merely empirical formula, and that, if it does not represent the physical law itself, it is at least a remarkably close approximation to it, sufficiently reliable, perhaps, to guide the theorist in his search for the ultimate cause of the spectral regularities here considered.

Before entering upon the first part of our investigation, viz. that dealing with the question how far equation (2) is capable of representing the observed wave-frequencies, it will be useful to derive other forms of this equation, which we shall employ later on. First of all, we see at once that (2) may be expressed by the following series:

$$\nu = \nu_\infty - \frac{1}{a_1(m+\mu)^2} - \frac{b_1}{a_1^2(m+\mu)^4} - \frac{b_1^2}{a_1^3(m+\mu)^6} + \cdots\cdots$$

and we notice that in this form it represents a more general case of RYDBERG's formula, into which it converges, for $b_1 = 0$. In order to express the fact of its belonging to this type, and at the same time to distinguish it from RYDBERG's more special equation, I propose for it the name "RYDBERG-THIELE" equation, recognising thereby Professor THIELE's merit in having first introduced its present form into spectroscopic science. Equation (2) assumes a simpler and in some cases a more convenient form by introducing ν_0, the wave-frequency corresponding to $m+\mu = 0$. In most cases there is no line referring to this special value of ν, which obviously must lie close to the "head" of the series. But for the sake of convenience we may be permitted to speak of ν_0 in the following formulæ as the wave-frequency of the "beginning" of the series. Introducing ν_0, we find from (2):

$$\frac{\nu - \nu_0}{\nu_\infty - \nu} = a_1(\nu_\infty - \nu_0) \cdot (m+\mu)^2 = a_2(m+\mu)^2 . \tag{3}$$

Similar equations are at once obtained if wave-lengths are substituted for wave-frequencies:

$$\frac{1}{\lambda - \lambda_\infty} = \frac{a_1}{\lambda_\infty^2}(m+\mu)^2 + \frac{1}{\lambda_0 - \lambda_\infty} = a_1(m+\mu)^2 + \beta_1$$

$$\frac{\lambda_0 - \lambda}{\lambda - \lambda_\infty} = a_1\frac{\lambda_0 - \lambda_\infty}{\lambda_\infty^2}(m+\mu)^2 = a_2(m+\mu)^2 \tag{4}$$

Still more convenient for computation are the following transformations, in which ν_x and λ_x, the wave-frequency and wave-length of the xth line of the spectrum, are introduced :

$$\begin{aligned}\frac{1}{\nu-\nu_x}&=\left(\frac{1}{\nu_\infty-\nu_x}\right)^2\cdot\frac{1}{a_1}\cdot\frac{1}{(m+\mu)^2-(x+\mu)^2}+\frac{1}{\nu_\infty-\nu_x}=\frac{a_3}{(m+\mu)^2-(x+\mu)^2}+b_3\\ \frac{1}{\lambda_x-\lambda}&=\left(\frac{\lambda_\infty}{\lambda_x-\lambda_\infty}\right)^2\cdot\frac{1}{a_1}\cdot\frac{1}{(m+\mu)^2-(x+\mu)^2}+\frac{1}{\lambda_x-\lambda_\infty}=\frac{a_3}{(m+\mu)^2-(x+\mu)^2}+\beta_3\,.\end{aligned}\qquad(5)$$

The advantage of using these two formulæ lies in the fact that we may choose for ν_x or λ_x the wave-frequency or wave-length of any *observed* line of the series, whereas in the preceding equations ν_0, ν_∞ and λ_0, λ_∞ are quantities not directly obtainable from the observations. The arithmetical process of evaluating the other three unknowns is thereby greatly facilitated. Lastly, we may write

$$\begin{aligned}\frac{1}{\nu-\nu_0}&=\frac{1}{a_1(\nu_\infty-\nu_0)^2(m+\mu)^2}+\frac{1}{\nu_\infty-\nu_0}=\frac{a_4}{(m+\mu)^2}+b_1\\ \frac{1}{\lambda_0-\lambda}&=\frac{1}{a_1}\left(\frac{\lambda_\infty}{\lambda_0-\lambda_\infty}\right)^2\frac{1}{(m+\mu)^2}+\frac{1}{\lambda_0-\lambda_\infty}=\frac{a_4}{(m+\mu)^2}+\beta_1\,.\end{aligned}\qquad(6)$$

equations which will be specially useful in the investigation of the band-series.

In comparing first our formula with the observed wave-frequencies of line-series, we shall make use of equation (2), and apply it in this form to all the series mentioned in Professor KAYSER'S *Handbuch der Spectroscopie*, vol. ii. I may state at once that in all cases the constant μ was found to be a proper fraction, the denominator of which is represented by an entire multiple of 5, or, expressed in algebraic symbols, $\mu=\frac{p}{5q}$, p and q being integers. As unit of wave-length the tenth-metre or Angström unit was adopted, while for ν the value $10^8\lambda^{-1}$ was taken, in accordance with KAYSER and RUNGE and others. When not mentioned otherwise, the wave-lengths and oscillation-frequencies are reduced to the Rowland scale.

A. LINE-SERIES.

Group of Alkali metals.—There are two distinct kinds of series of spectra in the elements of this group : the "principal" series, consisting of strong and sharply defined lines—the principal lines of the spectrum—which are easily reversible, and one or two fainter "subsidiary" series represented by hazy lines, with little or no tendency to reversal. These lines have been observed with very different degrees of accuracy. KAYSER'S tables contain a column indicating the estimated probable limits of the observational errors expressed in units of wave-length. These limits are given in the last columns of the following tables, but in units of wave-frequency. Although the estimated uncertainties of the observed wave-lengths cannot be considered as quite reliable, they may nevertheless convey an approximate idea of the accuracy to be expected in our comparison.

1. Lithium.

Principal Series.

$\log a_1 = 4{\cdot}95920 - 10$
$\log b_1 = 3{\cdot}55776 - 10$ $\quad \nu_\infty = 43498{\cdot}5$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	Limits of Error of Obs.
1·95	14907·1	14907·0	+0·1	0·4
2·95	30933·2	30932·8	+0·4	0·3
3·95	36477·8	36475·7	+2·1	0·4
4·95	39022·9	39022·5	+0·4	0·5
5·95	40401·9	40399·1	+2·8	1·7
6·95	41227·8	41226·0	+1·8	1·8
7·95	41761·7	41761·5	+0·2	3·5
8·95	42124·8	42127·7	−2·9	?
9·95	42383·7	42389·2	−5·5	?

1. Subsidiary Series.

$\log a_1 = 4{\cdot}95899 - 10$
$\log b_1 = -\infty$ $\quad \nu_\infty = 28594{\cdot}7$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	Limits of Error of Obs.
3·0	16383·3	16383·3	0·0	0·1
4·0	21727·9	21725·7	+2·2	0·5
5·0	24198·8	24198·6	+0·2	1·2
6·0	25541·5	25541·8	−0·3	1·3
7·0	26351·2	26351·8	−0·6	34·5
8·0	26889·7	26877·5	+12·2	36·5
9·0	27243·5	27237·9	+5·6	38·0

2. Subsidiary Series.

$\log a_1 = 4{\cdot}96060 - 10$
$\log b_1 = 3{\cdot}48865_n - 10$ $\quad \nu_\infty = 28583{\cdot}3$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	Limits of Error of Obs.
2·6	12304·0	12304·0	0·0	?
3·6	20112·2	20112·2	0·0	0·4
4·6	23400·4	23400·3	+0·1	1·1
5·6	25088·2	25087·9	+0·3	1·2
6·6	26053·2	26067·5	−14·3	21·0

2. Sodium.

Principal Series (1st Component).

$\log a_1 = 4{\cdot}95369 - 10$
$\log b_1 = 4{\cdot}43154_n - 10$ $\quad \nu_\infty = 41467{\cdot}8$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	Limits of Error of Obs.
2·2	16960·2	16960·2	0·0	...
3·2	30274·9	30274·9	0·0	0·3
4·2	35051·9	35051·7	+0·2	0·6
5·2	37307·0	37307·2	−0·2	1·3
6·2	38550·8	38550·9	−0·1	1·5
7·2	39310·5	39309·2	+1·3	1·5
8·2	39805·3	39804·8	+0·5	3·2

1. Subsidiary Series (1st Component).

$\log a_1 = 4{\cdot}95740 - 10$
$\log b_1 = 3{\cdot}16660_n - 10$ $\quad \nu_\infty = 24481{\cdot}2$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	Limits of Error of Obs.
3·0	12202·9	12202·9	0·0	?
4·0	17580·1	17580·2	−0·1	0·5
5·0	20066·1	20066·1	0·0	0·8
6·0	21416·0	21415·7	+0·3	2·2
7·0	22222·2	22229·3	−7·1	4·9
8·0	22759·9	22757·2	+2·7	?
9·0	23117·7	23119·2	−1·5	?

2. Subsidiary Series (1st Component).

$\log a_1 = 4{\cdot}95369 - 10$
$\log b_1 = 4{\cdot}29709_n - 10$ $\quad \nu_\infty = 24490{\cdot}4$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	Limits of Error of Obs.
3·7	16230·7	16230·8	−0·1	0·3
4·7	19403·5	19403·4	+0·1	0·4
5·7	21042·9	21042·7	+0·2	0·6
6·7	21997·2	21999·8	−2·6	1·0
7·7	22605·5	22607·0	−1·5	?
8·7	23021·8	23020·3	+1·5	?

3. Potassium.

PRINCIPAL SERIES (1st Component).

$\log a_1 = 4{\cdot}94821 - 10$
$\log b_1 = 4{\cdot}75966_n - 10$
$\nu_\infty = 35030{\cdot}0$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	Limits of Error of Obs.
2·4	12988·2	12991·0	– 2·8	8·5
3·4	24707·5	24705·2	+ 2·3	0·2
4·4	29006·6	29009·2	– 2·6	0·3
5·4	31077·5	31078·6	– 1·1	0·3
6·4	32233 4	32235·2	– 1·8	1·0
7·4	32949·6	32947·9	+ 1·7	1·1
8·4	33418·8	33418·5	+ 0·3	1·6
9·4	33746·2	33745·4	+ 0·8	2·4
10·4	33981·2	33982·1	– 0·9	11·6

1. SUBSIDIARY SERIES (2nd Component).

$\log a_1 = 4{\cdot}93963 - 10$
$\log b_1 = 4{\cdot}86438_n - 10$
$\nu_\infty = 22049{\cdot}8$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	Limits of Error of Obs.
4·0	14469·3	14469·3	0·0	1·0
5·0	17293·0	17293·0	0·0	0·2
6·0	18784·5	18781·5	+ 3·0	0·5
7·0	19667·7	19663·7	+ 4·0	0·8
8·0	20230·2	20230·4	– 0·2	4·2
9·0	20616·9	20616·3	+ 0·6	?
10·0	20883·8	20890·9	– 7·1	?

2. SUBSIDIARY SERIES (1st Component).

$\log a_1 = 4{\cdot}92834 - 10$
$\log b_1 = 4{\cdot}76493_n - 10$
$\nu_\infty = 21996{\cdot}9$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	Limits of Error of Obs.
5·0	17146·1	17146·2	– 0·1	0·2
6·0	18657·1	18657·1	0·0	0·5
7·0	19559·2	19555·8	+ 3·4	0·8
8·0	20139·0	20134·1	+ 4·9	4·1
9·0	20532·2	20528·4	+ 3·8	?
10·0	20796·9	20809·3	– 12·4	?
11·0	21011·1	21016·6	– 5·5	?

4. Rubidium.

PRINCIPAL SERIES $\left(\begin{matrix}1\text{st}\\2\text{nd}\end{matrix}\right\}$ Component$\Big)$.

$\log a_1 = \begin{cases} 4{\cdot}92680 - 10 \\ 4{\cdot}92935 - 10 \end{cases}$
$\log b_1 = \begin{cases} 4{\cdot}99618_n - 10 \\ 4{\cdot}98695_n - 10 \end{cases}$
$\nu_\infty = \begin{cases} 33762{\cdot}6 \\ 33754{\cdot}7 \end{cases}$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	Limits of Error of Obs.
2·6	12577·9	12578·1	– 0·2	0·5
	12810·7	12810·7	0·0	0·8
3·6	23720·7	23720·9	– 0·2	0·2
	23798·3	23798·2	+ 0·1	0·2
4·6	27841·7	27840·9	+ 0·8	0·4
	27876·7	27876·8	– 0·1	0·4
5·6	29841·6	29841·8	– 0·2	0·5
	29860·9	29860·9	0·0	0·5

SUBSIDIARY SERIES (2nd Component).

$\log a_1 = 4{\cdot}90803 - 10$
$\log b_1 = 4{\cdot}76895_n - 10$
$\nu_\infty = 21203{\cdot}3$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	Limits of Error of Obs.
4·0	13111·9	13111·9	0·0	0·9
5·0	16111·6	16111·9	– 0·3	0·5
6·0	17704·8	17699·6	+ 5·2	0·5
7·0	18646·5	18643·2	+ 3·3	0·7
8·0	19250·0	19250·0	0·0	?
9·0	19662·6	19663·6	– 1·0	?

5. Cæsium.

PRINCIPAL SERIES $\left(\begin{matrix}1\text{st}\\2\text{nd}\end{matrix}\right\}$ Component$\Big)$.

$\log a_1 = \begin{cases} 4{\cdot}90783 - 10 \\ 4{\cdot}91436 - 10 \end{cases}$
$\log b_1 = \begin{cases} 5{\cdot}15453_n - 10 \\ 5{\cdot}13949_n - 10 \end{cases}$
$\nu_\infty = \begin{cases} 31525{\cdot}6 \\ 31496{\cdot}5 \end{cases}$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	Limits of Error of Obs.
2·8	11173·3	11173·3	0·0	1·0
	11726·5	11726·5	0·0	0·5
3·8	21770·7	21770·9	– 0·2	0·2
	21951·8	21951·7	+ 0·1	0·2
4·8	25714·7	25714·0	+ 0·7	0·6
	25794·9	25794·5	+ 0·4	0·6
5·8	27646·6	27646·7	– 0·1	2·3
	27686·7	27685·7	+ 1·0	1·6

5. Cæsium—*continued.*

SUBSIDIARY SERIES (1st Component).

$\log a_1 = 4{\cdot}90651 - 10$
$\log b_1 = 5{\cdot}21884_n - 10$ $\qquad \nu_\infty = 19747{\cdot}6$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	Limits of Error of Obs.
4·0	10855·6	10855·6	0·0	0·8
5·0	14339·2	14343·1	– 3·9	10·0
6·0	16094·2	16094·3	– 0·1	1·3
7·0	17108·4	17106·0	+ 2·4	1·5
8·0	17745·9	17745·6	+ 0·3	1·6
9·0	18175·5	18176·7	– 1·2	?
10·0	18481·2	18481·4	– 0·2	?
11·0	18705·9	18705·0	+ 0·9	?

I have omitted the second subsidiary series of Rubidium, of which only three lines are known, the series being thus insufficient for determining all the constants. It is, I think, obvious, and scarcely requires to be mentioned, that the principal series of Rubidium and Cæsium, in which the four unknowns of our equation had to be computed from the only four lines available, can tell us nothing of the accuracy of the formula employed. But for obvious reasons I have made it a rule to compute every series from which all the four constants may be obtained.

Let us now, before we proceed to other groups of elements, investigate the residuals given in the columns [Obs. – Comp.], by comparing them with those of the hitherto best empirical formula, that proposed by KAYSER and RUNGE :

$$10^8\lambda^{-1} = a + bm^{-2} + cm^{-4}.$$

This equation apparently contains three unknown constants, a, b, and c, and seems therefore to possess in this respect an advantage over ours, which has four unknown quantities. But it is well to consider that a fourth unknown is implicitly involved in KAYSER'S formula, viz. the value of m for the first line of the series, which we may call m_1. Strictly speaking, the difference between KAYSER'S formula and the one here proposed is therefore this, that the fourth unknown m_1 in the former is an integer, while in the latter it is a compound fraction. But let us grant to Professor KAYSER'S equation the full advantage of having one unknown less than ours. If it would represent the observations equally well, it would doubtless be the superior formula. This condition, however, is far from being fulfilled. Let us, for instance, consider the principal series of Potassium. Professor KAYSER has computed the three unknowns of his equation by the method of least squares, and has found the residuals, expressed in units of wave-lengths, which are given in the first column of the following table,

while the second column contains, in the same units, the residuals computed from the RYDBERG-THIELE formula :

+2·7	+1·6	+161·3	+172
−3·6	−0·4	0·0	0·0
+1·6	+0·3	0·0	0·0
+1·3	+0·1	0·0	−0·1
+0·8	+0·2	+0·3	+0·1
−0·3	−0·1	+0·2	0·0
−0·4	0·0	+0·7	+0·4
−0·5	−0·1	+1·0	+0·8
−0·5	+0·1	+1·4	+1·1

A comparison of the two columns shows the considerable superiority of the RYDBERG-THIELE formula. Indeed, the representation of the observed wave-lengths by KAYSER's formula appeared to Professor KAYSER himself so little satisfactory that he abandoned this method of computing the constants. In order to obtain more reliable values of these, he disregarded the first line and computed his equation from the second, third and fourth lines only. The residuals in this case are given in the third column. In another calculation, the result of which is represented in the residuals of the fourth column, he computed the constants from all the lines except the first. We observe that in both cases his endeavour to improve the shorter wave-lengths was defeated, notwithstanding the sacrifice of the first line, which shows an enormous discordance. The computations demonstrate clearly that KAYSER's formula cannot be made to represent the observed wave-lengths nearly so well as the RYDBERG-THIELE equation, although the discrepancies of this latter, in the special case considered, are much greater than usual.

But it may be urged that the introduction of another unknown in KAYSER's formula might improve matters. To decide this point, let us consider the more general equation

$$10^8\lambda^{-1} = a + bm^{-2} + cm^{-4} + dm^{-6}.$$

I have computed the numerical forms of this equation for the principal series of Li, Na and K, by using first the 1., 3., 5. and 7. lines of the series, and obtained the following equations :

$$\text{Li}: \quad 10^8\lambda^{-1} = 43552{\cdot}5 - 130802\ m^{-2} - 1156838\ m^{-4} + 124237\ m^{-6}$$
$$\text{Na}: \quad 10^8\lambda^{-1} = 41516{\cdot}7 - 128378\ m^{-2} - 829617\ m^{-4} - 36437\ m^{-6}$$
$$\text{K}: \quad 10^8\lambda^{-1} = 35075{\cdot}6 - 126522\ m^{-2} - 618583\ m^{-4} - 287275\ m^{-6}$$

The residuals, expressed in units of wave-frequency, are shown in the subjoined table under the heading K-R, while those of the RYDBERG-THIELE formula are repeated from the preceding tables and given under Ry-Th.

Li.		Na.		K.	
K-R.	Ry-Th.	K-R.	Ry-Th.	K-R.	Ry-Th.
0·0	+0·1	0·0	0·0	0·0	−2·8
+44·4	+0·4	+31·5	0·0	+26·0	+2·3
0·0	+2·1	0·0	+0·2	0·0	−2·6
−6·3	+0·4	−2·7	−0·2	−0·1	−1·1
0·0	+2·8	0·0	−0·1	0·0	−1·8
+0·8	+1·8	+2·4	+1·3	+3·0	+1·7
0·0	+0·2	0·0	+0·5	0·0	+0·3
−4·1	−2·9			−2·0	+0·8
−8·8	−5·5			−6·3	−0·9

The enormous discrepancies in the second lines show that KAYSER's formula is still far from representing the actual phenomena, and hence, even in this extended form, is inferior to the RYDBERG-THIELE equation. To give full justice to the former I have repeated the computations by excluding the first lines, but I have come to the conclusion that while the fourth constant produces on the whole a slight improvement in the shorter wave-lengths, the discrepancies in the first lines are even increased. All these computations have convinced me that no general improvement can be expected by adding a further term to KAYSER's formula.

But besides we find other facts which speak in favour of the equation here employed. There exists a certain empirical law, first pronounced by RYDBERG and KAYSER, that if for a given element two or more subsidiary series exist, these series converge into one common tail. This law has now been confirmed approximately in so many instances, that we may accept its correctness. The presumption is warranted, therefore, that the better formula should also bear out this law more precisely. Now let us compare in this respect KAYSER's formula with the RYDBERG-THIELE equation. We find for the wave-frequencies of the tails of the 1st components:

		K-R.	Diff.	Ry-Th.	Diff.
Lithium	1. S.S.	28586·7		28594·7	
			+80·7		−11·4
	2. S.S.	28666·7		28583·3	
Sodium	1. S.S.	24492·3		24481·2	
			+56·8		+9·2
	2. S.S.	24549·1		24490·4	
Potassium	1. S.S.	21991·2		21991·8	
			+30·6		+5·1
	2. S.S.	22021·8		21996·9	

These figures leave no doubt that the law is represented much better by our formula than by that of KAYSER and RUNGE. We shall see later on that this applies to other spectra as well.

A scarcely less convincing test of the superiority of the RYDBERG-THIELE equation is afforded by the discussion of another empirical law which we owe to RYDBERG. This law may be expressed as follows: (see KAYSER, *H. d. S.*, p. 557). The wave-frequency of the first line of the principal series is equal to the difference of the wave-frequencies of the tails of the principal and subsidiary series. It must be remarked, however, that if we take the *first* components [*i.e.* those of greater wave-lengths] of the subsidiary series, the law refers to the *second* components [shorter wave-lengths] of the principal series. Here are the figures corresponding to the two formulæ:

	K-R.			Ry-Th.			Observed 1st Line of P.S.	Obs. – Comp.	
	P.S.	S.S.	Difference.	P.S.	S.S.	Difference.		K-R.	Ry-Th.
Li:	43584	28627	14957	43498	28589	14909	14907	– 50	– 2
Na:	41542	24521	17021	41468	24486	16982	16977	– 44	– 5
K:	35086	22006	13080	35030	21994	13036	13045	– 35	+ 9
Rb:	33762	20919	12843	33755	20965	12790	12811	– 32	+ 21
Cs:	31509	19743	11766	31496	19748	11748	11726	– 40	– 22

From the last two columns we see that RYDBERG's law is represented in a distinctly better manner if the RYDBERG-THIELE formula is used. More important advantages of the latter will, however, appear in a subsequent part of the investigation upon which we can enter only after having examined the line-series of the other elements.

Of the remaining elements of the first vertical column in MENDELEJEF's series we can investigate here only Silver, of which four lines of the 1st subsidiary series are known. This being the minimum number of lines necessary for computing the constants of our equation, we must omit those series in which less than four lines have been observed. Doubtless in some of these cases three lines would be sufficient, because we have seen already that in a considerable number of series $\mu = 0$ or $(m+\mu)$ an integer, so that our equation contains only three unknowns. But to decide whether the series belongs to this special group we still require at least four data.

6. Silver.

1st SUBSIDIARY SERIES (1st Component).

$\log a_1 = 4{\cdot}96094 - 10$
$\log b_1 = 4{\cdot}15268_n - 10$
$\nu_\infty = 30646{\cdot}4$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.
3·0	18275·8	18275·8	0·0
4·0	23741·1	23741·1	0·0
5·0	26242·6	26242·6	0·0
6·0	27593·7	27594·0	– 0·3

It is of course quite unnecessary to compute the equation for more than one component in those cases where the subsidiary series consists of double or triple lines, because it is well known that the difference between the wave-frequencies of corresponding components is constant throughout the series. We therefore obtain the equation of the second or third component from that of the first by simply adding a constant to ν_∞.

We now turn to the elements of the second vertical column of MENDELEJEF'S series. The tables are computed in the same manner as before, but I have added a column showing the residuals of KAYSER'S formula which have been taken from the data in his *Handbuch*, but have been converted into wave-frequencies. The structure of the triplets in the 1st subsidiary series is in many cases complicated, especially that of the first two components. In such instances my computations refer to the simpler third components :

7. Magnesium.

1. SUBSIDIARY SERIES (1st Component).

$\log a_1 = 4{\cdot}95065 - 10$
$\log b_1 = 4{\cdot}34688_n - 10$
$\nu_\infty = 39779{\cdot}3$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
2·9	26052·2	26051·3	+0·9	+0·1
3·9	32288·7	32290·9	−2·2	−0·1
4·9	35065·1	35064·2	+0·9	−3·9
5·9	36538·5	36537·6	+0·9	0·0
6·9	37412·6	37413·7	−1·1	+3·5
7·9	37977·6	37976·9	+0·7	+9·7

2. SUBSIDIARY SERIES (1st Component).

$\log a_1 = 4{\cdot}94696 - 10$
$\log b_1 = 4{\cdot}81358_n - 10$
$\nu_\infty = 39780{\cdot}2$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
2·5	19290·7	19290·2	+0·5	−198·0
3·5	29968·6	29967·5	+1·1	+ 0·3
4·5	33988·1	33990·2	−2·1	− 0·5
5·5	35951·4	35951·9	−0·5	+ 0·2
6·5	37058·5	37058·6	−0·1	− 1·5
7·5	37745·8	37744·9	+0·9	− 3·8

8. Calcium.

1. SUBSIDIARY SERIES (3rd Component).

$\log a_1 = 4{\cdot}95797 - 10$
$\log b_1 = 4{\cdot}76493_n - 10$
$\nu_\infty = 34073{\cdot}0$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
3·2	22595·9	22596·4	− 0·5	− 0·3
4·2	27592·5	27592·5	0·0	− 0·5
5·2	29900·0	29900·0	0·0	0·0
6·2	31155·0	31158·6	− 3·6	+ 3·8
7·2	31886·4	31921·4	−35·0	−21·8

2. SUBSIDIARY SERIES (1st Component).

$\log a_1 = 4{\cdot}94860 - 10$
$\log b_1 = 4{\cdot}92942_n - 10$
$\nu_\infty = 34001{\cdot}7$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
2·7	16227·4	16227·6	−0·2	−158·
3·7	25164·1	25161·3	+2·8	0·0
4·7	28671·3	28675·0	−3·7	0·0
5·7	30430·0	30431·9	−1·9	− 0·2
6·7	31432·9	31440·3	−7·4	− 21·6
7·7	32073·9	32072·0	+1·9	+ 5·1

9. Strontium.

1. Subsidiary Series (3rd Component).

$\log a_1 = 4{\cdot}93097 - 10$
$\log b_1 = 5{\cdot}11861_n - 10$
$\nu_\infty = 31640{\cdot}3$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
3·5	20694·3	20694·3	0·0	+ 0·7
4·5	25374·7	25374·8	– 0·1	– 2·6
5·5	27554·4	27557·1	– 2·7	– 11·7
6·5	28758·0	28760·6	– 2·6	– 9·5
7·5	29498·0	29497·5	+ 0·5	+ 3·1
8·5	29987·9*	29982·4	+ 5·5	+ 16·3

2. Subsidiary Series (1st Component).

$\log a_1 = 4{\cdot}93049 - 10$
$\log b_1 = 5{\cdot}10018_n - 10$
$\nu_\infty = 31077{\cdot}7$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
2·9	14142·9	14142·9	0·0	– 96·
3·9	22531·5	22530·1	+ 1·4	+ 0·1
4·9	25868·7	25868·7	0·0	– 2·2
5·9	27559·0	27556·5	+ 2·5	– 0·1
6·9	28533·3	28533·3	0·0	– 2·2

The enormous discrepancies in the first lines of the 2nd subsidiary series shown by Professor Kayser's formula might, as in similar cases before, be considerably reduced by employing these lines in the computation of the constants. But this would involve inadmissible residuals in the shorter wave-lengths. It may be shown by easy calculations that no numerical form of Kayser's equation can represent the observed wave-frequencies with the same accuracy as the Rydberg-Thiele formula. After the foregoing discussion of the relative merits of the two formulæ I need not dwell upon this point any longer. The shortest wave-length of the 1st series of calcium is badly represented by both equations; and although Kayser's formula is distinctly better than ours, neither formula is satisfactory. It is worthy of note that there are two other instances, viz. the first series of Aluminium and the second series of Thallium, where both equations distinctly fail.

10. Zinc.

1. Subsidiary Series (1st Component).

$\log a_1 = 4{\cdot}94561 - 10$
$\log b_1 = 4{\cdot}43373_n - 10$
$\nu_\infty = 42918{\cdot}6$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
3·0	29889·3	29889·2	+ 0·1	+ 0·7
4·0	35701·7	35701·6	+ 0·1	– 0·3
5·0	38333·3	38332·0	+ 1·3	– 0·9
6·0	39745·5	39745·5	0·0	+ 1·8
7·0	40592·7	40592·7	0·0	+ 6·5
8·0	41140·0	41140·6	– 0·6	...

2. Subsidiary Series.

$\log a_1 = 4{\cdot}93860 - 10$
$\log b_1 = 4{\cdot}84137_n - 10$
$\nu_\infty = 42924{\cdot}5$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
2·45	20787·0	20787·0	0·0	– 251·
3·45	32550·0	32550·5	– 0·5	0·0
4·45	36864·0	36863·1	+ 0·9	– 0·2
5·45	38940·0	38939·2	+ 0·8	+ 0·2
6·45	40101·0	40101·5	– 0·5	+ 0·6
7·45	40820·0	40818·4	+ 1·6	+ 3·3

* The line observed is the first component: $\nu = 29408{\cdot}0$.

11. Cadmium.

1. Subsidiary Series (3rd Component).

$\log a_1 = 4{\cdot}95027 - 10$
$\log b_1 = 4{\cdot}56211_n - 10$ $\quad \nu_\infty = 42450{\cdot}4$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
3·0	29379·3	29379·3	0·0	+0·1
4·0	35248·3	35248·3	0·0	+0·5
5·0	37884·5	37884·5	0·0	−4·1
6·0	39294·5	39299·8	−5·3	+1·0
7·0	40137*	40142·8	−5·8	+5·3

2. Subsidiary Series (1st Component).

$\log a_1 = 4{\cdot}93417 - 10$
$\log b_1 = 4{\cdot}92926_n - 10$ $\quad \nu_\infty = 40765{\cdot}7$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
2·55	19661·9	19661·0	+ 0·9	− 263·
3·55	30744·8	30746·5	− 1·7	− 0·1
4·55	34863·3	34862·9	+ 0·4	+ 0·2
5·55	36864·0	36862·7	+ 1·3	− 0·3
6·55	37989·0	37989·3	− 0·3	− 1·7
7·55	38716·7	38688·3	+28·4	+ 26·3
8·55	39141·0	39152·1	−11·1	...

12. Mercury.

1. Subsidiary Series (3rd Component).

$\log a_1 = 4{\cdot}95111 - 10$
$\log b_1 = 4{\cdot}44885_n - 10$ $\quad \nu_\infty = 46582{\cdot}5$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
3·0	33698·5	33696·8	+1·7	...
4·0	39449·0	39447·5	+1·5	...
5·0	42045·0	42048·9	−3·9	...
6·0	43448·0	43446·4	+1·6	...

2. Subsidiary Series (1st Component).

$\log a_1 = 4{\cdot}94401 - 10$
$\log b_1 = 4{\cdot}84592_n - 10$ $\quad \nu_\infty = 40168{\cdot}2$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
2·45	18311·7	18311·7	0·0	− 191·
3·45	29924·7	29923·9	+0·8	+ 0·1
4·45	34182·3	34182·5	−0·2	− 0·2
5·45	36234·2	36232·6	+1·6	− 0·6
6·45	37380·3	37380·3	0·0	− 3·4

Turning to the elements of the 3rd column of Mendelejef's series, we find Al, In and Tl each with two subsidiary series. I have mentioned already that unexplained anomalies exist in the first series of Al and the second series of Tl, and that neither Kayser's formula nor the present one represents the observed lines in a satisfactory manner. Professor Kayser has nevertheless partly reconciled his formula with the observations by neglecting in each case the first three lines, and by demonstrating that the remainder could thereby be brought into fairly satisfactory agreement. It might be easily shown that under these circumstances our formula would render equally good service. But as long as the nature of these quite exceptional discrepancies is unknown, I consider such computations as being of little value. I am inclined to think that the series referred to are not homogeneous, but are in fact the result of a superposition of several branches, perhaps two in each case, which coalesce in such a manner as to give the impression of one single series. We shall come upon such anomalous cases later on when treating of the band-series, where in at least one instance, viz. the cyanogen-band, the hetero-

* Computed from the 2nd component : $\nu = 39594{\cdot}5$.

geneous character of what to all appearance seems a single series can be proved. Leaving this point in abeyance for the present, we shall disregard the two discrepant series.

13. Aluminium.

2. Subsidiary Series (1st Component).

$\log a_1 = 4{\cdot}94086 - 10$
$\log b_1 = 4{\cdot}82757_n - 10$ $\quad \nu_\infty = 48208{\cdot}0$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
2·4	25242·2	25242·2	0 0	– 338·
3·4	37587·3	37587·8	– 0·5	0·0
4·4	42043·6	42043·9	– 0·3	0·0
5·4	44172·0	44171·4	+ 0·6	0·0
6·4	45356·7	45356·8	– 1·0	+ 1·7

14. Indium.

1. Subsidiary Series (2nd Component).

$\log a_1 = 4{\cdot}92091 - 10$
$\log b_1 = 4{\cdot}41282_n - 10$ $\quad \nu_\infty = 46706{\cdot}8$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
3·0	32900	32900	0	0
4·0	39060	39060	0	0
5·0	41847	41847	0	0
6·0	43350	43345	+ 5	+ 11
7·0	44236	44243	– 7	+ 2
8·0	44825	44823	+ 2	+ 16
9·0	45224	45220	+ 4	+ 20
10·0	45506	45502	+ 4	+ 19
11·0	45714	45713	+ 1	+ 17
12·0	45871	45872	– 1	+ 13

Indium—*continued.*

2. Subsidiary Series (2nd Component).

$\log a_1 = 4{\cdot}93998 - 10$
$\log b_1 = 4{\cdot}87482_n - 10$ $\quad \nu_\infty = 46709{\cdot}8$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
2·45	24379	24379	0	– 333
3·45	36311	36311	0	0
4·45	40648	40648	0	0
5·45	42730	42729	+ 1	0
6·45	43891	43891	0	– 2
7·45	44611	44609	+ 2	+ 2
8·45	45080	45082	– 2	– 6
9·45	45455	45412	+ 43	+ 39

15. Thallium.

1. Subsidiary Series (1st Component).

$\log a_1 = 4{\cdot}94304 - 10$
$\log b_1 = 4{\cdot}48629_n - 10$ $\quad \nu_\infty = 41511{\cdot}7$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
3·0	28332	28332	0	0
4·0	34228	34227	+ 1	+ 1
5·0	36887	36887	0	– 2
6·0	38316	38314	+ 2	0
7·0	39168	39168	0	+ 2
8·0	39722	39721	+ 1	+ 5
9·0	40096	40098	– 2	+ 3
10·0	40362	40363	– 1	– 1
11·0	40559	40567	– 8	– 4
12·0	40708	40718	– 10	– 7
13·0	40824	40836	– 12	– 11
14·0	40916	40929	– 13	– 13
15·0	40991	41005	– 14	– 16

To complete our investigation of the line-spectra we will now discuss the line-series observed by Runge and Paschen in the spectra of Oxygen, Sulphur and Selenium (*Astrophysical Journal*, vol. viii.), and in the spectrum of Helium (Kayser, *Handbuch*, pp. 560–1). The comparison of these series with the Rydberg-Thiele formula is specially interesting, not only on account of the extreme accuracy of the observations which thereby afford a valuable test of this formula, but still more because it had been

found that KAYSER's formula, in its original form, did not sufficiently satisfy the measurements, which had to be represented by an equation of the form

$$10^8\lambda^{-1} = a + bm^{-2} + cm^{-3}.$$

This modification must of necessity complicate the theoretical aspect of the problem, inasmuch as the line-series would appear to be of two different types, in which the structure of the series is seemingly determined by different conditions. But we shall see now that the series of all these elements are represented by the same type of the RYDBERG-THIELE equation which we had employed in the foregoing discussion, and hence that a separation in two different types, such as is demanded by KAYSER's formula, is not required. This must be considered as an important advantage in our equation, which will become more apparent in the later discussion of the results.

16. Oxygen.

1st TRIPLET SERIES (1st Component).

$\log a_1 = 4{\cdot}95404 - 10$
$\log b_1 = 3{\cdot}80598_n - 10$
$\nu_\infty = 23212{\cdot}5$

$(m+\mu)$	ν obs. (reduced to vacuum).	ν comp.	Obs. – Comp.	R-P. (*l.c.* p. 76).
4·0	16233·5	16233·9	– 0·4	0·0
5·0	18753·6	18753·4	+ 0·2	– 0·3
6·0	20119·5	20118·6	+ 0·9	+ 0·2
7·0	20941·3	20940·6	+ 0·7	+ 0·2
8·0	21473·9	21473·6	+ 0·3	+ 0·2
9·0	21838·4	21838·9	– 0·5	– 0·2
10·0	22099·7	22100·1	– 0·4	+ 0·5

2nd TRIPLET SERIES (1st Component).

$\log a_1 = 4{\cdot}95750 - 10$
$\log b_1 = 4{\cdot}16157_n - 10$
$\nu_\infty = 23207{\cdot}3$

$(m+\mu)$	ν obs. (reduced to vacuum).	ν comp.	Obs. – Comp.	R-P. (*l.c.* p. 77).
3·8	15484·6	15484·6	0·0	– 0·3
4·8	18387·3	18387·3	0·0	+ 1·0
5·8	19913·6	19913·4	+ 0·2	– 0·1
6·8	20813·8	20814·0	– 0·2	– 0·8
7·8	21389·6	21389·9	– 0·3	– 0·2
8·8	21780·2	21780·3	– 0·1	+ 0·9

Oxygen—*continued.*

1st PAIR SERIES (1st Component).

$\log a_1 = 4{\cdot}96523 - 10$
$\log b_1 = 4{\cdot}16168 - 10$
$\nu_\infty = 21201{\cdot}7$

$(m+\mu)$	ν obs. (red. to vac.)	ν comp.	Obs. – Comp.	R-P. (*l.c.* p. 81).
3·8	13781·2	13781·5	– 0·3	– 0·2
4·8	16533·8	16532·4	+ 1·4	+ 0·9
5·8	17996·4	17996·5	– 0·1	– 0·4
6·8	18865·7	18867·0	– 1·3	– 1·1
7·8	19426·4	19425·5	+ 0·9	+ 0·9
8·8	19804·9	19805·6	– 0·7	– 0·6
9·8	20075·9	20075·6	+ 0·3	+ 0·5

2nd PAIR SERIES (1st Component).

$\log a_1 = 4{\cdot}95587 - 10$
$\log b_1 = 3{\cdot}55793_n - 10$
$\nu_\infty = 21212{\cdot}7$

$(m+\mu)$	ν obs. (red. to vac.)	ν comp.	Obs. – Comp.	R-P. (*l.c.* p. 81).
4·0	14276·8	14276·7	+ 0·1	+ 0·2
5·0	16777·5	16777·7	– 0·2	– 0·4
6·0	18134·3	18134·3	0·0	– 0·5
7·0	18951·3	18951·8	– 0·5	– 0·8
8·0	19485·2	19482·0	+ 3·2	+ 3·0
9·0	19846·3	19845·4	+ 0·9	+ 0·8
10·0	20102·9	20105·4	– 2·5	– 2·3

17. Sulphur.

1st Triplet Series (1st Component).

$\log a_1 = 4{\cdot}96004 - 10$
$\log b_1 = 4{\cdot}63201 - 10$ $\quad \nu_\infty = 20085{\cdot}9$

$(m+\mu)$	ν obs. (red. to vac.)	ν comp.	Obs. – Comp.	R-P. (*l.c.* p. 87).
4·5	14794·6	14794·5	+0·1	+0·1
5·5	16516·3	16517·0	−0·7	−0·5
6·5	17519·3	17519·5	−0·2	+0·1
7·5	18153·1	18152·9	+0·2	+0·5
8·5	18578·2	18578·2	0·0	+0·1
9·5	18877·4	18877·4	0·0	+0·2

2nd Triplet Series (1st Component).

$\log a_1 = 4{\cdot}95431 - 10$
$\log b_1 = 4{\cdot}49826_n - 10$ $\quad \nu_\infty = 20089{\cdot}4$

$(m+\mu)$	ν obs. (red. to vac.)	ν comp.	Obs. – Comp.	R-P. (*l.c.* p. 88).
5·0	15582·6	15582·7	−0·1	+0·1
6·0	16973·1	16973·2	−0·1	−0·1
7·0	17806·2	17805·9	+0·3	+0·4
8·0	18343·6	18344·0	−0·4	+0·1

18. Selenium.

1st Triplet Series (1st Component).

$\log a_1 = 4{\cdot}97128 - 10$
$\log b_1 = 4{\cdot}81092 - 10$ $\quad \nu_\infty = 19257{\cdot}8$

$(m+\mu)$	ν obs. (red. to vac.)	ν comp.	Obs. – Comp.	R-P. (*l.c.* p. 94).
4·5	14156·2	14156·4	−0·2	0·0
5·5	15805·0	15804·8	+0·2	0·0
6·5	16769·2	16769·8	−0·6	0·0
7·5	17379·6	17381·6	−2·0	−1·2
8·5	17794·9	17793·3	+1·6	+2·2
9·5	18083·2	18083·0	+0·2	−0·6
10·5	18293·9	18294·7	−0·8	−1·8

Selenium—*continued.*

2nd Triplet Series (1st Component).

$\log a_1 = 4{\cdot}95145 - 10$
$\log b_1 = 3{\cdot}40040 - 10$ $\quad \nu_\infty = 19286{\cdot}2$

$(m+\mu)$	ν obs. (red. to vac.)	ν comp.	Obs. – Comp.	R-P. (*l.c.* p. 95).
5·0	14818·2	14818·2	0·0	0·0
6·0	16182·4	16182·3	+0·1	0·0
7·0	17005·4	17005·3	+0·1	0·0
8·0	17538·1	17539·5	−1·4	−1·7

19. Helium.

1st Principal Series.

$\log a_1 = 4{\cdot}95902 - 10$
$\log b_1 = 3{\cdot}11561 - 10$ $\quad \nu_\infty = 38465{\cdot}9$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
$2\frac{14}{15}$	25715·3	25715·3	0·0	0·0
$3\frac{14}{15}$	31369·2	31369·2	0·0	0·0
$4\frac{14}{15}$	33953·1	33953·2	−0·1	0·0
$5\frac{14}{15}$	35345·8	35345·6	+0·2	+0·2
$6\frac{14}{15}$	36180·7	36180·5	+0·2	−0·1
$7\frac{14}{15}$	36720·8	36720·2	+0·6	−0·1
$8\frac{14}{15}$	37088·3	37089·1	−0·8	−0·7
$9\frac{14}{15}$	37352·5	37353·2	−0·7	−0·3
$10\frac{14}{15}$	37547·5	37546·7	−0·8	+0·3

1st Subsidiary Series (2nd Component).

$\log a_1 = 4{\cdot}95913 - 10$
$\log b_1 = 2{\cdot}60695_n - 10$ $\quad \nu_\infty = 29231{\cdot}6$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
3·0	17018·8	17018·8	0·0	0·0
4·0	22363·5	22363·3	+0·2	0·0
5·0	24836·5	24836·3	+0·2	0·0
6·0	26180·0	26179·5	+0·5	−0·1
7·0	26989·4	26989·4	0·0	−0·3
8·0	27515·0	27514·9	+0·1	−0·5
9·0	27875·3	27875·2	+0·1	−0·7
10·0	28133·3	28133·0	+0·3	−1·1
11·0	28323·8	28323·6	+0·2	−1·0
12·0	28468·7	28468·6	+0·1	−1·0
13·0	28581·3	28581·5	−0·2	−1·0
14·0	28670·7	28670·9	−0·2	−1·2
15·0	28743·3	28743·3	0·0	−1·2
16·0	28802·7	28802·5	+0·2	−1·0
17·0	28851·3	28851·5	−0·2	−1·1
18·0	28890·0	28892·5	−2·5	−3·4
19·0	28927·3	28927·3	0·0	−0·7

Helium—*continued.*

2nd Subsidiary Series (2nd Component).

$\log a_1 = 4{\cdot}96066 - 10$
$\log b_1 = 3{\cdot}40436_n - 10$ $\quad \nu_\infty = 29229{\cdot}2$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
2·7	14153·3	14153·7	− 0·4	− 48·
3·7	21216·7	21216·0	+ 0·7	0·0
4·7	24266·1	24267·0	− 0·9	0·0
5·7	25855·9	25856·6	− 0·7	0·0
6·7	26788·1	26788·9	− 0·8	+ 1·4
7·7	27381·2	27381·8	− 0·6	+ 3·5
8·7	27781·9	27782·2	− 0·3	+ 5·5
9·7	28065·3	28065·3	0·0	+ 7·6
10·7	28272·7	28272·7	0·0	+ 9·3
11·7	28429·4	28429·3	+ 0·1	+ 10·7
12·7	28550·7	28550·3	+ 0·4	+ 12·7
13·7	28647·3	28645·8	+ 1·5	+ 14·1
14·7	28722·5	28722·5	0·0	+ 14·0

2nd Principal Series.

$\log a_1 = 4{\cdot}96161 - 10$
$\log b_1 = 3{\cdot}40531 - 10$ $\quad \nu_\infty = 32036{\cdot}9$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
3·0	19936·8	19936·4	+ 0·4	0·0
4·0	25221·1	25221·1	0·0	0·0
5·0	27671·3	27672·1	− 0·8	0·0
6·0	29004·5	29004·9	− 0·4	+ 0·2
7·0	29809·3	29808·8	+ 0·5	+ 0·7
8·0	30331·3	38330·8	+ 0·5	+ 0·7
9·0	30690·7	30688·7	+ 2·0	+ 1·5
10·0	30947·1	30944·8	+ 2·3	+ 1·8
11·0	31136·4	31134·3	+ 2·1	+ 2·1
12·0	31280·7	31278·4	+ 2·3	+ 2·0
13·0	...	...	...	...
14·0	31480·7	31479·6	+ 1·1	− 0·4

Helium—*continued.*

3rd Subsidiary Series.

$\log a_1 = 4{\cdot}95930 - 10$
$\log b_1 = 2{\cdot}60712_n - 10$ $\quad \nu_\infty = 27182{\cdot}4$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
3·0	14973·4	14973·6	− 0·2	0·0
4·0	20316·7	20316·5	+ 0·2	0·0
5·0	22789·0	22788·6	+ 0·4	0·0
6·0	24131·7	24131·4	+ 0·3	0·0
7·0	24941·2	24940·9	+ 0·3	0·0
8·0	25466·5	25466·3	+ 0·2	+ 0·1
9·0	25826·3	25826·5	− 0·2	− 0·2
10·0	26084·4	26084·2	+ 0·2	− 0·2
11·0	26274·7	26274·7	0·0	− 0·2
12·0	26420·0	26419·7	+ 0·3	− 0·3
13·0	26531·9	26532·5	− 0·6	− 0·4
14·0	26621·9	26622·0	− 0·1	− 0·1

4th Subsidiary Series.

$\log a_1 = 4{\cdot}95932 - 10$
$\log b_1 = 3{\cdot}67610_n - 10$ $\quad \nu_\infty = 27181{\cdot}7$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.	K-R.
$2\frac{13}{15}$	13732·8	13732·8	0·0	− 15·8
$3\frac{13}{15}$	19810·9	19810·7	+ 0·2	0·0
$4\frac{13}{15}$	22534·2	22534·6	− 0·4	0·0
$5\frac{13}{15}$	23986·1	23986·1	0·0	0·0
$6\frac{13}{15}$	24849·4	24850·1	− 0·7	+ 0·4
$7\frac{13}{15}$	25405·9	25405·6	+ 0·3	+ 1·0
$8\frac{13}{15}$	25784·7	25783·9	+ 0·8	+ 1·3
$9\frac{13}{15}$	26053·7	26053·0	+ 0·7	+ 1·8
$10\frac{13}{15}$	...	...	...	...
$11\frac{13}{15}$	26401·8	26401·6	+ 0·2	+ 2·3
$12\frac{13}{15}$	26520·0	26518·1	+ 1·9	+ 4·4

Reviewing the results of the foregoing computations, which embrace now all the line-series so far as known, we must admit that the Rydberg-Thiele formula is in many cases distinctly better than that proposed by Kayser and Runge, and is never inferior to that equation. Its advantages become still more obvious if we investigate more closely the constants. Let us begin by testing the law mentioned before, according to which two subsidiary series of the same element possess common tails. In the following table we give for all cases in which two subsidiary series have been found,

the differences of the wave-frequencies of their tails, computed on the one hand by KAYSER'S formula, and on the other by the RYDBERG-THIELE equation :

	K-R.	Ry-Th.		K-R.	Ry-Th.
Li:	+ 81	− 11	Hg:	+58	−13
Na:	+ 57	+ 9	In:	+20	+ 3
K:	+ 31	+ 5	O:	−14	− 5
Mg:	+ 41	+ 1		+ 8	+11
Ca:	+121	+83	S:	− 9	+ 3
Sr:	+ 35	+18	Se:	+20	+28
Zn:	+ 10	+ 5	He:	−26	− 3
Cd:	+ 42	+17		− 7	− 1

These figures show clearly that the law in question is more closely represented by the RYDBERG-THIELE formula. Errors of 10–15 units are possibly accounted for by the uncertainty of the data from which the constants have been derived, because it can be shown that an error of only one unit in one of the observed wave-frequencies may sometimes alter the computed position of the tail by more than ten times this amount. The discrepancies in the [Ry-Th.] column, except those for Ca and Se, may therefore be considered as admissible, if the still existing uncertainties of the observed wave-lengths are taken into account.

With regard to μ, it has doubtless been noticed that in a considerable number of cases this constant is zero. Indeed, among the 44 series mentioned above we have no less than 19 instances of this kind. We find that, with the exception of Mg, Ca and Sr, all the first subsidiary series belong to this particular group, which, as has been already pointed out, includes the first Hydrogen-series. There is also evidence of the existence of smaller groups, for instance $\mu = 0{\cdot}5$ [H, Mg, S, Se]. Now in any such group, if we denote by ν the wave-frequency of a line of one series and by n the wave-frequency of a line of another series, since μ is the same in both series, we have the relation

$$\frac{1}{\nu_x - \nu} = \frac{a_5}{n_x - n} + b_5 \qquad (7)$$

where a_5 and b_5 are constants while ν_x and n_x are the wave-frequencies of the xth lines of the two series. This relation obviously enables us to express one series of lines by means of another belonging to the same μ-group. A similar relation obtains for the wave-lengths :

$$\frac{1}{\lambda - \lambda_x} = \frac{a_5}{l - l_x} + \beta_5 \qquad (8)$$

It may be of interest to prove the existence of such a relation between the series of different spectra in some special cases. Let us take, for instance, the group $\mu = 0$, and let us assume as the series of comparison the well-known Hydrogen-series represented by BALMER'S formula. In accordance with the foregoing equations, we call the wave-frequencies and wave-lengths of the hydrogen-lines n and l respectively,

those of the series to be compared ν and λ. From the considerable number of series belonging to this group we select here the following: He, 3rd S.S.; In, 1st S.S. (2nd comp.); Tl, 1st S.S. (1st comp.); Zn, 1st S.S. (2nd comp.); Li, 1st S.S. The following table contains the results of the comparison. By using equation (8) and adopting for λ_x in each case the wave-lengths which I have bracketed in the table, we find the following values of log α_5 and log β_5:

	log α_5.	log β_5.
He:	9·99165 – 10	$4{\cdot}50246_n$ – 10
In:	0·43927	6·31899 – 10
Tl:	0·35204	6·13701 – 10
Zn:	0·39819	6·25993 – 10
Li:	0·03893	5·05185 – 10

while the computed wave-lengths as well as their discrepancies (Obs. – Comp.) are given as follows:

H.	He, 3rd S.S.	In, 1st S.S. (2nd Comp.)	Tl, 1st S.S. (1st Comp.)	Zn, 1st S.S. (2nd Comp.)	Li, 1st S.S.
6563·07	6678·37 0·00	3039·5 0·0	3529·6 0·0	3302·8 0·0	6103·8 0·0
4861·52	4922·10 0·00	2560·2 0·0	2921·6 0·0	2771·0 0·0	4602·4 0·0
4340·63	4388·07 + 0·03	2389·6 0·0	2711·1 – 0·3	2582·7 – 0·1	4132·3 + 0·1
4101·90	4143·90 + 0·02	2307·0 – 0·2	2610·2 – 0·3	2491·7 0·0	3915·1 + 0·1
3970·22	4009·41 + 0·01	2260·2 + 0·4	2553·3 – 0·2	2440·2 – 0·3	[3794·9] ...
3889·20	3926·70 – 0·02	[2230·9] ...	2517·9 – 0·4	[2408·0] ...	3720·8 – 1·9
3835·53	3871·96 – 0·01	2211·3 – 0·1	2494·2 – 0·2		3671·6 – 1·0
3798·04	3833·69 + 0·02	2197·5 0·0	[2477·6] ...		
3770·77	3805·88 + 0·02	2187·4 + 0·1	2465·4 + 0·1		
3750·30	3785·01 + 0·02	2179·9 + 0·1	2456·3 + 0·2		
3734·51	3768·91 + 0·04		2449·2 + 0·4		
3722·08	[3756·24] ...		2443·7 + 0·3		
3712·11			2439·2 + 0·4		

Although the method of computation scarcely needs an explanation, I will nevertheless illustrate it by one example. Suppose we want to find the 3rd helium-line (from the top). The corresponding H-line is $l = 4340{\cdot}63$, further $\lambda_x = 3756{\cdot}24$ and $l_x = 3722{\cdot}08$, hence

$$\frac{1}{\lambda - 3756{\cdot}24} = \frac{\alpha_5}{4340{\cdot}63 - 3722{\cdot}08} + \beta_5 = 0{\cdot}0015859 - 0{\cdot}0000032 = 0{\cdot}0015827$$

$$= 3756{\cdot}24 + 631{\cdot}83 = 4388{\cdot}07.$$

The existence of so remarkable a relation between the line-series of different elements is a novel and certainly important feature, the discovery of which we owe to the application of the RYDBERG-THIELE formula. In some cases the relation

between series belonging to different chemical atoms is still closer. If we write our original equation (2) in the form

$$\frac{1}{\nu_\infty - \nu} = a_1[(m+\mu)^2 \pm c^2], \quad c^2 = \frac{b_1}{a_1} \tag{9}$$

we find that in certain cases the expression within the brackets is the same for two series. Under this special condition we have obviously

$$\frac{\nu_\infty - \nu}{n_\infty - n} = \text{const.}$$

I have found the following four instances of this kind:

(1) Zinc, 1st S.S., and Indium, 1st S.S. Calling the difference $(\nu_\infty - \nu)$ in the case of Zn: $[\nu]_{Zn}$ and in the case of In: $[\nu]_{In}$ we find from the preceding tables

$[\nu]_{Zn}$	$[\nu]_{In}$	$0{\cdot}94374[\nu]_{In}$
13029	13807	13030
7217	7647	7217
4585	4860	4587
3173	3357	3168
2326	2471	2332
1779	1882	1776

Hence we have, with sufficient accuracy: $[\nu]_{Zn} = 0{\cdot}94374[\nu]_{In}$.

(2) Zinc, 1st S.S., and Mercury, 1st S.S.

$[\nu]_{Zn}$	$[\nu]_{Hg}$	$1{\cdot}01137[\nu]_{Hg}$
13029	12884	13031
7217	7134	7215
4585	4537	4588
3173	3134	3170

It obviously follows that $[\nu]_{Zn} = 0{\cdot}94374[\nu]_{In} = 1{\cdot}01137[\nu]_{Hg}$.

(3) Zinc, 2nd S.S., and Mercury, 2nd S.S.

$[\nu]_{Zn}$	$[\nu]_{Hg}$	$1{\cdot}01286[\nu]_{Hg}$
22138	21856	22138
10374	10243	10374
6061	5986	6063
3985	3934	3985
2824	2788	2824

(4) Lithium, 1st S.S., and Hydrogen, 1st S.S.

$[\nu]_{Li}$	$[\nu]_H$	$1{\cdot}00177[\nu]_H$
12211	12189	12211
6867	6856	6868
4396	4388	4395
3053	3047	3052
2243	2238	2242
1705	1714	1717
1351	1354	1356

The RYDBERG-THIELE equation in the form given in (9) presents some striking regularities of the constants which deserve to be mentioned. If we consider first the cases (by far the most frequent) where b_1 is negative, we may write the equation in the form:

$$[\nu_\infty - \nu]^{-1} = a_1(m+\mu+c)(m+\mu-c) = a_1(m+d)(m+e) \qquad (10)$$

Now it appears that in a certain group of elements, such for instance as the alkalis, the constants d and e may be represented by common fractions having the same denominator. Thus we find for the subsidiary series the following numerical equations:

Li	1. S.S.:	$[\nu_\infty - \nu]^{-1} = [4{\cdot}95899 - 10]m \cdot m$	$\nu_\infty = 28595$
	2. S.S.:	$= [4{\cdot}96101 - 10](m - \frac{3}{15})(m - \frac{9}{15})$	$= 28580$
Na	1. S.S.:	$= [4{\cdot}95748 - 10](m + \frac{2}{15})(m - \frac{2}{15})$	$= 24481$
	2. S.S.:	$= [4{\cdot}95512 - 10](m + \frac{3}{15})(m - \frac{12}{15})$	$= 24481$
K	1. S.S.:	$= [4{\cdot}94046 - 10](m + \frac{14}{15})(m - \frac{14}{15})$	$= 22049$
	2. S.S.:	$= [4{\cdot}92668 - 10](m + \frac{12}{15})(m - \frac{12}{15})$	$= 22005$
Rb	1. S.S.:	$= [4{\cdot}90881 - 10](m + \frac{13}{15})(m - \frac{13}{15})$	$= 21202$
Cs	S.S.:	$= [4{\cdot}90432 - 10](m + \frac{23}{15})(m - \frac{22}{15})$	$= 19746$

The coefficients a_1 are given as logarithms. The wave-frequencies computed by means of these equations leave the following errors in the observations:

Li_1	Li_2	Na_1	Na_2	K_1	K_2	Rb	Cs
0	+ 2	0	0	+1	+ 1	+1	+2
+ 2	− 1	−1	0	−3	0	−2	−7
0	0	0	+ 1	0	+ 2	+3	−2
0	+ 1	0	0	+2	+ 3	+1	+2
− 1	−13	−8	+ 3	−2	0	−1	+1
+12		+3	+10	−1	−16	−2	0
+ 6		−1		−8	−10		+1
							+3

These errors are sufficiently small to be explained by the uncertainties of the measurements; they are indeed of the same order as those previously met with and considerably less than the corresponding residuals of KAYSER's equation. Turning to

the principal series we find that here also the denominator 15 plays an important part. Thus, considering the first components, the values of c are for Li: 0·199, for Na: 0·548, for K: 0·805, for Rb: 1·083, and for Cs: 1·328, hence very nearly $\frac{3}{15}$, $\frac{8}{15}$, $\frac{12}{15}$, $\frac{16}{15}$ and $\frac{20}{15}$. Except Li, the values of c and μ form an arithmetic progression. For we notice that μ is represented by the fractions $-\frac{1}{20}$, $\frac{4}{20}$, $\frac{8}{20}$, $\frac{12}{20}$, $\frac{16}{20}$, the numerators of which are obtained from those of the quantities c by subtracting 4. Such regularities cannot be accidental, although we must confess that at present it is impossible to assign a meaning to them. Nor are they confined to the group of elements here considered. Thus we may easily convince ourselves that, for instance, the four series of Oxygen mentioned before are represented by the equations:

1. Triplet:	$(\nu_\infty - \nu)^{-1} = [4{\cdot}95404 - 10][m^2 - (\frac{4}{15})^2]$	$\nu_\infty = 23212{\cdot}5$
2. Triplet:	$= [4{\cdot}95750 - 10][(m - \frac{3}{15})^2 - (\frac{6}{15})^2]$	23207·3
1. Pair:	$= [4{\cdot}96523 - 10][(m - \frac{3}{15})^2 + (\frac{6}{15})^2]$	21201·7
2. Pair:	$= [4{\cdot}95587 - 10][m^2 - (\frac{3}{15})^2]$	21212·7

Similar conditions we find in three of the six Helium-series, viz.,

1st P.S.:	$[\nu_\infty - \nu]^{-1} = [4{\cdot}95884 - 10][(m - \frac{1}{15})^2 + (\frac{2}{15})^2]$	$\nu_\infty = 38466{\cdot}8$
1st S.S.:	$= [4{\cdot}95913 - 10][m^2 - (\frac{1}{15})^2]$	29231·6
3d S.S.:	$= [4{\cdot}95930 - 10][m^2 - (\frac{1}{15})^2]$	27182·4

whereas in the other three series the fractions contain the denominator 2×15, viz.,

2nd P.S.:	$[\nu_\infty - \nu]^{-1} = [4{\cdot}96161 - 10][m^2 + (\frac{5}{30})^2]$	$\nu_\infty = 32036{\cdot}9$
2nd S.S.:	$= [4{\cdot}96066 - 10][(m - \frac{9}{30})^2 - (\frac{5}{30})^2]$	29229·2
4th S.S.:	$= [4{\cdot}95943 - 10][(m - \frac{4}{30})^2 - (\frac{7}{30})^2]$	27181·4

The series of the three elements Mg, Ca and Sr, on the other hand, appear to be well represented by fractions with the denominator 14. I have found the following numerical equations:

Mg	1st S.S.:	$[\nu_\infty - \nu]^{-1} = [4{\cdot}95105 - 10](m + \frac{5}{14})(m - \frac{8}{14})$	$\nu_\infty = 39781$
	2nd S.S.:	$= [4{\cdot}94696 - 10](m + \frac{19}{14})(m - \frac{5}{14})$	39779
Ca	1st S.S.:	$= [4{\cdot}95797 - 10](m + \frac{15}{14})(m - \frac{9}{14})$	34067
	2nd S.S.:	$= [4{\cdot}94622 - 10](m + \frac{24}{14})(m - \frac{4}{14})$	34005
Sr	1st S.S.:	$= [4{\cdot}93097 - 10](m + \frac{26}{14})(m - \frac{11}{14})$	31636
	2nd S.S.:	$= [4{\cdot}92352 - 10](m + \frac{32}{14})(m - \frac{5}{14})$	31079

All these regularities seem to me interesting enough to be mentioned in this discussion, although I admit that without a theoretical foundation they are perhaps of but little importance. Nevertheless the mere fact that the RYDBERG-THIELE equation is capable of showing so many interesting links between the series of different chemical elements, of which we see no traces in other formulæ, speaks highly in its favour, especially if considered in conjunction with the no longer doubtful property of this formula, that it satisfies the observations much better than other empirical equations hitherto proposed. But the chief importance of the RYDBERG-THIELE formula seems to me to lie in some other properties which we are now to discuss.

It may be shown that an interesting geometrical relation exists between the line-series belonging to the same μ-group. *If, in fig. 1, the points* $a_1, a_2 \ldots a_m$, *are fixed upon the straight line* AB *so that the distance* $Aa_m = (m+\mu)^2$, *and if from any point* O *outside this line we draw the straight paths* $O\nu_0, O\nu_1, O\nu_2 \ldots O\nu_m \ldots O\nu_\infty$, *through* A, $a_1, a_2 \ldots a_m \ldots a_\infty$, *any line-series belonging to this particular group, which has been arranged along a transversal line* CD *so that the distance between consecutive points corresponds, on some arbitrary scale, to the differences of the wave-frequencies of consecutive lines, can be placed in such a position that all*

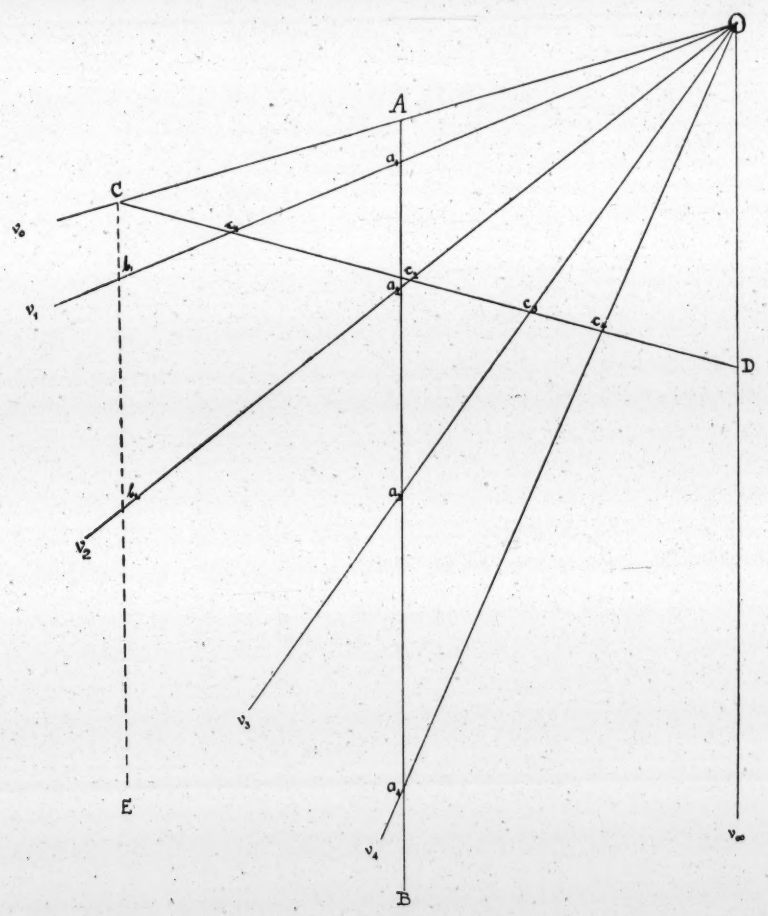


FIG. 1.

the spectral lines fall exactly upon the rays $O\nu_0, O\nu_1, O\nu_2 \ldots$. The proof of this theorem is very simple. If the dotted line, CE, be drawn parallel to AB, we have:

$$Cb_1 = \frac{CO}{AO} \cdot Aa_1$$

$$Cb_2 = \frac{CO}{AO} \cdot Aa_2$$

. . . . . .

and generally

$$Cb_m = \frac{CO}{AO} \cdot Aa_m = \frac{CO}{AO} \cdot (m+\mu)^2 .$$

Further we have from the similar triangles $Cc_m b_m$ and $Oc_m D$:

$$\frac{Cc_m}{c_m D} = \frac{Cb_m}{OD} = \frac{CO}{AO \cdot OD} \cdot (m+\mu)^2 .$$

On the other hand, from the RYDBERG-THIELE formula (3)

$$\frac{\nu_m - \nu_0}{\nu_\infty - \nu_m} = a_2(m+\mu)^2 .$$

Hence, if the transversal CD be drawn in such a manner that

$$\frac{CO}{AO \cdot OD} = a_2 ,$$

we find

$$Cc_m = \nu_m - \nu_0 \quad \text{and} \quad c_m D = \nu_\infty - \nu_m , \quad \text{and hence}$$
$$c_1c_2 = \nu_2 - \nu_1 , \; c_2c_3 = \nu_3 - \nu_2 , \; \ldots\ldots \; CD = \nu_\infty - \nu_0 \qquad Q.E.D.$$

If, now, all the series belonging to the same μ-group be arranged in our diagram, in every case the lines could be made to fall upon the rays $O\nu_0$, $O\nu_1$ . . . $O\nu_\infty$, so that

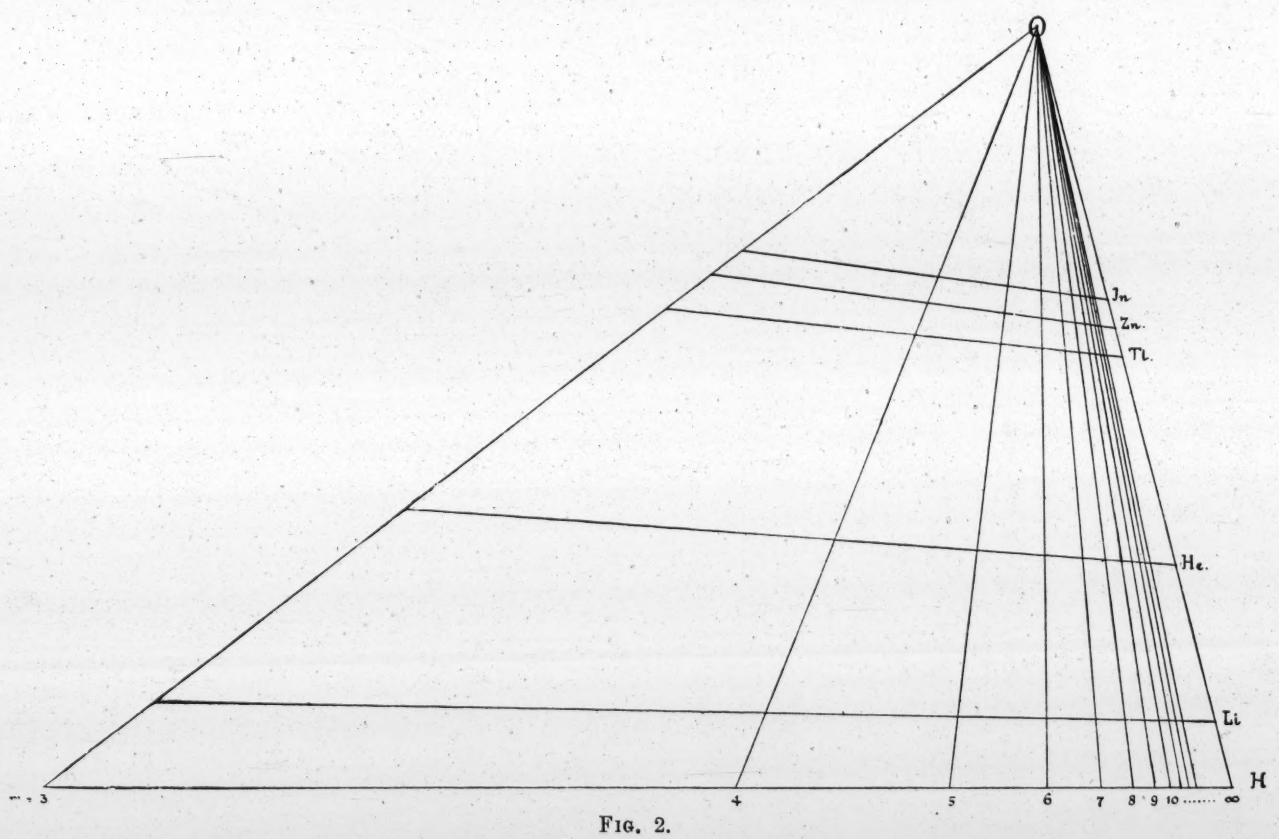


FIG. 2.

to an observer stationed at O all these series would seem to coalesce into one. In virtue of equation (4) the same geometrical property holds good, if we consider wave-lengths instead of wave-frequencies.

From the above theorem we deduce at once the following corollary :

If we fix upon a straight line, on any arbitrary scale, the lines of a given series in such a way that the distances between two lines express the differences of the corresponding wave lengths or frequencies, and if from any point outside we draw straight paths through these spectral lines, then the lines of any other series belonging to the same μ-group can be represented as the points of intersection of these straight paths with a certain transversal line.

An illustration of this geometrical relation is given in fig. 2 for some series

belonging to the group $\mu = 0$. On the lowest horizontal line the wave-lengths of the first Hydrogen-series (BALMER's formula) have been drawn on a certain scale (in the original drawing 10 t.m. = 1 mm.), beginning with $m = 3$ ($\lambda_3 = 6563{\cdot}07$) on the left to $m = \infty$ ($\lambda_\infty = 3646{\cdot}10$) on the right. The points 3, 4, 5, . . . thus obtained were then connected with a point O arbitrarily fixed in the plane of the paper. If now we draw, not necessarily but for the sake of convenience, on the *same* scale, the wave-lengths of, say, the 1st subsidiary series of Lithium on another straight line, for instance the edge of a ruler, we can bring this line on our diagram into such a position that its intersections with the rays O_3, O_4 . . . O_∞ mark precisely the positions of all the lines of the series. The line is indicated in the drawing by the transversal Li. In the same way the transversals He, Tl, Zn and In represent the 2nd principal series of Helium and the 1st subsidiary series of Thallium, Zinc and Indium, which, as we have seen before, belong to this particular group $\mu = 0$. Obviously the "tails" of all these series are situated upon the ray O_∞, while the "heads" lie upon O_0 which corresponds to $m = 0$. In line-series these "heads" are of no special interest, because they correspond to wave-lengths outside the range of observation. Nevertheless, they will be considered here on account of their great importance in relation to our investigation of the band-spectra in subsequent pages. The construction of fig. 2 was based on wave-lengths in order to show that the RYDBERG-THIELE formula may also be employed in the form given by equation (4). For theoretical purposes, however, an investigation founded on wave-frequencies must doubtless be preferred. This has been already pointed out by RYDBERG, KAYSER, SCHUSTER and others. We shall therefore now revert to our formulæ referring to wave-frequencies.

Since the position of O is quite arbitrary, we may conveniently choose it so that the two boundary rays $O\nu_0$ and $O\nu_\infty$ form a right angle. In this case the cotangent of the angle α between the rays $O\nu$ and $O\nu_\infty$ (fig. 3) is equal to $(m+\mu)^2$, if we make $AO = 1$. Let us further call β the angle formed by the ray $O\nu_0$ and the transversal which represents our series. Since $cc_\infty = \nu_\infty - \nu$ and $< Occ_\infty = 90^\circ - \alpha - \beta$, we have from triangle Occ_∞ :

$$\nu_\infty - \nu = Oc_\infty \cdot \frac{\sin \alpha}{\cos(\alpha+\beta)} = a_\infty \frac{\sin \alpha}{\cos(\alpha+\beta)} \cdot \tag{11a}$$

On the other hand, from triangle Occ_0 we find

$$\nu - \nu_0 = Oc_0 \cdot \frac{\cos \alpha}{\cos(\alpha+\beta)} = a_0 \cdot \frac{\cos \alpha}{\cos(\alpha+\beta)} \cdot \tag{11b}$$

Comparing the first of these formulæ with equation (2) we see at once that

$$Oc_\infty = \frac{\cos \beta}{a_1} = a_\infty$$

$$\tan \beta = -\frac{b_1}{a_1}$$

The RYDBERG-THIELE equation may therefore be written in the form

$$\nu = \nu_\infty - a_\infty \cdot \frac{\sin \alpha}{\cos(\alpha+\beta)} \cdot \tag{12}$$

This is probably the most convenient form for computation. It becomes identical with BALMER'S formula for $\mu = 0$ and $\beta = 0$, because we have then $\tan a = m^{-2}$ and

$$\nu = \nu_\infty - a_\infty \cdot \tan a = \nu_\infty - a_\infty \cdot m^{-2}.$$

If only $\beta = 0$, we find RYDBERG'S formula

$$\nu = \nu_\infty - a_\infty \cdot (m+\mu)^{-2}.$$

Hence we see that both BALMER'S and RYDBERG'S equations suppose the transversal to be parallel to the ray $0\nu_0$, whereas, according to the more general RYDBERG-THIELE formula, the two lines may form any angle β with each other.

Now, an important result is arrived at if we investigate more closely these angles β. We notice that while in some cases β is small, and hence the transversals are nearly parallel to $0\nu_0$, there are also instances where this angle is considerable. The extreme case

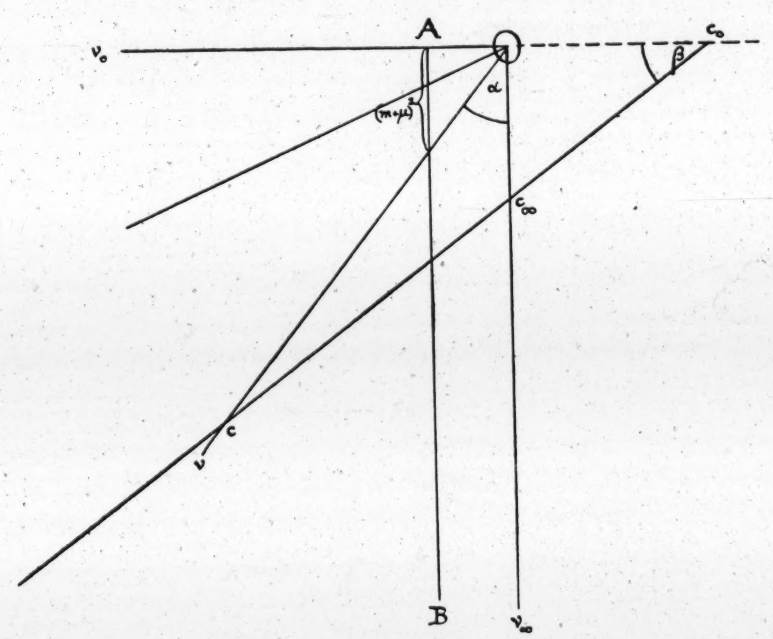


FIG. 3.

seems to be represented by Cæsium, where β is about 64°. Naturally the question arises, what would happen if β should become still greater and should finally be $\pm 90°$. In this case the transversal representing our series would be parallel to the ray $O\nu_\infty$, and hence a_∞ would become infinite. But from the equation (11*b*) we find that

$$\nu = \nu_0 \mp a_0(m+\mu)^2. \qquad (13)$$

Now obviously this equation is a more general form of DESLANDRES' formula for the *band*-spectra into which it converges for $\mu = 0$, viz.

$$\nu = \nu_0 \mp a_0 m^2,$$

where ν_0 represents the wave-frequency of the *first* line (the "head" of the band), while the upper or the lower sign indicates that the band "shades off" towards the red or the violet side respectively. We perceive, then, that the RYDBERG-THIELE formula includes as a special case also the DESLANDRES formula, and thus opens the prospect of

the two phenomena, the line- and band-series, being comprised under one and the same mathematical, and probably also physical, conception.

From what has been found in the case of the line-spectra it appears now extremely probable that DESLANDRES' formula represents only a limited number of band-spectra and is but a special case of our more general equation, just as BALMER's formula is a particular instance of the same equation in the case of line-spectra. A considerable part of the remainder of the present communication will be devoted to the investigation of this point. Before entering upon this new side of the question, however, I should like to discuss briefly, for the convenience of those readers who are interested in the foregoing computations, a method by which the four constants of the RYDBERG-THIELE formula may be determined from one of the two equations (5). If ν_x, ν_y, ν_z, ν_w are the wave-frequencies of any four lines of the series, we have from the second of (5):

$$\frac{1}{\nu_z-\nu_x}=\frac{a_3}{(z+\mu)^2-(x+\mu)^2}+b_3=\frac{a_3}{(z+x+2\mu)(z-x)}+b_3 \qquad (a)$$

$$\frac{1}{\nu_y-\nu_x}=\frac{a_3}{(y+x+2\mu)(y-x)}+b_3 \qquad (b)$$

$$\frac{1}{\nu_w-\nu_y}=\frac{a_3}{(w+y+2\mu)(w-y)}+b_3 \qquad (c)$$

$$\frac{1}{\nu_w-\nu_z}=\frac{a_3}{(w+z+2\mu)(w-z)}+b_3 \qquad (d)$$

Subtracting (c) from (a) and (d) from (b), and dividing, we obtain after slight transformations:

$$\frac{(x+y+2\mu)(z+w+2\mu)}{(x+z+2\mu)(y+w+2\mu)}=\frac{(z-x)(w-y)}{(y-x)(w-z)}\cdot\frac{(\nu_y-\nu_z)(\nu_w-\nu_z)}{(\nu_z-\nu_x)(\nu_w-\nu_y)}=c, \text{ say,} \qquad (e)$$

or by substituting

$$y-x=p$$
$$z-x=q$$
$$w-x=r$$

$$\frac{[p+2(\mu+x)][q+r+2(\mu+x)]}{[q+2(\mu+x)][p+r+2(\mu+x)]}=c, \quad \text{and}$$

$$(\mu+x)^2+\frac{p+q+r}{2}(\mu+x)+\frac{p(q+r)-q(p+r)c}{4(1-c)}=0.$$

The positive root of this quadratic equation is the desired value of $(\mu+x)$. Let us take as an example the principal series of Sodium and select the 1st, 3rd, 5th, and 7th observed line for our computation. From the observations we have therefore:

$\nu_x = 16960{\cdot}19$	$z-x=4$	$p=2$
$\nu_y=\nu_{x+2}=35051{\cdot}93$	$w-y=4$	$q=4$
$\nu_z=\nu_{x+4}=38550{\cdot}80$	$y-x=2$	$r=6$
$\nu_w=\nu_{x+6}=39805{\cdot}27$	$w-z=2$	

$$c=\frac{4\cdot4}{2\cdot2}\times\frac{18091{\cdot}74\times1254{\cdot}47}{21590{\cdot}61\times4753{\cdot}34}=0{\cdot}884582$$

$$(\mu+x)^2+6(\mu+x)-17{\cdot}992=0\,;\ \mu+x=2{\cdot}195\,.$$

The remaining computations are now simple. We have

$$\frac{1}{\nu_y - \nu_x} = \frac{1}{18091{\cdot}74} = \frac{a_3}{(4{\cdot}195)^2 - (2{\cdot}195)^2} + b_3$$
$$\frac{1}{\nu_z - \nu_x'} = \frac{1}{21590{\cdot}61} = \frac{a_3}{(6{\cdot}195)^2 - (2{\cdot}195)^2} + b_3,$$

and hence

$$\log a_3 = 6{\cdot}26686 - 10$$
$$\log b_3 = 5{\cdot}61075 - 10.$$

But from (5) and (2) we find

$$b_3 = \frac{1}{\nu_\infty - \nu_x}\,;\quad a_1 = \frac{b_3^2}{a_3}\,;\quad \frac{1}{\nu_0 - \nu_x} = -\frac{a_3}{(2{\cdot}195)^2} + b_3.$$

And finally from these relations:

$$\log a_1 = 4{\cdot}95464 - 10\,;\ \log b_1 = 4{\cdot}41367_n - 10\,;\ \nu_\infty = 41464{\cdot}9.$$

These constants differ slightly from those previously computed under the assumption that $x + \mu = 2{\cdot}200_n$, but they represent the observations almost equally well.

It may be remarked that the relation (e) can also be derived from a well-known geometrical theorem. If, in fig. 1, we take any four points α_x, α_y, α_z and α_w on the line AB and the four corresponding points c_x, c_y, c_z and c_w on the transversal CD, so that α_x and c_x lie on the same ray $O\nu_x$, α_y and c_y on the ray $O\nu_y$, etc., then we know from geometry that

$$\frac{\alpha_x\alpha_y}{\alpha_x\alpha_z} \cdot \frac{\alpha_z\alpha_w}{\alpha_y\alpha_w} = \frac{c_xc_y}{c_xc_z} \cdot \frac{c_zc_w}{c_yc_w}.$$

But we remember that $\alpha_x\alpha_y = (y+\mu)^2 - (x+\mu)^2$ and $c_xc_y = \nu_y - \nu_x$ etc., so that

$$\frac{\nu_y - \nu_x}{\nu_z - \nu_x} \cdot \frac{\nu_w - \nu_z}{\nu_w - \nu_y} = \frac{(y+\mu)^2 - (x+\mu)^2}{(z+\mu)^2 - (x+\mu)^2} \cdot \frac{(w+\mu)^2 \quad (z+\mu)^2}{(w+\mu)^2 - (y+\mu)^2}, \quad \text{or}$$
$$\frac{(z-x)(w-y)}{(y-x)(w-z)} \cdot \frac{(\nu_y - \nu_x)(\nu_w - \nu_z)}{(\nu_z - \nu_x)(\nu_w - \nu_y)} = \frac{(x+y+2\mu)(z+w+2\mu)}{(x+z+2\mu)(y+w+2\mu)},$$

which is identical with (e).

B. Band-Spectra.

The fact that the Rydberg-Thiele equation represents both line- and band-series is perhaps most strikingly demonstrated by the following computations where the wave-lengths of the lines of the Cyanogen-band (see Kayser, *Handbuch*, vol. ii. p. 479) are used for determining the wave-lengths of the first triplet series of Oxygen given on p. 564. If in equation (8)

$$\frac{1}{\lambda - \lambda_x} = \frac{\alpha_5}{l - l_x} + \beta_5$$

we take for l successively the wave-lengths of the 40th, 50th, etc., line of the Cyanogen-band, l_x being the wave-length for the 100th line for instance, and if we further make

$$\log \alpha_5 = 9{\cdot}60800 - 10$$
$$\log \beta_5 = 7{\cdot}63969_n - 10,$$

we obtain the values $(\lambda - \lambda_x)$ shown in the third column of the following table:

1	2	3	4	5	6
Cyanogen-band.	$l - l_x$	$\lambda - \lambda_x$	λ	1st Triplet Series of Oxygen (1st Comp.).	Difference Obs. – Comp.
$l_{40} = 3866{\cdot}95$	81·53	1634·70	6158·41	6158·41	0·00
$l_{50} = 3857{\cdot}82$	72·40	807·14	5330·84	5330·84	0·00
$l_{60} = 3846{\cdot}79$	61·37	445·31	4969·01	4968·94	– 0·07
$l_{70} = 3833{\cdot}93$	48·51	250·18	4773·88	4773·94	+ 0·06
$l_{80} = 3819{\cdot}36$	33·94	131·83	4655·53	4655·54	+ 0·01
$l_{90} = 3803{\cdot}16$	17·74	54·07	4577·77	4577·84	+ 0·07
$l_{100} = 3785{\cdot}42$	0·00	0·00	4523·70	4523·70	...

Adding to each figure of column 3 the constant 4523·70 we find the values of λ in column 4. The 5th column, on the other hand, contains the observed wave-lengths of the first triplet series (1st component) of Oxygen according to the measurements by RUNGE and PASCHEN (see p. 564). The very close agreement between 4 and 5 shows conclusively that the RYDBERG-THIELE equation satisfies the conditions of both series. Instead of the Oxygen-series we might, of course, have selected any series of the group $\mu = 0$.

In my introductory remarks I have alluded to Professor THIELE's investigation of the third band of the Carbon-spectrum (*Astrophysical Journal*, vol. viii. p. 1). I have mentioned that Professor THIELE found himself obliged to reject the simple formula here used, although he had been the first to notice some of its remarkable properties. His contention, however, that the formula does not sufficiently satisfy the observations, is not acceptable, as will be conclusively shown in the computations which follow. Indeed, if we study more closely the conditions under which Professor THIELE made use of the formula, we come to the conclusion that his negative result is due not so much to a deficiency in the equation employed, as to a particular extra demand imposed upon it. For Professor THIELE introduces an assumption which, though it may have some mathematical probability, has certainly so far no physical foundation. He assumes that all the series of the band should appear coupled in pairs, and that each pair should be represented by one and the same equation, the two branches being obtained by assigning to m either its positive or negative value. This hypothesis necessitates equations of great complexity, involving at least eight constants. It must appear difficult, if not altogether impracticable, to use cumbersome formulæ such as those obtained by Professor THIELE as a basis for theoretical investigations, nor are they of use as purely empirical expressions of the law of the spectral structures. But this complexity disappears as soon as we abandon Mr THIELE's assumption. If we consider by itself each of the many series which he has so successfully unravelled by his masterly treatment of the Carbon-band here considered, we

find that the RYDBERG-THIELE formula satisfies the observations well within their mean errors, the agreement being on the whole even closer than in Professor THIELE's computations. I have carried out the necessary computations for ten series, viz., for the five pairs $\alpha\pm$, $\beta\pm$, $\gamma\pm$, $\delta\pm$, and $\epsilon\pm$, but will present here, for the sake of brevity, only the results of the first two pairs. Since Professor THIELE employed wave-lengths in his calculations, I follow his example and use the second of equations (6):

$$\frac{1}{\lambda_0 - \lambda} = \frac{\alpha_4}{(m+\mu)^2} + \beta_4 = \frac{\alpha_4}{(m+\mu)^2} + \frac{1}{\lambda_0 - \lambda_\infty}$$

Series:	log α_4	λ_0	λ_∞
$\alpha+$	0·87883	5165·10	2241·1
$\alpha-$	0·88016	5165·10	2188·5
$\beta+$	0·87840	5165·18	2271·6
$\beta-$	0·87973	5165·18	2219·0

In the following comparisons I give now the results for every fifth line of the series:

$\alpha+$; $(m+\mu)$	λ obs.	λ comp.	O. – C.
5·0	5161·77	·81	– ·04
10·0	5151·87	·94	– ·07
15·0	5135·63	·66	– ·03
20·0	5113·17	·17	·00
25·0	5084·80	·76	+ ·04
30·0	5050·86	·79	+ ·07
35·0	5011·66	·67	– ·01
40·0	4967·84	·87	– ·03

$\alpha-$; $(m+\mu)$	λ obs.	λ comp.	O. – C.
4·5	5162·41	·43	– ·02
9·5	5153·21	·26	– ·05
14·5	5137·62	·65	– ·03
19·5	5115·84	·82	+ ·02
24·5	5088·11	·05	+ ·06
29·5	5054·73	·68	+ ·05
34·5	5016·12	·10	+ ·02
39·5	4972·78	·78	·00
45·5	4915·16	·20	– ·04

$\beta+$; $(m+\mu)$	λ obs.	λ comp.	O. – C.
5·0	5161·95	·88	+ ·07
10·0	5151·97	2·01	– ·04
15·0	5135·70	·71	– ·01
20·0	5113·17	·21	– ·04
25·0	5084·80	·78	+ ·02
30·0	5050·86	·81	+ ·05
35·0	5011·66	·69	– ·03
40·0	4967·84	·92	– ·08

$\beta-$; $(m+\ )$	λ obs.	λ comp.	O. – C.
4·5	5162·60	·51	+ ·09
9·5	5153·32	·32	·00
14·5	5137·72	·71	+ ·01
19·5	5115·84	·86	– ·02
24·5	5088·11	·07	+ ·04
29·5	5054·73	·69	+ ·04
34·5	5016·12	·12	·00
39·5	4972·78	·81	– ·03

We notice that all the four series possess "tails," a result which has already been obtained by Professor THIELE. Geometrically speaking, this means that the transversal representing the series is *not* parallel to the ray $O\nu_\infty$, but intersects this line at a certain point. We shall see that this is a general feature of the band-series.

In the *Astrophysical Journal*, vol. xx. No. 2, Mr LESTER has recently published most accurate measurements of the telluric Oxygen-bands of the solar spectrum. We shall proceed to investigate how far his observations are accounted for by the RYDBERG-THIELE formula. The general structure of the groups A, B, α and α', may be described as follows. Each group consists of two series of doublets called by Mr LESTER the first and second band, in each of which he distinguishes the two components as the first and second series. In the following tables I exhibit the results of my computations made on some of the series of the three groups A, B, and α.

OXYGEN-BAND. A group. First Band. First Series.

$\log a_4 = 0{\cdot}70266_n$
$\log b_4 = 6{\cdot}57928_n - 10$
$\nu_0 = 13168{\cdot}40$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.
0·75	13168·29	·29	·00
1·75	13167·81	·80	+ ·01
2·75	13166·89	·90	– ·01
3·75	13165·61	·61	·00
4·75	13163·94	·93	+ ·01
5·75	13161·88	·86	+ ·02
6·75	13159·39	·40	– ·01
7·75	13156·51	·54	– ·03
8·75	13153·30	·30	·00
9·75	13149·69	·68	+ ·01
10·75	13145·70	·68	+ ·02
11·75	13141·36	·30	+ ·06
12·75	13136·58	·55	+ ·03
13·75	13131·43	·43	·00

B Group. First Band. First Series.

$\log a_4 = 0{\cdot}90992_n$
$\log \beta_5 = 6{\cdot}68740 - 10$
$\lambda_0 = 6867{\cdot}379$

$(m+\mu)$	λ obs.	λ comp.	Obs. – Comp.
0·8	6867·458	·458	·000
1·8	67·794	·778	+ ·016
2·8	68·337	·344	– ·007
3·8	69·144	·157	– ·013
4·8	70·220	·218	+ ·002
5·8	71·528	·527	+ ·001
6·8	73·078	·085	– ·007
7·8	74·888	·893	– ·005
8·8	76·953	·952	+ ·001
9·8	79·275	·266	+ ·009
10·8	81·80	·83	– ·03
11·8	84·65	·66	– ·01
12·8	87·75	·74	+ ·01
13·8	91·05	·08	– ·03
14·8	6894·67	·69	– ·02

α Group. First Band. First Series.

$\log \alpha_4 = 0{\cdot}88746_n$
$\log \beta_4 = 7{\cdot}25370 - 10$
$\lambda_0 = 6276{\cdot}79$

$(m+\mu)$	λ obs.	λ comp.	Obs. – Comp.
0·4	6276·81	·81	·00
1·4	77·03	·04	– ·01
2·4	77·52	·54	– ·02
3·4	78·29	·29	·00
4·4	79·31	·31	·00
5·4	80·61	·60	+ ·01
6·4	82·16	·15	+ ·01
7·4	84·00	3·98	+ ·02
8·4	86·09	·09	·00
9·4	88·48	·48	·00
10·4	6291·14	·16	– ·02

I must remark here that the determination of μ in band-spectra is a somewhat uncertain operation. In the first series, for instance, the fraction 0·8 might have been used

instead of 0·75 without sensibly altering the differences of the last column. In all cases the agreement between the observed and computed wave-lengths and wave-frequencies is doubtless sufficiently close to justify the assertion that the RYDBERG-THIELE formula represents the measurements most accurately. The same agreement is shown in the following calculations, which refer to some of the second bands.

B Group. Second Band. First Series.

$\log \alpha_5 = 0{\cdot}90992_n$
$\log \beta_5 = 6{\cdot}68740 - 10$
$\lambda_0 = 6867{\cdot}379$

$(m+\mu)$	λ obs.	λ comp.	Obs. – Comp.
12·5	6886·004	·004	·000
13·5	89·183	·181	+ ·002
14·5	92·614	·605	+ ·009
15·5	96·282	·277	+ ·005
16·5	6900·196	·196	·000
17·5	04·363	·366	– ·003
18·5	08·785	·784	+ ·001
19·5	13·449	·452	– ·003
20·5	18·365	·371	– ·006
21·5	23·542	·545	– ·003
22·5	28·986	·970	+ ·016
23·5	34·669	·649	+ ·020
24·5	40·584	·584	·000
25·5	46·770	·771	– ·001

B Group. Second Band. Second Series.

$\log \alpha_4 = 0{\cdot}57250_n$
$\log \beta_4 = 5{\cdot}89220_n - 10$
$\nu_0 = 14558{\cdot}57$

$(m+\mu)$	ν obs.	ν comp.	Obs. – Comp.
11·0	14526·27	·27	·00
12·0	14520·15	·15	·00
13·0	14513·49	·50	– ·01
14·0	14506·29	·33	– ·04
15·0	14498·65	·64	+ ·01
16·0	14490·42	·43	– ·01
17·0	14481·70	·69	+ ·01
18·0	14472·45	·45	·00
19·0	14462·72	·70	+ ·02
20·0	14452·43	·41	+ ·02
21·0	14441·67	·63	+ ·04
22·0	14430·37	·35	+ ·02
23·0	14418·53	·55	– ·02
24·0	14406·26	·26	·00

α Group. Second Band. First Series.

$\log \alpha_4 = 0{\cdot}88756_n$
$\log \beta_4 = 6{\cdot}36303 - 10$
$\lambda_0 = 6276{\cdot}63$

$(m+\mu)$	λ obs.	λ comp.	Obs. – Comp.
10·0	6289·60	·62	– ·02
11·0	6292·35	·36	– ·01
12·0	6295·36	·37	– 0·1
13·0	6298·64	·64	·00
14·0	6302·18	·17	+ ·01
15·0	6306·00	5·98	+ ·02
16·0	6310·06	·05	+ ·01
17·0	6314·40	·40	·00
18·0	6319·02	·01	+ ·01
19·0	6323·92	·91	+ ·01
20·0	6329·10	·08	+ ·02
21·0	6334·55	·53	+ ·02
22·0	6340·28	·25	+ ·03
23·0	6346·27	·26	+ ·01

I have purposely computed wave-frequencies in some cases and wave-lengths in others in order to show that the RYDBERG-THIELE equation may be used in the same form in

both cases. It is not the object of this communication to enter upon a discussion of the constants, but merely to show in a number of typical cases the extraordinary accuracy of our formula. I shall therefore not discuss the other series of the Oxygen-bands, and reserve this particular investigation for another paper. One point, however, in connection with the second bands seems to require special mention. It is noticed that in these bands the index $(m + \mu)$ of the first visible line is a comparatively high number. According to the formula other lines before the one which "apparently" forms the head or beginning should be possible, but these lines are in fact not present in the spectrum. Instances of such "missing" lines are by no means rare among the band-spectra. Bands with "hypothetical" heads have indeed already been pointed out by Professor THIELE in his investigation of the Hydrocarbon-band. It is interesting to find at least one similar occurrence also among the line-series. According to Professor KAYSER the first observed double line of the 2nd Subsidiary Series of Potassium should be preceded by a strong pair at $\lambda = 6985$, which, however, has not been observed. Professor KAYSER remarks that this is the only case among all the line-series where computed lines seem to be actually missing. I am inclined to think that we have here a (so far) unique instance of a line-series with a hypothetical head. Perhaps the lines are not altogether absent, but are too faint to be noticed. It is quite a common feature in band-spectra that the intensity, instead of changing gradually from line to line, sometimes falls off abruptly. This abnormal phenomenon usually occurs in the tails, but there is no reason why it should not also be possible near the heads. We shall have to return to this interesting feature later on when we consider the structure of the Cyanogen-band.

Mr LESTER shows that the measured wave-frequencies of the Oxygen-band can be represented by an equation

$$\nu = \nu_x - am - bm^2 .$$

He lays particular stress on the existence of a term depending on the *first* power of m. Evidently his formula is a first approximation to the RYDBERG-THIELE equation, which may be written:

$$\nu - \nu_0 = \frac{(m+\mu)^2}{a_4\left(1+\frac{b_4}{a_4}(m+\mu)^2\right)} = \frac{(m+\mu)^2}{a_4} - \frac{b_4}{a_4^2}(m+\mu)^4 - \frac{b_4^2}{a_4^3}(m+\mu)^6 \ldots .$$

and can therefore be brought to Mr LESTER'S form if m be counted from the first *observed* line. Thus we find in the case of the second band (second series) of the B-group from the constants of the RYDBERG-THIELE equation:

$$\nu = 14526{\cdot}27 - 5{\cdot}86m - 0{\cdot}2635m^2 \ldots .$$

whereas Mr LESTER has:

$$\nu = 14526{\cdot}27 - 5{\cdot}86m - 0{\cdot}2611m^2 .$$

The examples here given are in my opinion quite sufficient to demonstrate the applicability of the RYDBERG-THIELE formula to both line- and band-spectra. The difference between these two types is seen to be solely due to the difference in the constants of our

equation. Specially striking is the contrast between the values of a_1, which for band-series is always very much smaller than for line-series.

Professor THIELE has pointed out that in a band-series, proceeding from the head towards the tail, the distances between consecutive lines should first increase and afterwards diminish, and therefore reach a maximum at a certain point between the head and the tail. If we differentiate the RYDBERG-THIELE equation

$$\frac{1}{\nu_\infty - \nu} = a_1(m+\mu)^2 + b_1$$

with regard to ν and m, we find

$$\frac{d\nu}{dm} = \frac{2a_1(m+\mu)}{[a_1(m+\mu)^2 + b_1]^2}$$

$$\frac{d^2\nu}{dm^2} = \frac{2a_1}{[a_1(m+\mu)^2 + b_1]^2}\left[1 - \frac{4a_1(m+\mu)^2}{a_1(m+\mu)^2 + b_1}\right].$$

Hence the particular point $m_x + \mu$, for which the distance is a maximum, is found from the equation

$$1 - \frac{4a_1(m_x+\mu)^2}{a_1(m_x+\mu)^2 + b_1} = 0\,; \quad (m_x+\mu)^2 = \frac{1}{3}\,\frac{b_1}{a_1}.$$

But from fig. 3 we know that $\frac{b_1}{a_1} = \tan\beta$, hence

$$(m_x+\mu)^2 = \tfrac{1}{3}\tan\beta\,.$$

We see at once that for a series, whose transversal is parallel to $O\nu_0$ (*e.g.* the first Hydrogen Series) the maximum distance is at $m_x + \mu = 0$, *i.e.* at the beginning (head of the series), since $\beta = 0$; whereas for a series whose transversal is parallel to $O\nu_\infty$ [band-series satisfying DESLANDRES' special equation], the maximum distance is at $m_x + \mu = \infty$ (tail of the series), since $\beta = \pm\frac{\pi}{2}$; but that in general it must be at a point between the head and the tail of the series. Thus we find without difficulty that in the $\alpha+$ series of the Hydrocarbon-band (see p. 579) the greatest distance between consecutive lines occurs between the 85th and 86th line.

Quite recently an important paper has been published by Dr JUNGBLUTH in the *Astrophysical Journal*, vol. xx. No. 4, in which the author discusses his exquisite measurements of the lines of the Cyanogen-band. I take this opportunity, therefore, of applying the RYDBERG-THIELE formula to this very extensive band, which, as Professor KAYSER has shown in his *Handbuch*, p. 479, seemed particularly inaccessible to a satisfactory representation by empirical formulæ. Before the appearance of Dr JUNGBLUTH'S paper I had already attempted to compute the wave-lengths of this band by means of the formula proposed in this paper. It was found that the RYDBERG-THIELE formula gave excellent results up to about the 80th line, that the agreement could still be called very satisfactory up to the 140th line, but that further on the discrepancies increased enormously, and showed the formula to be not applicable throughout the whole extent of Professor KAYSER'S observations. The question therefore arises, whether the formula, which has done such excellent work in the

shorter band-series, fails to account for the phenomena if they extend over a wider interval. If this could be proved, the RYDBERG-THIELE equation would certainly lose much of its value from the theoretical point of view. But there is an alternative. It is quite possible that the Cyanogen-band discussed by Professor KAYSER is not one single series, but really consists of several distinct parts apparently joined together so as to give the appearance of one continuous series. We have seen in the case of the Oxygen-band that two series of the same band need not necessarily exhibit their lines right down to their common head. It was noticed that while the one series which starts from the head may almost suddenly drop off on the tail side, the second series may show the inverse phenomenon, viz. a sudden decrease of intensity on the head-side. The effect of this peculiar behaviour is that the second series forms apparently the tail of the first series. Now in the Oxygen-bands both series consist of pairs of lines, but the distances between the components of the first (or head-) series are considerably smaller than those of the second (or tail-) series. Besides, in the head-series the distances decrease slightly towards the tail, while in the tail-series they increase very rapidly. To show this I subjoin a table giving Mr LESTER's wave-lengths of the B-group.

First (Head-) Series.	Second (Tail-) Series.	First (Head-) Series.	Second (Tail-) Series.
6867·46 } 1·00 68·46 }		6884·65 } 0·89 85·54 }	6884·08 } 1·92 86·00 }
67·79 } 0·99 68·78 }		87·75 } 0·85 88·60 }	86·98 } 2·20 89·18 }
68·37 } 0·97 69·34 }		91·05 } 0·82 91·87 }	90·14 } 2·47 92·61 }
69·14 } 0·99 70·13 }		94·67 } 0·83 6895·50 }	93·56 } 2·72 96·28 }
70·22 } 0·96 71·18 }			97·20 } 3·00 6900·20 }
71·53 } 0·96 72·49 }			01·12 } 3·24 04·36 }
73·08 } 0·96 74·04 }			05·26 } 3·52 08·78 }
74·89 } 0·94 75·83 }			09·68 } 3·77 13·45 }
76·95 } 0·93 77 88 }			14·33 } 4·03 18·36 }
79·27 } 0·90 80·17 }			19·24 } 4·30 23·54 }
81·80 } 0·92 6882·72 }			24·42 } 4·57 6928·99 }
			. . . . .

Now let us suppose for a moment that the distances between the components are very much smaller, perhaps $\frac{1}{50}$th only of those actually observed. The effect is that the first series appears as a single-line band, while the second, with sufficient dispersion, may still show its double character. Near the point where the two series coalesce, we would then have the following wave-lengths of the centres of the lines:

Single Line Series.	Double Line Series.
6877·41	
6879·72	
6882·26	
6885·09	6885·04
6888·18	6888·08
6891·46	6891·36
6895·08	6894·92
	6898·70
	. . . .

Here a remarkable phenomenon will be noticed, viz. that the positions of the centres of the double lines in the second series agree very nearly with those of the single lines of the first series. This is not accidental, because the other groups show the same feature. We have, for instance, in the A-group: (LESTER, *l.c.*, pp. 88 and 92)

Single Line Series.	Double Line Series.
. . . .	
7615·87	
7619·04	
7622·49	7622·40
7626·14	7626·04
7630·04	7629·90
	7633·98
	. . . . .

and for the a group: (*l.c.*, pp. 89 and 91)

Single Line Series.	Double Line Series.
. . . .	
6284·38	
6286·49	
6288·84	6288·77
(6291·48)	6291·38
	6294·26
	. . . .

We notice, then, that at the point where the second series begins the line of the first series falls between the components of the former, and if its intensity is still sufficiently great and the doubles close, the effect of superposition will be that the resulting line appears as a broadened single line. Hence the lines of the first series, which near the head are thin and sharply defined, will gradually widen out. But since the intensity of the first series declines rapidly, their vitiating effect will lessen, the further we

proceed towards the tail, so that finally only the double line series will survive. It is very probable that the interval between the double lines which in the above bands increases steadily, may in more extensive groups reach a maximum and then decrease again further towards the tail. Now the feature thus described seems indeed to agree closely with the phenomena observed in the Cyanogen-band according to Dr JUNGBLUTH. I may be allowed to quote the author's own remarks on this important point, *l.c.*, 241 :—"Following the series further, another peculiarity appears, which, strange to say, has not been heretofore observed, though it comes out distinctly in the first-order spectrum. The lines become gradually broader as they recede from the head, and each finally separates into two lines when it has reached a breadth of about 0·07 t.m. We have now *double lines* similar to those above and below the second head. As the series proceeds, the interval between components of the double lines increases to 0·09 – 0·1 t.m., and then decreases until the components unite again to form one line. The lines of all the observed series show this behaviour, so that in certain parts of the band structure, as for example above and below λ 3700, we have only double lines." He further remarks that this is "a noteworthy property and must be considered in forming any valid theory concerning the origin of spectra." The most remarkable feature seems to me to be the coincidence of the centres of the lines in both series at the point where the second series begins. In consequence of this agreement the band changes from the single line to the double line type steadily, *i.e.* without indication of an abrupt change in the positions of the lines, thus giving the impression of one continuous series. But in fact, as the Oxygen-bands show, the series consists of two separate branches. The reason why in the Oxygen-bands we see a marked discontinuity between the two series is obviously that the lines of the first series fade off too early, *i.e.* before the beginning of the second series. If the faint close doubles, which Mr LESTER calls the continuation of the first series, were more prominent, the appearance of the Oxygen-bands would agree with that of the Cyanogen-band, though the intervals between the components are here too great to produce the impression of widened single lines by the coalescence of the four superimposed components.

I think that this view explains sufficiently the peculiar behaviour of the lines pointed out by Dr JUNGBLUTH. It reveals also a hitherto unknown similarity in the structure of bands of different substances, which may have a bearing on the theoretical aspect of the problem. But what interests us most of all at present is the probability that the first part of the Cyanogen-series consisting of single lines differs essentially from the second part, the double-line series, just as the first series of the Oxygen-bands differs from the second. I am inclined to think that there is even a third branch containing the single lines near the tail, so that in fact each of the four series given by Dr JUNGBLUTH consists of three distinct parts, each of which satisfies a different numerical form of the RYDBERG-THIELE equation. The correctness of this view may appear from the following tables, where the three branches of "Series I.,"

which I shall call I_a, I_b and I_c, are discussed. For the series I_a and I_b the computations have been made on every fifth line, for I_c on every second.

I*a*.

$\log \alpha_4 = 1{\cdot}98171$, $\lambda_0 = 3883{\cdot}50$, $\lambda_\infty = 2199{\cdot}39$.

$(m+\mu)$	λ obs.	λ comp.	O. – C.
5·0	3883·25	·24	+ ·01
10·0	3882·44	·46	– ·02
15·0	3881·15	·16	– ·01
20·0	3879·33	·34	– ·01
25·0	3876·99	7·01	– ·02
30·0	3874·14	·16	– ·02
35·0	3870·80	·79	+ ·01
40·0	3866·94	·97	– ·03
45·0	3862·62	·64	– ·02
50·0	2857·81	·82	– ·01
55·0	3852·53	·53	·00
60·0	3846·79	·77	+ ·02
65·0	3840·59	·56	+ ·03
70·0	3833·95	·90	+ ·05
74·0	3828·32	·26	+ ·06

I*b* (first component).

$\log \alpha_4 = 1{\cdot}93237$, $\lambda_0 = 3876{\cdot}56$, $\lambda_\infty = 2708{\cdot}35$.

$(m+\mu)$	λ obs.	λ comp.	O. – C.
70·0	3823·98	·98	·00
75·0	3816·33	·33	·00
80·0	3808·29	·27	+ ·02
85·0	3799·82	·82	·00
90·0	3791·01	·00	+ ·01
95·0	3781·83	·82	+ ·01
100·0	3772·32	·32	·00
105·0	3762·52	·51	+ ·01
110·0	3752·41	·41	·00
115·0	3742·04	·05	– ·01
120·0	3731·44	·45	– ·01
125·0	3720·61	·62	– ·01
130·0	3709·63	·60	+ ·03
135·0	3698·50	·40	+ ·10

I*c*.

$\log \alpha_4 = 1{\cdot}53310$, $\lambda_0 = 3777{\cdot}78$, $\lambda_\infty = 3368{\cdot}78$.

$(m+\mu)$	λ obs.	λ comp.	O. – C.
64·0	3685·01	4·99	+ ·02
66·0	3680·51	·49	+ ·02
68·0	3676·00	·00	·00
70·0	3671·49	·50	– ·01
72·0	3667·00	·01	– ·01
74·0	3662·52	·54	– ·02
76·0	3658·05	·06	– ·01
78·0	3653·62	·62	·00
80·0	3649·21	·20	+ ·01
82·0	3644·83	·81	+ ·02
84·0	3640·45	·44	+ ·01

The smallness of the residuals in the last columns proves again the applicability of the RYDBERG-THIELE equation, if we admit the suggested subdivision of the series into three apparently coherent branches. With regard to the other series of the band mentioned by Dr JUNGBLUTH, analogous structures are apparent, but I have not yet found time to undertake the necessary computations.

In his paper Dr JUNGBLUTH suggests that the tails of the four series may probably be represented by four faint bands shading off towards the red at wave-lengths 3579, 3603, 3629 and 3658. This suggestion had been already made by Mr KING, who had first observed the peculiar shadings in question. For several reasons I doubt the correctness of this view, and rather incline to think that Mr KING's bands represent

the "heads" of an independent Cyanogen-band distinguished from the others by the fact that, like the Oxygen-bands, its components shade off towards the *red*. First we must admit that the appearance of the bands does not agree with what we should expect if they were "tails." On the plate accompanying Dr JUNGBLUTH's paper they by no means convey the impression of an "infinity" of lines, but appear to be composed of a limited number of lines at finite distances from each other. This feature is specially noticeable at $\lambda = 3603$ and 3629, and certainly agrees better with the supposition of "heads." It is also noteworthy that none of the tails of the 1st series computed above by means of the RYDBERG-THIELE formula agrees with any of the four wave-lengths of Mr KING's bands. Our experiences with the formula cannot but give us now some confidence that the structure of the band should be at least very nearly represented by it. On the other hand, however, Dr JUNGBLUTH'S view seems to be supported to a certain extent by two noteworthy relations between the wave-lengths of the four heads of the series and the supposed tails. He showed that if we form the differences between the wave-lengths of corresponding heads and tails, we find

$$\begin{array}{ll} 3884 - 3579 = 305 & \\ & 36 \\ 3872 - 3603 = 269 & \\ & 36 \\ 3862 - 3629 = 233 & \\ & 36 \\ 3855 - 3658 = 197 & \end{array}$$

It appears, then, that the lengths of the successive series form an arithmetical progression. Again, if we form the quotients

$$\begin{array}{ll} \frac{3884}{3579} = 1{\cdot}0852 & \\ & 0{\cdot}0105 \\ \frac{3872}{3603} = 1{\cdot}0747 & \\ & 0{\cdot}0105 \\ \frac{3862}{3629} = 1{\cdot}0642 & \\ & 0{\cdot}0103 \\ \frac{3855}{3658} = 1{\cdot}0539 & \end{array}$$

we notice that these quotients also form approximately an arithmetical progression. Hence a somewhat remarkable connection seems indeed to exist between the observed heads and the alleged tails. But on a closer view this connection is found to be only apparent. For it is seen that the same relations exist, if we combine Mr KING's "tails" with the successive heads of the other Cyanogen-bands. Thus we find:

$$\begin{array}{lll} 4606 - 3579 = 1027 & & \\ & 52 & \\ 4578 - 3603 = \ 975 & & \\ & 51 & \\ 4553 - 3629 = \ 924 & & \\ & 50 & \\ 4532 - 3658 = \ 874 & & \end{array} \qquad \begin{array}{lll} 4216 - 3579 = 637 & & \\ & 43 & \\ 4197 - 3603 = 594 & & \\ & 42 & \\ 4181 - 3629 = 552 & & \\ & 42 & \\ 4168 - 3658 = 510 & & \end{array} \qquad \begin{array}{lll} 3590 - 3579 = +11 & & \\ & 28 & \\ 3586 - 3603 = -17 & & \\ & 28 & \\ 3584 - 3629 = -45 & & \end{array}$$

These figures show therefore that Dr JUNGBLUTH's relation indicates not a specific connection between Mr KING's bands and the band at $\lambda = 3884$ only, but a far more general correspondence between the former and all the bands of the Cyanogen-spectrum. One might perhaps conclude that Mr KING's shadings may represent the common tail of

all the Cyanogen-bands, but this conclusion is evidently negatived by the last series at $\lambda = 3590$ mentioned in the preceding table, where the edges of the corresponding heads and tails would then be turned towards each other, *i.e.* the lines would not be within the space between head and tail. We notice also that Dr JUNGBLUTH's relation applies as well to the differences between the heads of the four Cyanogen-bands when compared one with the other. Thus:

4606 – 4216 = 390		4216 – 3884 = 332	
	9		7
4578 – 4197 = 381		4197 – 3872 = 325	
	9		6
4553 – 4181 = 372		4181 – 3862 = 319	
	8		6
4532 – 4168 = 364		4168 – 3855 = 313	

The most probable conclusion would therefore be that the alleged relation constitutes a property of the "heads" of the bands, and that Mr KING's shadings should be considered as the edges of a new band of the Cyanogen-spectrum, and not as the tails of the band at $\lambda = 3884$.

In concluding this section of my investigations dealing with the band-spectra, I may point out another form of the RYDBERG-THIELE formula which, though somewhat more complicated in appearance, reveals well its significant structure. Using wave-lengths we may write the equation in the form

$$\lambda = \frac{\lambda_0}{1 \pm \left(\frac{n}{\gamma}\right)^2} + \frac{\lambda_\infty}{1 \pm \left(\frac{\gamma}{n}\right)^2}, \tag{14}$$

where n stands for $(m+\mu)$ and where γ denotes a constant. We notice without difficulty that for $\lambda_0 = 0$, *i.e.* when

$$\lambda = \frac{\lambda_\infty}{1 \pm \left(\frac{\gamma}{n}\right)^2}, \tag{15}$$

the formula becomes identical with RYDBERG's equation for line-spectra, which again, on the further supposition that n represents integers, assumes the well-known form of BALMER's equation. So far, it is true, the observations have shown no evidence of series to which the positive sign in the denominator is applicable. In other words, no line-series have yet been found progressing from the violet towards the red, *i.e.* having their heads on the violet, and their tails on the red side of the spectrum. But the present investigation points now to the possibility of such regularities, and may perhaps induce physicists to search for series of this character.

If, on the other hand, we suppose $\lambda_\infty = 0$, equation (14) assumes the form

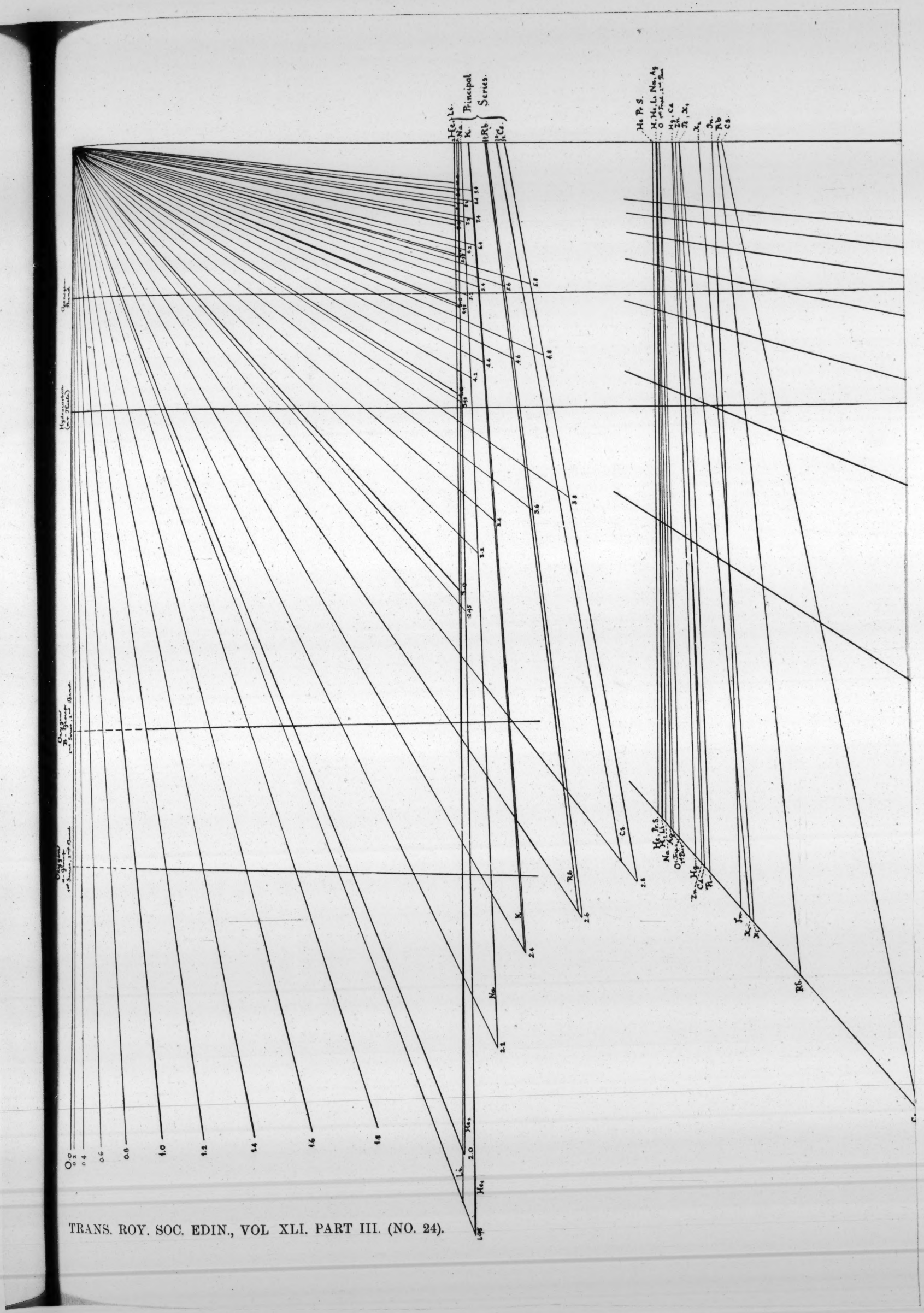
$$\lambda = \frac{\lambda_0}{1 \pm \left(\frac{n}{\gamma}\right)^2}, \tag{16}$$

which becomes identical with DESLANDRES' formula for band-series, if we assign to n integer values. The positive sign expresses that the band shades off towards the violet,

and *vice versâ*. In general, the wave-lengths of the lines of any line- or band-series appear to be made up by two terms, the one satisfying the RYDBERG formula (15) and the other the more general DESLANDRES equation (16). This interpretation of the RYDBERG-THIELE formula shows perhaps more concretely than any other the fundamental character of its structure, and also its importance as the universal expression of spectral regularities.

C. GENERAL CONCLUSIONS.

We are now prepared to enter upon the discussion of some results of a more general character. The geometrical property of the RYDBERG-THIELE formula, as already indicated, enables us to represent on one single diagram every possible line- or band-series as a transversal on which the successive lines of the series are indicated by the points of intersection with the rays $O\nu$. On fig. 4 accompanying this paper I have indicated the positions of these transversals for a limited number of cases. The construction of the diagram is made sufficiently clear by the explanations already given, and therefore requires little additional comment. After the rays O_0 and O_∞ had been constructed at right angles to each other, a line was drawn parallel to O_∞, and, starting from its point of intersection with O_0, the successive values of $(m+\mu)^2$, on a conveniently chosen scale, were measured off. Through O and the points thus obtained lines were then drawn which are marked at their ends in our figure by the corresponding values of $(m+\mu)$. Obviously, in order to obtain the true inclinations β of the transversals, the parallel should be drawn at unit distance from the ray O_∞. It was found, however, that under this condition the diagram would occupy too much space to be conveniently reproduced here, and I therefore decided to draw the parallel at a distance of 10 units. Consequently the inclinations of the transversals in the figure, which we may call β_0, are considerably smaller, the two angles being in the relation $\tan \beta_0 = \frac{1}{10} \tan \beta$. The rays O_1, O_2, O_3 . . . , which correspond to integral values of $(m+\mu)$, have been represented by slightly stronger lines. Now, in the lower part of the diagram we find nearly all the line-series of the group $\mu = 0$. The chemical elements to which the series belong are indicated at the two ends of the transversals, and, where not otherwise stated, the latter refer to the 1st subsidiary series. If on any of these transversals we measure off the distances between consecutive points of intersection, these distances will be found to be exactly proportional to the corresponding wave-frequencies of the series to which the transversal refers. In this arrangement, therefore, all the spectral lines lie precisely on the rays O_1, O_2, ... and all the tail-ends on the ray O_∞. Hence, if we imagine these rays to be represented by thin pencils of light, we may at once obtain the exact arrangement of the lines in any of these series by interposing a plane screen in a direction parallel to the corresponding transversal, since the centres of the luminous dots on the screen must then mark the true relative positions of the lines in the spectrum. There would perhaps be no difficulty in constructing an apparatus for lecture-purposes by which the correctness of


Principal Series.
Hydrogen (1st Spark)
Oxygen (1st Spark)
Oxygen (2nd Spark)
Hg, Li
Na
K
Rb
Cs
He P. S.
H, He, Li, Na, Ag
O (1st Spark)
H2, Cd
N, X1
X2
Jn
Rb
Cs
Hg
Na
K
Rb
Cs
Jn
Rb
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8

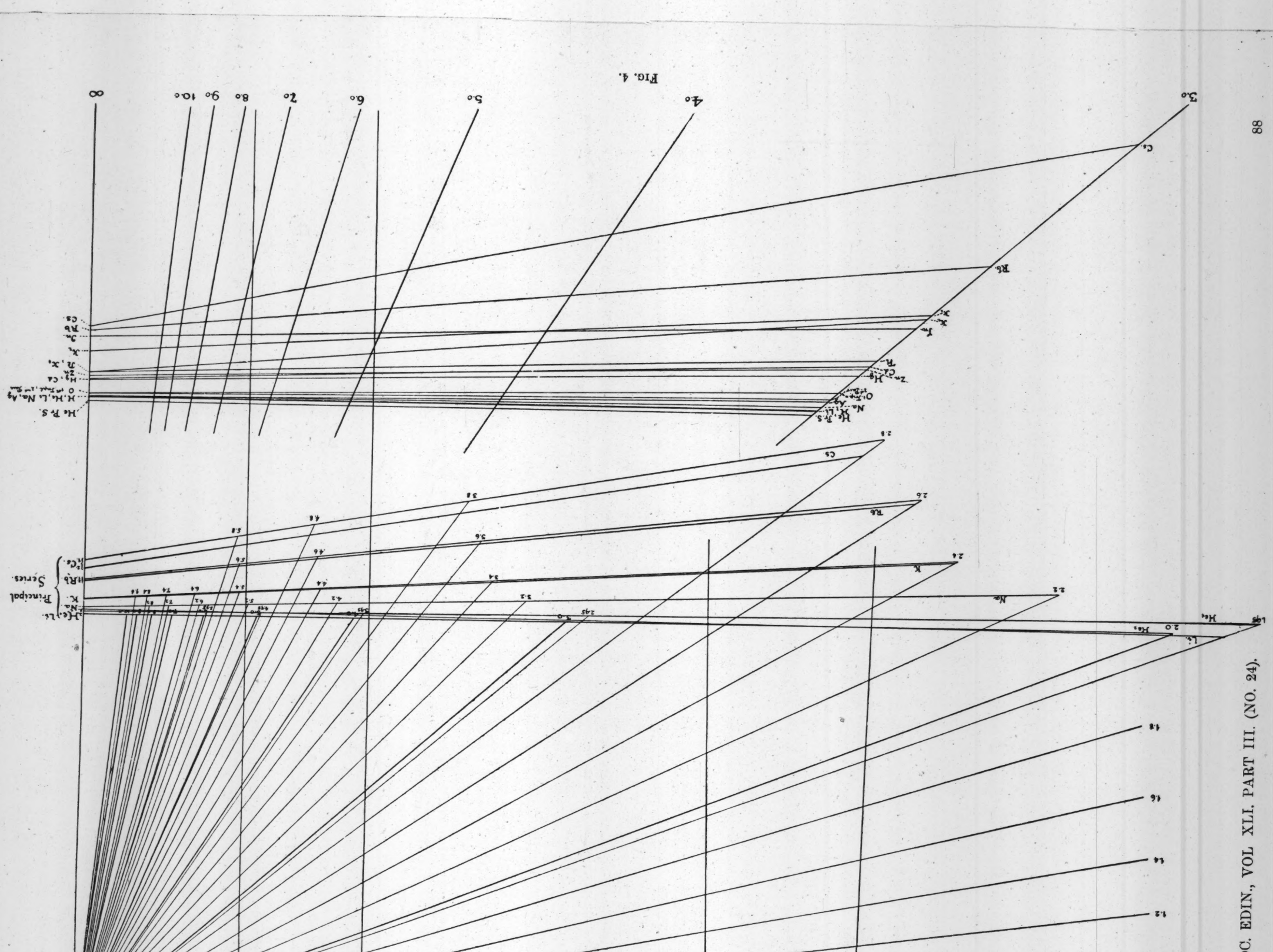

∞
10·0
9·0
8·0
7·0
6·0
5·0
4·0
3·0
Principal Series.
Fe
Na
K
Rb
Cs
Ca
Sr
Ba
Mg
Be
Li

FIG. 4.

the RYDBERG-THIELE formula could thus be experimentally demonstrated. Turning to the upper part of the diagram, we find the transversals exhibiting the arrangements of lines in all the principal series. A smaller scale has been adopted in this case in order to keep the drawing within convenient dimensions, but otherwise the lines have been constructed on exactly the same principle as before. With regard to the intensity of the lines it must be noted as a general rule that the lines are strongest near the head and gradually decrease in brightness towards the tail, and that this rule applies to line-series as well as to band-series. Lastly, four specimens of band-spectra are exhibited by the four transversals nearly parallel to the ray O_∞. In these cases, however, the scale had to be considerably enlarged, since otherwise the transversals would have been too close to O_∞. The four series belong to the group $\mu = 0$, and the drawing is so arranged that the points of intersection with the rays O_1, O_2, O_3 . . . . represent the 10th, 20th, 30th . . . . line of the band.

In his researches on the band-series M. DESLANDRES points out that the wave-frequencies of the lines of such a series are arranged in a manner similar to those of the sound-vibrations produced by an elastic transversely vibrating rod. Indeed we recognise without difficulty that the series of sound-vibrations are represented in our diagram by transversals parallel to O_∞. For it is well known from the mathematical investigations of POISSON, SEEBECK and others, that the wave-frequencies of the stationary transverse oscillations in a vibrating rod, with the exception of the two lowest vibrations, can be expressed by the relation

$$\frac{\nu}{\nu_x} = \frac{(m+\mu)^2}{(x+\mu)^2},$$

where μ depends on certain conditions under which the vibrations take place. But this equation is obviously of the form

$$\frac{1}{\nu - \nu_x} = \frac{a}{(m+\mu)^2 - (x+\mu)^2}, \tag{17}$$

and therefore agrees with the first of (5), if we assume $\nu_\infty = \infty$, *i.e.* if the transversal is parallel to O_∞. The remarkable analogy between the sound-vibrations of an elastic body and the light-vibrations of a radiating atom or molecule is at least suggestive. Is it not, for instance, conceivable that the latter are caused by "standing waves" in the elastic system of electrons which constitutes the atom? If it were possible to find an elastic body of such shape and internal conditions that its transverse vibrations would satisfy the equation

$$\frac{1}{\nu - \nu_x} = \frac{a}{(m+\mu)^2 - (x+\mu)^2} + b, \tag{18}$$

where b is a constant, instead of the simpler relation (17), which refers to the special conditions in a uniform rod, the series of transverse sound-vibrations emitted by such a body would be exactly analogous to the series of light-vibrations emitted by the radiating atoms of a gas or vapour. We could then, by varying the conditions on which b depends, represent the acoustic analogies to the whole range of spectral phenomena

from the band-series to the line-series. At present we are, of course, ignorant of the conditions which lead to the more general equation (18), but at least this remarkable and highly suggestive feature has been brought out by our investigation, that the addition of a single constant to equation (17), the acoustic theory of which is well established, leads to the RYDBERG-THIELE formula, which, as we have demonstrated in the preceding sections, represents so completely all the phenomena of spectral regularities.

It is well known that if the three dimensions of the elastic body are altered in the same proportion, the wave-frequencies of its transverse vibrations change also in the same but inverse proportion, higher frequencies corresponding to smaller dimensions. Thus if we compare two rods of the same material, the one of length l, breadth b, and width w, and the other of length $a.l$, breadth $a.b$, and width $a.w$, the wave-frequencies of their transversal vibrations are in the ratio $1 : a$, while the volumes are in the proportion $1 : a^3$. Hence the wave-frequencies are inversely proportional to the cube roots of the volumes. Now, it appears that in the case of vibrating atoms a similar, although not quite so simple, relation obtains. In each group of chemically related elements, such as the alkalis for instance, the wave-frequencies of the tails, ν_∞, can be approximately represented by an equation

$$\nu_\infty = m + \frac{n}{\sqrt[3]{v}}, \qquad (19)$$

where m and n are constants and v denotes the atomic volume of the element. As an example, let us take the wave-frequencies ν_∞ of the following five elements, referring to the first components of the subsidiary series :

	v	$1/v_1$	ν_∞	Computed.
Li :	11·8	0·43925	28589	28514
Na :	23·7	0·34814	24486	24660
K :	45·0	0·28115	21994	21827
Rb :	56·0	0·26138	20965	20990
Cs :	79·0	0·23306	19748	19792

The values of the last column are those computed from the equation

$$\nu_\infty = 9934 + \frac{42300}{\sqrt[3]{v}}.$$

The atomic volumes have been taken from the data given in the article "Chemistry" in the *Enc. Brit.* To expect more than a merely approximate agreement seems scarcely warranted, considering the uncertainties in the values of v here adopted. In spite of the doubtless large discrepancies between the values of the last two columns, I am inclined to think that the asserted relation between the wave-frequencies ν_∞ and the linear dimensions of the atoms expresses a real physical law. The view is supported by the following interesting fact. In the vertical column of MENDELEJEF'S system

which contains the five elements here considered we find also Hydrogen, which with regard to chemical valency is certainly related to our group of metals. Supposing, then, that Hydrogen belongs to the same group, we may, since we know the wave-frequency of the tail of its subsidiary series, compute its atomic volume. With $\nu_\infty = 27426$ we find from the preceding equation

$$v = \left(\frac{42300}{17492}\right)^3 = 14{\cdot}1\,.$$

Since the atomic volume is defined as the atomic weight divided by the specific gravity of the substance in the solid state, we conclude that solid hydrogen should be 14 times lighter than water. It is interesting to see that for the specific gravity of the liquid at the lowest attainable temperature the same value was found by Professor DEWAR in his celebrated experiments on the liquefaction of gases. The close agreement may perhaps be accidental, especially since we do not know how much the specific gravity may change in the transformation of the gas from the liquid to the solid state, but nevertheless it seems that the new relation assigns to Hydrogen spectroscopically its correct position in the group of elements to which it is chemically related. The difficulty which appears in the attempt to connect, in this group, the position of the tails with the atomic *weights*, no longer comes in when atomic *volumes* are considered.

In other groups the same relation, with altered constants, is noticeable. Thus we find :

	v	$1/v_i$	ν_∞	Comp.		v	$1/v_i$	ν_∞	Comp.
Mg :	13·7	0·41792	39780	39850	Zn :	9·1	0·47898	42925	42889
Ca :	25·4	0·34019	33919	33779	Cd :	12·9	0·42638	40766	40840
Sr :	34·8	0·30629	31060	31130	Hg :	14·7	0·40822	40168	40130
	$\nu_\infty = 7203 + \frac{78112}{\sqrt[3]{v}}$					$\nu_\infty = 24228 + \frac{38960}{\sqrt[3]{v}}$			

On the other hand, however, the relation is not fulfilled for the elements Al, In, Tl, which Professor KAYSER considers as a group of the third vertical column of MENDELEJEF'S system. But whether the three metals are indeed chemically co-ordinated, and should therefore be grouped together, is still somewhat doubtful.

I am not contending, however, that the existence of the asserted relation between the wave-frequencies of the tails and the atomic volumes is conclusively proved by the preceding figures. So far as we may judge from the scanty materials at our disposal, we can only venture to say that indications in its favour seem to be present among the observations. Unfortunately it is not very probable that we shall learn more on this point in the future, as far as observations are concerned. Nevertheless the suggestion is perhaps valuable from the theoretical point of view, and for this reason a reference to it was thought advisable at this stage of the investigation. With regard to

the constant n of equation (19) some evidence may be brought forward which seems to point to the conclusion that in the various groups n is always an integral multiple of one and the same number. For instance, we have seen that in the Mg-group n had almost exactly twice the value of that in the Zn-group. If, in the various groups, we assume for m (equation 19) the values

Li-group :	$m = 10580$
Mg-group :	6536
Zn-group :	23772

the following comparison will show that by assuming $n = 4 \times 10^4$ or 8×10^4 we can approximately represent the wave-frequencies of the tails of the subsidiary series :

	ν_∞	Comp. ($n = 4 \times 10^4$)		ν_∞	Comp. ($n = 8 \times 10^4$)
Li :	28589	28150	Mg :	39780	39970
Na :	24486	24506	Ca :	33919	33752
K :	21994	21826	Sr :	31060	31040
Rb :	20965	21035			
Cs :	19748	19902		ν_∞	Comp. ($n = 4 \times 10^4$)
			Zn :	42925	42931
			Cd :	40766	40827
			Hg :	40168	40101

Applying the relation (19) to the principal series of the alkali-group, we find for the two elements of lowest atomic weight Li and Na, $n = 2 \times 10^4$, but for the three others K, Rb and Cs, $n = 8 \times 10^4$, with the corresponding values of m : 34610 and 12758.

	ν_∞	Comp.		ν_∞	Comp.
Li :	43498	43395	K :	35030	35250
Na :	41468	41573	Rb :	33762	33668
			Cs :	31526	31402

Again we have to confess, however, that the materials at our disposal are too limited to demonstrate the alleged property of the quantity n conclusively, and hence that it is useless to enter upon further comment. The history of the subject here discussed must warn us to state such regularities with due reserve, and not to rush to hasty conclusions, however tempting they may be.

I shall now discuss in a few words some interesting results with regard to the constant a_1 of the RYDBERG-THIELE equation. It was shown at the beginning of this investigation that we can write

$$\nu = \nu_\infty - \frac{1}{a_1}(m+\mu)^{-2} - \frac{b_1}{a_1^2}(m+\mu)^{-4} \ldots\ldots$$

an equation which assumes the form of RYDBERG's formula when $b_1 = 0$. As is well

known, Professor RYDBERG had assumed that the factor $\frac{1}{a_1}$ should be a constant for all the line-series. KAYSER, however, showed that this assumption had to be abandoned because it led to quite inadmissible discrepancies in the computed wave-lengths. In his own formula the value of $\frac{1}{a_1}$ ranges between 109625 for Li and 155562 for Al, and thus "varies only within narrow limits for the various elements" (*Handbuch*, vol. ii. p. 516). At another place he remarks that "probably in the true formula, which neither he nor RYDBERG had found, this factor may indeed be a constant." Let us now see how far $\frac{1}{a_1}$ changes if the RYDBERG-THIELE formula is employed. In this comparison between the various elements we must confine our attention to the subsidiary series, because so far principal series are only known for the elements of the group of alkalis and for Helium. Now we notice at once that the values of $\frac{1}{a_1}$ are certainly not the same for different spectra, since they range between 109575 for Li and 124020 for Cs. But the variations are doubtless much smaller than in Professor KAYSER's formula. Since the *a priori* presumption may perhaps be admitted that the changes of $\frac{1}{a_1}$ may be connected with the position of the element in MENDELEJEF's system, I have arranged the following table, which shows the constants in this order :

I.	II.	III.	VI.
1. H: 109704			
2. Li: 109575			O: 110118
3. Na: 110788	Mg: 112512	Al: 114590	S: 110367
4. K: 116430	Ca: 111363		
5. Cu: 109726	Zn: 114265		Se: 109345
6. Rb: 123572	Sr: 117292		
7. Ag: 109410	Cd: 114250	In: 117398	
8. Cs: 124020			
9. ...			
10. ...			
11. ...	Hg: 112838	Tl: 114015	

The system is that published in vol. 71 of *Nature*, p. 66. In all cases where more than one subsidiary series is known, I have taken the arithmetical mean of the constants computed from each series. One interesting fact is at once revealed by the figures of this table, viz. that the changes are greatest in the first vertical column, the difference between the largest and smallest values of $\frac{1}{a_1}$ being respectively, column I. : 14445 ; column II. : 5929 ; column III. : 2808 ; column VI. : 1022. But another important feature is shown if we compare the horizontal rows 1, 2, 3 . . . 11. In the odd rows 3, 5, 7 and 11 the numbers increase at first, reach a maximum, and then

decrease again. In the even rows 4 and 6, on the other hand, there is a rapid decrease from column I. to column II.

Now, by a course of reasoning upon which I shall not enter here, I have been led to investigate a phenomenon which curiously shows the same regularities, although at first sight it may appear difficult to perceive its causal connection with the changes of our constants. I refer to the effects of pressure on the displacements of the spectral lines. If we take these displacements from the results of HUMPHREY and MOHLER's well-known researches and arrange them in MENDELEJEF's system, we obtain the following table of values, in which the displacements are given in $\frac{1}{1000}$th of a tenth-metre, and correspond in each case to a change of pressure of 12 atmospheres:

	I.	II.	III.	IV.	V.	VI.	VII.	VIII.		
1										
2										
	Na	Mg	Al	Si						
3	25	46	55	43						
	K	Ca		Ti	V	Cr	Mn	Fe	Ni	Co
4	130	27/54		19	25	26	23	25	28	24
	Cu	Zn			As					
5	33	57			35					
	Rb	Sr	Y	Zr	Nb	Mo			Rh	Pd
6	130	37/65	15	28	34	40			30	33
	Ag	Cd	In	Sn	Sb					
7	39	80	88	55	49					
	Cs	Ba	La	Ce						
8	160	34/58	32	27						
9										
							W	Os		Pt
10							19	19		20
	Au	Hg	Tl	Pb	Bi					
11	49	81	61	60	49					

Evidently these figures show the same features as the values of $\frac{1}{a_1}$ discussed before. The amplitude of the changes is again decidedly greatest in column I., and decreases rapidly, being quite insignificant in columns VII. and VIII. We notice further that in the odd horizontal rows 3, 5, 7 and 11 the displacements tend from small values towards a maximum and decrease afterwards, whereas the even rows 4, 6 and 8 begin with high figures, which diminish rapidly and asymptotically approach a constant minimum value. It is also worth mentioning that those cases where two kinds of displacements have been observed by Messrs HUMPHREY and MOHLER, *e.g.* Ca, Sr, and Ba, belong to this second class. Are we allowed, then, to conclude that a connection exists between the displacements of the lines through changes of pressure and the constants a_1 of the RYDBERG-THIELE equation? Obscure and seemingly unfathomable as these phenomena are at present, they cannot but open new vistas of thought and instigate theoretical research.

In my demonstration of the superiority of the RYDBERG-THIELE equation over that of KAYSER and RUNGE I hope that I may not have conveyed the impression that I was insensible of the undoubtedly great merits of the latter. Everyone who has studied the problem here discussed cannot but recognise how much we are indebted to Professor KAYSER and his co-workers for the enormous advance of our knowledge in this field of spectroscopic research, an advance which without his empirical formula would not have been possible. Our feeling of admiration and gratitude for this great physicist's work in a novel and difficult sphere of optics, to which he has so long and successfully devoted his high scientific abilities, will doubtless stand unabated even if his formula should not be finally accepted as the mathematical expression of the law of those marvellous spectral regularities, the knowledge of which we owe chiefly to his ingenious investigations.

XXV.—On the Hydrodynamical Theory of Seiches. By Professor Chrystal.

With a Bibliographical Sketch.

(Read June 19, 1905. Issued separately July 3, 1905.)

PART I.

General Summary.*

§ 1. The variations of the surface-level of lakes due to the direct action of wind and rain, and the smaller disturbances caused by surface waves, of small or moderate length, due to the action of the wind and the movement of boats and animals, must have been familiar phenomena at all times. The first accurately recorded observation, that lake-levels are subject to a rhythmic variation, similar in some respects to the ocean tides, seems to have been made at Geneva in 1730 by Fatio de Duillier, a well-known Swiss engineer. Owing to the peculiar configuration of the Geneva end of Lake Léman, these variations occasionally reach a magnitude of 5 or even 6 feet; and Duillier mentions that they were known in his time by the local name of "Seiches," which has now been applied to rhythmic alterations of the level of lakes in general.

From Duillier's time onwards various observations and speculations regarding the seiches of Lake Léman are recorded. It seems to have been J. P. E. Vaucher, Pastor, and Professor successively of Botany and Church History at Geneva, who, in a memoir written between 1802 and 1804, and published in the memoirs of the Physical Society of Geneva in 1833, first pointed out that seiches are not confined to Léman, but are to be found more or less in all lakes; that they may be of all degrees of amplitude up to 5 feet; and may occur at all seasons of the year, although their occurrence seems to be affected by the state of the atmosphere. He also pointed out that the amplitude of the seiches in Léman increases towards its western end; and that the seiches at its eastern end are not more marked than those observed in other lakes.

These and other early observations of seiches are mentioned by Forel in his great monograph on the Lake of Geneva, vol. ii. p. 50. In particular, he cites one observed at Kenmore on Loch Tay in 1784, which lasted several hours, and is said to have had a period of seven minutes and a maximum amplitude of nearly 5 feet.

A still earlier example is given in the *Scots Magazine* for 1755, p. 593, from which it appears that seiches were caused in several of the lakes of Scotland by the great earthquake of Lisbon on 1st November 1755.

As the source is not easily accessible to everyone, an extract may be printed here :—

* For the convenience of those who are more interested in the observation of seiches than in the purely mathematical theory, I have separated the mathematics, so far as possible, from the general statement of the conclusions arrived at and the suggestions of further problems to be solved by experiment or observation.

"On the first of November last, Loch Lomond, all of a sudden, and without the least gust of wind, rose against its banks with great rapidity, and immediately retiring, in about five minutes subsided as low, in appearance, as ever it used to be in the greatest drought of summer. In about five minutes after, it returned again, as high and with as great rapidity as before. The agitation continued in the same manner, from half an hour past nine till fifteen minutes after ten in the morning; the waters taking five minutes to subside and as many to rise again. From ten to eleven the agitation was not so great; and every rise was somewhat less than the immediately preceding one; but taking the same time, viz., five minutes to flow and five to ebb, as before. About eleven the agitation ceased. The height the waters rose was measured immediately after and was found to be 2 feet 6 inches (76$^{cm.}$) perpendicular. The same day, at the same hour, Loch Lung and Loch Keatrin were agitated in much the same manner; and we are informed from Inverness, that the agitation in Loch Ness was so violent as to threaten destruction to some houses built on the sides of it."

From this clear description there can be no doubt that the phenomenon observed was a longitudinal seiche of Loch Lomond of exceptional amplitude, having a period of ten minutes, probably the trinodal or quadrinodal seiche of that lake. The longest dimension of both Lomond and Ness is nearly in a straight line with the centre of disturbance at Lisbon, and a plurinodal seiche is the result we should expect. The greater disturbance in Loch Ness may be due to the fact that one of the seiche periods of that lake is about nine minutes.

But our really accurate knowledge of the phenomena of seiches dates from the commencement of Forel's own observations at the harbour of Morges, on the Lake of Geneva, in 1869. He may with justice be called the Faraday of seiches. He worked at first with a small portable apparatus (plemyrameter), and later (1876) with a self-registering limnograph installed at Morges, and a portable limnograph of simpler construction. In 1877 Plantamour established a magnificent self-registering limnograph at his villa at Sécheron, near Geneva, which has been in continuous operation since. In 1879 Sarasin devised his portable limnograph, with which observations were made at Tour de Peilz, Chillon, Rolle, and various other stations on Léman, and also upon other Swiss lakes. In 1880 the French Government engineers also installed a fixed limnograph at Thonon, with which observations have been made under the superintendence of Delebecque, Du Boys, and Lauriol.

During the last twenty years a large number of enthusiastic observers have followed the lead given by Forel and his fellow-countrymen; and we have now a great mass of information regarding the seiches in lakes in various parts of the world,* from the 15-hour seiches observed by Henry in Lake Erie, which is 396 km. long, to the seiches of 14 seconds, observed by Endrös in a small pond whose length was only 111 m.

§ 2. The accurate theoretical discussion and co-ordination of the results has scarcely

* See the extension of Forel's bibliography appended to this paper.

kept pace with their accumulation. In the main, the original theory of FOREL has been clearly established, viz., that a seiche is a standing oscillation of a lake, usually in the direction of its longest dimension, but occasionally transverse. In a motion of this kind every particle of the lake oscillates synchronously with every other, the periods and phases being the same for all; and the orbits similar (in fact, rectilinear), but of different dimensions, and not similarly situated. Taking, for simplicity, a longitudinal seiche in a lake of uniform breadth and rectangular section, but varying depth, the horizontal and vertical displacements of any particle on the surface originally at a distance x from a fixed point of reference would be given by $\xi = \phi_n(x) \sin n(t-\tau)$, $\zeta = \chi_n(x) \sin n(t-\tau)$; where t is the time measured from any fixed epoch, τ an arbitrary constant determining the phase of the oscillation, and $T = 2\pi/n$ is the period of the oscillation.

For a lake of given configuration, an infinite number of different values of n (but not any value) are admissible, say $n_1, n_2, n_3, \ldots$; and the functions $\phi_n(x)$ and $\chi_n(x)$ are determined when n is given.

For any given value of n, say n_ν, the function $\chi_{n_\nu}(x)$ vanishes for ν different values of x. At these points, which are called *nodes*, the level of the surface is unaltered by the seiche. Corresponding to $\nu = 1, 2, 3$, etc., we have uninodal, binodal, trinodal, etc., seiches. Any number of these may coexist; and the total seiche displacement is obtained by adding these. When only one of these harmonic components is present we shall call the seiche *pure*.

For a number of values of x, intermediate between the nodal values, $\phi_n(x)$ vanishes, and there is no horizontal motion of the surface particles. These points are called *ventral points*. Four times the distance between a node and the next ventral point is called the wave length. Obviously the wave length is not in general the same at all points of the lake. When the wave length is large compared with the depth, which is always the case in a seiche, the wave is spoken of as a *long wave*; and the hydrodynamical theory in that case admits, as is well known,* of considerable simplification.

§ 3. When the depth of the lake is constant, the theory of long waves leads to the well-known result

$$\xi = A_\nu \sin \frac{\nu\pi x}{l} \sin \frac{\nu\pi \sqrt{(gd)}}{l} (t-\tau),$$

$$\zeta = -\frac{d\nu\pi}{l} A_\nu \cos \frac{\nu\pi x}{l} \sin \frac{\nu\pi \sqrt{gd)}}{l} (t-\tau);$$

where l is the length of the lake, d its depth, x the initial distance of the surface particle in question from one end, all measured in feet, and $g = 32{\cdot}2 : t$ denotes the time, and A_ν and τ are arbitrary constants (amplitude and epoch).

It follows that in a lake of uniform depth the period of the uninodal seiche is $2l/\sqrt{(gd)}$; and the periods of the uninodal, binodal, trinodal, etc. seiches are proportional to the terms of the harmonic series $1, \frac{1}{2}, \frac{1}{3}, \ldots$. Also the uninode

* See AIRY, art. "Tides and Waves," §§ 187 *et seqq.*, *Encyclopædia Metropolitana*, 1848.

is given by $x_1 = l/2$; the binodes by $x_1 = l/4$, $x_2 = 3l/4$; the trinodes by $x_1 = l/6$, $x_2 = 3l/6$, $x_3 = 5l/6$; and so on.

In this case the wave length for each pure seiche is the same at all parts of the lake; the ventral points are midway between the nodes; and the uninode, middle trinode, middle quinquinode, etc. are all at the middle of the lake. In fact, the periods, nodes, and ventral points follow the same law as the periods, nodes, and ventral points of the fundamental and over tones of an organ pipe open at both ends.

§ 4. When, however, the depth of the lake varies, this acoustic analogy is in some important respects misleading. The theory of long waves applied to a longitudinal seiche in a lake of uniform breadth and rectangular cross section, but varying depth, leads to the following among other general results.

§ 5. In any given lake, seiches of all degrees of nodality, *i.e.* uninodal, binodal, trinodal, etc., are possible; and any actual seiche is either one of these or a superposition of several of them. Perhaps the most commonly occurring case is what FOREL calls a dicrote seiche, whose components are uninodal and binodal.

§ 6. The periods of the series of pure seiches are not in general proportional to the terms of the harmonic series $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. The ratios of the periods are in general incommensurable; in general, not even algebraic numbers, although in certain special cases the periods are inversely proportional to the square roots of integral numbers. Thus, for a lake of symmetric complete parabolic longitudinal section, we have $T_\nu = \pi l / \sqrt{\{\nu(\nu+1)gh\}}$; so that the ratios are

$$T_1 : T_2 : T_3 : \ldots = 1/\sqrt{(1\times 2)} : 1/\sqrt{(2\times 3)} : 1/\sqrt{(3\times 4)} : \ldots$$

Indeed, it follows from a result* which I obtained for lakes whose longitudinal section is part of the quartic curve $z = h(1 - x^2/a^2)^2$ that concave lakes can be imagined in which $T_1, T_2, T_3, \ldots$ all approach as nearly to equality as we please. Hence, for example, it may very well happen that it is the trinodal, and not the binodal seiche whose period is half the period of the uninodal,—a result wholly in contradiction with the acoustic analogy suggested by the consideration of lakes of uniform depth.

§ 7. As this is a matter which seems to have caused some perplexity, it may be well to give some numerical illustrations. Let us take a quartic lake of the kind discussed in the paper to which I have referred. The period of the ν-nodal seiche is given by $T_\nu = 2\pi l / \gamma \sqrt{\{gd(4\nu^2\pi^2/k^2 + 1)\}}$, where γ and k depend on the configuration of the lake. Suppose that $T_3/T_1 = 1/2$, then we find that we must have $4\pi^2/k^2 = 3/5$. It then follows that $T_2/T_1 = \cdot 686$. For a lake of this kind we should therefore have $T_1 : T_2 : T_3 = 1 : \cdot 686 : \cdot 5$. In a communication to the Société Vaudoise des Sciences Naturelles, FOREL† mentions that three periods have been determined from observations on the Lake of Constance, viz., $T_1 = 55\cdot 8$, $T_2 = 39\cdot 1$, $T_3 = 28\cdot 1$, which give $T_1 : T_2 : T_3 = 1 : \cdot 701 : \cdot 504$. It would appear, therefore, that the Lake of Constance

* See *Proc. Roy. Soc. Edin.*, vol. xxv. p. 328, Mar. 20, 1905.

† *Ext. Bull.*, vol. xl. 149, Feb. 3, 1904.

behaves not very differently from a quartic lake of the kind supposed. Of course it by no means follows that the normal curve* of the Lake of Constance is even approximately like the quartic curve supposed.† The point I wish to make is that, from the hydrodynamical point of view, there is nothing surprising in the relation of its periods. Farther investigation of the phases of the seiche at different parts of the lake will probably show that the *seiche à la quinte*, as FOREL calls it, for which $T = 39'$, is really the binodal seiche; and that the seiche for which $T = 28{\cdot}1'$ is a trinodal.

§ 8. The formula for the periods of a quartic lake given above may be written $T_\nu = \rho/\sqrt{(\nu^2+\epsilon)}$, where $\rho = kl/\gamma\sqrt{(gd)}$, $\epsilon = \pm k^2/4\pi^2$, the minus sign corresponding to convex lakes. As this formula applies to quartic lakes of every variety, concave or convex, symmetric or asymmetric, it may be conjectured that it will give rough approximations at least to the periods of any actual lake of fairly regular configuration. The constants could not be determined *à priori* without discussing the bathymetric data for each lake. The ratios of the periods, however, depend only on the single constant ϵ. We have, in fact, $T_\nu/T_1 = \sqrt{\{(1+\epsilon)/(\nu^2+\epsilon)\}}$. In general the two longest periods are best known. If we take these as given, the equation $T_2^2/T_1^2 = (1+\epsilon)/(4+\epsilon)$ gives the value of ϵ; and then $T_\nu = T_1\sqrt{\{(1+\epsilon)/(\nu^2+\epsilon)\}} = \rho/\sqrt{(\nu^2+\epsilon)}$, where $\rho = T_1\sqrt{(1+\epsilon)}$.‡ For very large values of ν, we have approximately $T_\nu = \rho/\nu$; in other words, the periods of the seiches of higher nodality approximate to a harmonic series. By means of these formulæ—which I shall call the *Quartic approximation*—the table on p. 604 has been calculated for a number of lakes whose periods are fairly well determined. I have gone in most cases as far as $T = 4{\cdot}7'$, which is the period found by FOREL for the longest progressive surface waves ever observed on Léman, calculated for infinite depth of water.

As a control, I have added at the end of the table, under A, B, C, D, the periods for a complete symmetric rectilinear, semirectilinear, complete parabolic, and semiparabolic lake respectively, as calculated by the quartic approximation, and as calculated by the accurate formulæ of §§ 27, 34, 49, and 51 below.

* See § 12 below.

† Nevertheless it is curious to pursue this numerical case a little further. Referring to my paper already quoted, and calculating k as above, we get $k = 8{\cdot}112$. If we assume the longitudinal section to be symmetrical, then $\gamma = 2\tanh(k/4)$; and we have $\gamma = 1{\cdot}932$. Hence $r = \{1-(\gamma/2)^2\}^2 d$, gives, if we put $d = 252^{\text{m.}}$ (the maximum depth of Constance), $r = 1{\cdot}1^{\text{m.}}$ If then we take a symmetric quartic lake having the same length as Constance, viz., $65^{\text{km.}}$, the same maximum depth, and end depths of $1{\cdot}1^{\text{m.}}$, we find $T_1 = \pi 65\times 10^5/{\cdot}966\sqrt{\{981\times 25200\times 8\div 5\}} = 56'{\cdot}0$. Hence $T_1 = 56'{\cdot}0$, $T_2 = 38'{\cdot}4$, $T_3 = 28'$. The agreement with the observed periods of Constance is curiously close, and is, no doubt, partly accidental. It will be of great interest to work out the normal curve for Constance, and calculate the periods by a rigorous application of the theory, as has been done by Mr WEDDERBURN and myself for Treig and Earn.

‡ It is interesting to notice that in the case of a concave lake $\rho/\sqrt{\epsilon}$ is the period of the "anomalous seiche." See *Proc. R.S.E.*, xxv. (1905), p. 645.

Quartic Approximations to Lake Periods.

Lake.		T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}	T_{17}	T_{18}	T_{19}	T_{20}
Constance * $\epsilon=1\cdot89$ $\rho=94\cdot92$	Calc.	...	...	28·76	22·44	18·30	15·42	13·31	11·69	10·43	9·40	8·56	7·86	7 26	6·75	6·30	5·91	5·57	5·26	4·98	4·74
	Obs.	55·80	39·10	28·10	...	...	15·00														
Garda $\epsilon=1\cdot84$ $\rho=72\cdot53$	Calc.	...	...	22·02	17·17	14·00	11·79	10·17	8·94	7·97	7·19	6·54	6·01	5·55	5·16	4·82	4·52				
	Obs.	43·00	30·00	23·00																	
Madüsee $\epsilon=\cdot416$ $\rho=42\cdot24$	Calc.	...	...	13·76	10·43	8·38	7·00	6·01	5·26	4·68											
	Obs.	35·50	20·10	{13·7 14·6}	...	{8·0 8·4}	...	...	{5·3 5·5}												
Neuchâtel † $\epsilon=4\cdot13$ $\rho=112\cdot4$	Calc.	...	...	31·01	25·04	20·82	17·74	15·41	13·61	12·18	11 01	10·05	9·23	8·54	7·94	7·42	6·97	6·56	6·20	5·88	5·60
	Obs.	49·6	39·4	...	24·20	20 60	...	...	{14 15}	...	...	...	9·04	8·3	{7·7 7·8}	...	...	...	...	...	5·7
Starnberg $\epsilon=\cdot991$ $\rho=35\cdot2$	Calc.	...	...	11·15	8·55	6·92	5·80	4·99	4·37												
	Obs.	24·98	15·78																		
Geneva $\epsilon=-\cdot0871$ $\rho=70\cdot23$	Calc.	...	...	23·52	17·61	14·09	11·72	10·04	8·79	7·80	7·03	6·38	5·86	5·41	5·00	4·68					
	Obs.	73·50	35·50	...	20 ?	...	...	10·3	...	...	7	...	...	...	5						
Ness $\epsilon=-\cdot0738$ $\rho=30\cdot32$	Calc.	...	...	10·14	7·60	6·07	5·05	4·33													
	Obs.	31·5	15·3	8·8 ?																	
A $\epsilon=\cdot9497$ $\rho=139\cdot6$	Quart.	...	...	44·27	33·92	27·41	22·97	19·76	17·33	15·43	13·90										
	Acc.	100	62·76	43·57	34·28	27·79	23·65	20·40	18·05	16·09	14·60										
B $\epsilon=\cdot2753$ $\rho=112\cdot9$	Quart.	...	...	37·08	27·99	22·46	18·75	16·09	14·09	12·53	11·28										
	Acc.	100	54·62	37·67	28·76	23·27	19·54	16·84	14·79	13·19	11·90										
C $\epsilon=\cdot5000$ $\rho=122\cdot5$	Quart.	...	...	39·74	30·15	24·25	20·27	17·41	15·25	13·57	12·22										
	Acc.	100	57·74	40·82	31·62	25·81	21·82	18·90	16·67	14·91	13·48										
D $\epsilon=\cdot2857$ $\rho=113\cdot4$	Quart.	...	...	37·21	28·10	22·55	18·83	16·15	14·14	12·58	11 32										
	Acc.	100	54·77	37·80	28·87	23·35	19·61	16·90	14·85	13·24	11·95										

* Also Calc. $T_{23}=4\cdot13'$, $T_{24}=3\cdot96'$, Obs. $T=4\cdot00'$.

† Also Calc. $T_{26}=4\cdot3'$, Obs. $4\cdot3'$; Calc. $T_{38}=2\cdot9'$, Obs. $2\cdot9'$ (due to steamers); Calc. $T_{102}=1\cdot1'$, Obs. $1\cdot1'$ (due to steamers). For Neuchâtel $T_{150}=\cdot75'$. For a wave-length of $510^{m.}$ the values of T for progressive waves at depths of $10^{m.}$ and $50^{m.}$ are $24\cdot6''$ and $51\cdot7''$ as compared with $T_{150}=45''$.

Authorities for the various lakes are given in Part III.

This table must, of course, be used with great caution. In the first place, as we increase the number of the nodes we trench more and more upon the limits of the hypothesis that the waves are "long." Then it cannot possibly represent the case of convex (*i.e.* in practice concavo-convex lakes) so accurately as the case of wholly concave lakes. It will be noticed, for example, that in Geneva and Ness, which are similar in their seiche-character, there are seiches of 20′ and 8·8′, probably trinodal or quadrinodal, which do not fit into the scheme suggested. As the nodality increases, the periods become more and more nearly equal, and therefore more difficult to distinguish, either by a rough calculation, or, for that matter, by observation, from one another, and from progressive surface waves of purely local significance which are not seiches at all.

Subject to these qualifications, the table is very interesting and suggestive. It shows the greater variety of possible periods in some lakes as compared with others. It shows that there is nothing surprising, from the hydrodynamical point of view, in the fact that the three longest periods for Constance are 55·8′, 39·1′, and 28·1′. The *seiche à la quinte* of which FOREL speaks in the cases of Constance, Garda, and Starnberg is in all probability simply the binodal seiche; and the seiche whose period is approximately half the longest period is a trinodal. Such questions cannot be finally settled until the phases of the seiches have been determined by simultaneous limnographic observations at different parts of the lake, as has been done by ENDRÖS in his admirable investigation of the seiches of the Chiemsee, *Seespiegelschwankungen beobachtet am Chiemsee*, Traunstein, 1903.

I have included in the table as longitudinal seiches some which have been held by observers to be transversal. This I have done for two reasons. In the first place, the absolute identification of these by means of phase observations has not in all cases been satisfactorily accomplished.* Again, it is possible that a transversal seiche might coexist with a longitudinal one of nearly the same period throughout a considerable part of the lake. The interference of these at the ventral points of the longitudinal seiche would produce the phenomenon of seiche beats† at various points along the shore. At the ends of the lake, which are ventral points for all the different pure seiches, all these seiches interfere. It follows, equally from observation and from the analogy of the vibrating string of varying density, presently to be mentioned, that the average amplitudes of the seiches that occur in nature diminish rapidly as their nodality increases. Hence the chief features of the limnographic trace at the shallow ends of a lake will in general be the periodic configuration due to the interference of the uninodal and binodal seiches; the effect of the others will merely be to produce an embroidery on the main outline. Also, since the periods become more nearly equal as the nodality increases, this embroidery will have an

* See FOREL, *Le Léman*, t. ii. p. 148.

† Since this was written I have noticed that ENDRÖS, in his able analysis of the seiches of the Chiemsee, cites examples of variations in the phases and amplitudes of nearly pure seiches, which he regards as due to the interference of seiches of the same period differing in phase. He suggests, with great probability, that such seiches are generated by a common but intermittent cause of disturbance.

irregular character, due to the beats of seiches not very different in period. The ends of a lake are therefore the worst places for distinguishing seiches of higher nodality. If a lake had three periods 10′, 11′, 12′, the proper place to observe at, in order to establish clearly the seiche corresponding to the first or last of these periods, would be at a node of the seiche whose period is 11′. Thus FOREL's argument, that the 10′ seiche of Léman is a transversal and not a plurinodal longitudinal one, because it is not detected at Geneva, is by no means conclusive. I may also add, although I have little theoretical or experimental ground as yet for the opinion, that it seems to me unlikely that any transversal seiche would be so stable, especially in a lake of the shape of Léman, as the beautiful records of FOREL's limnograph seem to indicate. Nevertheless, great weight must be attached to the inclination of so sagacious an observer, who has all the data before him.

The table also raises many interesting subjects of inquiry. Why, for example, are no seiches observed in Constance of the periods 22·4′ and 18·3′? Is this an accident, due to the position at which the limnograph was placed; or are these seiches unstable, owing to irregularities of the lake-bottom near one or more of the corresponding nodes?

§ 9. In a purely concave lake the ratio of the uninodal period to the binodal period is less than a half. In a purely convex lake, if such a thing could be found in nature, the corresponding ratio would be greater than a half. In lakes which are neither purely concave nor purely convex the value of T_2/T_1 will be greater or less than a half according as the concavity or convexity predominates.

§ 10. In the case of parabolic and quartic lakes the rule given by DU BOYS for calculating the periods, viz., ${}_aT_\nu = (2/\nu)\int dl/\sqrt{(gh)}$, where h is the depth corresponding to the element dl of the line of maximum depth, gives too high a value for purely concave and too low a value for purely convex lakes; but it gives in many cases a good first approximation to the periods. This approximation is better for concavo-convex lakes than for purely concave or purely convex lakes; and for purely concave or purely convex lakes, the approximation is better the higher the nodality of the seiche. For a purely concave symmetric parabolic lake DU BOYS' rule would be considerably out; in fact, for such a case ${}_aT_1/T_1 = 1{\cdot}414$. It may also err greatly in cases where there are great variations of the breadth of the lake, as the method of applying the formula takes no account of such peculiarities.*

§ 11. In a lake of varying depth the uninode is not in general in the middle of the lake, and the uninode, middle trinode, middle quinquinode, etc. are not coincident. Also the ventral points are not midway between the nodes; and the wave length varies from node to node. Thus, for example, in a symmetric parabolic lake the uninode is of course in the middle, but the binodes are displaced towards the shallow ends. It results from the calculations *à priori*, made by myself and Mr WEDDERBURN, that in

* Dr ENDRÖS has found a striking example in the uninodal seiche of the Waginger See, of which he was good enough to tell me by letter.

Earn, which is asymmetric, the uninode is very near the middle; but in Treig this is not so nearly the case: in neither is the trinode coincident with the uninode. In both these lakes the deep binode is not far from the point of greatest depth; in both the shallow binode is nearer the end than a fourth of the length of the lake; but, whereas in Earn the binodes are on opposite sides of the deepest point, in Treig they are on the same side. In neither lake are the binodes equidistant from the uninode. It remains to be seen how far these results of theory will agree with observation.

§ 12. When the breadth and the form of the transverse section of a lake vary as well as the depth, provided these variations are not too abrupt, it can be submitted to calculation by introducing two new variables, viz., σ, which is the product of the area of the transverse section by the breadth of this section at the surface; and v, which is the area of the surface of the lake between the trace on the surface of the transverse section corresponding to σ, and any other similar line chosen for reference. In order to submit the lake to calculation, its line of maximum depth is taken and laid out straight; and practically the lake is treated as if it were a lake of uniform breadth and rectangular cross section, whose longitudinal section is the curve, the abscissa and ordinate of any point on which are v and σ respectively. This curve I call the *normal curve* of the lake. If we may judge by our results for Treig and Earn, these assumptions are sufficiently correct for ordinary concave lakes at least.

§ 13. In my calculations no account is taken of the dissipative forces which damp the seiche oscillation. In some cases the damping is hardly sensible during the period for which a seiche is observed to be pure, or even simply dicrote. Forel quotes an observation of Plantamour's, in which a pure uninodal seiche in Léman, whose maximum double amplitude was about $169^{mm.}$, lasted for seven and a half days, and consisted of 148 oscillations. The mean double amplitude of 20 oscillations was at first $167^{mm.}$, and at the 140th, $80^{mm.}$. It was finally disturbed by the appearance of a binodal component, which turned it into a dicrote seiche; otherwise Forel calculates that it might have lasted two days more.* In other cases the damping of some of the pure seiches seems to be considerable, owing probably to the fact that the lake is, so to speak, not well tuned for particular periods. This is seen by studying the form of the limnograph trace, by the elegant method suggested by Soret.† A serious difficulty is thus raised if we attempt to determine the seiche periods from the dicrote trace; and it is not unlikely that some of the divergence between the results of observations at different times on the same lake may be due to this cause. Calculation and observation alike have led me to the conclusion that the best way to determine with final accuracy the seiche constants for a lake is to deduce approximately the periods and the positions of its uninode and binode beforehand

* It is just possible that this seiche may have been maintained by some continuing but partly intermittent external cause. The limnogram given by Forel seems to show traces of the interference to which Endrös has called attention.

† *Arch. de Sc. Phys. et Nat. Genève*, 3me Pér. t. iii. p. 1, 1880. The method has been elaborated and used with great effect by Endrös, *l.c.*, p. 15.

by calculation, from the contours of the lake basin, or by using rough preliminary observations, if such are available; and then to deduce the uninodal and binodal periods from limnographic traces obtained at the binode and uninode respectively, where these seiches are very nearly pure.*

§ 14. It will be obvious that a seiche, properly so called, differs essentially from an ocean tide. The origin of a seiche, and the absolute and relative magnitudes of the pure seiches of which it is composed, no doubt depend on external circumstances; but the periods and the positions of the nodes of the component seiches depend merely on the configuration of the lake basin, and on the surface-level of the water at the time. In a tide, on the other hand, the periods are dependent on external disturbing agencies, chiefly the sun and moon. In the language of physicists, a seiche is a *free oscillation*; a tide a *forced oscillation*.

It is by no means the case that the limnograph traces always indicate a seiche, properly so called.

True and unmistakable seiches are generally seen in comparatively calm weather, or in the lull after storm. At the commencement of a seiche the trace is often irregular; and during storm, oscillations of great amplitude are found, the intervals between the successive maxima of which are not equal. This inequality might, it is true, be due to a jumble of different seiche components, only some of which survive in the subsequent steady oscillation. In the traces for Lake Erie, which does not show the ordinary seiche phenomena with remarkable clearness, but, on the other hand, is remarkable for oscillations of enormous amplitude, associated with the passage of tornadoes, I have noticed indications of the retardation and acceleration of phase which characterise the initial stages of the action of an intermittent external disturbing force, whose period differs from any seiche-period of the lake. The same thing is, I think, visible in the trace of PLANTAMOUR'S limnograph during the cyclone of 20th February 1879, which is reproduced by FOREL on p. 194 of his monograph on Léman. Seiches of this kind I propose, for distinction, to call *forced seiches*, and I propose to discuss them farther in a subsequent communication, dealing with the external causes of seiche phenomena.

§ 15. In the following purely mathematical part of this memoir my purpose is twofold:—first, to establish generally, and also by means of special instances, the leading principles stated above; secondly, to furnish formulæ and methods which can be applied in the investigation *à priori* of the periods and nodes of a lake whose length is considerably greater that its breadth, and which does not present excessive or abrupt variations in the configuration of its basin. I believe that in many such cases a sufficiently close approximation can be obtained by finding the normal curve of the lake, and replacing this curve by a combination of parabolas, straight lines, or simple quartic curves. Thus, for example, as will be seen in a paper by Mr E.

* This was written before I had access to the monograph of ENDRÖS, whose plan of observation for the Chiemsee is in many ways a model.

MACLAGAN-WEDDERBURN and myself, presently to be submitted to the Society, good results for Lochs Earn and Treig have been obtained by using two parabolas.

§ 16. In the course of the mathematical investigation, an abstract of which was published in the *Proceedings* of the Society in October last, I have had occasion to introduce certain new functions, which I have called the Seiche-cosine, Seiche-sine, Seiche-cotangent, and the Lake-function. My reason for venturing to give special names to these functions is, that I believe that some of them, especially the Seiche-cosine and the Seiche-sine, which are independent synectic integrals of a special case of the hyper-geometric equation, will be found interesting theoretically, and very useful in applications. Preoccupation with the physical side of the present problem has prevented me from following the pure mathematics which it suggests. Happily, this part of the question has been taken up by a younger mathematician, Dr HALM, who, in a paper which will be submitted to the Society along with the present communication, has obtained a number of interesting results, which tend to justify my belief that the seiche functions are not unlikely to occupy a permanent place in applied mathematics. He has also given tables for calculating $C(c, 1)$ and $S(c, 1)$, and graphs of these functions.

§ 17. I have said that the acoustic analogy with an organ pipe is unsatisfactory in the general case. There is, however, an acoustic analogy which is perfect. Consider a stretched string, fixed at both ends, whose length in conveniently chosen units represents the median line of the lake, which we take to be of uniform breadth and rectangular cross section, but of variable depth, $h(x)$, at any distance x along the median line. Suppose the string so constructed that its density at x is $1/h(x)$; and let $u = P(x) \sin nt$ be the transverse displacement in any fundamental (normal) mode of vibration at time t of the point which was originally at x. Then, if the tension of the string be properly adjusted, we shall have

$$\xi h(x) = u, \qquad \zeta = -du/dx,$$

ξ and ζ denoting the horizontal and vertical displacements of the seiche. The motion of the string, therefore, exactly represents the seiche movement, the transverse displacement of the string corresponding to the horizontal displacement of the seiche; and the gradient of the curve formed by the string at any moment to the vertical displacement of the seiche.

It will be noticed that *nodes of the string correspond to ventral points of the lake*;* and the ventral points on the string to nodes on the lake. This analogy is most useful, both in explanation and in suggesting methods of calculation. To a uniform string with its binodal, trinodal, quadrinodal, etc. modes of vibration, corresponds a lake of uniform depth with its biventral, triventral, quadriventral, etc., *i.e.* uninodal, binodal, trinodal, quadrinodal, etc. seiches.

Starting with a uniform string, let us increase its density by adding a small

* There seems to have been a good deal of confusion and some false analogy in this respect.

weight at the middle. The result is to lengthen the periods of the binodal, quadrinodal, etc. modes, but to leave unaltered the periods of the trinodal, quinquinodal, etc. modes. Hence, if we start with a lake of uniform depth, and diminish its depth a little in the middle, the result will be to lengthen the periods of the uninodal, trinodal, etc. seiches, and to leave the periods of the binodal, quadrinodal, etc. seiches unaltered. One consequence of this is, that in the modified lake the period of the binodal seiche is less than half the period of the uninodal. This accords with what we have said before, since the lake is now convex. Similarly, by diminishing the density in the middle of the string, we get an explanation of the corresponding peculiarity of concave lakes.

A slight modification of MELDE'S experiment will give us another illustration of the use of this acoustic analogy. Fix a tuning-fork with its prongs horizontal, so that it vibrates in a vertical plane. If we stretch a fine string vertically upwards from a point of attachment on one of the prongs, then, with the proper length and tension, when the fork is excited it will vibrate steadily with a beautifully sharp node in the middle, its lower ventral point being at a distance of about one-fourth of the length from the lower end. If a very small pellet of beeswax be fixed on the string at a distance of about one-eighth of its length from the lower end, it will be found, after slightly adjusting the tension, if necessary, so as to re-establish full resonance, that the string will vibrate with one node as before; but that node and also the lower ventral point will be nearer the lower end than before.* Hence, if we start with a lake of uniform depth, the nodes of the binodal seiche will be each at a distance of one-fourth of the length from its respective end; but if we make a shallow at one end, say about one-eighth of the length of the lake from that end, the corresponding node will be displaced towards the shallow. A depression of the lake bottom would, of course, have the opposite effect.

In a similar way, we can see that an irregularity of the lake bottom will seriously disturb the seiches that have nodes there, while it may affect very little those seiches which have ventral points there. This suggests a very natural explanation why, of the many theoretically possible seiches for any given lake, we may find traces of only a limited number on the limnogram. To this point I shall return when we come to consider special lakes in detail, and the causes which may excite the seiches observed in them.

A very important practical conclusion suggested by the acoustic analogy is that, while the boundary conditions at the end of the lake may seriously affect seiches of higher nodality, they have comparatively little influence on the seiches of lower nodality, in particular on the uninodal and binodal seiches.

§ 18. So far as I am aware, the only serious attempt hitherto made to submit seiche phenomena in detail to mathematical calculation is contained in an interesting paper

* With a Koenig's fork (256 Ut_2), 140 cm. of No. 30 thread, which weighs ·00058 gm. per cm., and a tension of 45 gm., 2·5 mg. of wax produced a displacement of the ventral point exceeding 8 cm.

by Du Boys, entitled "Essai Théorique sur les Seiches" (*Arch. d. Sc. Phys. et Nat. Genève*, t. xxv., June 15, 1891). In this paper he reproduces the well-known theory for a lake of uniform depth and section. Then, by a confessedly inexact application of the theory of progressive waves ("onde solitaire de translation"), he arrives at the formula $T_\nu = (2/\nu)\int dl/\sqrt{(gh)}$, to which we have already referred. Du Boys' theory gives, for reasons explained elsewhere,* a good approximation in certain cases for T_1, and also for the position of the uninode, but it gives no account of the incommensurability of the seiche periods (it would give $T_1 : T_2 : T_3 : \ldots = 1 : 1/2 : 1/3 : \ldots$ in all cases); nor does it lead to a correct determination of the binodes. Moreover, no good reason can be given for the fact that in order to get a good approximation for T_1 we must integrate along the line of greatest depth, instead of using Kelland's formula † for the wave velocity, and putting $T_1 = 2\int dl\sqrt{(b/ag)}$, where b is the surface breadth, and a the area of the cross section. Nevertheless Du Boys' work was a notable initial step in a difficult investigation.

§ 19. In an address, an abstract of which was published in *Il Nuovo Cimento*, 4, vol. viii. p. 270, 1898, Volterra, at the instigation of Forel, pointed out the great interest of the phenomena of seiches from a physico-mathematical point of view; and promised to develop a mathematical theory in a separate paper. It is, however, impossible to gather the exact nature of his views from the brief abstract referred to; and his promised memoir has not, so far as I can learn, yet appeared.

At a meeting of the Royal Society of Edinburgh on 16th February 1903, I gave, at the request of Sir John Murray, and chiefly for the sake of his Lake Surveyors, a similar address, in which I stated most of the general principles above enunciated on the strength of approximative calculations, which I have since replaced by the more rigorous and more effective methods of the present paper. During this address, Mr E. Maclagan-Wedderburn demonstrated the uninodal and binodal standing waves in a long rectangular trough by means of the following elegant apparatus devised by himself.

C P is a long rectangular trough, in which there is a fixed partition, D, pierced at the bottom, dividing C P into a small and a large compartment, with a communication at the bottom of moderate size. E is a movable partition, by means of which the longer part of the trough can be shortened at will. A strong spiral spring, S, suitably supported and weighted with a heavy ball, B, to give sufficient inertia, is used to make a plunger execute a vertical simple harmonic motion, of period T, in the compartment C D. So long as D E is not adjusted so that the period of the standing wave in that part of the trough is T or a multiple of T, the forced oscillations in D E attain no particular magnitude; but when E is so adjusted that the periods are very nearly equal, or the one double the other, a large standing wave is set up, which can be maintained for any length of time by giving properly timed impulses to B. In this

* See *Proc. R.S.E.*, vol. xxv. p. 646; also § 49 below.

† *Trans. Roy. Soc. Edin.*, xiv. p. 524, 1839.

way Mr WEDDERBURN was able to verify with very fair accuracy MERIAN'S law for the uninodal and binodal periods in a trough of uniform depth: to show that for a trough of the same length and same maximum depth, T_1 is greater when the bottom is caused to slope by inserting a wedge * inside it, or, what comes to the same thing, that when T_1 is unaltered, C E must be made shorter; and that then $T_2 > \frac{1}{2}T_1$. The displacement of the uninode towards the shallow end and the greater amplitude of the wave there were also readily demonstrated. The foregoing experiments were made with water 8·18 cm. deep, in a trough of length C E = 39·8 cm.; and the periods were $T_1 = \cdot 96''$,

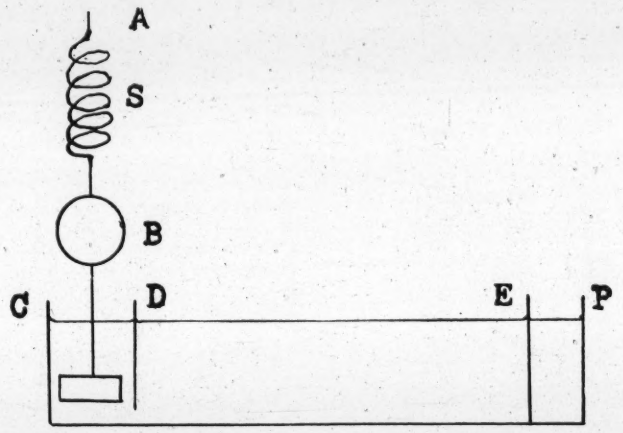


FIG. 1.

$T_2 = \cdot 52''$. With a depth of 4·1 cm. and C E = 30·7 cm. it was still possible to maintain a uninodal seiche of period ·97″, which agrees very well with the formula $T = 2l/\sqrt{(gd)}$ applicable to a long wave.

FOREL based his theory of seiches in part upon an interesting series of experiments of this kind made in 1870.†

* To incline the trough, keeping the volume of water the same, is not the same thing.

† ENDRÖS commenced his investigation of the complicated seiche-phenomena in the Chiemsee with a series of experiments on the oscillation of mercury in a vessel imitating the configuration of the lake, the results of which, on the whole, were in remarkable agreement with his subsequent observations, *l.c.*, p. 8.

PART II.

Mathematical Theory.

GENERAL THEORY OF A SMALL LONGITUDINAL SEICHE IN A LAKE OF VARYING DEPTH AND CROSS SECTION.

§ 20. Let O X be a longitudinal axis in the undisturbed surface of the lake. Observation seems to show that this axis should be as nearly as possible in the average direction of the channel of greatest depth. Take O Z vertical, and O Y horizontal and perpendicular to O X.

Consider any cross section at a distance O P $= x$ from the origin. Let the area of this section be A(x), and its breadth at the surface $b(x)$. Take a section parallel to A(x) at a distance dx from A(x). The volume of this slice (S) will be, to the first order of small quantities, A(x)dx.

Suppose that, after a time, t, the slice, S, has moved into a new position, so that the distance of its posterior face from O is now $x+\xi$. Then the breadth of S in its new position will be $dx(1+\partial\xi/\partial x)$; and the part of its volume below the normal level of the lake will be $A(x+\xi)dx(1+\partial\xi/\partial x)$.

If we suppose the rise in level of the slice to be the same throughout, say ζ, which involves the assumption that there is no flow parallel to O Y, and that all the water particles in the same transverse vertical plane have the same velocity parallel to the plane Z O X, then we may take the increment of the slice owing to the rise of the water above the original level to be $b(x)\zeta dx(1+\partial\xi/\partial x)$. In so doing we neglect the effect of the shelving of the shore; so that our calculation would certainly not apply in cases where the seiche causes a large horizontal displacement of the high-water mark.

With these assumptions the equation of continuity is

$$A(x)dx = \{A(x+\xi)+b(x)\zeta\}dx(1+\partial\xi/\partial x):$$

that is,

$$\zeta b(x) = A(x)/(1+\partial\xi/\partial x) - A(x+\xi) \quad . \quad . \quad . \quad . \quad (1).$$

Since the amplitude of the seiche is small, we neglect vertical and consider only horizontal acceleration. The difference between the pressures on the two sides of the slice in its disturbed position will therefore be simply that due to the difference of level at its two ends, viz., $g\rho d\zeta$ per unit of area, ρ being the density of the liquid.* The equation of motion for S, regarded as a whole, is therefore

$$dx(1+\partial\xi/\partial x)\rho\frac{\partial^2\xi}{\partial t^2} = -g\rho\frac{\partial\zeta}{\partial x}\,dx;$$

that is

$$\frac{\partial^2\xi}{\partial t^2} = -g\frac{\partial\zeta}{\partial x}\Big/\left(1+\frac{\partial\xi}{\partial x}\right) \quad . \quad . \quad . \quad . \quad . \quad (2).$$

* In order that these assumptions may be justified, the square of the ratio of the depth to the wave length must be negligible at every part of the lake. See Lamb's *Hydrodynamics* (1895), § 169.

The amplitudes being small, we shall neglect quantities of the order of $\xi^2, (\partial\xi/\partial x)^2$. The above equation then becomes

$$b(x)\zeta = \mathrm{A}(x)\left(1 - \frac{\partial\xi}{\partial x}\right) - \mathrm{A}(x) - \xi\frac{\partial \mathrm{A}(x)}{\partial x},$$
$$= -\mathrm{A}(x)\frac{\partial\xi}{\partial x} - \xi\frac{\partial \mathrm{A}(x)}{\partial x};$$

or

$$\zeta = -\frac{1}{b(x)}\frac{\partial}{\partial x}\{\mathrm{A}(x)\xi\} \quad . \quad . \quad . \quad . \quad . \quad (3);$$

and

$$\mathrm{A}(x)\frac{\partial^2\xi}{\partial t^2} = -g\mathrm{A}(x)\frac{\partial\zeta}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad (4).$$

If we substitute the value of ζ in (4), we get

$$\mathrm{A}(x)\frac{\partial^2\xi}{\partial t^2} = g\mathrm{A}(x)\frac{\partial}{\partial x}\left[\frac{1}{b(x)}\frac{\partial}{\partial x}\{\mathrm{A}(x)\xi\}\right].$$

Now this last equation may be written

$$\frac{\partial^2\{\mathrm{A}(x)\xi\}}{\partial t^2} = g\mathrm{A}(x)b(x)\cdot\frac{1}{b(x)}\frac{\partial}{\partial x}\left[\frac{1}{b(x)}\frac{\partial}{\partial x}\{\mathrm{A}(x)\xi\}\right] \quad . \quad . \quad . \quad . \quad (5).$$

If we determine new variables u and v by the equations

$$u = \mathrm{A}(x)\xi, \quad v = \int dx b(x) \quad . \quad . \quad . \quad . \quad . \quad (6),$$

then (5) may be written

$$\frac{\partial^2 u}{\partial t^2} = g\mathrm{A}(x)b(x)\frac{\partial^2 u}{\partial v^2}, \quad = g\sigma(v)\frac{\partial^2 u}{\partial v^2} \quad . \quad . \quad . \quad . \quad (7),$$

where x is to be determined as a function of v by the second equation of (6); and $\sigma(v) = \mathrm{A}(x)b(x)$. Also (3) becomes

$$\zeta = -\frac{\partial u}{\partial v} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

The curve whose ordinate and abscissa are σ and v we shall call the *normal curve* of the lake.

Since a seiche is a standing oscillation, ξ, and therefore u, is a periodic function of the time. We may suppose this periodic function analysed into simple harmonic terms, and write

$$u = \Sigma \mathrm{P}\sin n(t - \tau) \quad . \quad . \quad . \quad . \quad . \quad (9),$$

where P is a function of v alone and τ is constant. The values of n admissible depend on the circumstances of each case: but, in order that (9) may satisfy (7) we must have

$$-n^2\mathrm{P} = g\sigma(v)\frac{\partial^2\mathrm{P}}{\partial v^2}.$$

The mathematical theory of a seiche of small amplitude depends therefore essentially on the differential equation

$$\frac{d^2\mathrm{P}}{dv^2} + \frac{n^2}{g\sigma(v)}\mathrm{P} = 0 \quad . \quad . \quad . \quad . \quad . \quad (10),$$

where x is determined in terms of v by the equation $v = \int dx b(x)$; and $\sigma(v) = \mathrm{A}(x)b(x)$.

As (10) is simply the canonical form of the linear differential equation of the second order, a variety of cases can be devised in which the seiche problem can be solved in finite terms; and in any case where $A(x)$ and $b(x)$ are given slowly varying functions, approximate solutions can be found, with more or less labour.

§ 21. In all the seiche problems considered in this paper we have at the ends of the lake either $A(x) = 0$ or else $\xi = 0$; that is $u = 0$.

Moreover, the equation (7) may be regarded as the equation of motion of a vertical string vibrating in one plane, the ratio of whose tension to the longitudinal density is $g\sigma(v)$, v being the distance of any point P of the string from one end when the whole is at rest. The variable u denotes the lateral displacement of the point P at time t; and, in view of the conditions $u = 0$ at both ends of the lake, we may suppose both ends of the string fixed. We can then deduce the seiche displacements from the motion of the string by the equations

$$\xi = u/A(x), \quad \zeta = -\partial u/\partial v.$$

It will be observed that the nodes of the string correspond to ventral points of the seiche, and *vice versa*; and it appears that we could, by experimenting with a string loaded so that its density is inversely proportional to the product of the area and surface breadth of the cross section of a lake, roughly determine in the laboratory the periods and nodes of the pure seiches that might occur in the lake.

It follows from STURM's Oscillation theorem * that in any given lake seiches are possible which have

$$2, 3, 4, \ldots, \nu+1, \ldots\ldots$$

ventral points and

$$1, 2, 3, \ldots, \nu, \ldots\ldots$$

nodes respectively.

In other words, pure seiches of all degrees of nodality are possible; and the most general seiche disturbance is a sum of such pure seiches with arbitrary amplitudes and phases. We regard the ends of the lake as ventral points because u always vanishes there, although in most cases the horizontal displacement does not vanish, as it should do at a ventral point properly so called.

The identification of the seiche problem with the theory of a vibrating string is not only very instructive from the physical point of view, but is very helpful mathematically. For example, when we have worked out the periods and nodes of a seiche for any simple configuration approximately fitting a given lake, we can correct for the divergence of the actual lake from the assumed mathematical form by means of the beautiful method described by LORD RAYLEIGH in his *Theory of Sound*, vol. i. § 90.

* See RAYLEIGH'S *Sound* (1877), vol. i. § 142.

Case of a Lake of constant Breadth, rectangular Section, but varying Depth.

§ 22. Let $b(x)=b$, $A(x)=bh(x)$, where b is the constant breadth, and $h(x)$ the varying depth.

Then the equations (7) and (8) may be written

$$\frac{\partial^2 u}{\partial t^2}=gh(x)\frac{\partial^2 u}{\partial x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7'),$$

$$\zeta=-\frac{\partial u}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8');$$

where $u=\xi h(x)$.

In the case of a stationary oscillation, we shall have

$$\xi h(x)=u=\mathrm{P}\sin n(t-\tau) \quad . \quad . \quad . \quad . \quad . \quad (9')$$

where

$$\frac{\partial^2 \mathrm{P}}{\partial x^2}+\frac{n^2}{gh(x)}\mathrm{P}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (10').$$

It will be useful in this case to give the displacement ζ_1 for a point P in the water not lying on the surface. If the co-ordinates of P be (x, y, z) before and $(x+\xi, y, z+\zeta_1)$ after disturbance, ξ and ζ denoting as before the displacements of the point $(x, y, 0)$ vertically over (x, y, z), then, applying the equation of continuity to a small parallelopiped reaching from (x, y, z) to the surface and having sides parallel to the axes, we get, supposing z measured downwards, ζ and ζ_1 upwards, $zdxdy=(z-\zeta_1+\zeta)dx(+\partial\xi/\partial x)dy1$. Therefore, to the order of approximation contemplated, we have

$$-\zeta_1=-\zeta-z\frac{\partial \xi}{\partial x}$$

that is

$$-\zeta_1=\frac{\partial}{\partial x}(h(x)\xi)-z\frac{\partial \xi}{\partial x}$$

$$=h'(x)\xi+(h(x)-z)\frac{\partial \xi}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

It follows that

$$-\frac{\zeta_1}{\xi}=h'(x)+\frac{h(x)(h(x)-z)}{\mathrm{P}}\frac{d}{dx}\left(\frac{\mathrm{P}}{h(x)}\right)$$

$$=\frac{zh'(x)}{h(x)}+\frac{h(x)-z}{\mathrm{P}}\frac{d\mathrm{P}}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

Hence as we pass vertically upwards we find all the particles of water oscillating in rectilinear orbits the inclinations of which vary gradually from tangency with the bottom to tangency with the free surface of the lake.

§ 23. If we compare equations (7), (8), (9), (10) with (7'), (8'), (9'), (10'), it is at once obvious that mathematically there is no difference between the general case and the special one where the breadth of the lake is constant, its cross section rectangular, and the depth alone varies. We can pass from the one case to the other by obvious changes in the meanings of the variables and constants involved. We shall therefore in future confine ourselves to the more special case, of which it is easier to form a clear mental picture.

GENERAL SOLUTION FOR A PARABOLIC LONGITUDINAL SECTION.

Introduction of the Seiche Functions.

§ 24. Consider the equation

$$\frac{d^2P}{dv^2}+\frac{c}{1-\lambda v^2}P=0 \quad . \quad . \quad . \quad . \quad . \quad (13).$$

Assuming $y=a_0+a_1v+a_2v^2+\ldots\ldots$, we find in the usual way

$$\begin{aligned} &ca_0+1.2a_2=0, && ca_1+2.3a_3=0,\\ &(c-1.2\lambda)a_2=3.4a_4=0, && (c-2.3\lambda)a_3+4.5a_5\cdot=0,\\ &\quad\ldots\ldots\ldots && \quad\ldots\ldots\ldots\\ &\{c-(n-3)(n-2)\lambda\}a_{n-2}+(n-1)na_n=0. \end{aligned}$$

Hence we have

$$\begin{aligned} P=A\left\{1-\frac{c}{1.2}v^2+\frac{c(c-1.2\lambda)}{1.2\times3.4}v^4+\frac{c(c-1.2\lambda)(c-3.4\lambda)}{1.2\times3.4\times5.6}v^6+\ldots\ldots\right\}\\ +B\sqrt{c}\left\{v-\frac{c}{2.3}v^3+\frac{c(c-2.3\lambda)}{2.3\times4.5}v^5+\frac{c(c-2.3\lambda)(c-4.5\lambda)}{2.3\times4.5\times6.7}v^7+\ldots\ldots\right\} \quad . \quad . \quad (14); \end{aligned}$$

where A and B are arbitrary constants. The series in the brackets are obviously convergent if $|v|<1/\sqrt{|\lambda|}$; and divergent if $|v|>1/\sqrt{|\lambda|}$. They are also convergent if $v=\pm1/\sqrt{\lambda}$; for the general term, of say the first series, may be written

$$\pm u_n=\left(1-\frac{c/\lambda}{1.2}\right)\left(1-\frac{c/\lambda}{3.4}\right)\ldots\ldots\left(1-\frac{c/\lambda}{(2n-1)2n}\right)\frac{c}{(2n+1)(2n+2)};$$

and, since the infinite product

$$\left(1-\frac{c/\lambda}{1.2}\right)\left(1-\frac{c/\lambda}{3.4}\right)\ldots\ldots$$

is convergent, u_n is ultimately of the same order as $1/n^2$.

If we introduce the notation

$$\begin{aligned} C(c,\lambda,v)&=1-\frac{c}{1.2}v^2+\frac{c(c-1.2\lambda)}{1.2\times3.4}v^4-\ldots\ldots\\ S(c,\lambda,v)&=\sqrt{c}v\left\{1-\frac{cv^2}{2.3}+\frac{c(c-2.3\lambda)}{2.3\times4.5}v^4-\ldots\ldots\right\}, \end{aligned}$$

we see that the functions C and S have certain properties in common with the circular functions. For example, we have

$$\begin{aligned} &C(c,\lambda,-v)=\ C(c,\lambda,v);\\ &S(c,\lambda,-v)=-S(c,\lambda,v);\\ &C(c,\lambda,0)=1,\ \ S(c,\lambda,0)=0. \end{aligned}$$

Also

$$C(c,0,v)=\cos(\sqrt{c}v),\quad S(c,0,v)=\sin(\sqrt{c}v);$$

so that the cosine and sine are particular cases of the functions just defined.

From EULER's identity, viz.

$$(1-u_1)(1-u_2)\ldots\ldots-(1-u_n)$$
$$=1-u_1+u_2(u_1-1)-u_3(u_1-1)(u_2-1)+\ldots\ldots$$
$$\ldots\ldots(-)^n u_n(u_1-1)(u_2-1)\ldots\ldots(u_{n-1}-1),$$

we see that

$$C(c,\lambda,1/\sqrt{\lambda})=\left(1-\frac{c/\lambda}{1.2}\right)\left(1-\frac{c/\lambda}{3.4}\right)\ldots\ldots ad\ \infty;$$

$$S(c,\lambda,1/\sqrt{\lambda})=\frac{\sqrt{c}}{\sqrt{\lambda}}\left(1-\frac{c/\lambda}{2.3}\right)\left(1-\frac{c/\lambda}{4.5}\right)\ldots\ldots ad\ \infty.$$

In our applications we shall for the most part put $\lambda=1$, or $\lambda=-1$. Then we may omit the argument λ and write

$$C(c,v)=1-\frac{c}{1.2}v^2+\frac{c(c-1.2)}{1.2\times 3.4}v^4-\ldots\ldots \quad (15);$$

$$S(c,v)=v-\frac{cv^3}{2.3}+\frac{c(c-2.3)}{2.3\times 4.5}v^5-\ldots\ldots \quad (16).$$

We shall call $C(c,v)$ and $S(c,v)$ the Seiche-cosine and the Seiche-sine respectively. Also

$$\mathfrak{C}(c,v)=1-\frac{cv^2}{1.2}+\frac{c(c+1.2)}{1.2\times 3.4}v^4-\ldots\ldots \quad (17);$$

$$\mathfrak{S}(c,v)=v-\frac{cv^3}{2.3}+\frac{c(c+2.3)}{2.3\times 4.5}v^5-\ldots\ldots \quad (18).$$

These may be called the hyperbolic seiche-cosine and the hyperbolic seiche-sine.

When c has one of the integral values $1.2, 3.4, \ldots\ldots$, $C(c,v)$ reduces to a rational integral function of v; and when c has one of the integral values $2.3, 4.5, \ldots\ldots$, the like happens to $S(c,v)$.

The same holds for $\mathfrak{C}(c,v)$ and $\mathfrak{S}(c,v)$ with regard to the negative integral values $-1.2, -3.4, \ldots\ldots$, and $-2.3, -4.5, \ldots\ldots$; but this is of little interest in the seiche problem, for which the values of c must be positive.

By a well-known property of the solutions of a linear equation of the second order, we must have $C(c,v)\,S'(c,v)-C'(c,v)\,S(c,v)=$ constant, where the dash denotes differentiation with respect to v. In the present case this constant is easily seen to be unity; and we have

$$C(c,v)\,S'(c,v)-C'(c,v)\,S(c,v)=1 \quad (19);$$

$$\mathfrak{C}(c,v)\mathfrak{S}'(c,v)-\mathfrak{C}'(c,v)\mathfrak{S}(c,v)=1 \quad (20).$$

These are the analogues of the relation $\cos^2\theta+\sin^2\theta=1$ for the circular functions; and they are very useful in seiche calculations. We might also define a seiche-tangent, seiche-cotangent, seiche-secant, and seiche-cosecant. We shall only have occasion to use the seiche-cotangent, viz., $C(c,v)/S(c,v)$, which we shall denote by $K(c,v)$. These functions have many curious properties more or less analogous to those of the circular functions: *e.g.* $K'(c,v)=-1/S^2(c,v)$, but it is needless to encumber the present paper with details of this description.

§ 25. We have now, of course,

$$C(c,1)=\left(1-\frac{c}{1.2}\right)\left(1-\frac{c}{3.4}\right)\ldots\ldots ad\ \infty \quad . \quad . \quad . \quad (21);$$

$$S(c,1)=\left(1-\frac{c}{2.3}\right)\left(1-\frac{c}{4.5}\right)\ldots\ldots ad\ \infty \quad . \quad . \quad . \quad (22);$$

$$\mathfrak{C}(c,i)=\left(1+\frac{c}{1.2}\right)\left(1+\frac{c}{3.4}\right)\ldots\ldots ad\ \infty \quad . \quad . \quad . \quad (23);$$

$$\mathfrak{S}(c,i)=i\left(1+\frac{c}{2.3}\right)\left(1+\frac{c}{4.5}\right)\ldots\ldots ad\ \infty \quad . \quad . \quad . \quad (24).$$

We shall have frequent occasion to use $C(c,1)$ and $S(c,1)$; and it is convenient for purposes of calculation to express them in terms of the gamma function.

We have

$$C(c,1)=\left(1-\frac{c}{2}\right)\prod_{n=1}^{n=\infty}\frac{4n^2+6n+2-c}{(2n+1)(2n+2)},$$

$$=\left(1-\frac{c}{2}\right)\Pi\frac{\{n+\frac{1}{4}(3+\alpha)\}\{n+\frac{1}{4}(3-\alpha)\}}{(n+\frac{1}{2})(n+1)},$$

where $\alpha=\sqrt{(4c+1)}$.

Since $\frac{1}{4}(3+\alpha)+\frac{1}{4}(3-\alpha)=\frac{1}{2}+1$, it follows by a well-known theorem * that

$$C(c,1)=\left(1-\frac{c}{2}\right)\cdot\tfrac{1}{2}\cdot 1\Gamma(\tfrac{1}{2})\Gamma(1)\Big/\left(\frac{3+\alpha}{4}\right)\left(\frac{3-\alpha}{4}\right)\Gamma\left(\frac{3+\alpha}{4}\right)\Gamma\left(\frac{3-\alpha}{4}\right).$$

Since $\Gamma(\frac{1}{2})=\sqrt{\pi}$, $\Gamma(1)=1$, and $\alpha^2=4c+1$, we get finally

$$C(c,1)=\pi^{\frac{1}{2}}/\Gamma\left(\frac{3+\alpha}{4}\right)\Gamma\left(\frac{3-\alpha}{4}\right) \quad . \quad . \quad . \quad . \quad (25);$$

and in exactly the same way

$$S(c,1)=\pi^{\frac{1}{2}}/2\Gamma\left(\frac{5+\alpha}{4}\right)\Gamma\left(\frac{5-\alpha}{4}\right) \quad . \quad . \quad . \quad . \quad (26).$$

It follows that

$$K(c,1)=2\Gamma\left(\frac{5+\alpha}{4}\right)\Gamma\left(\frac{5-\alpha}{4}\right)\Big/\Gamma\left(\frac{3+\alpha}{4}\right)\Gamma\left(\frac{3-\alpha}{4}\right) \quad . \quad . \quad . \quad (27),$$

a formula which has been much used by Mr WEDDERBURN and myself in the numerical calculation of seiche periods. If we recollect that

$$\Gamma\left(\frac{1+\alpha}{4}\right)\Gamma\left(\frac{3-\alpha}{4}\right)=\pi/\sin\left(\frac{1+\alpha}{4}\pi\right),$$

$$\Gamma\left(\frac{1-\alpha}{4}\right)\Gamma\left(\frac{3+\alpha}{4}\right)=\pi/\sin\left(\frac{1-\alpha}{4}\pi\right),$$

we can put (27) into the form

$$K(c,1)=-\frac{c}{\frac{1}{4}\pi^2}\cos\left(\alpha\frac{\pi}{2}\right)\Gamma^2\left(\frac{1+\alpha}{4}\right)\Gamma^2\left(\frac{1-\alpha}{4}\right) \quad . \quad . \quad . \quad (28)$$

which is useful for determining the sign of $K(c,1)$.

* See *Whittaker's Modern Analysis* (1902), § 96.

§ 26. By STURM's Oscillation theorem, applied to the solutions of the differential equation (13), we see that for any real value of v not exceeding unity the equations

$$C(c,v)=0, \quad S(c,v)=0, \quad . \quad . \quad . \quad . \quad (29).$$
$$\mathfrak{C}(c,v)=0, \quad \mathfrak{S}(c,v)=0, \quad . \quad . \quad . \quad . \quad (30),$$

have each an infinite number of real roots, and that the roots of each equation of either pair separate the roots of the other equation of the same pair.

In particular, the roots of

$$C(c,1)=0, \quad \text{are} \quad c=1.2, 3.4, \ldots\ldots;$$

of

$$S(c,1)=0, \quad \text{are} \quad c=2.3, 4.5, \ldots\ldots;$$

unfortunately the roots of

$$\mathfrak{C}(c,1)=0, \qquad \mathfrak{S}(c,1)=0$$

are not commensurable; and, owing to the slow convergence of the series involved, they are very difficult to calculate directly. By a very laborious calculation, I find:— for the smallest root of $\mathfrak{C}(c,1)=0$, $c=2{\cdot}77\ldots$; and for the smallest root of $\mathfrak{S}(c,1)=0$, $c=12{\cdot}34\ldots$ As these figures agree with the approximations given by Dr HALM in the paper on the seiche functions above referred to, and with calculations which Dr BURGESS, Professor GIBSON, and Mr HORSBURGH have been kind enough to make for me, probably they are correct. It would, however, be hopeless to calculate the higher roots, or even these two to greater accuracy, by direct use of the series as it stands in (17) and (18).

SEICHES IN A CONCAVE SYMMETRIC COMPLETE PARABOLIC LAKE.

$$h(x)=h\times(1-x^2/a^2).$$

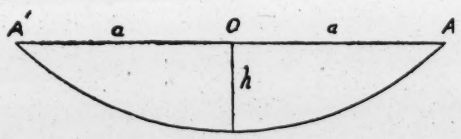


FIG. 2.

§ 27. The equation for determining P is, by § 22,

$$\frac{\partial^2 P}{\partial x^2}+\frac{n^2}{gh(1-x^2/a^2)}P=0;$$

or, if $w=x/a$,

$$(1-w^2)\frac{d^2P}{dw^2}+\frac{n^2a^2}{gh}P=0,$$

say,

$$(1-w^2)\frac{d^2P}{dw^2}+cP=0, \quad . \quad . \quad . \quad . \quad . \quad (31),$$

where

$$c=n^2a^2/gh, \quad . \quad . \quad . \quad . \quad . \quad (32).$$

We have therefore

$$\xi h(1-w^2) = u = \{A\,C(c,w) + B\,S(c,w)\} \sin n(t-\tau),$$

where A and B are arbitrary constants.

Also

$$\zeta = -\frac{\partial u}{\partial x} = -\frac{1}{a}\frac{du}{dw},$$
$$= -\frac{1}{a}\left\{A\,C'(c,w) + B\,S'(c,w)\right\} \sin n(t-\tau);$$

where the dashes denote differentiation with respect to w.

Since ξ is finite, we have $u=0$, when $w=\pm 1$. Therefore, since $C(c,-1)=C(c,1)$, and $S(c,-1)=-S(c,1)$, the following boundary conditions must be satisfied:

$$A\,C(c,1) + B\,S(c,1) = 0,$$
$$A\,C(c,1) - B\,S(c,1) = 0.$$

These are equivalent to

$$A\,C(c,1) = 0, \quad B\,S(c,1) = 0.$$

Now we see from the relation

$$C(c,w)S'(c,w) - C'(c,w)S(c,w) = 1,$$

that $C(c,1)$ and $S(c,1)$ cannot vanish simultaneously. Therefore either

$$B=0, \quad C(c,1)=0;$$

or

$$A=0, \quad S(c,1)=0.$$

As we have seen, the roots of $C(c,1)=0$ are $c_1=1.2$, $c_3=3.4$, ... $c_{2s-1}=(2s-1)2s$, ... ; and the roots of $S(c,1)=0$, $c_2=2.3$, $c_4=4.5$, ... , $c_{2s}=2s(2s+1)$, ... Hence we have the two sets of solutions

$$\left.\begin{aligned} \xi &= \frac{A}{h}\frac{C(c_{2s-1},w)\sin n_{2s-1}(t-\tau)}{1-w^2}; \\ \zeta &= -\frac{A}{a}C'(c_{2s-1},w)\sin n_{2s-1}(t-\tau) \end{aligned}\right\} \quad . \quad . \quad . \quad (33).$$

Here $C(c_{2s-1},w)$ is a polynomial of degree $2s$; and $C'(c_{2s-1},w)$ a polynomial of degree $2s-1$.

Also

$$\left.\begin{aligned} \xi &= \frac{B}{h}\frac{S(c_{2s},w)\sin n_{2s}(t-\tau)}{1-w^2}; \\ \zeta &= -\frac{B}{a}S'(c_{2s},w)\sin n_{2s}(t-\tau) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (34),*$$

where $S(c_{2s},w)$ and $S'(c_{2s},w)$ are polynomials of degrees $2s+1$ and $2s$ respectively.

* Since the abstract of this paper was published, I have discovered that the solution for the particular case of a symmetric parabolic basin was given by LAMB in the new edition of his *Hydrodynamics* (1895), § 182. He arrives at his result by means of LEGENDRE's function, which is closely allied to the seiche functions.

In either case, if T_ν be the period of the ν-nodal seiche, we have

$$\begin{aligned} T_\nu &= 2\pi/n_\nu = 2\pi a/\sqrt{(c_\nu gh)}\,, \\ &= \pi l/\sqrt{\{\nu(\nu+1)gh\}}\,, \end{aligned} \qquad \ldots \quad (35),$$

if l denote the whole length of the lake.

§ 28. Uninodal Seiche.

$$c_1 = 1.2\,. \qquad T_1 = \pi l/\sqrt{(2gh)}$$
$$C(c_1, w) \equiv 1 - w^2, \quad C'(c_1, w) = -2w\,.$$
$$\xi = \frac{A}{h}\sin n_2(t-\tau), \quad \zeta = \frac{2Ax}{a^2}\sin n_1(t-\tau)\,.$$

One node

$$x/a = 0\,.$$

In this case the amplitude of the horizontal displacement is constant; and the free surface is a plane which oscillates about the line of the uninode.

If $\bar{\zeta}$ be the maximum rise of the water at the end of the lake above the undisturbed level, then $\bar{\zeta} = 2A/a = 4A/l$. Hence $A = l\bar{\zeta}/4$. Hence the maximum horizontal displacement of a water particle from its mean position is $\bar{\xi} = l\bar{\zeta}/4h$; and the maximum velocity of the horizontal stream is $nl\bar{\zeta}/4h = \pi l\bar{\zeta}/2hT_1$. For example, if Loch Ness were a symmetric parabolic lake, every inch of maximum vertical seiche at one end would give over 40 inches of maximum horizontal displacement; and a maximum horizontal stream velocity of over 8 inches per minute.

§ 29. Binodal Seiche.

$$c_2 = 2.3, \qquad T_2 = \pi l/\sqrt{(6gh)},$$
$$S(c_2, w) = w - w^3, \quad S'(c_2, w) = 1 - 3w^2,$$
$$\xi = \frac{Bx}{ah}\sin n_2(t-\tau), \quad \zeta = \frac{B(3x^2 - a^2)}{a^3}\sin n_2(t-\tau).$$

Two nodes at

$$x/a = \pm\tfrac{1}{3}\sqrt{3} = \pm\,.5774\ldots.$$

The amplitude of ξ increases uniformly from the centre to the ends of the lake; and the free surface is parabolic,

$$T_2/T_1 = \sqrt{2}/\sqrt{6} = \tfrac{1}{3}\sqrt{3} = .5774\ldots.$$

Hence the period of the binodal seiche is greater than half the period of the uninodal seiche.

Also the nodes are more than half way from the middle of the lake towards the ends; *i.e.* they are displaced towards the shallows.

If $\bar{\xi}$ and $\bar{\zeta}$ be the maximum horizontal and vertical displacements at the end of the lake, we find $\bar{\xi}/\bar{\zeta} = l/4h$, as before, also $\bar{\xi}/\bar{\zeta} = 2\pi\bar{\xi}/T_2\bar{\zeta}$. For Loch Ness we get

$\xi/\zeta = 40 \ldots$, $\bar{\xi}/\bar{\zeta} = 16{\cdot}4 \ldots$. As a binodal seiche of $3\frac{1}{2}$ in amplitude has been observed, maximum horizontal displacements of about 12 feet and stream velocities of about 5 feet per minute may occur near the end of the loch; these would be reduced in the ratio of ·57 : 1 at the two nodes of the seiche. As the centre of the loch is a ventral segment, the horizontal displacement vanishes there at all times, and the value of $\bar{\zeta}$ is half its value at the ends of the loch.

Trinodal Seiche.

§ 30.

$$c_3 = 3.4\,; \qquad T_3 = \pi l/\sqrt{(12gh)}\,.$$

$$C(c_3, w) = 1 - 6w^2 + 5w^4,$$

$$= (1 - w^2)(1 - 5w^2)\,.$$

$$C'(c_3, w) = -12w + 20w^3\,.$$

$$\xi = \frac{A}{ha^2}(a^2 - 5x^2)\sin n_3(t - \tau),$$

$$\zeta = \frac{A}{a^4}(12a^2x - 20x^3)\sin n_3(t - \tau)\,.$$

Three nodes :—$x = 0$, and $x/a = \pm\sqrt{3}/\sqrt{5} = \pm{\cdot}7746 \ldots$.
Four ventral points :—$x/a = \pm{\cdot}4472$, ± 1.

$$T_3/T_1 = \sqrt{2}/\sqrt{12} = \frac{1}{6}\sqrt{6} = {\cdot}4082 \ldots.$$

Quadrinodal Seiche.

§ 31.

$$c_4 = 4.5\,; \qquad T_5 = \pi l/\sqrt{(20gh)}\,.$$

$$S(c_4, w) = w - \frac{10}{3}w^3 + \frac{7}{3}w^5,$$

$$= \frac{1}{3}(3w - 10w^3 + 7w^5),$$

$$= \frac{1}{3}w(1 - w^2)(3 - 7w^2)\,;$$

$$S'(c_4, w) = \frac{1}{3}(3 - 30w^2 + 35w^4)\,.$$

$$\xi = \frac{B}{ha^3}x(3a^2 - 7x^2)\sin n_4(t - \tau)\,;$$

$$\zeta = \frac{B}{a^5}(-3a^4 + 30a^2x^2 - 35x^4)\sin n_4(t - \tau)\,.$$

Four nodes :—$x/a = \pm{\cdot}3400$, $\pm{\cdot}8621$.
Five ventral points :—$x/a = 0$, $\pm{\cdot}6546$, ± 1.

$$T_4/T_1 = \frac{1}{10}\sqrt{10} = {\cdot}3162\,.$$

Quinquinodal Seiche.

§ 32.

$$c_5 = 5.6\,, \qquad T_5 = \pi l/\sqrt{(30gh)}\,.$$

$$C(c_5, w) = 1 - 15w^2 + 35w^4 - 21w^6 = (1 - w^2)(1 - 14w^2 + 21w^4)\,;$$

$$C'(c_5, w) = -30w + 140w^3 - 126w^5\,.$$

$$\xi = \frac{A}{ha^4}(a^4 - 14a^2x^2 + 21x^4)\sin n_5(t - \tau)\,;$$

$$\zeta = \frac{A}{a^6}(30a^4x - 140a^2x^3 + 126x^5)\sin n_5(t - \tau)\,.$$

Five nodes :—$x = 0$, $x/a = \pm\cdot5384$, $\pm\cdot9062$.
Six ventral points :—$x/a = \pm\cdot2853$, $\pm\cdot7650$, ± 1 .

$$T_5/T_1 = \frac{1}{15}\sqrt{15} = \cdot2582\,.$$

§ 33. The following conspectus of the numerical distances from the centre of the nodes and ventral points will give a clear idea of the shortening of the wave length towards the ends of the lake :—

I. Binodal.

	x/a	$\frac{1}{4}\lambda$
V	·0000	·5774
N	·5774	·4226
V	1·0000	

II. Trinodal.

	x/a	$\frac{1}{4}\lambda$
N	·0000	·4472
V	·4472	·3274
N	·7746	·2254
V	1·0000	

III. Quadrinodal.

	x/a	$\frac{1}{4}\lambda$
V	·0000	·3400
N	·3400	·3146
V	·6546	·2075
N	·8621	·1379
V	1·0000	

IV. Quinquinodal.

	x/a	$\frac{1}{4}\lambda$
N	·0000	·2853
V	·2853	·2531
N	·5384	·2266
V	·7650	·1412
N	·9062	·0938
V	1·0000	

It will thus be seen that the wave length near the centre is greater, and near the end is less than it would be in a lake of the same length but of uniform depth. All the nodes and ventral points which are not central are displaced towards the shallows.

As we see from § 8 that the amplitudes of the various pure seiches at the end of the lake is of special interest, the following table may be given, in which R denotes the

ratio for any seiche of the amplitude at the end of the lake to the amplitude at the ventral point at or nearest to the centre :—

ν	2	3	4	5
R	2·00	2·24	2·66	2·88

SEICHES IN A CONCAVE SEMIPARABOLIC LAKE.

§ 34. Since all the pure seiches in a symmetric parabolic lake have ventral points at the ends, and the seiches of even nodality have also a ventral point at the centre, where there is no horizontal displacement, we could build a wall across the middle of the lake without disturbing these seiches. It follows that the pure seiches of a semi-

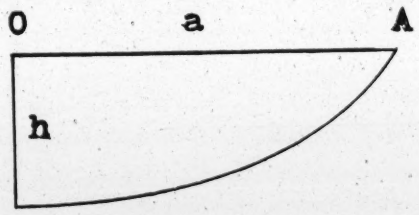


FIG. 3.

parabolic lake have the same periods as the seiches of even nodality in a complete parabolic lake of double the length. The nodes and ventral segments will also be the same as in one of the halves of the complete parabolic lake.

If, therefore, T_ν' be the period of the ν-nodal seiche in a semiparabolic lake of length l and maximum depth h, we shall have

$$T_\nu' = 2\pi l / \sqrt{\{2\nu(2\nu+1)gh\}} \quad . \quad . \quad . \quad . \quad (36).$$

If T_ν have the meaning of § 27, we find

$$T_\nu'/T_\nu = \sqrt{\{(\nu+1)/(\nu+\tfrac{1}{2})\}}.$$

Hence every period of a semiparabolic lake is longer than the corresponding period of a complete parabolic lake of the same length and the same maximum depth ; but the ratio of the periods comes nearer unity the higher the nodality.

The nodes and ventral points for the uninodal and binodal, etc. seiches will be given by Tables I. and III. of § 33, prŏvided we remember that x is now measured from the deeper end of the lake, and no longer from the middle, and that a is now the whole length of the lake, and not half the length as before.

§ 35. The results for parabolic and semiparabolic lakes are of great use in forming rough estimates of the constants and periods, either for experimental purposes, or in order to get first approximations to the roots of the transcendental equations which in

general determine these constants. We assume, as in general probable, that any concave lake whose form is not unusual will be intermediate in character between a complete parabolic and a semiparabolic lake. It follows that the periods, nodes, and ventral points will be intermediate; and it is found in practice that in many cases a good first approximation is obtained by taking the arithmetic mean between the two extreme cases. As an example, we should expect that the distances of the uninode and deep binode from the deeper end would lie between ·5 and ·58, and between ·78 and ·87 respectively.

In this connection we have found the following table of the ratios of the periods useful :—

	T_2/T_1	T_3/T_1	T_4/T_1	T_5/T_1	T_6/T_1	T_7/T_1	T_8/T_1	T_9/T_1
Parabolic Lake . .	·577	·408	·316	·258	·218	·189	·167	·149
Semiparabolic Lake .	·548	·378	·289	·234	·196	·169	·1485	·134

SEICHES IN A TRUNCATED PARABOLIC LAKE.

§ 36. By means of the seiche functions we can readily find the solution for a parabolic lake which is bounded by vertical cross walls at distances $x=p$, $x=q$ from

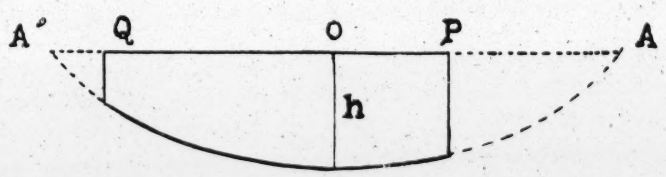


FIG. 4.

its deepest point. The formulæ are, in our previous notation,

$$\xi h(1-w^2) = \frac{A}{S(c,p/a)}\{S(c,p/a)C(c,w) - C(c,p/a)S(c,w)\}\sin n(t-\tau)\,;$$

$$\zeta \quad = -\frac{A}{aS(c,p/a)}\{S(c,p/a)C'(c,w) - C(c,p/a)S'(c,w)\}\sin n(t-\tau)\,.$$

And the values of c are given by the period equation

$$C(c,p/a)S(c,q/a) - S(c,p/a)C(c,q/a) = 0\,.$$

In the case of a symmetric lake $q=-p$; and the period equation reduces to

$$C(c,p/a)S(c,p/a) = 0\,.$$

If, further, $p=a$, we get

$$C(c,1)S(c,1) = 0\,;$$

and return to the case of a complete parabolic lake already discussed.

SEICHES IN A CONVEX SYMMETRIC PARABOLIC LAKE.

$$h(x) = h \times (1 + x^2/a^2).$$

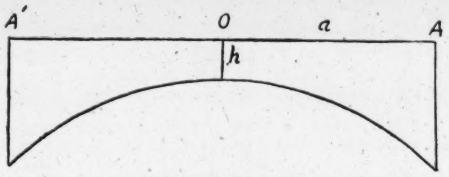


FIG. 5.

§ 37. The equation for P is

$$\frac{\partial^2 \mathrm{P}}{\partial x^2} + \frac{n^2}{gh(1 + x^2/a^2)}\mathrm{P} = 0;$$

that is, if $w = x/a$,

$$(1 + w^2)\frac{d^2\mathrm{P}}{dw^2} + c\mathrm{P} = 0, \quad . \quad . \quad . \quad . \quad . \quad (37),$$

where

$$c = n^2a^2/gh, \quad . \quad . \quad . \quad . \quad . \quad . \quad (38).$$

We have therefore

$$\xi h(1 + w^2) = u = \{\mathrm{A}\,\mathfrak{C}(c, w) + \mathrm{B}\,\mathfrak{S}(c, w)\}\sin nt;$$

$$\zeta = -\frac{1}{a}\{\mathrm{A}\,\mathfrak{C}'(c, w) + \mathrm{B}\,\mathfrak{S}'(c, w)\}\sin nt.$$

Since ξ must vanish at the vertical ends corresponding to $w = \pm 1$, we have, although for a different reason, the same boundary conditions as before, viz.—

$$\mathrm{A}\,\mathfrak{C}(c, w) + \mathrm{B}\,\mathfrak{S}(c, w) = 0,$$
$$\mathrm{A}\,\mathfrak{C}(c, w) - \mathrm{B}\,\mathfrak{S}(c, w) = 0.$$

Hence, as before, we arrive at the two sets of solutions

$$\left.\begin{aligned} \xi &= \frac{\mathrm{A}}{h}\frac{\mathfrak{C}(c_{2s-1}, w)\sin n_{2s-1}t}{1 + w^2}, \\ \zeta &= -\frac{\mathrm{A}}{a}\mathfrak{C}'(c_{s-1}, w)\sin n_{2s-1}t; \end{aligned}\right\} \quad . \quad . \quad . \quad (39);$$

and

$$\left.\begin{aligned} \xi &= \frac{\mathrm{B}}{h}\frac{\mathfrak{S}(c_{2s}, w)\sin n_{2s}t}{1 + w^2}, \\ \zeta &= -\frac{\mathrm{B}}{a}\mathfrak{S}'(c_{2s}, w)\sin n_{2s}t; \end{aligned}\right\} \quad . \quad . \quad . \quad (40);$$

where $c_1, c_3, \ldots, c_{2s-1}, \ldots$ are the roots of the equation

$$\mathfrak{C}(c, 1) = 0;$$

and $c_2, c_4, \ldots, c_{2s}, \ldots$ are the roots of

$$\mathfrak{S}(c, 1) = 0.$$

$\mathfrak{C}(c_{2s-1}, w)$ and $\mathfrak{S}(c_{2s}, w)$ are, of course, no longer polynomials, but transcendental functions of w.

Uninodal and Binodal Seiches.

§ 38. Since, as has already been mentioned, $\mathfrak{c}_1 = 2{\cdot}77\ \ldots\ .$ and $\mathfrak{c}_2 = 12{\cdot}34\ \ldots\ .$, if $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be the periods of the uninodal and binodal seiches respectively, we have

$$\mathfrak{T}_2/\mathfrak{T}_1 = \sqrt{(2{\cdot}77/12{\cdot}34)} = {\cdot}474\,.$$

The distances of the binodes from the centre of the lake are given approximately by

$$x/a = {\cdot}472.$$

SEICHES IN A CONCAVE ASYMMETRIC BIPARABOLIC LAKE.

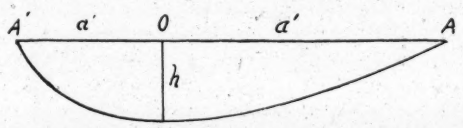


Fig. 6.

§ 39. Let the equations for the portions O A and A′O be $h(x) = h \times (1 - x^2/\alpha^2)$; and $h(x) = h \times (1 - x^2/\alpha'^2)$. Then, if $w = x/a$, $w' = x/\alpha'$; $c = n^2\alpha^2/gh$, $c' = n^2\alpha'^2/gh$, we have for the two portions

$$\xi h(1 - w^2) = \{\mathrm{A\ C}(c, w) + \mathrm{B\ S}(c, w)\} \sin nt\,,$$

$$\zeta = -\frac{1}{a}\{\mathrm{A\ C'}(c, w) + \mathrm{B\ S'}(c, w)\} \sin nt\,;$$

and

$$\xi' h(1 - w'^2) = \{\mathrm{A'\ C}(c', w') + \mathrm{B'\ S}(c', w')\} \sin nt\,,$$

$$\zeta' = -\frac{1}{a'}\{\mathrm{A'\ C'}(c', w') + \mathrm{B'\ S'}(c', w')\} \sin nt\,.$$

The boundary conditions at A and A′ give

$$\mathrm{A\ C}(c, 1) + \mathrm{B\ S}(c, 1) = 0\,;$$
$$\mathrm{A'\ C}(c', 1) - \mathrm{B'\ S}(c', 1) = 0\,.$$

The conditions $\xi = \xi'$, $\zeta = \zeta'$ at O give obviously

$$\mathrm{A} = \mathrm{A'}\,;$$
$$\mathrm{B}/a = \mathrm{B'}/a'.$$

From these we deduce

$$\left.\begin{aligned} &a'\mathrm{C}(c, 1)\mathrm{S}(c', 1) + a\mathrm{C}(c', 1)\mathrm{S}(c, 1) = 0\,, \\ \text{or}\quad &a'\mathrm{K}(c, 1) + a\mathrm{K}(c', 1) = 0\,, \end{aligned}\right\} \quad \ldots\ldots\ (41),$$

which is the equation that enables us to calculate c or c', if we remember that $c/c' = \alpha^2/\alpha'^2$.

If we put $a'^2c = a^2c' = n^2a^2a'^2/gh = z$, the equation (41) may be written

$$a\left(1-\frac{z}{1.2a^2}\right)\left(1-\frac{z}{3.4a^2}\right)\ldots\left(1-\frac{z}{2.3a'^2}\right)\left(1-\frac{z}{4.5a'^2}\right)\ldots$$
$$+a'\left(1-\frac{z}{1.2a'^2}\right)\left(1-\frac{z}{3.4a'^2}\right)\ldots\left(1-\frac{z}{2.3a^2}\right)\left(1-\frac{z}{4.5a^2}\right)\ldots = 0 \quad . \quad (42).$$

Then the period of the ν-nodal seiche is given by

$$T_\nu = 2\pi/n_\nu = 2\pi a a'/\sqrt{(z_\nu gh)}; \quad . \quad . \quad . \quad . \quad . \quad (43);$$

where z_ν is the corresponding root of (42).

For some purposes it is convenient to put $a/a' = \rho$. The equation (41) then becomes

$$(1/\rho)\mathrm{K}(\rho^2c', 1) + \mathrm{K}(c', 1) = 0; \quad . \quad . \quad . \quad . \quad . \quad (44);$$

and we have

$$\left.\begin{aligned} T_\nu &= 2\pi a'/\sqrt{(c_\nu' gh)} \\ &= 2\pi l/(1+\rho)\sqrt{(c_\nu' gh)}, \\ &= 2\pi l/(\sqrt{c_\nu} + \sqrt{c_\nu'})\sqrt{(gh)}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (45).$$

The equations for the seiche displacement in the two portions of the lake may now be written

$$\left.\begin{aligned} \xi h(1-w^2) &= \frac{A}{S(c,1)}\{S(c,1)C(c,w) - C(c,1)S(c,w)\}\sin nt; \\ \zeta &= -\frac{A}{aS(c,1)}\{S(c,1)C'(c,w) - C(c,1)S'(c,w)\}\sin nt; \end{aligned}\right\} \quad . \quad . \quad (46);$$

$$\left.\begin{aligned} \xi' h(1-w'^2) &= \frac{A}{S(c',1)}\{S(c',1)C(c',w') + C(c',1)S(c',w')\}\sin nt; \\ \zeta' &= -\frac{A}{a'S(c',1)}\{S(c',1)C'(c',w') + C(c',1)S'(c',w')\}\sin nt. \end{aligned}\right\} \quad . \quad . \quad (47).$$

It is obvious from (44) that when ρ is given the value of c' is determined. Now ρ is the ratio of the distances of the deepest section of the lake from the ends. Hence, if this ratio remain unaltered, we see that T_ν is proportional directly to the length of the lake, and inversely to the square root of its maximum depth.*

In particular, it follows that, if the basins of two lakes be geometrically similar, the seiche periods are directly proportional to the square roots of the linear dimensions; a result obvious by NEWTON's principle of dynamic similarity.

A graphic picture of the solution of the equation (44) may be obtained as follows:—

If we trace the curves whose equations are $y = \mathrm{K}(c', 1)$, $\rho y = -\mathrm{K}(\rho^2c', 1)$, c' being the common abscissa, then the values of c' corresponding to the intersection of these curves are the roots of (44). The latter of these curves is deducible from the former by diminishing all the abscissæ in the ratio $1 : \rho^2$ and all the ordinates in the ratio $1 : \rho$, and then taking the image of this deformed curve in the axis of c'. It is thus easy to see by drawing a schematic diagram that the effect of increasing ρ is to diminish c'.

The period depends on the value of $(1+\rho)\sqrt{c'}$ or $\sqrt{c} + \sqrt{c'}$; and the effect of the increase of ρ upon this is not so easy to trace by direct analysis. Since, however, the

* In the general case h is the maximum value of the product of the area of a cross section by its surface breadth; and a and a' the areas of the lake surface between the corresponding section and the ends.

shifting of the deepest point of the biparabolic lake without alteration of the length or maximum depth does not alter the whole volume of water, general dynamical considerations regarding energy would lead us to expect that increase of ρ would lengthen all the seiche periods; and, in point of fact, in the semiparabolic lake, which may be regarded as the limiting case of a biparabolic lake when $\rho = \infty$, all the periods are greater than in the complete parabolic lake, which corresponds to $\rho = 1$.

SEICHES IN AN UNSYMMETRICAL LAKE WITH ONE SHALLOW AND TWO MAXIMUM DEPTHS.

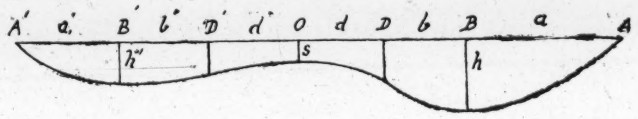


FIG. 7.

§ 40. A good approximation to the form in many cases that occur in nature can be obtained by piecing together six parabolæ, so as to form a continuous curve.

Let s be the minimum; and h, h' the two maximum depths. D and D′ the points of inflexion (the depths at which cannot be arbitrarily assigned). Let $AB = a_1$, $BD = b$, $DO = d$, $D'O' = d'$, $B'D' = b'$, $A'B' = a_1'$; then, for the continuity of the curve of longitudinal section at D and D′ we have the following conditions, the laws of depth being $h(x) = h \times (1 - x^2/a_1^2)$ for AB, $h(x) = h \times (1 - x^2/a_2^2)$ for BD, $h(x) = s \times (1 + x^2/a_3^2)$ for OD, $h(x) = s \times (1 + x^2/a_3'^2)$ for OD′, etc. :—

$$hb/a_2^2 - sd/a_3^2 = 0, \qquad hb^2/a_2^2 + sd^2/a_3^2 = h - s;$$
$$h'b'/a_2'^2 - sd'/a_3'^2 = 0, \qquad h'b'^2/a_2'^2 + sd'^2/a_3'^2 = h' - s.$$

These lead to

$$\left.\begin{aligned} a_2^2 &= hb(d+b)/(h-s), & a_3^2 &= sd(d+b)/(h-s); \\ a'^2 &= h'b'(d'+b')/(h'-s), & a_3'^2 &= sd'(d'+b')/(h-s). \end{aligned}\right\} \quad . \quad . \quad . \quad (48).$$

With the exception of a_2, a_3, a_2', a_3', and the depths at D and D′, the other quantities may be arbitrarily chosen.

If now

$$v_1 = x/a_1, \qquad v_2 = x/a_2, \qquad v_3 = x/a_3;$$
$$v_1' = x/a_1', \qquad v_2' = x/a_2', \qquad v_3' = x/a_3';$$
$$w_2 = b/a_2, \qquad w_3 = d/a_3;$$
$$w_2' = b'/a_2', \qquad w_3' = d'/a_3';$$
$$c_1 = n^2a_1^2/gh, \qquad c_2 = n^2a_2^2/gh, \qquad c_3 = n^2a_3^2/gh;$$

then we have for the various sections A, B, etc.

$$\xi h(1 - v_1^2) = \{A_1C(c_1, v_1) + B_1S(c_1, v_1)\}\sin n(t - \tau),$$
$$\zeta = -\frac{1}{a_1}\{A_1C'(c_1, v_1) + B_1S'(c_1, v_1)\}\sin n(t - \tau),$$

etc.

The origin for x being in each case the vertex of the corresponding parabola.

The boundary conditions are then as follows :—

$$A_1C(c_1, 1) + B_1S(c_1, 1) = 0;$$
$$A_1 = A_2,\ B_1/a_1 = B_2/a_2;$$
$$A_2C(c_2, w_2) - B_2S(c_2, w_2) = A_3\mathfrak{C}(c_3, w_3) + B_3\mathfrak{S}(c_3, w_3),$$
$$-A_2C'(c_2, w_2) + B_2S'(c_2, w_2) = \frac{a_2}{a_3}\left\{ A_3\mathfrak{C}'(c_3, w_3) + B_3\mathfrak{S}'(c_3, w_3) \right\};$$
$$A_3 = A_3',\ B_3/a_3 = B_3'/a_3';$$
$$A_2'C(c_2', w_2') + B_2'S(c_2', w_2') = A_3'\mathfrak{C}(c_3', w_3') - B_3'\mathfrak{S}(c_3', w_3'),$$
$$A_2'C'(c_2', w_2') + B_2'S'(c_2', w_2') = \frac{a_2'}{a_3'}\left\{ -A_3'\mathfrak{C}'(c_3', w_3') + B_3'\mathfrak{S}'(c_3', w_3') \right\};$$
$$A_1' = A_2',\ B_1'/a_1' = B_2'/a_2';$$
$$A_1'C(c_1', 1) - B_1'(c_1', 1) = 0.$$

From these, since $C(c_2, w_2)S'(c_2, w_2) - C'(c_2, w_2)S(c_2, w_2) = 1$, we derive

$$a_2A_2C(c_1, 1) + a_1B_2S(c_1, 1) = 0,$$
$$A_2 = \left\{ S'(c_2, w_2)\mathfrak{C}(c_3, w_3) + \frac{a_2}{a_3}S(c_2, w_2)\mathfrak{C}'(c_3, w_3) \right\} A_3$$
$$+ \left\{ S'(c_2, w_2)\mathfrak{S}(c_3, w_3) + \frac{a_2}{a_3}S(c_2, w_2)\mathfrak{S}'(c_3, w_3) \right\} B_3,$$
$$= \lambda A_3 + \mu B_3 \text{ (say)}.$$
$$B_2 = \left\{ C'(c_2, w_2)\mathfrak{C}(c_3, w_3) + \frac{a_2}{a_3}C(c_2, w_2)\mathfrak{C}(c_3, w_3) \right\} A_3$$
$$+ \left\{ C'(c_2, w_2)\mathfrak{S}(c_3, w_3) + \frac{a_2}{a_3}C(c_2, w_2)\mathfrak{S}'(c_3, w_3) \right\} B_3$$
$$= \nu A_3 + \rho B_3 \text{ (say)}.$$

Hence, using the first of these last equations, we have

$$a_2C(c_1, 1)\{\lambda A_3 + \mu B_3\} + a_1S(c, 1)\{\nu A_3 + \rho B_3\} = 0,$$

that is,

$$\{a_2\lambda C(c_1, 1) + a_1\nu S(c_1, 1)\}A_3$$
$$+ \{a_2\mu C(c_1, 1) + a_1\rho S(c_1, 1\}B_3 = 0.$$

Again,

$$a_2'A_2'C(c_1', 1) - a_1'B_2'S(c_1', 1) = 0$$
$$A_2' = \left\{ S'(c_2', w_3')\mathfrak{C}(c_3', w_3') + \frac{a_2'}{a_3'}S(c_2', w_2')\mathfrak{C}'(c_3', w_3') \right\} A_3'$$
$$- \left\{ S'(c_2', w_3')\mathfrak{S}(c_3', w_3') + \frac{a_2'}{a_3'}S(c_2', w_2')\mathfrak{S}'(c_3', w_2') \right\} B_3',$$
$$= \lambda' A_3' - \mu' B_3' \text{ (say)}.$$
$$B_2' = -\left\{ C'(c_2', w_2')\mathfrak{C}(c_3', w_3',) + \frac{a_2'}{a_3'}C(c_2', w_2')\mathfrak{C}'(c_3', w_3') \right\} A_3'$$
$$+ \left\{ C'(c_2', w_2')\mathfrak{S}(c_3', w_3') + \frac{a_2'}{a_3'}C(c_2', w_2')\mathfrak{S}'(c_3', w_3') \right\} B_3'$$
$$= -\nu' A_3' + \rho' B_3' \text{ (say)}.$$

Hence

$$a_2'C(c_1', 1)\{\lambda' A_3' - \mu' B_3'\} - a_1'S(c_1', 1)\{-\nu' A_3' + \rho' B_3'\} = 0$$

that is,

$$\{a_2'\lambda' C(c_1', 1) + a_1'\nu' S(c_1', 1)\}A_3' - \{a_2'\mu' C(c_1', 1) + a_1'\rho' S(c_1', 1)\}B_3' = 0.$$

Since $A_3' = A_3$, $B_3' = \alpha_3' B_3/\alpha_3$, the last equation may be written

$$a_3\{a_2'\lambda' C(c_1', 1) + a_1'\nu' S(c_1', 1)\}A_3 - a_3'\{a_2'\mu' C(c_1', 1) + a_1'\rho' S(c_1', 1)\}B_3 = 0.$$

Eliminating A_3 and B_3, we get finally

$$a_3'\{a_2\lambda C(c_1, 1) + a_1\nu S(c_1, 1)\}\{a_2'\mu' C(c_1', 1) + a_1'\rho' S(c_1', 1)\}$$
$$+ a_3\{a_2'\lambda' C(c_1', 1) + a_1'\nu' S(c_1', 1)\}\{a_2\mu C(c_1 1) + a_1\rho S(c_1, 1)\} = 0\,;$$

which is the period equation for the lake. When the lake is symmetrical, when $a_1 = a_1'$, $a_2 = a_2'$, etc., this equation simplifies, and breaks up into t following :—

$$a_2 C(c_1, 1)\{a_3 S'(c_2, w_2)\mathfrak{C}(c_3, w_3) + a_2 S(c_2, w_2)\mathfrak{C}'(c_3, w_3)\}$$
$$+ a_1 S(c_1, 1)\{a_3 C'(c_2, w_2)\mathfrak{C}(c_3, w_3) + a_2 C(c_2, w_2)\mathfrak{C}'(c_3, w_3)\} = 0\,,$$

and

$$a_2 C(c_1, 1)\{a_3 S'(c_2, w_2)\mathfrak{S}(c_3, w_3) + a_2 S(c_2, w_2)\mathfrak{S}'(c_3, w_3)\}$$
$$+ a_1 S(c_1, 1)\{a_3 C'(c_2, w_2)\mathfrak{S}(c_3, w_3) + a_2 C(c_2, w_2)\mathfrak{S}'(c_3, w_3)\} = 0\,.$$

ALTERNATIVE SOLUTION FOR PARABOLIC LAKES.

INTRODUCTION OF THE LAKE FUNCTION.

§ 41. For certain purposes a modification of the solution for parabolic l convenient. This is obtained by shifting the origin to the positive end, a convenience, halving the scale of the new variable; that is, we put $w = 1 - 2z$ equation

$$(1 - w^2)\frac{d^2P}{dw^2} + cP = 0$$

then becomes

$$(z - z^2)\frac{d^2P}{dz^2} + cP = 0 \quad . \quad . \quad . \quad . \quad .$$

If we attempt to solve this by the assumption

$$P = A_0 + A_1 z + A_2 z^2 + \ldots\ldots,$$

we find in the usual way that we must have

$$A_0 = 0,$$
$$cA_1 + 1.2A_2 = 0,$$
$$(c - 1.2)A_2 + 2.3A_3 = 0,$$
$$(c - 2.3)A_3 + 3.4A_4 = 0,$$
$$\ldots\ldots\ldots$$
$$(c - \overline{n-2}\,.\,\overline{n-1})A_{n-1} + \overline{n-1}\,.\,nA_n = 0.$$

Therefore $A_2 = -A_1 c/1^2.2$;

$$A_n = (-1)^{n-1}\frac{c(c-1.2)(c-2.3)\ldots\ldots(c-\overline{n-2}\,.\,\overline{n-1})}{1^2\,.\,2^2\ldots\ldots(n-1)^2\,.\,n}A_1.$$

That is to say, we find

$$P = A_1\left(z - \frac{c}{1^2}\frac{z^2}{2} + \frac{c(c-1.2)}{1^2\,2^2}\frac{z^3}{3} - \frac{c(c-1.2)(c-2.3)}{1^2\,2^2\,3^2}\frac{z^4}{4} + \ldots\ldots\right),$$

where the series within the brackets is obviously convergent for all real valu between -1 and $+1$, both included.

We have thus obtained only one synectic integral of (50). A second integral may be derived in the usual way, but it is not synectic. It has, in fact, an essential singularity at $z=0$; and may be expressed in the form $P\log z+\psi(z)$, where $\psi(z)$ is a power series. It is needless to state the actual result here, as it is of no use for our present purpose.

Dropping the multiplicative constant, we write

$$L(c,z)=z-\frac{c}{1^2}\frac{z^2}{2}+\frac{c(c-1.2)}{1^2\,2^2}\frac{z^3}{3}-\frac{c(c-1.2)(c-2.3)}{1^2\,2^2\,3^2}\frac{z^4}{4}+\cdots\cdots \quad (51);$$

or

$$\begin{aligned}L(c,z)=z&-c\frac{z^2}{1.2}-c(1-c/1.2)\frac{z^3}{2.3}\\&-c(1-c/1.2)(1-c/2.3)\frac{z^4}{3.4}\\&-\cdots\cdots\end{aligned} \quad (51').$$

The function thus defined will be called the *Lake Function*. It is obvious that

$$P=L(c,z)$$

is that synectic integral of the equation (50) which vanishes when $z=0$, *i.e.* when $w=1$; and it will be valid for $1\geqslant z\geqslant 0$, that is, for $-1\leqslant w\leqslant +1$, in short, throughout the whole length of the complete parabolic lake.

In particular, we have

$$L(c,1)=1-\frac{c}{1.2}-\left(1-\frac{c}{1.2}\right)\frac{c}{2.3}-\left(1-\frac{c}{1.2}\right)\left(1-\frac{c}{2.3}\right)\frac{c}{3.4}-\cdots\cdots;$$

or

$$L(c,1)=\left(1-\frac{c}{1.2}\right)\left(1-\frac{c}{2.3}\right)\left(1-\frac{c}{3.4}\right)\cdots\cdots \quad (52),$$

by EULER's Identity.

Hence, again, we get

$$\left.\begin{aligned}L(c,1)&=C(c,1)S(c,1);\\&=-\frac{2}{c\pi}\cos\left\{\frac{\pi}{2}\sqrt{(4c+1)}\right\};\end{aligned}\right\} \quad (53),$$

if we use the Γ-expressions for $C(c,1)$ and $S(c,1)$ given in § 25.

Since $L(c,z)$ is a synectic integral function of the general equation for P, which has the two synectic integral functions $C(c,w)$ and $S(c,w)$, that is, $C(c,1-2z)$ and $S(c,1-2z)$, it must be possible to determine two constants A and B, so that

$$L(c,z)\equiv A\,C(c,1-2z)+B\,S(c,1-2z).$$

Thus, in particular, we must have, for $z=0$ and $z=1$,

$$\begin{aligned}&A\,C(c,1)+B\,S(c,1)=0;\\&A\,C(c,1)-B\,S(c,1)=L(c,1)=C(c,1)S(c,1).\end{aligned}$$

Whence it appears that $A=\frac{1}{2}S(c,1)$, and $B=-\frac{1}{2}C(c,1)$. Therefore

or

$$\left.\begin{aligned}2L(c,z)&\equiv S(c,1)C(c,1-2z)-C(c,1)S(c,1-2z);\\2L\left(c,\frac{1-w}{2}\right)&\equiv S(c,1)C(c,w)-C(c,1)S(c,w).\end{aligned}\right\} \quad (54).$$

Also, if the dash on $L(c, z)$ denote differentiation with respect to z,

$$\left.\begin{aligned} -L'(c,z) &\equiv S(c,1)C'(c,1-2z) - C(c,1)S'(c,1-2z) \\ \text{or} \quad -L'\left(c,\frac{1-w}{2}\right) &\equiv S(c,1)C'(c,w) - C(c,1)S'(c,w) \end{aligned}\right\} \quad . \quad . \quad . \quad (55),$$

we may also notice the particular relations

$$L(c,\tfrac{1}{2}) = \tfrac{1}{2}S(c,1), \qquad L'(c,\tfrac{1}{2}) = C(c,1) \quad . \quad . \quad . \quad (56).$$

§ 42.

THE L-SOLUTION FOR A COMPLETE PARABOLIC LAKE

is obviously

$$\left.\begin{aligned} \xi h(1-w^2) &= AL(c,z)\sin n(t-\tau); \\ \zeta &= \frac{A}{a}L'(c,z)\sin n(t-\tau). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (57).$$

The values of c are the roots of the period equation

$$\left.\begin{aligned} L(c,1) &= 0, \\ \text{that is to say,} \quad c_\nu &= \nu(\nu+1); \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (58)$$

and

$$T_\nu = 2\pi a/\sqrt{\{c\nu(\nu+1)gh\}},$$

as before.

The values of z corresponding to the nodes of the ν-nodal seiche are the roots of the equation

$$L'(c_\nu, z) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (59).$$

THE L-SOLUTION FOR A BIPARABOLIC LAKE.

§ 43. For the part between $w = 0$ and $w = +1$,

$$\left.\begin{aligned} \xi h(1-w^2) &= \frac{A}{L(c,\frac{1}{2})}L(c,z)\sin n(t-\tau); \\ \zeta &= \frac{A}{2aL(c,\frac{1}{2})}L'(c,z)\sin n(t-\tau); \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (60);$$

where $z = \frac{1}{2}(1-w)$.

For the part between $w' = 0$ and $w' = -1$,

$$\left.\begin{aligned} \xi' h(1-w'^2) &= \frac{A}{L(c',\frac{1}{2})}L(c',z')\sin n(t-\tau); \\ \zeta &= -\frac{A}{2a'L(c',\frac{1}{2})}L'(c',z')\sin n(t-\tau); \end{aligned}\right\} \quad . \quad . \quad . \quad (61);$$

where $z' = \frac{1}{2}(1+w')$.

The period equation is now

$$aL(c,\tfrac{1}{2})/L'(c,\tfrac{1}{2}) + a'L(c',\tfrac{1}{2})/L'(c',\tfrac{1}{2}) = 0 \quad . \quad . \quad . \quad . \quad (62);$$

where

$$c = n^2a^2/gh, \quad c' = n^2a'^2/gh \quad . \quad . \quad . \quad . \quad . \quad (63);$$

and

$$T_\nu = 2\pi/n_\nu = 2\pi a/\sqrt{(c_\nu gh)} = 2\pi a'/\sqrt{(c'_\nu gh)} \quad . \quad . \quad . \quad (64).$$

The nodes in the two parts are given by

$$L'(c_\nu, z)=0 \quad \text{and} \quad L'(c'_\nu, z)=0 \quad . \quad . \quad . \quad . \quad (65).$$

It may, of course, happen that one of the two equations (65) has no real root.

Ratios of the Seiche Amplitudes at various Parts of a Biparabolic Lake.

§ 44. It is convenient for purposes of calculation to tabulate the following formulæ; where ζ_0, ζ_Δ, $\zeta_{\Delta'}$, ζ_z denote the amplitudes of a seiche at the deepest part, the positive and negative end, and any point z respectively; and $\alpha=\sqrt{(4c+1)}$.

$$\zeta_\Delta/\zeta_0=1/C(c,1)=\Gamma\left(\frac{3+\alpha}{4}\right)\Gamma\left(\frac{3-\alpha}{4}\right)\Big/\sqrt{\pi}=1/L'(c,\tfrac{1}{2}). \quad . \quad . \quad (66).$$

$$\zeta_{\Delta'}/\zeta_0=1/C(c',1)=\Gamma\left(\frac{3+\alpha'}{4}\right)\Gamma\left(\frac{3-\alpha'}{4}\right)\Big/\sqrt{\pi}=1/L'(c',\tfrac{1}{2}). \quad . \quad . \quad (67).$$

$$\begin{aligned}\zeta_\Delta/\zeta_{\Delta'}&=\alpha' S(c',1)/\alpha S(c,1)=C(c',1)/C(c,1),\\ &=\Gamma\left(\frac{3+\alpha}{4}\right)\Gamma\left(\frac{3-\alpha}{1}\right)\Big/\Gamma\left(\frac{3+\alpha'}{4}\right)\Gamma\left(\frac{3-\alpha'}{4}\right),\\ &=L'(c',1/2)/L'(c,1/2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (68).\end{aligned}$$

$$\begin{aligned}\zeta_z/\zeta_0&=L'(c,z)/L'(c,\tfrac{1}{2}),\\ &=L'(c,z)\Gamma\left(\frac{3+\alpha}{4}\right)\Gamma\left(\frac{3-\alpha}{4}\right)\Big/\sqrt{\pi},\\ &=L'(c,z)(\zeta_\Delta/\zeta_0). \quad . \quad . \quad . \quad . \quad . \quad (69).\end{aligned}$$

All irrespective of algebraic sign.

Owing to the want of a simple companion fundamental integral, the Lake Function is not convenient when the parabolic lake is truncated. In this respect it has the same defect as the Legendre and Bessel Functions. Its practical advantage is that it gives highly convergent series at points where the series for $C(c, w)$ and $S(c, w)$ converge slowly. Unhappily, the corresponding function for a convex lake has an imaginary argument.

SEICHES IN RECTILINEAR LAKES.

§ 45. If we take the origin of x at a point where the depth is h, then the law of depth will be $h(x)=h\times(1-x/a)$, where a is a constant, positive or negative according as the lake bottom slopes upwards or downwards in the direction in which x increases.

We have, therefore, with the previous notation

$$\left.\begin{aligned}\xi h(1-x/a)=u&=P\sin n(t-\tau),\\ \zeta&=-\frac{\partial u}{\partial x};\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (70),$$

where P is determined by

$$\frac{d^2P}{dx^2}+\frac{n^2P}{gh(1-x/a)}=0 \quad . \quad . \quad . \quad . \quad (71).$$

If we transform (71) by putting $w = 2n\alpha\sqrt{(1-x/\alpha)}/\sqrt{(gh)}$, and $P = Rw$, it becomes

$$\frac{d^2R}{dw^2} + \frac{1}{w}\frac{dR}{dw} + \left(1 - \frac{1}{w^2}\right)R = 0, \quad . \quad . \quad . \quad . \quad (72).$$

which is a particular case of the BESSEL Equation.*

If $J_n(w)$ and $Y_n(w)$ denote the BESSEL and NEUMANN Functions, as defined in GRAY and MATHEWS' treatise, the general solution of (72) is

$$R = A\,J_1(w) + B\,Y_1(w).$$

Hence, with a slight adaptation of the constants, we find

$$\xi w = \{A\,J_1(w) + B\,Y_1(w)\}\sin n(t-\tau); \quad . \quad . \quad . \quad . \quad . \quad (73);$$

$$\zeta = \frac{2a}{h}\left\{\frac{A}{w}\frac{d}{dw}\left(wJ_1(w)\right) + \frac{B}{w}\frac{d}{dw}\left(wY_1(w)\right)\right\}\sin n(t-\tau).$$

Now, by one of the fundamental properties of $J_1(w)$ and $Y_1(w)$, we have

$$\frac{1}{w}\frac{d}{dw}\left(wJ_1(w)\right) = J_0(w), \quad \frac{1}{w}\frac{d}{dw}\left(wY_1(w)\right) = Y_0(w).$$

Hence

$$\zeta = \frac{2a}{h}\{A\,J_0(w) + B\,Y_0(w)\}\sin n(t-\tau) \quad . \quad . \quad . \quad . \quad . \quad (74).$$

From (73) and (74) solutions for the following particular cases are readily obtained.

RECTILINEAR LAKE WITH TWO SLOPES TRUNCATED AT BOTH ENDS.

§ 46. The laws of depth for the two parts will be given by $h(1-x/\alpha)$ and $h'(1+x/\alpha')$, if we take the origin at the junction of the two slopes, and choose the standard case to be that where the bottom slopes upwards on both sides of the junction.

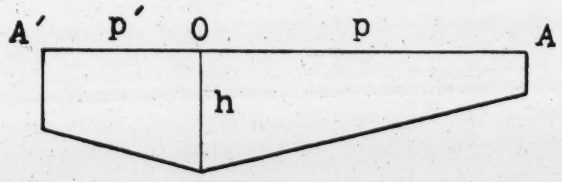


FIG. 8.

Let

$$w = 2n\alpha\sqrt{(1-x/\alpha)}/\sqrt{(gh)}, \; w' = 2n\alpha'\sqrt{(1+x/\alpha')}/\sqrt{(gh)} \quad . \quad . \quad . \quad . \quad (75);$$

and

$$\alpha = 2\alpha/\sqrt{(gh)}, \quad \beta = 2\alpha\sqrt{(1-p/\alpha)}/\sqrt{(gh)};$$
$$\alpha' = 2\alpha'/\sqrt{(gh)}, \quad \beta' = 2\alpha'\sqrt{(1-p'/\alpha')}/\sqrt{(gh)}.$$

* Readers unacquainted with the properties of BESSEL Functions will find all that is here required in a few pages of the treatise by GRAY and MATHEWS (1895), ch. ii., and pp. 241–292 containing the tables.

The boundary conditions are, that ξ vanish at the ends, and that $\xi = \xi, \zeta = \zeta'$ at O. These lead readily to the following :—

$$\left.\begin{aligned} w\xi &= aA\frac{Y_1(n\beta)J_1(w) - J_1(n\beta)Y_1(w)}{Y_1(n\beta)J_1(na) - J_1(n\beta)Y_1(na)}\sin n(t-\tau); \\ \zeta &= \frac{2a^2}{h}A\frac{Y_1(n\beta)J_0(w) - J_1(n\beta)Y_0(w)}{Y_1(n\beta)J_1(na) - J_1(n\beta)Y_1(na)}\sin n(t-\tau). \\ w'\xi' &= a'A\frac{Y_1(n\beta')J_1(w') - J_1(n\beta')Y_1(w')}{Y_1(n\beta')J_1(na') - J_1(n\beta')Y_1(na')}\sin n(t-\tau); \\ \zeta &= -\frac{2a'^2}{h}A\frac{Y_1(n\beta')J_0(w') - J_1(n\beta')Y_0(w')}{Y_1(n\beta')J_1(na') - J_1(n\beta')Y_1(na')}\sin n(t-\tau) \end{aligned}\right\} \quad . \quad (76).$$

And the period equation is

$$\begin{aligned} &a^2\{Y_1(n\beta)J_0(na) - J_1(n\beta)Y_0(na)\}\{Y_1(n\beta')J_1(na') - J_1(n\beta')Y_1(na')\} \\ &\quad + a^2\{Y_1(n\beta')J_0(na') - J_1(n\beta')Y_0(na')\}\{Y_1(n\beta)J_1(na) - J_1(n\beta)Y_1(na) = 0 \quad . \quad (77). \end{aligned}$$

Unsymmetric Lake shelving at Both Ends.

§ 47.

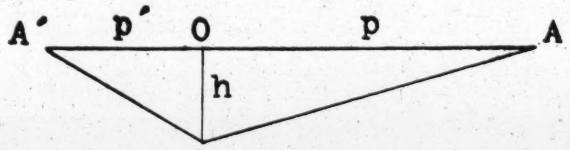


Fig. 9.

In this case $\beta = 0$, $\beta' = 0$.

Therefore, since $\underset{w=0}{L}\, J_1(w)/Y_1(w) = 0$, the equations of § 46 reduce to

$$\begin{aligned} w\xi &= aA\frac{J_1(w)}{J_1(na)}\sin n(t-\tau), \\ \zeta &= \frac{2a^2A}{h}\frac{J_0(w)}{J_1(na)}\sin n(t-\tau); \\ w'\xi' &= a'A\frac{J_1(w')}{J_1(na')}\sin(t-\tau); \\ \zeta' &= -\frac{2a'^2A}{h}\frac{J_0(w')}{J_1(na')}\sin(t-\tau). \quad . \quad . \quad . \quad . \quad (78). \end{aligned}$$

Period equation

$$a^2J_0(na)J_1(na') + a'^2J_0(na')J_1(na) = 0 \quad . \quad . \quad . \quad . \quad . \quad (79).$$

The nodes are given by *

$J_0(w) = 0$ in the part O A ;
$J_0(w') = 0$ . . . . . O A' :

where for the ν-nodal seiche

$$w = n_\nu a, \quad w' = n_\nu a'.$$

* *Roots of the Bessel Functions.*—In what follows I shall denote the positive roots of the equation $J_0(z) = 0$ by j_1, j_3, j_5 . . . . . ; and the positive roots of $J_1(z) = 0$ (excluding the zero root $j_0 = 0$) by j_2, j_4, j_6, ... So that we have approximately $j_1 = 2{\cdot}405$, $j_2 = 3{\cdot}832$, $j_3 = 5{\cdot}520$, $j_4 = 7{\cdot}016$, $j_5 = 8{\cdot}654$, $j_6 = 10{\cdot}173$, $j_7 = 11{\cdot}792$, $j_8 = 13{\cdot}323$, $j_9 = 14{\cdot}931$, $j_{10} = 16{\cdot}471$, etc.

For large values of n, $j_n = (2n+1)\pi/4$, approximately : *e.g.* this formula gives $j_{11} = 18{\cdot}064$ instead of the correct value $18{\cdot}071$; so that the error after $n = 11$ is less than $\cdot 1\ \%$.

If ${}_\nu x_r$ denote the distance from O of the r^{th} node in O A counting from A to O, and ${}_\nu x_r$ the like for O A′, we have

$$n_\nu a \sqrt{(1 - {}_\nu x_r/a)} = j_{2r-1}, \quad j_{2r-1} < n_\nu a .$$

Hence

$$1 - {}_\nu x_r/a = j_{2r-1}^2 \mathrm{T}_\nu^2/4\pi^2 a^2, \quad j_{2r-1} < n_\nu a ;$$
$$1 - {}_\nu x_r'/a' = j_{2r-1}^2 \mathrm{T}_\nu^2/4\pi^2 a'^2, \quad j_{2r-1} < n_\nu a' . \quad . \quad . \quad . \quad (80).$$

There will be ν roots altogether; but the distribution between the two formulæ will depend on circumstances. In some cases the nodes are all on one side of O. These formulæ lead to some remarkable relations which are true accurately for complete rectilinear lakes, and approximately for such as are approximately rectilinear. For example, we have

$$(a - {}_\mu x_1)/(a - {}_\nu x_1) = \mathrm{T}_\mu^2/\mathrm{T}_\nu^2 . \quad . \quad . \quad . \quad . \quad (81).$$

In other words, the distance from the ends of the lake of the first node of any pure seiche in a given complete rectilinear lake is proportional to the square of the period of the seiche.

Symmetric Truncated Lake.

§ 48. Here $\alpha = \alpha'$, $\beta = \beta'$. The equations can be simplified by the suppression of unnecessary constants; and the period equation breaks up into two. We have then

$$w\xi = \mathrm{A}\{\mathrm{Y}_1(n\beta)\mathrm{J}_1(w) - \mathrm{J}_1(n\beta)\mathrm{Y}_1(w)\} \sin n(t-\tau),$$
$$\zeta = \frac{2a\mathrm{A}}{h}\left\{ \mathrm{Y}_1(n\beta)\mathrm{J}_0(w) - \mathrm{J}_1(n\beta)\mathrm{Y}_0(w) \right\} \sin n(t-\tau);$$
$$w'\xi' = \mathrm{A}\{\mathrm{Y}_1(n\beta)\mathrm{J}_1(w') - \mathrm{J}_1(n\beta)\mathrm{Y}_1(w')\} \sin n(t-\tau),$$
$$\zeta = -\frac{2a\mathrm{A}}{h}\left\{ \mathrm{Y}_1(n\beta)\mathrm{J}_0(w') - \mathrm{J}_1(n\beta)\mathrm{Y}_0(w') \right\} \sin n(t-\tau). \quad . \quad . \quad (82).$$

The period equation is:—for odd nodality,

$$\mathrm{Y}_1(n\beta)\mathrm{J}_0(n\alpha) - \mathrm{J}_1(n\beta)\mathrm{Y}_0(n\alpha) = 0 ;$$

for even nodality,

$$\mathrm{Y}_1(n\beta)\mathrm{J}_1(n\alpha) - \mathrm{J}_1(n\beta)\mathrm{Y}_1(n\alpha) = 0 . \quad . \quad . \quad . \quad . \quad (83).$$

Symmetric Lake shelving at Both Ends.

§ 49. Starting with the formulæ of § 47, we have to put $a = a' = p = p'$; and therefore $\alpha = \alpha'$. Suppressing unnecessary constants, we may now write

$$w\xi = \mathrm{AJ}_1(w) \sin n(t-\tau),$$
$$\zeta = \frac{2a\mathrm{A}}{h}\mathrm{J}_0(w) \sin n(t-\tau);$$
$$w'\xi' = \mathrm{AJ}_1(w') \sin n(t-\tau);$$
$$\zeta' = -\frac{2a\mathrm{A}}{h}\mathrm{J}_0(w') \sin n(t-\tau) \quad . \quad . \quad . \quad . \quad (84).$$

The period equation breaks up into

$$\mathrm{J}_0(n\alpha) = 0, \quad \mathrm{J}_1(n\alpha) = 0 .* \quad . \quad . \quad . \quad . \quad (85).$$

Hence we have

$$\mathrm{T}_\nu = 4\pi a/j_\nu \sqrt{(gh)} . \quad . \quad . \quad . \quad . \quad . \quad (86).$$

* These formulæ are given by Lamb in his *Hydrodynamics* (1895), § 182.

For large values of ν, we have

$$T_\nu = 16a/(2\nu+1)\sqrt{(gh)} \quad . \quad . \quad . \quad . \quad . \quad (87).$$

And, when $1/2\nu$ is negligible, simply

$$\begin{aligned} T_\nu &= 8a/\nu\sqrt{(gh)}, \\ &= 4l/\nu\sqrt{(gh)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (88). \end{aligned}$$

Hence, as the nodality increases, the periods of the pure seiches tend more and more to follow the harmonic law; and ultimately are the same as the periods in a uniform lake of double the length.

If ${}_\nu x_r$ have the same meaning as before, and ${}_\nu X_r$ have a corresponding meaning for the ventral points, we find

$$\begin{aligned} 1 - {}_\nu x_r/a &= j_{2r-1}^2/j_\nu^2 = T_\nu^2/T_{2r-1}^2; \\ 1 - {}_\nu X_r/a &= j_{2r}^2/j_\nu^2 = T_\nu^2/T_{2r}^2 \quad . \quad . \quad . \quad . \quad . \quad (89). \end{aligned}$$

Hence, if we compare the different nodes of the same seiche, the distances of the nodes from the end of the lake are inversely proportional to the squares of the periods of the lower seiches of odd nodality.

If we compare nodes of the same order for different seiches, the distances from the end are directly proportional to the squares of the periods of the corresponding seiches.

It is also easy to see from the above formulæ that, when the nodality of the seiche is high, the wave lengths near the ends of the lake increase at first in arithmetic progression.

If we apply the rule of Du Boys to the present case we get

$${}_dT_1 = 4\sqrt{\left(\frac{l}{2gh}\right)}\int_0^{l/2} dx/\sqrt{x} = \frac{4l}{\sqrt{(gh)}}.$$

Since $T_1 = 2\pi l/j_1\sqrt{(gh)}$, we have

$$\frac{{}_dT_1}{T_1} = \frac{2j_1}{\pi} = \frac{4{\cdot}810}{3{\cdot}142} = 1{\cdot}531;$$

that is, Du Boys' rule gives too great a period, as it does in the case of parabolic or quartic concave lakes: the deviation is even greater than in the case of a symmetric parabolic lake.

As the present case is an interesting one, serving as a standard of comparison for other cases, I add some numerical data.

Table of Ratios of Periods for a Complete Symmetric Rectilinear Lake.

T_2/T_1	T_3/T_1	T_4/T_1	T_5/T_1	T_6/T_1	T_7/T_1	T_8/T_1	T_9/T_1	T_{10}/T_1
·6276	·4357	·3428	·2779	·2365	·2040	·1805	·1609	·1460

Positions of Nodes in one-half of the Lake.

1	0									
2	...	...	...	...	·6057					
3	0	...	...	...	...	...	8102			
4	...	...	·3809	...	...	...	...	·8825		
5	0	...	...	·5930	...	...	...	...	·9228	
6	...	·3763	...	...	...	...	...	...	...	·9441

Positions of Nodes when Depth is uniform.

1	0									
2	...	...	...	...	·5000					
3	0	...	...	...	...	...	·6667			
4	...	...	·2500	...	...	...	...	·7500		
5	0	...	...	·4000	...	...	...	...	·8000	
6	...	·1667	...	...	·5000	...	...	...	...	·8333

Lake with One Slope.

§ 50.

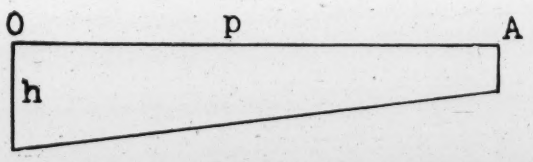


Fig. 10.

Obviously the seiches are the same as the seiches of even nodality in the symmetric truncated lake of § 48.

Hence we have $$\xi_w = A\{Y_1(n\alpha)J_1(w) - J_1(n\alpha)Y_1(w)\}\sin n(t-\tau);$$

$$\zeta = \frac{2\alpha A}{h}\{Y_1(n\alpha)J_0(w) - J_1(n\alpha)Y_0(w)\}\sin n(t-\tau) \quad . \quad . \quad . \quad . \quad (90).$$

The period equation is $$Y_1(n\alpha)J_1(n\beta) - J_1(n\alpha)Y_1(n\beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad (91).$$

For the nodes and ventral points

$$1 - {}_{r}x_r/\alpha = j_{2r-1}^2/j_{2\nu}^2$$
$$1 - {}_{\nu}X_r/\alpha = j_{2r}^2/j_{2\nu}^2 = T_\nu^2/T_r^2 .$$

Lake with One Slope shelving at One End.

§ 51.

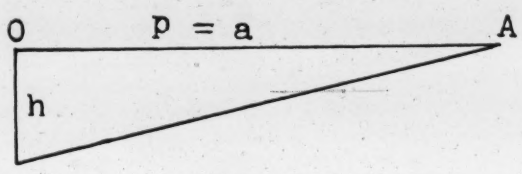


Fig. 11.

Here we have simply to take the seiches of even nodality from the case of § 49.

$$\xi w = \mathrm{A}\mathrm{J}_1(w) \sin n(t-\tau);$$

$$\zeta = \frac{2a\mathrm{A}}{h}\mathrm{J}_0(w) \sin n(t-\tau) \quad . \quad . \quad . \quad . \quad (92).$$

The period equation is

$$\mathrm{J}_1(na) = 0;$$

so that

$$\mathrm{T}_\nu = 4\pi a/j_{2\nu} \cdot \sqrt{(gh)} \quad . \quad . \quad . \quad . \quad . \quad (93).$$

The nodes are, of course, the nodes of the seiches of even nodality in the right-hand half of the complete symmetric rectilinear lake discussed in § 49.

The following data are useful for reference in lake calculations :—

Ratios of Periods in a Semicomplete Rectilinear Lake.

T_2/T_1	T_3/T_1	T_4/T_1	T_5/T_1	T_6/T_1	T_7/T_1	T_8/T_1	T_9/T_1	T_{10}/T_1
·5462	·3767	·2883	·2327	·1954	·1684	·1479	·1319	·1190

Positions of the Nodes.

1	...	...	·6057			
2	...	·3809	...	...	·8825	
3	·3763	...	...	·7056	...	·9441

Positions of Nodes for Uniform Depth.

1	...	...	·5000			
2	...	·2500	...	...	·7500	
3	·1667	...	·5000	...	...	·8333

SEICHES IN QUARTIC LAKES.

§ 52. In a paper recently published in the Society's *Proceedings* (vol. xxv. p. 638, May 11, 1905) I gave the solution of the seiche problem for lakes whose normal curve is a quartic of the form $\sigma = h(1 \mp v^2/a^2)^2$. For convenience of reference I recapitulate the results here, supposing as usual, for simplicity, that the lake has uniform breadth and rectangular cross section, so that the expression for the depth is $h \times (a^2 \mp x^2)^2$.

Concave Truncated Quartic Lake.

The origin is at O over the deepest point (see fig. 12). The length P Q is l, and

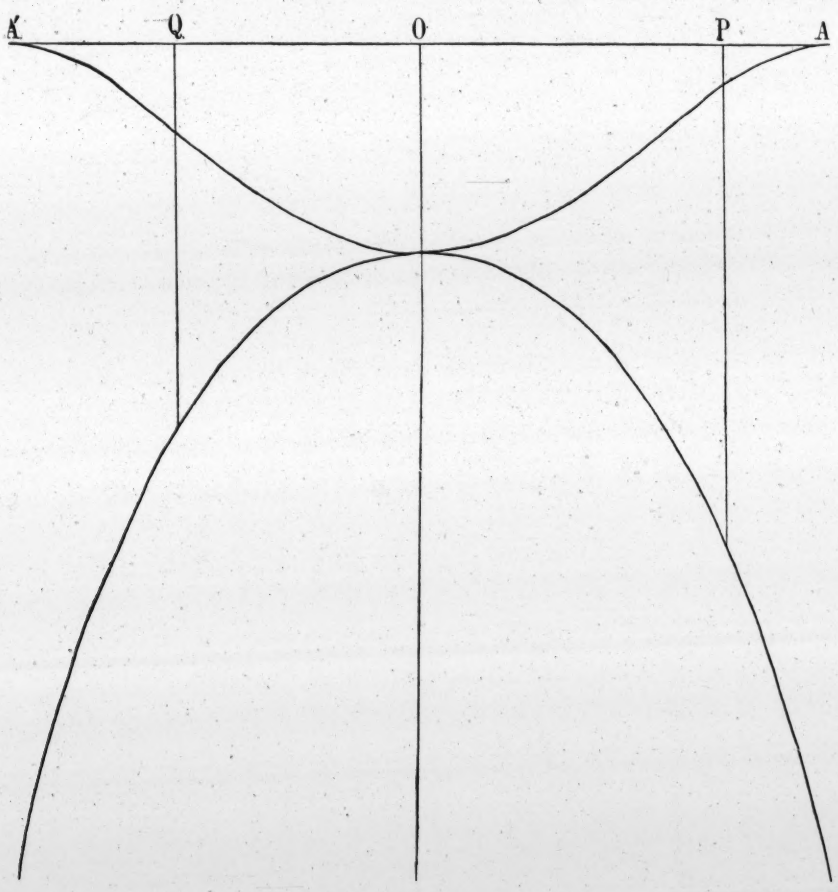


Fig. 12.

P and Q correspond to $x = p$, $x = q$. The depths at P, Q, and O are r, s, d respectively; so that

$$p = a\sqrt{\left(1 - \sqrt{\frac{r}{d}}\right)}, \quad q = \mp a\sqrt{\left(1 - \sqrt{\frac{s}{d}}\right)};$$

$$l = a\left\{\sqrt{\left(1 - \sqrt{\frac{r}{d}}\right)} \pm \sqrt{\left(1 - \sqrt{\frac{s}{d}}\right)}\right\} = a\gamma, \text{ say} \quad . \quad . \quad . \quad (94)$$

The upper signs correspond to the case figured, where P and Q are on opposite sides of O.

Then

$$\xi h(a^2-x^2)^2 = u = \mathrm{A}(a^2-x^2)^{\frac{1}{2}} \sin\left\{\frac{\nu\pi}{k}\left(\log\frac{a+x}{a-x} - \log\frac{a+p}{a-p}\right)\right\} \sin n_\nu(t-\tau),$$

$$\zeta = -\frac{\partial u}{\partial x};$$

and

$$\mathrm{T}_\nu = 2\pi l/\gamma \sqrt{\{gd(4\nu^2\pi^2/k^2+1)\}};$$

where

$$\gamma = \sqrt{\left(1-\sqrt{\frac{r}{d}}\right)} \pm \sqrt{\left(1-\sqrt{\frac{s}{d}}\right)};$$

$$k = \log\left\{\frac{1+\sqrt{\left(1-\sqrt{\frac{r}{d}}\right)}}{1-\sqrt{\left(1-\sqrt{\frac{r}{d}}\right)}} \Bigg/ \frac{1\mp\sqrt{\left(1-\sqrt{\frac{s}{d}}\right)}}{1\pm\sqrt{\left(1-\sqrt{\frac{s}{d}}\right)}}\right\}.$$

When the end barriers P and Q approach more and more nearly to the infi shallow theoretical ends A and A′, the periods of all the seiches become more and nearly equal to each other and to $\pi l/\sqrt{(gd)}$, which I have called *the period o anomalous seiche.*

Convex Quartic Lake.

§ 53. The symbols being defined as in § 52, we now have

$$p = a\sqrt{\left(\sqrt{\frac{r}{d}}-1\right)}, \quad q = \mp a\sqrt{\left(\sqrt{\frac{s}{d}}-1\right)};$$

$$l = a\left\{\sqrt{\left(\sqrt{\frac{r}{d}}-1\right)} \pm \sqrt{\left(\sqrt{\frac{s}{d}}-1\right)}\right\} = a\gamma, \text{ say.}$$

$$\xi h(a^2+x^2)^2 = u = (a^2+x^2)^{\frac{1}{2}} \sin\left\{\frac{2\nu\pi}{k}\left(\tan^{-1}\frac{x}{a} - \tan^{-1}\frac{p}{a}\right)\right\} \sin n_\nu(t-\tau),$$

$$\zeta = -\frac{\partial u}{\partial x};$$

and $$\mathfrak{T}_\nu = 2\pi l/\gamma \sqrt{\{gd(4\nu^2\pi^2/k^2-1)\}};$$

where

$$\gamma = \sqrt{\left(\sqrt{\frac{r}{d}}-1\right)} \pm \sqrt{\left(\sqrt{\frac{s}{d}}-1\right)};$$

$$k = 2\tan^{-1}\sqrt{\left(\sqrt{\frac{r}{d}}-1\right)} - 2\tan^{-1}\sqrt{\left(\sqrt{\frac{s}{d}}-1\right)}.$$

§ 54. In constructing a theoretical curve to represent the normal curve of any deduced from bathymetric data we can, of course, combine pieces of parabolas, str lines or quartics at will; and the variety of formulæ above given is probably suffi for most practical purposes, although the labour of the calculation, even in simple is not small. To this part of the subject I shall have occasion to return in subse communications to the Society.

PART III.

A Sketch of the Bibliography of Seiches.

The following list of books and memoirs dealing with Seiches does not claim to be exhaustive; and in many cases I have not been able to verify the references directly. It must, therefore, be looked on merely as an endeavour to help others in a matter where I myself have felt the need of help.

What may be called the ancient history of seiche observations is fully dealt with by Forel; and it is to his bibliography, extending from 1873 to 1891, that I owe most of the information for that period. I must also acknowledge obligations to papers by Halbfass, mentioned below, and to Messrs Chumley and Maclagan Wedderburn, both connected with the Scottish Lake Survey, for many of the later references.

The following abbreviations are used :—

A.G., *Archives des Sciences Physiques et Naturelles, Genève.*
A.H., *Actes de la Société Helvétique des Sciences Naturelles.*
A.Hy., *Annalen der Hydrographie.*
A.W., *Sitzungsberichte der K.K. Akademie der Wissenschaft, Wien.*
B.A., *British Association Reports.*
B.V., *Bulletin de la Société Vaudoise des Sciences Naturelles.*
C.R., *Comptes Rendus de l'Académie des Sciences, Paris.*
P.M., *Petermann's Mittheilungen.*
Z.G., *Zeitschrift für Gewässerkunde.*
Z.I., *Zeitschrift für Instrumentenkunde.*

The Roman numeral indicates the volume, the Arabic the page.

1755. Disturbances of the Levels of Lakes in Scotland and elsewhere caused by the Earthquake of Lisbon, *Scots Magazine* for 1755.

These notices are interesting, because there is, as yet, little evidence connecting seiches with seismic disturbances; in fact, none at all in the case of ordinary seiches.

1776. Laplace. "Sur les Ondes. Suite des Recherches sur plusieurs points du Système du Monde," § xxxvii., *Hist. de l'Ac. Roy. d. Sc. Paris*, Année 1776.

The modern mathematics of wave motion may be said to date from Laplace's researches on the tides. In the memoir quoted he considers waves in a canal of uniform depth to be caused by the immersion of a given object, and arrives at the expression $\sqrt{\{g \tanh mh/m\}}$ for the velocity of wave propagation. But, as he does not consider oscillatory waves, the connection of m with the wave length is not made clear.

1781. Lagrange. "Sur la Théorie du Mouvement des Fluides," *Mém. Ac. Berl.*

In §§ 48 and 49 he gives for the first time the theory of long waves, with the formula $\sqrt{(gh)}$ for the velocity of propagation in a canal of uniform depth, h.

1804. Young. *Lectures on Natural Philosophy*, xxiii. Also Works (ed. Peacock), ii. pp. 141, 262.

1815. Cauchy. "Sur la Théorie des Ondes." *Ouvres*, 1e sér., i. 175.

1815. Poisson. "Sur la Théorie des Ondes," *Mém. d. l'Inst.*, i. (1816), etc.

In the works of Cauchy and Poisson the mathematics of wave motion has already taken its modern form. Both have Laplace's formula for velocity of propagation, but both are preoccupied with the difficult problem of Laplace, and do not consider oscillatory surface waves, either progressive or stationary.

1825. WEBER, W. E. and E. H. *Die Wellenlehre auf Experimente gegrundet.* Leipzig.
This treatise is a classic in the experimental part of our subject. Here for the first time direct observations of the motion of the fluid particles in wave motions are described A great variety of different cases of wave motion are discussed; and in particular, standing waves in a canal, of which seiches are a particular example.

1828. MERIAN, J. R. *Ueber die Bewegung tropfbarer Flussigkeiten in Gefässen.* Basel.
This memoir was really the first after LAGRANGE that dealt fully and effectively with the problem of stationary waves in a canal of uniform depth (h) and length (l). In MERIAN'S paper the formula $T = \sqrt{\{\pi l/g \tanh(\pi h/l)\}}$ appears for the first time in connection with stationary waves. Its relation to the formula of LAPLACE, CAUCHY, and POISSON for the velocity of propagation is obvious. Unfortunately the memoir attracted no attention when it was published, and was forgotten until it was reproduced by the author's grandnephew, VON DER MÜHL, *Math. Ann.*, xxviii. 575, 1885. Meanwhile MERIAN'S results had been rediscovered by other mathematicians.

1837. RUSSELL, JOHN SCOTT. His beautiful experiments on canal waves, described in the *B.A. Reports* from 1837 onwards, were the starting-point of a long series of English researches by GREEN, KELLAND, AIRY, STOKES, KELVIN, RAYLEIGH, and others.

1837. GREEN, G. "On the Motion of Waves in a variable Canal of small Depth and Width," *Trans. Camb. Phil. Soc.*, 1838. Also *ib.*, 1839.

1839. KELLAND, P. "On the Theory of Waves," *Trans. Roy. Soc. Edin.*, xiv., 1839. Also *ib.*, xv., 1841.

1843. MILNE, D. "On a Remarkable Oscillation of the Sea," July 1843. *Trans. Roy. Soc. Edin.*, xv. 609.

1845. AIRY, G. B. "Tides and Waves," *Ency. Metrop.*

1846. STOKES, G. G. "On Recent Researches in Hydrodynamics," *B.A.*, 1846.
AIRY'S Article and STOKES' Report are classics on the analytical side of our subject.

1873. FOREL, F. A. "Première Étude sur les Seiches," *B.V.*, xii. 213.

1874. STAHLBERGER. *Ebbe und Flut in der Rhede von Fiume*, Budapest.

1875. FOREL, F. A. "Deuxième Étude sur les Seiches," *B.V.*, xiii. 510. Also *A.G.*, liii. 281.

1875. GUTHRIE, F. "Periods of Oscillation of Water in Small Tanks," *Proc. Phys. Soc.*, i. Also *Phil. Mag.*, Oct. and Nov. 1875.

1876. RAYLEIGH. "Periods of Oscillation of Water in Tanks," *Phil. Mag.*, 5th ser., vol. i. p. 275.

1876. FOREL, F. A. "Les Seiches, Vague d'Oscillation fixe des lacs," *A.H.*, Andermatt, 157. Also *Ann. Chim. Phys.*, ix., 1876.

1876. FOREL, F. A. "Le Limnimètre enregistreur de Morges," *A.G.*, lvi. 305.

1876. FOREL, F. A. "La Formule des Seiches," *A.G.*, lvii. 278.

1877. AIRY, G. B. "On the Tides at Malta" (Seiches in the Sea at Malta and Swansea), *Phil. Trans.*, 169, pp. 123–138.

1877. PLANTAMOUR, PH. "Notes sur quelques Observations limnimétriques faites à Sécheron," *A.G.*, lviii. 302.

1877. GIESEN. "Versuch einer Mathematischen Darstellung der Flüssigkeitswellen," *Schl. Zeitsch. f. Math.*, xxii. 133.

1877. FOREL, F. A. "Essai Monographique sur les Seiches du Léman," *A.G.*, lix. 50.

1877. PLANTAMOUR, PH. "Note sur la Limnimétrie, à l'occasion du tremblement de Terre du 8 Oct. 1877," *A.G.*, lx. 511.

1877. FOREL, F. A. "Contributions à la limnimétrie du Léman," *B.V.*, xv. 160.

1878. FOREL, F. A. "Les Causes des Seiches," *A.G.*, lxiii. 113, 189.

1878. FOREL, F. A. "Les Seiches des Lacs et leurs Causes," *C.R.*, lxxxvi. 1500.

1878. FOREL, F. A. "Seiches and Earthquakes," *Nature*, xvii. 281.

1878. Plantamour, Ph. "Le Limnographe de Sécheron," *A.G.*, lxiv. 318.

1878. Gautier, E. "The Seiches of the Lake of Geneva," *Nature*, xviii. 100.

1879. Forel, F. A. "Le Problème de l'Euripe," *C.R.*, lxxxix. 859. Also *La Nature*, VIII., i. 35.

1879. Forel, F. A. "Les Seiches, Vague d'Oscillation fixe des Lacs," *A.H.*, Berne.

1879. Plantamour, Ph. "Seiche Occasionnée par le Cyclone du 20 Fev. 1879," *A.H.*, Berne, i. 335.

1879. Sarasin, Ed. "Limnimètre enregistreur transportable. Observations à la Tour-de-Peilz, près Vevey," *A.H.*, Berne, ii. 724.

1880. Kirchhoff, G. "Ueber stehende Schwingungen einer Schweren Flussigkeit," *Wied. Ann.*, x. 34. Along with G. Hansemann, "Versuche über stehende Schwingungen des Wassers," *ib.*, 337.

They do not deal with long waves, but it is interesting to compare their results with those given in § 49 of this Memoir.

1880. Forel, F. A., et Soret, J. L. "Les Seiches Dicrotes," *A.H.*, Berne, iii. 15.

1880. Forel, F. A. "Seiches et Vibrations des Lacs et de la Mer" (Seiches on the Sea), *C.R.*, viii^{e} session Assoc. Fr. pour l'Av. d. Sc., 493.

1880. Sarasin, Ed. "Tracés Limnographiques dans diverses Stations du Léman," *A.G.*, iv. 383.

1880. Capper, S. J. "Tidal Phenomenon in Lake Constance," *Nature*, vol. xxi. p. 397.

1881. Hann, J. *Allgemeine Erdkunde*, Prag.

1881. Hicks, W. M. "Recent Progress in Hydrodynamics," *B.A.*

1882. Μιαούλης, A. A. Περὶ τῆς Παλλιρροίας τοῦ Εὐρίπου. Ἐν Αθήναις.

1885. Günther, S. *Lehrbuch der Geophysik*, ii. Stuttgart.

1885. Russell, H. C. "On the Seiches of Lake George, Australia," *Roy. Soc. N.S.W. Ann. Add.*, 13.

1885. Forel, F. A. "La Formules des Seiches (Les Seiches du Lac George, les Seiches longitudinales du Léman)," *A.G.*, xiv. 203.

1886. Sarasin, Ed. "Tracés Limnographiques du Lac de Zurich," *A.G.*, xvi. 210.

1887. Krümmel, O. *Handbuch der Oceanographie*, Stuttgart.

1887. Greenhill, A. G. "Wave Motion in Hydrodynamics," *Am. Jour. Math.*, ix.

1888. Krümmel, O. "Zum Problem des Euripus," *P.M.*, xi. 331.

1888. Günther, S. "Von den rhythmischen Schwankungen des Spiegels geschlossener Meeresbecken," *Mitth. d. K.K. Geog. Soc. Wien*, 497.

1888. Sieger, R. *Die Schwankungen der hocharmenischen Seen seit* 1880, Wien.

1890. Grablovitz, G. "Ricerche sulle Maree d'Ischia," *Rend. Acc. d'Lincei*, 29, 359.

1891. Grablovitz, G. "Le Isorachie della Marea nel Mediterraneo," *Rend. Acc. d. Lincei*, 1891, 135.

1891. Marinelli. "Lo Studio delle Sesse nei Laghe Italiani," *Riv. Geog. Ital.*, viii. 10.

1891. Plantamour, Ph. "Effets du Cyclone du 19 Août 1890 sur le Lac," *A.G.*, xxv. 302.

1891. Forel, F. A. "Note sur la Formule des Seiches," *A.G.*, xxv. 599.

1891. Du Boys, P. "Sur le mouvement de Balancement rythmé de l'Eau des Lacs (Seiches)," *C.R.*, cxii. 1202.

1891. Du Boys, P. "Essai Théorique sur les Seiches, avec Appendice par F. A. Forel," *A.G.*, xxv. 628.

1891. Sarasin, Ed. "Remarques sur les Seiches Binodales, à propos de l'Essai Théorique de M. du Boys," *A.G.*, xxvi. 81.

1891. Pollisadov. "Sur les changements périodiques en niveau des lacs du district de Bourowitchy, Gouvernement de Novgorod" (in Russian), *Rev. Sci. Nat.*, 293.

1892. Sarasin, Ed. "Les Seiches du Lac de Neuchâtel," *A.G.*, xxviii. 356.

1892. Burton, W. K. "Notes on Seiches observed at Hakone Lake," *Trans. Seism. Soc. Japan*, xvi. 49.

1892. Horn, A. von. "Ueber den Einfluss von Windrichtung und Luftdruck auf den Seespiegel," *A.Hy.*, xix. 498.

1892. Carson, A. "The Rise and Fall of Lake Tanganyika," *Quart. Journ. Geol. Soc.*, xlviii. 401.

1892. Grablovitz, G. "Sulle Observazioni Mareographici in Italia e specialmente su quelle fatte in Ischia," *Att. d.* 1° *Cong. Geog. Ital.*

1892. Sieger, R. "Niveauveränderungenan skandinavischen Seen und Küsten," *Verh. 9th Deutsch Geogr. Wien*, 224.

1892. Brückner, E. "Ueber Schwankungen der Seen und Meere," *Verh. 9th Deutsch Geographentag, Wien*, 1892, 209.

1892. Sziklay, J. "Oscillations de niveau du lac Balaton, 1890" (in Hungarian), *Bull. Soc. Hong. Geog.*, xix. 366.

1892. Filipof, N. "Ueber die Schwankungen des Spiegels des Kaspischen Meeres" (in Russian), *Russ. Geogr. Soc.*, xx.

1893. Sieger, R. "Seeschwankungen und Strandverschiebungen in Skandinavien," *Zeitsch. Ges. Erdk. Berlin*, xxviii.

1893. Sieger, R. "The Rise and Fall of Lake Tanganyika," *Quart. Jour. Geol. Soc.*, xlix. 579.

1893. F. A. Forel. "Die Schwankungen des Bodensees," *Schriften des Vereins für Geschichte des Bodensees*, xxii., Lindau.

1893. Perkins, E. A. "The Seiche in America," *Amer. Met. Journ.*, x. 251.

1893. Sarasin, E. "Des seiches de Neuchâtel, *A.H.*, lxxv. 38.

1894. Cholnoky, E. von. "Bericht über die Ergebnisse des selbstregistrirenden Wasserstandmessers am Plattensee" (in Hungarian), *Földr Közl*, Budapest, xxii. 148. Abrégé, *Bull. Soc. Hongr. Geogr.*, xxiii. 39.

1894. Belck, W. "Die Niveau-Schwankungen des Geoktschai-Sees," *Globus*, lxv. 301.

1894. Ule, W. "Beitrag zur Instrumentenkunde auf dem Gebiete der Seenforschung," *P.M.*, xl. 213.

1894. Perkins, E. A. "Seiches in Lake Michigan," *Am. Meteor. Jour.*

1894–5. Sarasin, E., and Pasquin, L. du. "Les Seiches du Neuchâtel," *A.G.*, xxxi. 213, xxxiii. 193; *Bull. Soc. Sci. Nat. Neuchâtel*, xxi., xxiii.

1895. Sarasin, E. "Les seiches du lac de Thoune," *A.G.*, xxxiv. 368.

1895. Dawson, W. Bell. "Notes on Secondary Undulations," *Proc. Roy. Soc. Canada*, May.

1895. Macfarlane, J. H. R. "The occurrence of Seiches in Lake Derravaragh, Co. Westmeath, 1893–4," *Sci. Proc. Roy. Dublin Soc.*, N.S., viii. 288.

1895. Lamb, H. *Hydrodynamics.* Cambridge.

1895. Forel, F. A. *Le Léman, Monographie Limnologique. Lausanne*, t. ii., sixième partie, pp. 1–288.

These pages form a treatise on Seiches, recording the work of the author and his followers during a period of more than thirty years; they must always remain the great classic of Seiche literature.

1897. Denison, F. N. "Secondary Undulations of Tide Gauges," *Proc. Can. Inst.*, i. 28.

1897. Denison, F. N. "The Origin of Tidal Secondary Undulations," *ib.*, i. 134.

1897. Denison, F. N. "The Great Lakes as a Sensitive Barometer," *ib.*, i. 55.

1897. Denison, F. N. "Seiches in Lakes Ontario and Huron," *Rep. Brit. Ass.*

1897. Cholnoky, E. *Limnologie des Plattensees.* Wien, 1897.

Besides giving an account of the seiches in a long and very shallow lake, the paper contains observations and conclusions of great importance regarding the non-rhythmic denivellation of a lake by the wind. It also describes observations of great interest on lake currents; the first, so far as I am aware, made with a self-recording apparatus.

1897. Sarasin, Éd. "Les Seiches du lac des Quatre Cantons," *A.G.*, iv. 458.

1897. Forel, F. A. "Les Seiches des Lacs et les Variations Locales de la Pression Atmosphérique," *A.G.*, iv.

1898. Darwin, G. H. *The Tides and Kindred Phenomena in the Solar System.* London.

1898. Sarasin, Éd. "Les Seiches du lac des Quatre-Cantons," *A.G.*, v. 389, vi. 382.

1898. Volterra, V. "Sul fenomeno delle seiches," *Il n. Cimento* (Pisa), viii. 270.

1898. Kleiber, W. "Studien zur Wasserstandsprognose," *Z.G.*, i. 10, 129.

1898. Denison, F. H. "Periodic Fluctuations on the Great Lakes," *Monthly Weather Rev. Washington*, xxvi. 261.

1898. Wheeler, W. H. "Undulations in Lakes and Inland Seas due to Wind and Atmospheric Pressure," *Nature*, lvii. 321.

1898. Holmsen, A. "Seiches i norske Indsjøer," *Arch. Math. Naturvid* (Kristiania), xx., No. 1, p. 28.

1898. "Steigen des Wasserspiegels im Urmia-See," *Met. Zeitschr.* (Wien), xv. 80.

1899. Schulz, K. *Beiträge zur Kentniss des Gmunden Sees*, Gymnasial program, Gmünden.

1899. Sarasin, Ed. "Les Seiches du lac des Quatre-Cantons," *A.G.*, viii. 382, 517.

1899. Forel, F. A. "Les Seiches des Lacs," *Verhandl. d. vii. internationalen Geographen-Kongresses in Berlin*, 1899.

1899. Richter, E. "Stehende Seespiegelschwankungen (Seichen) auf dem Traunsee," *P.M.*, xlv. 41.

1900. Ebert, H. "Periodische Seespiegelschwankungen beobachtet am Starnberger See," *Sitzungsberichte der Math. Phy. Classe der k. bayer. Akademie d. Wissenschaft*, xxx. 435.

Starnberg is a highly interesting example of a concave lake ($T_2/T_1 = \cdot 632$, which exceeds the corresponding ratio for a complete symmetric rectilinear lake). As its configuration is well known from Ule's *Atlas*, it is to be hoped that the Bavarian observers, to whom we already owe so much, will return to the investigation, work out the uninode and binodes, and also the seiches of higher nodality, and trace the meteorological conditions under which the various seiches occur.

1900. Bettoni, P. "Studî Limnografici sulle Sesse del Lago di Garda," *Comm. d. Ateneo d. Brescia.*

1900. Fulleborn. "Seiches on Nyassa," quoted by Forel from *Verh. d. Gesellsch. f. Erdkunde*, xxviii., Berlin, 1900.

1900. Berg, L., and Ignatov, P. "Sur les variations du niveau des lacs en Asie centrale et en Sibérie occidentale," *Russ. Geogr. Soc. St Petersburg*, xxxvi. 111.

1900. Henry, A. J. "Lake Levels and Wind Phenomena," *Monthly Weather Rev. Washington*, xxviii. 203.

1900. Forel, F. A., et Sarasin, Éd. *Les Oscillations des Lacs. Rapport présenté au Congrès international de Physique.* Paris, 1900.

1900. Sarasin, Éd. "Les Seiches du lac des Quatre-Cantons," *A.G.*, x. 454.

1901. Sarasin, Éd. "Les Seiches du lac des Quatre-Cantons," *A.G.*, xi. 161, xii. 254.

1901. Ebert, H. "Periodische Seespiegelschwankungen," *Sitzungsber. Math.-Phys. Kl. Akad. Wiss. München.*

1901. Ebert, H. "Sarasin's neues selbstregistrirendes Limnimeter," *Z.I.*, xxi. 193.

1902. Sarasin, Éd. "L'Histoire de la Théorie des Seiches," *Discours d'Ouverture de la* 85^{me} *Session Annuelle de la Soc. Helv. d. Sc. Nat. à Genève*, Sept. 8, 1902.

1902. HALBFASS, W. "Stehende Seespiegelschwankungen (Seiches) im Madüsee in Pommern" (2 parts), *Z.G.*, v. 15, vi. 65.

This Pomeranian lake is concave ($T_2/T_1 = \cdot 566$), and of comparatively regular form. HALBFASS's results are of great interest; and it would be of importance to complete the bathymetric data, so that the seiche phenomena could be more thoroughly discussed.

1902. HENRY, A. J. "Wind Velocity and Fluctuations of Water-Level on Lake Erie," *U.S. Weather Bureau Bull. J.*, No. 262.

1902. NAKAMURA, S., and YOSHIDA, Y. "On the Seiches of Lakes Biwa and Hakone," *Tokyo Phys. Math. Soc.*, No. 15, p. 115.

1903. NAKAMURA, S., and YOSHIDA, Y. "On the Seiches of Lakes Biwa and Hakone, Tokyo," *A.G.*, 559.

1903. ENDRÖS, A. *Seeschwankungen beobachtet am Chiemsee.* (Dissertation.) Traunstein, 1903.

One of the most complete examples of the exact observation of seiches known to me. Also *Z.I.*, June 1904, 180.

1903. BRUYANT. "Les Seiches du lac Pavin," *Rev. d'Auvergne.*

1903. VALENTIN, J. "Seiches in Riva on lake Garda," *Wiener Anzeiger*, 1903, p. 93. Also *A. W.*, April 3, 1903.

1903. MACLAGAN-WEDDERBURN, E. "Seiches observed in Loch Ness," *Proc. Roy. Soc. Ed.*, xxv. 1.

1904. HALBFASS, W. "Seiches oder Stehende Seespiegelschwankungen," *Naturwissenschaftliche Wochenschrift*, iii. 881. See also i. 127.

An excellent popular account of the present state of our knowledge of seiche phenomena.

1904. FOREL, F. A. "Sur les Seiches," *Ext. B.V.*, xl. 149.

1904. PALAZZO, L. "La Stazione Limnologica de Bolsena," *Boll. Soc. Geogr. Ital.*, v.

1904. HALBFASS, W. "Les Seiches du Madusee en Pomèranie," *A.G.*, xvii. 281.

1904. HALBFASS, W. "Eine bemerkens-werthe Verbesserung des Sarasinschen Limnimètre enregistreur portatif," *P.M.*, heft v.

1904. GÜNTHER, R. T. "The Limnological Stations on the Lake of Bolsena," *Nature*, lxx. 455.

1904. ENDRÖS, A. "Seiches Kleiner Wasserbecken," *P.M.*, heft xii. 294.

1904. CHRYSTAL, G. "Some Results in the Mathematical Theory of Seiches," *Proc. Roy. Soc. Ed.*, xxv. 328.

A brief abstract of part of the present paper.

1904. CROSTHWAIT, H. L. "Seiches in Lake San Martin, Patagonia," *R. Geog. Soc. Lond.*, March 1904.

1905. CHRYSTAL, G. "Some further Results in the Mathematical Theory of Seiches," *Proc. Roy. Soc. Ed.*, xxv. 637.

1905 MAGRINI, G. P. "I Recenti Studi sulle Sesse; e le Sesse nei Laghi Italiani," *Riv. Geog. Ital.*, xii.

XXVI.—On a Group of Linear Differential Equations of the 2nd Order, including Professor Chrystal's Seiche-equations.

By J. Halm, Ph.D., Lecturer on Astronomy in the University of Edinburgh.

(MS. received May 20, 1905. Read June 19, 1905. Issued separately July 31, 1905.)

It is readily seen that the two differential equations

$$(1-w^2)\frac{d^2y}{dw^2}+n(n-1)y=0 \tag{1}$$

$$(1+x^2)^2\frac{d^2y}{dx^2}+n(n+2)y=0 \tag{2}$$

which play an important rôle in Professor CHRYSTAL's mathematical theory of the Seiches, are special cases of the more general type

$$(1-w^2)\frac{d^2y}{dw^2}-(2a+1)w\frac{dy}{dw}+n(n+2a)y=0\,. \tag{3}$$

With regard to the first, the Seiche-equation, this becomes at once apparent by writing $a=-\frac{1}{2}$. Equation (2), on the other hand, which we may briefly call the STOKES-equation [see Professor CHRYSTAL's paper on "Some further Results in the Mathematical Theory of Seiches," *Proc. Roy. Soc. Edin.*, vol. xxv.] will be recognised as a special case ($a=+1$) of the equation

$$(1+x^2)^2\frac{d^2y}{dx^2}-(2a-2)x(1+x^2)\frac{dy}{dx}+n(n+2a)y=0 \tag{4}$$

which is transformed into (3) by the substitution $x=\frac{w}{(1-w^2)^{\frac{1}{2}}}$.

It appears, therefore, that the Seiche- as well as the STOKES-equation belong to the same family of differential equations whose general form is given by (3). We may write the latter also

$$\frac{d^2y}{dz^2}-2a\tan z\frac{dy}{dz}+n(n+2a)y=0 \tag{5}$$

if we substitute $w=\sin z$ or $x=\tan z$. Corresponding to this equation we have further:

$$\frac{d^2y}{dz^2}+2a\tanh z\frac{dy}{dz}+n(n+2a)y=0 \tag{6}$$

which for $w=\sinh z$ and $a=-\frac{1}{2}$ leads to the hyperbolic Seiche-equation:

$$(1+w^2)\frac{d^2y}{dw^2}+n(n-1)y=0 \tag{7}$$

and for $x=\tanh z$ and $a=+1$ to the "hyperbolic" STOKES-equation:

$$(1-x^2)^2\frac{d^2y}{dx^2}+n(n+2)y=0\,. \tag{8}$$

From (5) and (6) the solutions of the STOKES-equation may at once be derived. If we write, in accordance with Professor CHRYSTAL'S notations, $n(n+2)=4\partial^2-1$, and consider that (5) is identical with

$$\frac{d^2[y\cos z]}{dz^2}+(n+1)^2[y\cos z]=0,$$

we find the general solution

$$y=\frac{1}{\cos z}\left[\mathrm{C}.\cos(2\partial z)+\mathrm{D}\sin(2\partial z)\right],$$

or since $z=\tan^{-1}x$,

$$y=(1+x^2)^{\frac{1}{2}}\left[\mathrm{C}.\cos(2\partial\tan^{-1}x)+\mathrm{D}\sin(2\partial\tan^{-1}x)\right]. \tag{9}$$

In the same way we find from (6)

$$\frac{d^2[y\cosh z]}{dz^2}+(n^2-2n-1)[y\cosh z]=0,$$

and, writing $n^2+2n-1=4\partial^2$,

$$y=\frac{1}{\cosh z}\left[\mathrm{C}'\cos(2\partial z)+\mathrm{D}'\sin(2\partial z)\right]$$

which, since $z=\tanh^{-1}x=\frac{1}{2}\log\frac{1+x}{1-x}$, becomes

$$y=(1-x^2)^{\frac{1}{2}}\left[\mathrm{C}'\cos\left(\delta\log\frac{1+x}{1-x}\right)+\mathrm{D}'\sin\left(\partial\log\frac{1+x}{1-x}\right)\right]. \tag{10}$$

(9) and (10) are identical with (26) and (12) of Professor CHRYSTAL'S paper quoted above if $\frac{x}{a}$ is substituted for x.

It is also at once evident that if we express the STOKES-functions by means of the variable w instead of x, we find the general solutions:

$$y_1=\mathrm{A}\frac{d}{dw}\left[\cos(2\partial\sin^{-1}w)\right]+\mathrm{B}\frac{d}{dw}\left[\sin(2\partial\sin^{-1}w)\right]=\frac{d\mathrm{Y}_1}{dw},\ \text{say,}$$

and

$$y_2=\mathrm{A}'\frac{d}{dw}\left[\cos(2\partial\sinh^{-1}w)\right]+\mathrm{B}'\frac{d}{dw}\left[\sin(2\partial\sinh^{-1}w)\right]=\frac{d\mathrm{Y}_2}{dw}.$$

This result is made evident if we consider that Y_1 and Y_2 satisfy the well-known differential equations

$$(1-w^2)\frac{d^2\mathrm{Y}_1}{dw^2}-w\frac{d\mathrm{Y}_1}{dw}+4\partial^2\mathrm{Y}_1=0$$

$$(1+w^2)\frac{d^2\mathrm{Y}_2}{dw^2}+w\frac{d\mathrm{Y}_2}{dw}+4\partial^2\mathrm{Y}_2=0,$$

which, by differentiation with regard to w, assume the form:

$$(1-w^2)\frac{d^2y_1}{dw^2}-3w\frac{dy_1}{dw}+(4\partial^2-1)y_1=0$$

$$(1+w^2)\frac{d^2y_2}{dw^2}+3w\frac{dy_2}{dw}+(4\partial^2+1)y_2=0,$$

and become identical with the STOKES-equations if x is introduced instead of w.

Seeing that it is possible to refer the four differential equations of the Seiche-problem to one and the same general equation, viz., that given under (3), it may be of interest to discuss here the properties of this in many ways remarkable general type, and to derive its particular solutions. The results obtained seem, on the one hand, to be of practical value in regard to the computations involved in the evolution of the periods and nodes of the seiches, while, on the other hand, the mathematical investigation here given establishes an important relation between the Seiche-functions discovered by Professor CHRYSTAL and other functions frequently used in mathematical and physical problems, notably those of LEGENDRE, a relation which increases the importance of the Seiche-functions from the mathematical point of view. It will be recognised at once that equation (3) is a special case of the hypergeometric differential equation

$$v(1-v)\frac{d^2y}{dv^2}+[\gamma-(\alpha+\beta+1)v]\frac{dy}{dv}-\alpha\beta y=0\,, \tag{11}$$

whose solution is represented by the contour-integral

$$y=\text{const}\times\int_C u^{\alpha-\gamma}(1-u)^{\gamma-\beta-1}(u-v)^{-\alpha}du \tag{12}$$

Indeed, if we substitute $v=\frac{1-w}{2}$ and $\alpha=n+2\alpha$, $\beta=-n$, $\gamma=\alpha+\frac{1}{2}$ we have

$$(1-w^2)\frac{d^2y}{dw^2}-(2\alpha+1)\frac{dy}{dw}+n(n+2\alpha)y=0 \tag{3}$$

which, by introducing $u=\frac{1-t}{2}$ under the sign of integration in (12), is satisfied by the integral

$$y=\text{const}\times\int_C\frac{(1-t^2)^{n+\alpha-\frac{1}{2}}}{(t-w)^{n+2\alpha}}dt\,. \tag{13}$$

If we write

$$y=(1-w^2)^{\frac{1}{2}-\alpha}\cdot Y\,,$$

we find easily that Y is a solution of the equation :

$$(1-w^2)\frac{d^2Y}{dw^2}+(2\alpha-3)\frac{dY}{dw}+(n+1)(n+2\alpha-1)Y=0\,,$$

and hence, substituting in (11) and (12) $\alpha=n+1$, $\beta=1-n-2\alpha$, $\gamma=\frac{3}{2}-\alpha$:

$$Y=\text{const}\times\int_C\frac{(1-t^2)^{n+\alpha-\frac{1}{2}}}{(t-w)^{n+1}}dt\,,$$

so that we find as another solution of (3) :

$$y=\text{const}\times(1-w^2)^{\frac{1}{2}-\alpha}\int_C\frac{(1-t^2)^{n+\alpha-\frac{1}{2}}}{(t-w)^{n+1}}dt\,. \tag{14}$$

For $\alpha=\frac{1}{2}$, the integrals (13) and (14) become identical, viz.,

$$y=\text{const}\times\int_C\frac{(1-t^2)^n}{(t-w)^{n+1}}dt\,.$$

This is SCHLÄFLI's contour-integral of the LEGENDRE-functions, which thus is seen to satisfy the differential equation:

$$(1-w^2)\frac{d^2y}{dw^2}-2\frac{dy}{dw}+n(n+1)y=0\,. \tag{15}$$

Our general equation (3) includes therefore also the LEGENDRE-functions. If we denote the solution of (3) by the symbol $C_n^{\alpha}(w)$, the Seiche-functions are represented by $C_n^{-\frac{1}{2}}(w)$, the LEGENDRE-functions by $C_n^{\frac{1}{2}}(w)$, and the STOKES-functions by $C_n^{1}(w)$.

Since (3) is a special case of the hypergeometric differential equation, its integrals can be at once expressed by hypergeometric series of the type $F(\alpha, \beta, \gamma; w)$. From JACOBI's schematic table of particular solutions given in his "Untersuchungen über die Differentialgleichung der hypergeometrischen Reihe," *Crelle's Journal*, vol. lvi., we obtain thus the following 24 possible integrals of equation (3):—

GROUP I.

1. $\quad F\left(n+2a,\ -n,\ a+\tfrac{1}{2};\ \frac{1-w}{2}\right)$

2. $\left(\frac{1+w}{2}\right)^{\frac{1}{2}-a} \times F\left(\tfrac{1}{2}-n-a,\ \tfrac{1}{2}+n+a,\ a+\tfrac{1}{2};\ \frac{1-w}{2}\right)$

3. $\left(\frac{1+w}{2}\right)^{-n-2a} \times F\left(n+2a,\ \tfrac{1}{2}+n+a,\ a+\tfrac{1}{2};\ \frac{w-1}{w+1}\right)$

4. $\left(\frac{1+w}{2}\right)^{n} \times F\left(-n,\ \tfrac{1}{2}-n-a,\ a+\tfrac{1}{2};\ \frac{w-1}{w+1}\right)$

GROUP II.

1. $\left(\frac{1-w}{2}\right)^{-n-2a} \times F\left(n+2a,\ \tfrac{1}{2}+n+a,\ a+\tfrac{1}{2};\ \frac{w+1}{w-1}\right)$

2. $\left(\frac{1-w}{2}\right)^{n} \times F\left(-n,\ \tfrac{1}{2}-n-a,\ a+\tfrac{1}{2};\ \frac{w+1}{w-1}\right)$

3. $\quad F\left(n+2a,\ -n,\ a+\tfrac{1}{2};\ \frac{1+w}{2}\right)$

4. $\left(\frac{1-w}{2}\right)^{\frac{1}{2}-a} \times F\left(\tfrac{1}{2}+n+a,\ \tfrac{1}{2}-n-a,\ a+\tfrac{1}{2};\ \frac{1+w}{2}\right)$

GROUP III.

1. $\left(\frac{1-w}{2}\right)^{-n-2a} \times F\left(n+2a,\ \tfrac{1}{2}+n+a,\ 2n+2a+1;\ \frac{2}{1-w}\right)$

2. $\left(\frac{1-w}{2}\right)^{-n-a-\frac{1}{2}}\left(\frac{1+w}{2}\right)^{\frac{1}{2}-a} \times F\left(1+n,\ \tfrac{1}{2}+a+n,\ 2n+2a+1;\ \frac{2}{1-w}\right)$

3. $\left(\frac{1+w}{2}\right)^{-n-2a} \times F\left(n+2a,\ \tfrac{1}{2}+n+a,\ 2n+2a+1;\ \frac{2}{1+w}\right)$ (16)

4. $\left(\frac{1-w}{2}\right)^{\frac{1}{2}-a}\left(\frac{1+w}{2}\right)^{-n-a-\frac{1}{2}} \times F\left(\tfrac{1}{2}+n+a,\ 1+n,\ 2n+2a+1;\ \frac{2}{1+w}\right)$

Group IV.

1. $\left(\frac{1-w}{2}\right)^{n} \times \mathrm{F}\left(-n,\ \tfrac{1}{2}-n-a,\ 1-2n-2a;\ \frac{2}{1-w}\right)$

2. $\left(\frac{1-w}{2}\right)^{n+a-\frac{1}{2}}\left(\frac{1+w}{2}\right)^{\frac{1}{2}-a} \times \mathrm{F}\left(1-n-2a,\ \tfrac{1}{2}-n-a,\ 1-2n-2a;\ \frac{2}{1-w}\right)$

3. $\left(\frac{1+w}{2}\right)^{n} \times \mathrm{F}\left(-n,\ \tfrac{1}{2}-n-a,\ 1-2n-2a;\ \frac{2}{1+w}\right)$

4. $\left(\frac{1-w}{2}\right)^{\frac{1}{2}-a}\left(\frac{1+w}{2}\right)^{n+a-\frac{1}{2}} \times \mathrm{F}\left(\tfrac{1}{2}-n-a,\ 1-n-2a,\ 1-2n-2a;\ \frac{2}{1+w}\right)$

Group V.

1. $\left(\frac{1-w}{2}\right)^{\frac{1}{2}-a} \times \mathrm{F}\left(\tfrac{1}{2}+n+a,\ \tfrac{1}{2}-n-a,\ \tfrac{3}{2}-a;\ \frac{1-w}{2}\right)$

2. $\left(\frac{1-w}{2}\right)^{\frac{1}{2}-a}\left(\frac{1+w}{2}\right)^{\frac{1}{2}-a} \times \mathrm{F}\left(1-n-2a,\ 1+n,\ \tfrac{3}{2}-a;\ \frac{1-w}{2}\right)$

3. $\left(\frac{1-w}{2}\right)^{\frac{1}{2}-a}\left(\frac{1+w}{2}\right)^{-\frac{1}{2}-n-a} \times \mathrm{F}\left(\tfrac{1}{2}+n+a,\ 1+n,\ \tfrac{3}{2}-a;\ \frac{w-1}{w+1}\right)$

4. $\left(\frac{1-w}{2}\right)^{\frac{1}{2}-a}\left(\frac{1+w}{2}\right)^{n+a-\frac{1}{2}} \times \mathrm{F}\left(\tfrac{1}{2}-n-a,\ 1-n-2a,\ \tfrac{3}{2}-a;\ \frac{w-1}{w+1}\right)$

Group VI.

1. $\left(\frac{1-w}{2}\right)^{n+a-\frac{1}{2}}\left(\frac{1+w}{2}\right)^{\frac{1}{2}-a} \times \mathrm{F}\left(\tfrac{1}{2}-n-a,\ 1-n-2a,\ \tfrac{3}{2}-a;\ \frac{w+1}{w-1}\right)$

2. $\left(\frac{1-w}{2}\right)^{-n+a-\frac{1}{2}}\left(\frac{1+w}{2}\right)^{\frac{1}{2}-a} \times \mathrm{F}\left(\tfrac{1}{2}+n+a,\ 1+n,\ \tfrac{3}{2}-a;\ \frac{w+1}{w-1}\right)$

3. $\left(\frac{1+w}{2}\right)^{\frac{1}{2}-a}\left(\frac{1-w}{2}\right)^{\frac{1}{2}-a} \times \mathrm{F}\left(1-n-2a,\ 1+n,\ \tfrac{3}{2}-a;\ \frac{1+w}{2}\right)$

4. $\left(\frac{1+w}{2}\right)^{\frac{1}{2}-a} \times \mathrm{F}\left(\tfrac{1}{2}-n-a,\ \tfrac{1}{2}+n+a,\ \tfrac{3}{2}-a;\ \frac{1+w}{2}\right)$

That these integrals cannot all exist at the same time is evident, the convergence of the F-series depending on the values of n and c. It is easy, however, to find in each single case those series which represent convergent solutions of the differential equation. With regard to this point I may refer to Jacobi's treatise, from which the conditions of convergence may at once be obtained. We notice that the above solutions may be arranged in pairs, which differ only by $-w$ being substituted for the positive value. The necessity of the existence of such pairs is obvious, since the differential equation (3) remains unaltered if $-w$ is substituted for $+w$. Since we are also permitted to write $-n-2c$ for n without changing the equation (3), we have on the whole the following eight particular solutions, from which the others are obtained by the substitutions just mentioned.

$$\mathrm{F}\left(n+2a,\ -n,\ a+\tfrac{1}{2};\ \frac{1-w}{2}\right)$$

$$\left(\frac{1+w}{2}\right)^{\frac{1}{2}-a}\mathrm{F}\left(\tfrac{1}{2}+n+a,\ \tfrac{1}{2}-n-a,\ a+\tfrac{1}{2};\ \frac{1-w}{2}\right)$$

$$\left(\frac{1+w}{2}\right)^{n}\mathrm{F}\left(-n,\ \tfrac{1}{2}-n-a,\ a+\tfrac{1}{2};\ \frac{w-1}{w+1}\right)$$

$$\left(\frac{1+w}{2}\right)^{n}\mathrm{F}\left(-n,\ \tfrac{1}{2}-n-a,\ 1-2n-2a;\ \frac{2}{1+w}\right)$$

$$\left(\frac{1-w}{2}\right)^{\frac{1}{2}-a}\left(\frac{1+w}{2}\right)^{n+a-\frac{1}{2}}\mathrm{F}\left(\tfrac{1}{2}-n-a,\ 1-n-2a,\ 1-2n-2a;\ \frac{2}{1+w}\right) \qquad (16a)$$

$$\left(\frac{1-w}{2}\right)^{\frac{1}{2}-a}\times\mathrm{F}\left(\tfrac{1}{2}+n+a,\ \tfrac{1}{2}-n-a,\ \tfrac{3}{2}-a;\ \frac{1-w}{2}\right)$$

$$\left(\frac{1-w}{2}\right)^{\frac{1}{2}-a}\left(\frac{1+w}{2}\right)^{\frac{1}{2}-a}\times\mathrm{F}\left(1-n-2a,\ 1+n,\ \tfrac{3}{2}-a;\ \frac{1-w}{2}\right)$$

$$\left(\frac{1-w}{2}\right)^{\frac{1}{2}-a}\left(\frac{1+w}{2}\right)^{n+a-\frac{1}{2}}\times\mathrm{F}\left(1-n-2a,\ \tfrac{1}{2}-n-a,\ \tfrac{3}{2}-a;\ \frac{w-1}{w+1}\right)$$

Note added on June 30.—The sixth integral in (16a) agrees with Professor CHRYSTAL's Lake function, which is obtained by substituting $a=-\frac{1}{2}$, $n(n-1)=c$ and $\frac{1-w}{2}=z$, so that $\mathrm{L}(c\,,\,z)=z\,\mathrm{F}(n\,,\,1-n\,,\,2\,;\,z)$. The other corresponding integral is represented by No. 1 of (16a), viz., $\mathrm{F}(n-1\,,\,n\,,\,0\,;\,z)$, but it belongs to the exceptional class $\gamma=0$ of the hypergeometric series, and has a logarithmic form (see Professor CHRYSTAL "On the Hydrodynamical Theory of Seiches," § 41). The corresponding solution of the LEGENDRE-function ($a=\frac{1}{2}$) is $\mathrm{F}\left(n+1\,,\,-n\,,\,1\,;\,\frac{1-w}{2}\right)$, a well-known expression for the LEGENDRE-function of the first kind. (See WHITTAKER, *Modern Analysis*, § 118.) In this case Nos. 1 and 6 of (16a) are identical.

Reverting to equation (11), we notice that it may be transformed into (3) by still another substitution, viz., by $v=w^2$, $\alpha=\frac{n}{2}+a$, $\beta=-\frac{n}{2}$, and $\gamma=\frac{1}{2}$. Hence we obtain 24 other particular integrals of the following type :—

GROUP I.

1. $\mathrm{F}\left(\frac{n}{2}+a,\ -\frac{n}{2},\ \frac{1}{2};\ w^2\right)$

2. $(1-w^2)^{\frac{1}{2}-a}\times\mathrm{F}\left(\frac{1}{2}-\frac{n}{2}-a,\ \frac{1}{2}+\frac{n}{2},\ \frac{1}{2};\ w^2\right)$

3. $(1-w^2)^{-\frac{n}{2}-a}\times\mathrm{F}\left(\frac{n}{2}+a,\ \frac{1}{2}+\frac{n}{2},\ \frac{1}{2};\ \frac{w^2}{w^2-1}\right)$

4. $(1-w^2)^{\frac{n}{2}}\times\mathrm{F}\left(-\frac{n}{2},\ \frac{1}{2}-\frac{n}{2}-a,\ \frac{1}{2};\ \frac{w^2}{w^2-1}\right)$

GROUP II.

1. $w^{-n-2a} \times \mathrm{F}\left(\frac{n}{2}+a,\ \frac{1}{2}+\frac{n}{2}+a,\ \frac{1}{2}+a;\ \frac{w^2-1}{w^2}\right)$
2. $w^{n} \times \mathrm{F}\left(-\frac{n}{2},\ \frac{1}{2}-\frac{n}{2},\ \frac{1}{2}+a;\ \frac{w^2-1}{w^2}\right)$
3. $\mathrm{F}\left(\frac{n}{2}+a,\ -\frac{n}{2},\ \frac{1}{2}+a;\ (1-w^2)\right)$
4. $w \times \mathrm{F}\left(\frac{1}{2}+\frac{n}{2}+a,\ \frac{1}{2}-\frac{n}{2},\ \frac{1}{2}+a;\ (1-w^2)\right)$

GROUP III.

1. $w^{-n-2a} \times \mathrm{F}\left(\frac{n}{2}+a, \frac{1}{2}+\frac{n}{2}+a,\ n+a+1;\ \frac{1}{w^2}\right)$
2. $w^{-n-1}(1-w^2)^{\frac{1}{2}-a} \times \mathrm{F}\left(1+\frac{n}{2}, \frac{1}{2}+\frac{n}{2},\ n+a+1;\ \frac{1}{w^2}\right)$
3. $(1-w^2)^{-\frac{n}{2}-a} \times \mathrm{F}\left(\frac{n}{2}+a, \frac{1}{2}+\frac{n}{2},\ n+a+1;\ \frac{1}{1-w^2}\right)$
4. $w(1-w^2)^{-\frac{n}{2}-a-\frac{1}{2}} \times \mathrm{F}\left(\frac{1}{2}+\frac{n}{2}+a,\ 1+\frac{n}{2},\ n+a+1;\ \frac{1}{1-w^2}\right)$

GROUP IV.

1. $w^{n} \times \mathrm{F}\left(-\frac{n}{2}, \frac{1}{2}-\frac{n}{2},\ 1-n-a;\ \frac{1}{w^2}\right)$ (17)
2. $w^{n+2a-1}(1-w^2)^{\frac{1}{2}-a} \times \mathrm{F}\left(1-\frac{n}{2}-a, \frac{1}{2}-\frac{n}{2}-a,\ 1-n-a;\ \frac{1}{w^2}\right)$
3. $(1-w^2)^{\frac{n}{2}} \times \mathrm{F}\left(-\frac{n}{2}, \frac{1}{2}-\frac{n}{2}-a,\ 1-n-a;\ \frac{1}{1-w^2}\right)$
4. $w(1-w^2)^{\frac{n}{2}-\frac{1}{2}} \times \mathrm{F}\left(\frac{1}{2}-\frac{n}{2},\ 1-\frac{n}{2}-a,\ 1-n-a;\ \frac{1}{1-w^2}\right)$

GROUP V.

1. $w \cdot \times \mathrm{F}\left(\frac{1}{2}+\frac{n}{2}+a, \frac{1}{2}-\frac{n}{2},\ \frac{3}{2};\ w^2\right)$
2. $w(1-w^2)^{\frac{1}{2}-a} \times \mathrm{F}\left(1-\frac{n}{2}-a,\ 1+\frac{n}{2},\ \frac{3}{2};\ w^2\right)$
3. $w(1-w^2)^{-\frac{n}{2}-a-\frac{1}{2}} \times \mathrm{F}\left(\frac{1}{2}+\frac{n}{2}+a,\ 1+\frac{n}{2},\ \frac{3}{2};\ \frac{w^2}{w^2-1}\right)$
4. $w(1-w^2)^{\frac{n}{2}-\frac{1}{2}} \times \mathrm{F}\left(\frac{1}{2}-\frac{n}{2},\ 1-\frac{n}{2}-a,\ \frac{3}{2};\ \frac{w^2}{w^2-1}\right)$

GROUP VI.

1. $w^{n+2a-1}(1-w^2)^{\frac{1}{2}-a} \times \mathrm{F}\left(\frac{1}{2}-\frac{n}{2}-a,\ 1-\frac{n}{2}-a,\ \frac{3}{2}-a;\ \frac{w^2-1}{w^2}\right)$
2. $w^{-n-1}(1-w^2)^{\frac{1}{2}-a} \times \mathrm{F}\left(\frac{1}{2}+\frac{n}{2},\ 1+\frac{n}{2},\ \frac{3}{2}-a;\ \frac{w^2-1}{w^2}\right)$
3. $(1-w^2)^{\frac{1}{2}-a} \times \mathrm{F}\left(\frac{1}{2}-\frac{n}{2}-a, \frac{1}{2}+\frac{n}{2},\ \frac{3}{2}-a;\ (1-w^2)\right)$
4. $w(1-w^2)^{\frac{1}{2}-a} \times \mathrm{F}\left(1-\frac{n}{2}-a,\ 1+\frac{n}{2},\ \frac{3}{2}-a;\ (1-w^2)\right)$

The first of Group I. corresponds to Professor CHRYSTAL's Seiche Cosine, and the first of Group V. to the Seiche Sine. Since for the Seiche-functions $\alpha = -\frac{1}{2}$, we have

$$\text{Seiche Cosine} = \mathrm{F}\left(\frac{n-1}{2},\ -\frac{n}{2},\ \tfrac{1}{2};\ w^2\right)$$

$$\text{Seiche Sine} = w\,\mathrm{F}\left(\frac{n}{2},\ \frac{1}{2}-\frac{n}{2},\ \tfrac{3}{2};\ w^2\right)$$

or

$$\text{Seiche Cosine} = 1 - \frac{n(n-1)}{1.2}w^2 + \frac{n(n-1)[n(n-1)-1.2]}{1.2.3.4}w^4 - \frac{n(n-1)[n(n-1)-1.2][n(n-1)-3.4]}{1.1\times 3.4\times 5.6}w^6 + \ldots.$$

$$\text{Seiche Sine} = w - \frac{n(n-1)}{2.3}w^3 + \frac{n(n-1)[n(n-1)-2.3]}{2.3\times 4.5}w^5 - \frac{n(n-1)[n(n-1)-2.3][n(n-1)-4.5]}{2.3\times 4.5\times 6.7}w^7 + \ldots.$$

or finally, if we write for $n(n-1)$, which is the factor of y in the Seiche-equation, the symbol c, in accordance with Professor CHRYSTAL's notation:

$$\text{Seiche Cosine} = 1 - \frac{c}{1.2}w^2 + \frac{c(c-1.2)}{1.2\times 3.4}w^4 - \frac{c(c-1.2)(c-3.4)}{1.2\cdot\times 3.4\times 5.6}w^6 + \ldots.$$

$$\text{Seiche Sine} = w - \frac{c}{2.3}w^3 + \frac{c(c-2.3)}{2\cdot 3\times 4.5}w^5 - \frac{c(c-2.3)(c-4.5)}{2.3\times 4.5\times 6.7}w^7 + \ldots.$$

Let us denote generally

$$\mathrm{Cos}_a(w) = \mathrm{F}\left(\frac{n}{2}+a,\ -\frac{n}{2},\ \tfrac{1}{2};\ w^2\right)$$

$$\mathrm{Sin}_a(w) = w\mathrm{F}\left(\frac{1}{2}+\frac{n}{2}+a,\ \frac{1}{2}-\frac{n}{2},\ \tfrac{3}{2};\ w^2\right) \qquad (18)$$

then we have:

$$\mathrm{Cos}_{-\frac{1}{2}}(w) = \text{Seiche Cosine} = 1 - \frac{(c+1.0)}{1.2}w^2 + \frac{(c+1.0)(c-1.2)}{1.2\times 3.4}w^4 - \frac{(c+1.0)(c-1.2)(c-3.4)}{1.2\times 3.4\times 5.6}w^6 + \ldots.$$
$$c = n(n-1)$$

$$\mathrm{Cos}_0(w) = \mathrm{Cos}(n\sin^{-1}w) = 1 - \frac{(c+0.0)}{1.2}w^2 + \frac{(c+0.0)(c-2.2)}{1.2\times 3.4}w^4 - \frac{(c+0.0)(c-2.2)(c-4.4)}{1.2\times 3.4\times 5.6}w^6 + \ldots$$
$$c = n^2$$

$$\mathrm{Cos}_{\frac{1}{2}}(w) = \text{Legendre Cosine} = 1 - \frac{(c-1.0)}{1.2}w^2 + \frac{(c-1.0)(c-3.2)}{1.2\times 3.4}w^4 - \frac{(c-1.0)(c-3.2)(c-5\cdot 4)}{1.2\times 3.4\times 5.6}w^6 \ldots.$$
$$c = n(n+1)$$

$$\mathrm{Cos}_1(w) = \text{Stokes Cosine} = 1 - \frac{(c-2.0)}{1.2}w^2 + \frac{(c-2.0)(c-4.2)}{1.2\times 3.4}w^4 - \frac{(c-2.0)(c-4.2)(c-6.4)}{1.2\times 3.4\times 5\times 6}w^6 \ldots.$$
$$c = n(n+2) \qquad (19)$$

$$\mathrm{Sin}_{-\frac{1}{2}}(w) = \text{Seiche Sine} = w - \frac{(c-0.1)}{2.3}w^3 + \frac{(c-0.1)(c-2.3)}{2.3\times 4.5}w^5 - \frac{(c-0.1)(c-2.3)(c-4.5)}{2.3\times 4.5\times 6.7}w^7 \ldots.$$
$$c = n(n-1)$$

$$\mathrm{Sin}_0(w) = \frac{1}{n}\sin(n\sin^{-1}w) = w - \frac{(c-1.1)}{2.3}w^3 + \frac{(c-1.1)(c-3.3)}{2.3\times 4.5}w^5 - \frac{(c-1.1)(c-3.3)(c-5.5)}{2.3\times 4.5\times 6.7}w^7 \ldots.$$
$$c = n^2$$

$$\mathrm{Sin}_1(w) = \text{Legendre Sine} = w - \frac{(c-2.1)}{2.3}w^3 + \frac{(c-2.1)(c-4.3)}{2.3\times 4.5}w^5 - \frac{(c-2.1)(c-4.3)(c-6.5)}{2.3\times 4.5\times 6.7}w^7 \ldots.$$
$$c = n(n+1)$$

$$\mathrm{Sin}_1(w) = \text{Stokes Sine} = w - \frac{(c-3.1)}{2.3}w^3 + \frac{(c-3.1)(c-5\cdot 3)}{2.3\times 4.5}w^5 - \frac{(c-3.1)(c-5.3)(c-7.5)}{2.3\times 4.5\times 6.7}w^7 + \ldots.$$
$$c = n(n+2)$$

From (18) we conclude that

$$\mathrm{Cos}_a(1) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-a)}{\Gamma\left(\frac{1}{2}-\frac{n}{2}-a\right)\Gamma\left(\frac{1}{2}+\frac{n}{2}\right)} \qquad (20)$$

$$\mathrm{Sin}_a(1) = \frac{1}{2}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-a)}{\Gamma\left(1-\frac{n}{2}-a\right)\Gamma\left(1+\frac{n}{2}\right)}$$

Hence

$$\mathrm{Cos}_0(1) = \cos n\frac{\pi}{2} = \frac{(\Gamma(\frac{1}{2}))^2}{\Gamma\left(\frac{1}{2}-\frac{n}{2}\right)\Gamma\left(\frac{1}{2}+\frac{n}{2}\right)}\ ;\ \Gamma\left(\frac{1}{2}-\frac{n}{2}\right)\Gamma\left(\frac{1}{2}+\frac{n}{2}\right) = \frac{\pi}{\cos\frac{n}{2}\pi}$$

$$\mathrm{Sin}_0(1) = \frac{1}{n}\sin n\frac{\pi}{2} = \frac{(\Gamma(\frac{1}{2}))^2}{n\Gamma\left(1-\frac{n}{2}\right)\Gamma\left(\frac{n}{2}\right)}\ ;\ \Gamma\left(1-\frac{n}{2}\right)\Gamma\left(\frac{n}{2}\right) = \frac{\pi}{\sin\frac{n}{2}\pi}$$

which are well-known relations between the Γ-functions.

From (20) we see at once that

$$\mathrm{Cos}_a(1) = 0, \text{ if } n(n+2a) = 1.(1-2a),\ 3(3-2a),\ 5(5-2a)\ \ldots\ldots$$
$$\mathrm{Sin}_a(1) = 0, \text{ if } n(n+2a) = 2\,(2-2a),\ 4(4-2a),\ 6(6-2a)\ \ldots\ldots$$

Applying this result to the Seiche-functions, we find

$$\mathrm{Cos}_{-\frac{1}{2}}(c, 1) = 0 \text{ for } c = 1.2, 3.4, 5.6, \ldots\ldots$$
$$\mathrm{Sin}_{-\frac{1}{2}}(c, 1) = 0 \text{ for } c = 2.3, 4.5, 6.7, \ldots\ldots$$

If in equation (3) n represents a positive integer, the functions $\mathrm{C}_n^a(w)$ have a peculiar significance. They are then the coefficients of the powers h^n in the series

$$(1-2hw+h^2)^{-a} = \sum_0^\infty h^n \mathrm{C}_n^a(w).$$

Now, according to an important theorem

$$\frac{d^n f(w)}{dw^n} = \frac{n!}{2\pi i}\int_C \frac{f(t)}{(t-w)^{n+1}}dt,$$

if $f(t)$ represents a function which is regular within the contour C (see WHITTAKER, *Modern Analysis*, p. 53). Since we had before (by (13) and (14)) that

$$\mathrm{C}_n^a(w) = \text{const} \times \int_C \frac{(1-t^2)^{n+a-\frac{1}{2}}dt}{(t-w)^{n+2a}} = \text{const}\,(1-w^2)^{\frac{1}{2}-a}\int_C \frac{(1-t^2)^{n+a-\frac{1}{2}}}{(t-w)^{n+1}}dt,$$

we notice that $\mathrm{C}_n^a(w)$ is proportional to

$$\frac{d^{n+2a-1}}{dw^{n+2a-1}}\left\{(1-w^2)^{n+a-\frac{1}{2}}\right\}$$

and also to

$$(1-w^2)^{\frac{1}{2}-a}\frac{d^n}{dw^n}\left\{(1-w^2)^{n+a-\frac{1}{2}}\right\}.$$

We verify without difficulty the following relations :—

$$\mathrm{C}_n^a(w) = (-2)^n \frac{a(a+1)(a+2)\ \ldots\ (a+n-1)}{1.2.3\ \ldots\ (2a+2n-1)}(-1)^{a-\frac{1}{2}}\frac{d^{n+2a-1}}{dw^{n+2a-1}}\left\{(1-w^2)^{n+a-\frac{1}{2}}\right\}$$
$$= (-2)^n \frac{a(a+1)(a+2)\ \ldots\ (a+n-1)}{n!\,(2n+2a-1)(2n+2a-2)\ \ldots\ (n+2a)}(1-w^2)^{\frac{1}{2}-a}\frac{d^n}{dw^n}\left\{(1-w^2)^{n+a-\frac{1}{2}}\right\} \qquad (21)$$

The Seiche-polynomials $C_n^{-\frac{1}{2}}(w)$ are therefore represented by the coefficients of the ascending powers in the series

$$(1-2hw+h^2)^{\frac{1}{2}} = \sum_0^\infty h^n C_n^{-\frac{1}{2}}(w)$$

where

$$C_n^{-\frac{1}{2}}(w) = (-1)^{n-1}\frac{1.3.5.7\ .\ .\ .\ .\ (2n-3)}{1.2.3\ .\ .\ .\ .\ (2n-2)}\frac{d^{n-2}}{dw^{n-2}}\left\{(1-w^2)^{n-1}\right\}$$

$$= \frac{(-1)^{n-1}}{(n-1)n}\cdot\frac{1.3.5.7\ .\ .\ .\ .\ (2n-3)}{1.2.3\ .\ .\ .\ .\ (2n-2)}(1-w^2)\frac{d^n}{dw^n}\left\{(1-w^2)^{n-1}\right\} \qquad (22)$$

The following relations between contiguous C-functions may be mentioned (see WHITTAKER, *l.c.*, p. 236).

$$wC_{n-1}^{1+a}(w) - C_{n-2}^{1+a}(w) = \frac{n}{2a}C_n^a(w)$$

$$\frac{dC_n^a(w)}{dw} = 2aC_{n-1}^{1+a}(w) \qquad (23)$$

$$C_n^{1+a}(w) - wC_{n-1}^{1+a}(w) = \frac{n+2a}{2a}C_n^a(w)$$

$$nC_n^a(w) = (n-1+2a)wC_{n-1}^a(w) - 2a(1-w^2)C_{n-2}^{a-1}(w).$$

In the case $a=0$ the functions $C_n^0(w)$ are of course zero, since

$$(1-2hw+h^2)^0 = 1.$$

But it can be shown that the limiting values of $\frac{C_n^0(w)}{a}$ $(\lim a=0)$ represent the coefficients in the expansion

$$\tfrac{1}{2}\log(1-2hw+h^2),$$

and we derive the following relation:

$$\log\sqrt{1-2h\sin z+h^2} = \sum_{n=1}^\infty(-1)^n\left[\frac{h^{2n-1}}{2n-1}\sin(2n-1)z - \frac{h^{2n}}{2n}\cos 2nz\right] \qquad (24)$$

$$w = \sin z.$$

The relations (21) are of particular interest in the Seiche-theory because they lead to elegant expressions for the horizontal and vertical displacements ξ and ζ. If we write the Seiche-equation referring to a parabolic concave lake

$$(1-w^2)\frac{d^2P}{dw^2} + n(n-1)P = 0$$

we have, using Professor CHRYSTAL's notations,

$$(-1)^{n-1}2.4\ .\ .\ .\ .\ (2n-2).\ \overline{\sin n_i t} = -\frac{2A}{h(1-w^2)}\frac{d^{n-2}}{dw^{n-2}}\left\{(1-w^2)^{n-1}\right\} = \frac{2A}{hn(n-1)}\frac{d^n}{dw^n}\left\{(1-w^2)^{n-1}\right\}$$

$$(-1)^{n-1}2.4\ .\ .\ .\ .\ (2n-2)\ \frac{\zeta}{\sin n_i t} = \frac{2A}{a}\frac{d^{n-1}}{dw^{n-1}}\left\{(1-w^2)^{n-1}\right\}$$

Uninodal Seiche.

$$n=2\,,\ n(n-1)=1.2$$

$$2\xi = -\frac{2A}{2h}\cdot\frac{d^2}{dw^2}\left\{(1-w^2)\right\}\sin n_1 t\,,\ \xi=\frac{A}{h}\sin n_1 t$$

$$2\zeta = -\frac{2A}{a}\cdot\frac{d}{dw}\left\{(1-w^2)\right\}\sin n_1 t\,,\ \zeta=\frac{2Aw}{a}\sin n_1 t\,.$$

Binodal Seiche.

$$n=3\,,\ n(n-1)=2.3$$

$$2.4\xi=\frac{2A}{6h}\frac{d^3}{dw^3}\left\{(1-w^2)^2\right\}\sin n_2 t\,,\ \xi=\frac{A}{h}w\sin n_2 t$$

$$2.4\zeta=\frac{2A}{a}\frac{d^2}{dw^2}\left\{(1-w^2)^2\right\}\sin n_2 t\,,\ \zeta=\frac{A}{a}(3w^2-1)\sin n_2 t\,.$$

Trinodal Seiche.

$$n=4\,,\ n(n-1)=4.3$$

$$2.4.6\ \xi=-\frac{A}{6}\frac{d^4}{dw^4}\left\{(1-w^2)^3\right\}\sin n_3 t\,;\ \xi=-\frac{A}{4h}(1-5w^2)\sin n_3 t \qquad (25)$$

$$2.4.6\ \zeta=-\frac{2A}{a}\frac{d^3}{dw^3}\left\{(1-w^2)^3\right\}\sin n_3 t\,;\ \zeta=-\frac{A}{4a}(12-20w^2)w\sin n_3 t\,.$$

Quadrinodal Seiche.

$$n=5\,,\ n(n-1)=5.4$$

$$2.4.6.8\ \xi=\frac{A}{10h}\cdot\frac{d^5}{dw^5}\left\{(1-w^2)^4\right\}\sin n_4 t\,;\ \xi=\frac{A}{4h}w(7w^2-3)\sin n_4 t$$

$$2.4.6.8\ \zeta=\frac{2A}{a}\cdot\frac{d^4}{dw^4}\left\{(1-w^2)^4\right\}\sin n_4 t\,;\ \zeta=\frac{A}{4a}(35w^4-30w^2+3)\sin n_4 t\,.$$

The positions of the nodes are found from the equation $\zeta=0$, and hence $\frac{d^n}{dw^n}\left\{(1-w^2)^n\right\}=0$ for n-nodal seiche, and since this differential-quotient is proportional to the Legendre-polynomial $P_n(w)$, the nodes are also determined by the equation

$$P_n(w)=0\,,$$

where $P_n(w)$ may be defined as the coefficient of h^n in the expansion

$$(1-2hw+h^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}P_n(w)h^n\,.$$

Turning now to the hyperbolic Seiche-functions, we obtain convenient expressions in the form of hypergeometric series by substituting in equation (3) wi for w and $ni-a$ for n, i being the imaginary unit root $\sqrt{-1}$. We have then

$$(1+w^2)\frac{d^2y}{dw^2}+(2a+1)w\frac{dy}{dw}+(n^2+a^2)y=0\,, \qquad (26)$$

the particular solutions of which may at once be taken from (17). Most of these series

are, however, unsuitable on account of the presence of complex factors. The hyperbolic Sine- and Cosine-functions are again represented by the first series of Groups I. and V. These series contain only real terms, as was shown already by Professor CHRYSTAL. We find for the hyperbolic Cosine-function:

$$\mathrm{F}\left(\frac{a}{2}+\frac{ni}{2},\ \frac{a}{2}-\frac{ni}{2},\ \tfrac{1}{2};\ -w^2\right)$$

$$=1-\frac{n^2+a^2}{1.2}w^2+\frac{(n^2+a^2)\left[(n^2+a^2)+4(a+1)\right]}{1.2.3.4}w^4-\frac{(n^2+c^2)\left[n^2+a^2+4(a+1)\right]\left[n^2+a^2+8(a+2)\right]}{1.2.3.4.5.6}w^6-\ldots.$$

The Seiche Cosine is obtained by substituting $a=-\frac{1}{2}$ and $n^2+\frac{1}{4}=c$, hence

$$\text{hyp. Seiche Cosine } \mathfrak{C}(c,w)=1-\frac{c}{1.2}w^2+\frac{c(c+1.2)}{1.2.3.4}w^4-\frac{c(c+1.2)(c+3.4)}{1.2.3.4.5.6}w^6+\ldots. \tag{27}$$

while for the hyperbolic Seiche Sine we have

$$\mathfrak{S}(c,w)=w\mathrm{F}\left(\frac{1}{4}+\frac{ni}{2},\ \frac{1}{4}-\frac{ni}{2},\ \tfrac{3}{2};\ -w^2\right)$$

$$=w-\frac{c}{2.3}w^3+\frac{c(c+2.3)}{2.3.4.5}w^5-\frac{c(c+2.3)(c+4.5)}{2.3.4.5.6.7}w^7+\ldots\ldots$$

Similar expressions are obtained for the corresponding hyperbolic LEGENDRE- and STOKES-functions. The series, by which the functions are represented, converge very slowly. For this reason the theory of convex lakes with parabolic floor has so far remained incomplete, chiefly owing to the difficulty in determining the roots of the equations $\mathfrak{C}(c\ ,\ 1)=0$ and $\mathfrak{S}(c\ ,\ 1)=0$. To avoid this difficulty Professor CHRYSTAL, in his second communication, proposed a different assumption as to the contour of the floor of the lakes, by which he was enabled to express the problem by the solutions of the STOKES-equation

$$(1\pm x^2)^2\frac{d^2y}{dx^2}+cy=0,$$

which, as we have seen, are represented by elementary transcendents. In this case, curiously, the convex lake offers the least difficulty, doubtless owing to the fact that its equation belongs to the class which also contains the more tractable parabolic concave lakes. A disadvantage of the STOKES-equations, however, is that the quartic lake profile seems to be a less close approximation to the actual conditions than the parabolic. Even apart from this, it is certainly of importance to discuss the problem on at least two different assumptions in order to estimate the influence of the form of the lake on the periods and nodes. Now the preceding investigation enables us to find any of the infinite number of roots of the equations $\mathfrak{C}(c\ ,\ 1)=0$ and $\mathfrak{S}(c\ ,\ 1)=0$ with at least a sufficient degree of approximation, and thus places us in a position to find also without any difficulty the periods and nodes of the seiches in lakes with *parabolic convex* floor.

From equation (26) we see that the Seiche-equation, for which $\alpha = -\frac{1}{2}$, lies between the two equations :

$$\left.\begin{aligned} (1+w^2)\frac{d^2y}{dw^2} - w\frac{dy}{dw} + (n^2+1)y = 0 \qquad & c_{-1} = n^2+1 \\ (1+w^2)\frac{d^2y}{dw^2} + w\frac{dy}{dw} + n^2 y = 0 \qquad & c_0 = n^2 \end{aligned}\right\} \tag{28}$$

The particular solutions of these corresponding to the hyperbolic Seiche Cosine and Seiche Sine are respectively :

$$\left.\begin{aligned} \mathfrak{C}_{-1}(w) &= -\frac{w}{n}\sin\left[n\log\left(w+\sqrt{1+w^2}\right)\right] + \sqrt{1+w^2}\cos\left[n\log\left(w+\sqrt{1+w^2}\right)\right] \\ \mathfrak{C}_0(w) &= \cos\left[n\log\left(w+\sqrt{1+w^2}\right)\right] \end{aligned}\right\}$$

and

$$\left.\begin{aligned} \left(n+\frac{1}{n}\right)\mathfrak{S}_{-1}(w) &= \frac{w}{n}\cos\left[n\log\left(w+\sqrt{1+w^2}\right)\right] + \sqrt{1+w^2}\sin\left[n\log\left(w+\sqrt{1+w^2}\right)\right] \\ n\,\mathfrak{S}_0(w) &= \sin\left[n\log\left(w+\sqrt{1+w^2}\right)\right] \end{aligned}\right\} \tag{29}$$

We may also at once write down the corresponding particular integrals of the two further differential equations ($\alpha = 1$ and 2) :

$$\begin{aligned} (1+w^2)\frac{d^2y}{dw^2} + 3w\frac{dy}{dw} + (n^2+1)y = 0 \qquad & c_1 = n^2+1 \\ (1+w^2)\frac{d^2y}{dw^2} + 5w\frac{dy}{dw} + (n^2+4)y = 0\,, \qquad & c_2 = n^2+4 \end{aligned} \tag{30}$$

which are respectively :

$$\begin{aligned} \mathfrak{C}_1(w) &= (1+w^2)^{-\frac{1}{2}}\cos\left[n\log\left(w+\sqrt{1+w^2}\right)\right] \\ \mathfrak{C}_2(w) &= (1+w^2)^{-\frac{3}{2}}\left\{\sqrt{1+w^2}\cos\left[n\log\left(w+\sqrt{1+w^2}\right)\right] - \frac{w}{n}\sin\left[n\log\left(w+\sqrt{1+w^2}\right)\right]\right\} \end{aligned}$$

and (31)

$$\begin{aligned} n\,\mathfrak{S}_1(w) &= (1+w^2)^{-\frac{1}{2}}\sin\left[n\log\left(w+\sqrt{1+w^2}\right)\right] \\ \left(n+\frac{1}{n}\right)\mathfrak{S}_2(w) &= (1+w^2)^{-\frac{3}{2}}\left\{\sqrt{1+w^2}\sin\left[n\log\left(w+\sqrt{1+w^2}\right)\right] + \frac{w}{n}\cos\left[n\log\left(w+\sqrt{1+w^2}\right)\right]\right\}. \end{aligned}$$

On the other hand, the solutions (29) and (31) are expressed by the following hypergeometric series :

$\mathfrak{C}$-FUNCTIONS.

$$1 - \frac{c_{-1}}{1.2}w^2 + \frac{c_{-1}\cdot c_{-1}}{1.2\,3.4}w^4 - \frac{c_{-1}\cdot c_{-1}\cdot(c_{-1}+2.4)}{1.2.3.4.5.6}w^6 + \ldots.$$

$$1 - \frac{c_0}{1.2}w^2 + \frac{c_0(c_0+2.2)}{1.2.3\,4}w^4 - \frac{c_0(c_0+2.2)(c_0+4.4)}{1.2.3.4.5.6}w^6 + \ldots.$$

$$1 - \frac{c_1}{1.2}w^2 + \frac{c_1(c_1+4.2)}{1.2.3.4}w^4 - \frac{c_1(c_1+4.2)(c_1+6.4)}{1.2.3.4.5.6}w^6 + \ldots.$$

$$1 - \frac{c_2}{1.2}w^2 + \frac{c_2(c_2+6.2)}{1.2.3.4}w^4 - \frac{c_2(c_2+6.2)(c_2+8.4)}{1.2.3.4.5.6}w^6 + \ldots.$$

$\mathfrak{S}$-FUNCTIONS.

$$w - \frac{(c_{-1}-1.1)}{2.3}w^3 + \frac{(c_{-1}-1.1)(c_{-1}+1.3)}{2.3.4.5}w^5 - \frac{(c_{-1}-1.1)(c_{-1}+1.3)(c_{-1}+3.5)}{2.3.4.5.6.7}w^7 + \ldots . \quad (32)$$

$$w - \frac{(c_0+1.1)}{2.3}w^3 + \frac{(c_0+1.1)(c_0+3.3)}{2.3.4.5}w^5 - \frac{(c_0+1.1)(c_0+3.3)(c_0+5.5)}{2.3.4.5.6.7}w^7 + \ldots .$$

$$w - \frac{(c_1+3.1)}{2.3}w^3 + \frac{(c_1+3.1)(c_1+5.3)}{2.3.4.5}w^5 - \frac{(c_1+3.1)(c_1+5.3)(c_1+7.5)}{2.3.4.5\,6.7}w^7 + \ldots .$$

$$w - \frac{(c_2+5.1)}{2.3}w^3 + \frac{(c_2+5.1)(c_1+7.3)}{2.3.4.5}w^5 - \frac{(c_2+5.1)(c_2+7.3)(c_2+9.5)}{2.3.4.5.6.7}w^7 + \ldots .$$

Let us now find the roots of the equations $\mathfrak{S}_a(c_a, 1) = 0$ and $\mathfrak{S}_a(c_a, 1) = 0$ for the eight functions given in (29) and (31). For the $\mathfrak{S}$-functions we have the conditions:

$$\begin{aligned}
\tan[n\log(1+\sqrt{2})] &= n\sqrt{2}\,;\ c_{-1} = n^2+1\,.\\
\cos[n\log(1+\sqrt{2})] &= \ \ 0\ \ ;\ c_0 = n^2\,.\\
\cos[n\log(1+\sqrt{2})] &= \ \ 0\ \ ;\ c_1 = n^2+1\,.\\
\tan[n\log(1+\sqrt{2})] &= n\sqrt{2}\,;\ c_2 = n^2+4\,.
\end{aligned} \quad (33)$$

and for the $\mathfrak{S}$-functions:

$$\begin{aligned}
\operatorname{cotan}[n\log(1+\sqrt{2})] &= -n\sqrt{2}\,;\ c_{-1} = n^2+1\,.\\
\sin[n\log(1+\sqrt{2})] &= \ \ 0\ \ ;\ c_0 = n^2\,.\\
\sin[n\log(1+\sqrt{2})] &= \ \ 0\ \ ;\ c_1 = n^2+1\,.\\
\operatorname{cotan}[n\log(1+\sqrt{2})] &= -n\sqrt{2}\,;\ c_2 = n^2+4\,.
\end{aligned}$$

From these conditions the roots c_a are easily obtained. We find

$\mathfrak{C}$-FUNCTIONS.	$\mathfrak{S}$-FUNCTIONS.
$c_{-1} = 2{\cdot}35012\,;\ 27{\cdot}9681\,;\ 78{\cdot}7976\,;\ \ldots .$	$c_{-1} = 12{\cdot}06756\,;\ 50{\cdot}2082\,;\ \ldots .$
$c_0 = 3{\cdot}17627\,;\ 28{\cdot}5865\,;\ 79{\cdot}4068\,;\ \ldots .$	$c_0 = 12{\cdot}70508\,;\ 50{\cdot}8204\,;\ \ldots .$
$c_{+1} = 4{\cdot}17627\,;\ 29{\cdot}5865\,;\ 80{\cdot}4068\,;\ \ldots .$	$c_{+1} = 13{\cdot}70508\,;\ 51{\cdot}8204\,;\ \ldots .$
$c_{+2} = 5{\cdot}35012\,;\ 30{\cdot}9681\,;\ 81{\cdot}7976\,;\ \ldots .$	$c_{+2} = 15{\cdot}06756\,;\ 53{\cdot}2082\,;\ \ldots .$

(34)

But obviously the roots in each vertical row may be considered as special values of a certain unknown function of a, so that generally

$$c_a = f(a),$$

from which equation, if $f(a)$ were known, we would obtain the values c_{-1}, c_0, etc. by substituting $a = -1$, 0, etc. Now, as long as $f(a)$ may be considered as finite and continuous, the well-known formulæ of numerical interpolation allow us to find intermediate values of c from the given data, without knowing the analytical character of $f(a)$, if a sufficient number of equidistant values of c are at our disposal. We are thus in a position to determine, at least approximately, the roots $c_{-\frac{1}{2}}$ of the hyperbolic Seiche-functions $\mathfrak{C}$ and $\mathfrak{S}$ by interpolating between the numbers of the above table (34). Performing the necessary calculations, we obtain the following roots:

$$\begin{aligned}
&\mathfrak{C}_{-\frac{1}{2}}(c, 1) = 0\,;\ c_{-\frac{1}{2}} = \ 2{\cdot}74\ldots ., \ 28{\cdot}23\ldots ., \ 79{\cdot}05\ldots .\\
&\mathfrak{S}_{-\frac{1}{2}}(c, 1) = 0\,;\ c_{-\frac{1}{2}} = 12{\cdot}34\ldots ., \ 50{\cdot}46\ldots .
\end{aligned} \quad (35)$$

These interpolated values of the roots of the hyperbolic Seiche-functions are in all probability sufficiently accurate for practical purposes. The smallest roots of each

function have also been computed directly from the series by Professor CHRYSTAL, assisted by Dr BURGESS, Professor GIBSON, and Mr HORSBURGH. They were found to be 2·76 and 12·34, and hence are remarkably close to the values given in (35). The root 28·23 was also verified by Dr BURGESS, Professor GIBSON, and myself. I have convinced myself that the exact value must certainly lie between 28·229 and 28·230, so that the second decimal place of the interpolated value appears to be correct. The method of interpolation here adopted places us therefore in a position to determine the infinite number of roots of the two transcendental equations $\mathfrak{C}_{-\frac{1}{3}}(c, 1) = 0$, and $\mathfrak{S}_{-\frac{1}{3}}(c, 1) = 0$, and thus enables us to determine the periods of the seiches in lakes with convex parabolic floors from Professor CHRYSTAL'S equation :—$T_r = 2\pi a / \sqrt{c_r g}$ (see *Proc. R.S.E.*, vol. XXV. p. 332).

A similar method of interpolation may be employed in the calculation of the position of the nodes. These are found by first forming the equations

$$\begin{aligned} &\frac{d\mathfrak{C}_{-1}(w)}{dw} = 0, \quad \frac{d\mathfrak{C}_0(w)}{dw} = 0, \quad \frac{d\mathfrak{C}_1(w)}{dw} = 0, \quad \frac{d\mathfrak{C}_2(w)}{dw} = 0, \\ &\frac{d\mathfrak{S}_{-1}(w)}{dw} = 0, \quad \frac{d\mathfrak{S}_0(w)}{dw} = 0, \quad \frac{d\mathfrak{S}_1(w)}{dw} = 0, \quad \frac{d\mathfrak{S}_2(w)}{dw} = 0, \end{aligned} \tag{36}$$

and by determining the values of w satisfying these equations under the condition that c_a has the values mentioned in (35). By interpolating between the w thus obtained, we find with sufficient approximation the successive values which satisfy the two Seiche-equations

$$\frac{d\mathfrak{C}_{-\frac{1}{3}}(w)}{dw} = 0 \quad \text{and} \quad \frac{d\mathfrak{S}_{-\frac{1}{3}}(w)}{dw} = 0,$$

and which therefore fulfil the required condition that the vertical displacement $\zeta = 0$. In this way I have found the following positions of the nodes in convex parabolic lakes:

Uninodal Seiche $w = 0$
Binodal Seiche $w = \pm 0{\cdot}473 \ldots$.
Trinodal Seiche $w = 0$; $\pm 0{\cdot}632 \ldots$.
Quadrinodal Seiche $w = \pm 0{\cdot}224 \ldots$; $\pm 0{\cdot}717 \ldots$.

It is interesting now to investigate the positions of the nodes under the various assumptions which have been made with regard to the curvature of the floor of the lake. From Professor CHRYSTAL'S investigations and the preceding discussion we find for

Lake with	Uninodal Seiche.	Binodal Seiche.	Trinodal Seiche.	Quadrinodal Seiche.
concave parabolic floor, . .	$w = 0$	$\pm$ ·577	0; $\pm$ ·775	$\pm$ ·340; $\pm$ ·862
plain horizontal ,, . .	0	$\pm$ ·500	0; $\pm$ ·667	$\pm$ ·250; $\pm$ ·750
convex parabolic ,, . .	0	$\pm$ ·473	0; $\pm$ ·632	$\pm$ ·224; + ·717
convex quartic ,, . .	0	$\pm$ ·447	0; $\pm$ ·600	$\pm$ ·202; $\pm$ ·684

(37)

The figures show clearly that in lakes with curved floors the nodes are always displaced towards the *shallow* water.

As regards the constants c_ν which determine the periods of the various seiches, we have the following values:

Lake with	Uninodal Seiche.	Binodal Seiche.	Trinodal Seiche.	Quadrinodal Seiche.
concave parabolic floor, . .	2·00	6·00	12·0	20·0
plain horizontal „ . .	2·47 . .	9·87 . .	22·2 . .	39·5 . .
convex parabolic „ . .	2·76 . .	12·34 . .	28·2 . .	50·5 . .
convex quartic „ . .	3·00	15·00	35·0	63·0

(38)

and by virtue of the equation $T_\nu/T_1 = \sqrt{c_1/c_\nu}$:

	T_2/T_1	T_3/T_1	T_4/T_1
concave parabolic:	·577	·408	·317
plain horizontal:	·500	·333	·250
convex parabolic:	·472	·312	·234
convex quartic:	·447	·293	·218

(39)

We recognise here, in a more general form, the law found by Professor CHRYSTAL, that in concave lakes the ratio T_ν/T_1 is greater and in convex lakes smaller than the corresponding ratio in a lake with plain horizontal floor

The positions of the nodes may be represented in a convenient graphical form, which not only shows clearly their dependence on the curvature of the lake, but at the same time enables us to find the nodes for the curves lying between those here discussed, which are not amenable to direct analytical treatment. In fig. 1 are shown the halves of the vertical longitudinal sections of symmetric lakes. O B represents a, the half-length, and O A the central depth, h, of the lake, whereas A B, A C, A D, and A E signify the intersections of the vertical plane with the concave-parabolic, the plane-horizontal, the convex-parabolic, and the convex-quartic floors. Now, on each of these curves the nodes have been marked by the points B_2, B_3, B_4, etc., in such a way that for instance the distance of B_2 from A O agrees with the value of w in (43) which refers to the binodal seiche in a lake with concave parabolic floor, *i.e.* $w = \frac{x}{a} = 0{\cdot}577$. In the same way E_2 is drawn at a distance 0·447 from A O, thus representing the position of the binode in a convex-quartic lake. Having secured the corresponding four points on each of the curves A B, A C, A D, and A E, we draw the curved lines B_2E_2, B_3E_3, and B_4E_4, and these lines are obviously the *loci* of the nodes. We recognise at once in all cases the displacements of the nodal points towards the shallow water, a phenomenon specially marked in concave lakes. Let us now take, for instance, a convex lake whose depth is represented by $h\sqrt{1+w^2}$ and is indicated in our diagram by the dotted curve A F. The solutions of the corresponding differential equation

$$\sqrt{1+w^2}\frac{d^2y}{dw^2} + cy = 0$$

are not known, and hence we are not in a position to compute the nodes and periods of these particular seiches by analytical methods. But approximately the nodes may be now directly found from the diagram, being represented by the points of intersection between the *loci* B E and the curve A F. It would seem, therefore, that by the preceding

investigation we have gained considerable freedom in the selection of curves representing the curvature-conditions of the lake floors, so that we may, by the suggested interpolatory method, assign approximate values to the nodes, and by means of (38) also to the periods, even under conditions which are unmanageable from the rigorous mathematical point of view.*

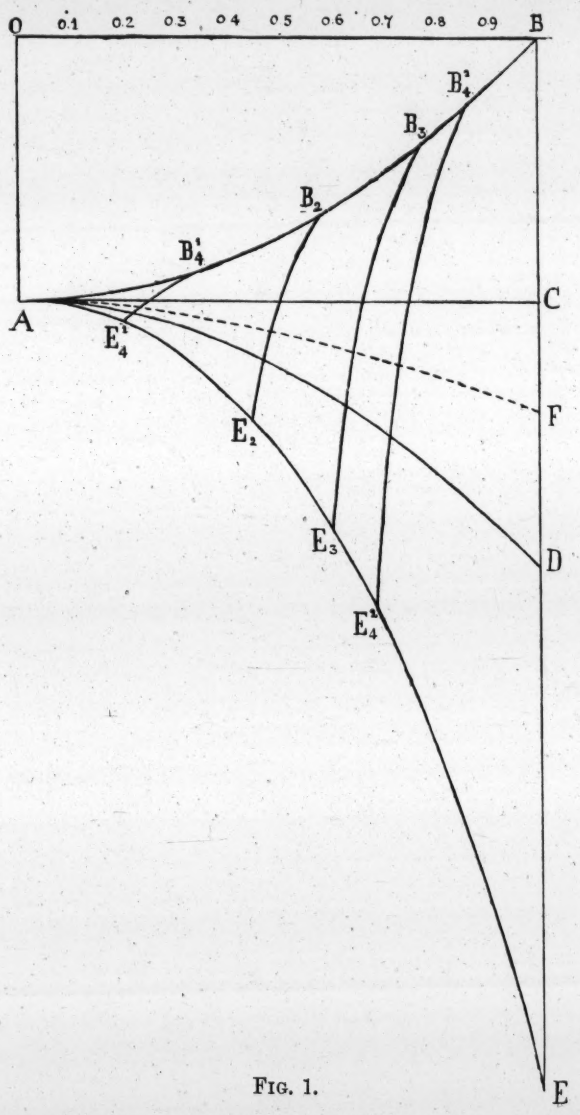


FIG. 1.

The advantage of the preceding analysis of Professor CHRYSTAL's Seiche-equations lies in the fact that they belong to a class which includes cases where the solutions are expressible by means of simple transcendents. If we write the general differential equations in their most symmetric form :

$$\begin{aligned}(1-w^2)\frac{d^2y}{dw^2} - (\gamma+1)w\frac{dy}{dw} + \left(n^2 - \frac{\gamma^2}{4}\right)y = 0 \\ (1+w^2)\frac{d^2y}{dw^2} + (\gamma+1)w\frac{dy}{dw} + \left(n^2 + \frac{\gamma^2}{4}\right)y = 0\end{aligned} \qquad (40)$$

* In § 45 of his Hydrodynamical Theory, Professor CHRYSTAL investigates the case of a rectilinear lake, the floor of which would be represented in the above diagram by the straight line A B. The directions of the nodal loci E B in this figure suggest that the nodes in rectilinear lakes should be more displaced towards the shallow water than in any of the cases here considered, and this conclusion is indeed supported by the numerical results given in § 49 of Professor CHRYSTAL's paper.—(*Note added on June* 30.)

we may say quite generally, that the solutions can always be expressed by simple transcendents if γ is an *even* integer. The Seiche- and LEGENDRE-functions, on the other hand, belong to differential equations of the same class for which γ is an *odd* integer. Hence each equation of the latter class is included between two equations of the former, and we have seen that through this remarkable property we were enabled, by studying the behaviour of the neighbouring simple transcendents, to form conclusions with regard to the far more difficult Seiche-functions.

In the theory of lakes, the floors of which are composed of two or more parabolæ with different parameters, the evaluation of the two functions $C(c, 1)$ and $S(c, 1)$ for any given value of c becomes important. The calculation of the series

$$C(c, 1) = 1 - \frac{c}{1.2} + \frac{c(c-2)}{1.2.3.4} - \frac{c(c-2)(c-12)}{1.2.3.4.5.6} + \cdots\cdots$$

$$S(c, 1) = 1 - \frac{c}{2.3} + \frac{c(c-6)}{2.3.4.5} - \frac{c(c-6)(c-20)}{2.3.4.5.6.7} + \cdots\cdots$$

is always an exceedingly troublesome process, especially for great values of c. In many cases even the calculation of 100 terms is not sufficient. But the foregoing investigation immediately suggests a rigorous and extremely simple method of computing these quantities. Reverting to equations (18) and (20) we find for the two Seiche-functions $(\alpha = -\frac{1}{2})$ the relations:

$$C(c, 1) = \frac{\Gamma(\frac{1}{2})}{\Gamma\left(1 - \frac{n}{2}\right)\Gamma\left(\frac{1}{2} + \frac{n}{2}\right)} \quad ; \; n = \tfrac{1}{2} + \sqrt{c + \tfrac{1}{4}}.$$

$$S(c, 1) = \frac{1}{2}\frac{\Gamma(\frac{1}{2})}{\Gamma\left(\frac{3}{2} - \frac{n}{2}\right)\Gamma\left(1 + \frac{n}{2}\right)}$$

But considering that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

we find

$$C(c, 1) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \frac{\sin n \cdot \frac{\pi}{2}}{\sqrt{\pi}}$$

$$S(c, 1) = -\frac{2}{n(n-1)} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \cdot \frac{\cos n \cdot \frac{\pi}{2}}{\sqrt{\pi}}. \qquad (41)$$

These expressions are very convenient for computation. Let us take, for instance, $c = 105{\cdot}0176$, to which corresponds $n = 10{\cdot}76$. We have

$$C(105{\cdot}0176\,;\,1) = -\frac{\Gamma(5{\cdot}38)}{\Gamma(5{\cdot}88)} \frac{\sin 68^\circ{\cdot}4}{\sqrt{\pi}}$$

$$= -\frac{1{\cdot}38 \times 2{\cdot}38 \times 3{\cdot}38 \times 4{\cdot}38\, \Gamma(1{\cdot}38)}{1{\cdot}88 \times 2{\cdot}88 \times 3{\cdot}88 \times 4{\cdot}88\, \Gamma(1{\cdot}88)} \cdot \frac{\sin 68^\circ{\cdot}4}{\sqrt{\pi}}$$

$$S(105{\cdot}0176\,;\,1) = \frac{2}{105{\cdot}0176} \cdot \frac{1{\cdot}88 \times 2{\cdot}88 \times 3{\cdot}88 \times 4{\cdot}88\, \Gamma(1{\cdot}88)}{1{\cdot}38 \times 2{\cdot}38 \times 3{\cdot}38 \times 4{\cdot}38\, \Gamma(1{\cdot}38)} \frac{\cos 68^\circ{\cdot}4}{\sqrt{\pi}};$$

and, since

$$\log\Gamma(1{\cdot}38) = \bar{1}{\cdot}94868 \quad \log\Gamma(1{\cdot}88) = \bar{1}{\cdot}98004,$$

we find $\quad C(105{\cdot}0176\,;\,1) = -0{\cdot}231467\,;\ S(105{\cdot}0176\,;\,1) = +0{\cdot}008964$

I need scarcely say that in this case a satisfactory calculation by means of the series is quite impossible. For smaller values of c the computations are of course still more convenient. If, for instance, $c = 14{\cdot}96$, or $n = 4{\cdot}4$, we have

$$C(14{\cdot}96\,;\,1) = \frac{\Gamma(2{\cdot}2)}{\Gamma(2{\cdot}7)} \cdot \frac{\sin 36^\circ}{\sqrt{\pi}} = \frac{1{\cdot}2}{1{\cdot}7}\,\frac{\Gamma(1{\cdot}2)}{\Gamma(1{\cdot}7)}\,\frac{\sin 36^\circ}{\sqrt{\pi}} = +0{\cdot}23654\,.$$

We notice also the relation

$$S(c\,,\,1) = -\frac{\sin n\pi}{\pi} \cdot \frac{1}{n(n-1)C(c\,,\,1)}\,. \tag{42}$$

Note added on June 30.—The relations (41) and (42), which follow here immediately from a well-known property of the hypergeometric series $F(\alpha\,,\beta\,,\gamma\,;\ 1)$, were originally derived by Professor CHRYSTAL from a different point of view. (See §§ 25 and 41 of the Hydrodynamical Theory.) His important relation (53) in § 41 may be also obtained from the equation

$$L(c\,,\,1) = F(n\,,\,1-n\,,\,2\,;\ 1) = 1/\Gamma(1+n)\Gamma(2-n) = -\frac{\sin n\pi}{n(n-1)\pi},$$

and, in consequence of (42):

$$L(c\,,\,1) = C(c\,,\,1)\,.\,S(c\,,\,1).$$

At the request of Professor CHRYSTAL I subjoin tables from which the numerical values of $C(c\,,\,1)$ and $S(c\,,\,1)$ may be taken for any value of c. If we write

$$a = \tfrac{1}{4} + \tfrac{1}{2}\sqrt{c+\tfrac{1}{4}}$$

$$\Theta(a) = \frac{\Gamma(a)}{\Gamma(a+\frac{1}{2})}\,\frac{\sin a\pi}{\sqrt{\pi}}$$

$$\Sigma(a) = -\frac{\Gamma(a+\frac{1}{2})}{\Gamma(a)}\,\frac{\cos a\pi}{\sqrt{\pi}},$$

we find easily from the above formulæ the following relations:

$$0<a<1 : C(c\,,\,1) = -\frac{a+\frac{1}{2}}{a} \cdot \Theta(a+1)\,;\ \frac{c}{2}\,S(c\,,\,1) = -\frac{a}{a+\frac{1}{2}}\,\Sigma(a+1)$$

$$1<a<2 : C(c\,,\,1) = \Theta(a) \qquad \frac{c}{2}\,S(c\,,\,1) = \Sigma(a)$$

$$2<a<3 : C(c\,,\,1) = -\frac{a-1}{a-\frac{1}{2}}\Theta(a-1)\,;\ \frac{c}{2}\,S(c\,,\,1) = -\frac{a-\frac{1}{2}}{a-1}\Sigma(a-1)\,;$$

$$3<a<4 : C(c\,,\,1) = \frac{(a-2)(a-1)}{(a-\frac{3}{2})(a-\frac{1}{2})}\Theta(a-2)\,;\ \frac{c}{2}\,S(c\,,\,1) = \frac{(a-\frac{3}{2})(a-\frac{1}{2})}{(a-2)(a-1)}\Sigma(a-2)\,,$$

. . . . . .

where the values of θ and Σ are to be taken from the following Table I.

TABLE I.

a	$\Theta(a)$	a	$\Theta(a)$	a	$\Theta(a)$	a	$\Theta(a)$	a	$\Theta(a)$
1·00	0·00000	1·20	– 0·33511	1·40	– 0·49501	1·60	– 0·45815	1·80	– 0·26474
1·01	– 0·01988	1·21	– 0·34769	1·41	– 0·49775	1·61	– 0·45163	1·81	– 0·25237
1·02	– 0·03949	1·22	– 0·35984	1·42	– 0·49999	1·62	– 0·44471	1·82	– 0·23983
1·03	– 0·05884	1·23	– 0·37152	1·43	– 0·50171	1·63	– 0·43741	1·83	– 0·22712
1·04	– 0·07790	1·24	– 0·38271	1·44	– 0·50294	1·64	– 0·42975	1·84	– 0·21430
1·05	– 0·09666	1·25	– 0·39346	1·45	– 0·50366	1·65	– 0·42171	1·85	– 0·20133
1·06	– 0·11512	1·26	– 0·40370	1·46	– 0·50391	1·66	– 0·41332	1·86	– 0·18824
1·07	– 0·13326	1·27	– 0 41347	1·47	– 0·50365	1·67	– 0·40457	1·87	– 0·17505
1·08	– 0·15106	1·28	– 0·42275	1·48	– 0·50292	1·68	– 0·39552	1·88	– 0·16177
1·09	– 0·16852	1·29	– 0·43153	1·49	– 0·50171	1·69	– 0·38612	1·89	– 0·14842
1·10	– 0·18563	1·30	– 0·43982	1·50	– 0·50000	1·70	– 0·37643	1·90	– 0·13499
1·11	– 0·20237	1·31	– 0·44762	1·51	– 0·49785	1·71	– 0·36641	1·91	– 0·12151
1·12	– 0·21874	1·32	– 0·45490	1·52	– 0·49520	1·72	– 0·35612	1·92	– 0·10800
1·13	– 0·23473	1·33	– 0·46170	1·53	– 0·49211	1·73	– 0·34555	1·93	– 0·09445
1·14	– 0·25032	1·34	– 0·46798	1·54	– 0·48858	1·74	– 0·33472	1·94	– 0·08090
1·15	– 0·26551	1·35	– 0·47375	1·55	– 0·48458	1·75	– 0·32362	1·95	– 0·06734
1·16	– 0·28029	1·36	– 0·47902	1·56	– 0·48014	1·76	– 0·31228	1·96	– 0·05380
1·17	– 0·29464	1·37	– 0·48378	1·57	– 0·47527	1·77	– 0·30071	1·97	– 0·04028
1·18	– 0·30855	1·38	– 0·48804	1·58	– 0·46998	1·78	– 0·28892	1·98	– 0·02680
1·19	– 0·32206	1·39	– 0·49177	1·59	– 0·46426	1·79	– 0·27692	1·99	– 0·01337

a	$\Sigma(a)$	a	$\Sigma(a)$	a	$\Sigma(a)$	a	$\Sigma(a)$	a	$\Sigma(a)$
1·00	+ 0·50000	1·20	+ 0·45171	1·40	+ 0·18899	1·60	– 0·20419	1·80	– 0·57178
1·01	+ 0·50281	1·21	+ 0·44336	1·41	+ 0·17133	1·61	– 0·22463	1·81	– 0·58638
1·02	+ 0·50510	1·22	+ 0·43447	1·42	+ 0·15336	1·62	– 0·24499	1·82	– 0·60047
1·03	+ 0·50687	1·23	+ 0·42503	1·43	+ 0·13507	1·63	– 0·26525	1·83	– 0·61406
1·04	+ 0·50811	1·24	+ 0·41505	1·44	+ 0·11649	1·64	– 0·28536	1·84	– 0·62708
1·05	+ 0·50880	1·25	+ 0·40453	1·45	+ 0·09765	1·65	– 0·30533	1·85	– 0·63956
1·06	+ 0·50894	1·26	+ 0·39347	1·46	+ 0·07855	1·66	– 0·32513	1·86	– 0·65148
1·07	+ 0·50853	1·27	+ 0·38190	1·47	+ 0·05921	1·67	– 0·34473	1·87	– 0·66279
1·08	+ 0·50758	1·28	+ 0·36982	1·48	+ 0·03966	1·68	– 0·36410	1·88	– 0·67348
1·09	+ 0·50604	1·29	+ 0·35723	1·49	+ 0·01992	1·69	– 0·38325	1·89	– 0·68355
1·10	+ 0·50396	1·30	+ 0·34416	1·50	0·00000	1·70	– 0·40213	1·90	– 0·69303
1·11	+ 0·50130	1·31	+ 0·33060	1·51	– 0·02007	1·71	– 0·42072	1·91	– 0·70183
1·12	+ 0·49807	1·32	+ 0·31657	1·52	– 0·04028	1·72	– 0·43901	1·92	– 0·70998
1·13	+ 0·49428	1·33	+ 0·30207	1·53	– 0·06060	1·73	– 0·45697	1·93	– 0·71744
1·14	+ 0·48991	1·34	+ 0·28716	1·54	– 0·08101	1·74	– 0·47456	1·94	– 0·72423
1·15	+ 0·48496	1·35	+ 0·27179	1·55	– 0·10149	1·75	– 0·49181	1·95	– 0·73032
1·16	+ 0·47946	1·36	+ 0·25601	1·56	– 0·12203	1·76	– 0·50866	1·96	– 0·73570
1·17	+ 0·47335	1·37	+ 0·23982	1·57	– 0·14259	1·77	– 0·52511	1·97	– 0·74037
1·18	+ 0·46671	1·38	+ 0·22324	1·58	– 0·16315	1·78	– 0·54111	1·98	– 0·74432
1·19	+ 0·45948	1·39	+ 0·20630	1·59	– 0·18369	1·79	– 0·55667	1·99	– 0·74755

More convenient, however, for practical use may be Table II., from which the numerical values of $C(c, 1)$ and $S(c, 1)$ are directly obtained for values of c included between 0 and 30·0. The intervals chosen are sufficiently close to permit an easy

interpolation of these functions for intermediate values of c. If required, the table may be extended by means of Table I. and the preceding formulæ.

TABLE II.

c	$C(c, 1)$	$S(c, 1)$	c	$C(c, 1)$	$S(c, 1)$
0·0	+1·00000	+1·00000	5·6	−0·50392	+0·02851
0·2	+0·86568	+0·94030	5·8	−0·50292	+0·01375
0·4	+0·74008	+0·88215	6·0	−0·50000	0·00000
0·6	+0·62247	+0·82674	7·0	−0·46052	−0·05615
0·8	+0·51257	+0·77390	8·0	−0·38976	−0·09397
1·0	+0·41034	+0·72302	9·0	−0·29986	−0·11695
1·2	+0·31593	+0·67440	10·0	−0·20033	−0·12809
1·4	+0·22694	+0·62778	11·0	−0·09837	−0·13006
1·6	+0·14511	+0·58331	12·0	0·00000	−0·12500
1·8	+0·06956	+0·54071	13·0	+0·09082	−0·11479
2·0	0·00000	+0·50000	14·0	+0·17106	−0·10101
2·2	−0·06385	+0·46111	15·0	+0·23802	−0·08524
2·4	−0·12221	+0·42406	16·0	+0·29323	−0·06753
2·6	−0·17561	+0·38864	17·0	+0·33377	−0·04971
2·8	−0·22395	+0·35492	18·0	+0·36035	−0·03222
3·0	−0·26758	+0·32282	19·0	+0·37382	−0·01550
3·2	−0·30673	+0·29227	20·0	+0·37500	0·00000
3·4	−0·34177	+0·26317	21·0	+0·36508	+0·01404
3·6	−0·37273	+0·23555	22·0	+0·34531	+0·02640
3·8	−0·39987	+0·20933	23·0	+0·31710	+0·03698
4·0	−0·42343	+0·18442	24·0	+0·28192	+0·04572
4·2	−0·44360	+0·16083	25·0	+0·24103	+0·05266
4·4	−0·46051	+0·13853	26·0	+0·19611	+0·05779
4·6	−0·47438	+0·11741	27·0	+0·14830	+0·06123
4·8	−0·48542	+0·09744	28·0	+0·09889	+0·06308
5·0	−0·49366	+0·07865	29·0	+0·04919	+0·06345
5·2	−0·49941	+0·06088	30·0	0·00000	+0·06250
5·4	0·50277	+0·04422			

As regards the hyperbolic functions $\mathfrak{C}(c, 1)$ and $\mathfrak{S}(c, 1)$, it seems, however, impossible to calculate their values by means of the corresponding relations, because the Γ-functions, which appear in the expressions, have imaginary arguments. But fortunately, in this case, the series are far more manageable than those of the C- and S-functions, so that the labour involved in their direct calculation is not nearly so stupendous as it may appear at first sight, even if an accuracy within the fifth decimal place is desired. To show this let us first consider the series

$$1 - \frac{c+1}{2.3} + \frac{(c+1)(c+9)}{2.3.4.5} - \frac{(c+1)(c+9)(c+25)}{2.3.4.5.6.7} + \ldots\ldots$$

which, as we know from the preceding investigations (see (29)), is rigorously represented by the transcendent $\frac{1}{\sqrt{c}} \sin(\sqrt{c} \log(1+\sqrt{2}))$. We want to inquire how many terms of the series are necessary to obtain the accurate value of this transcendent within, say, five places after the decimal point. If we call $\Sigma(x)$ the sum of the x first terms, and if

we plot the successive Σ as ordinates, taking the corresponding x as abscissæ, we find that, owing to the continuous change of sign, the Σ are represented as the corner-points of a zigzag line; and it can be shown that for sufficiently great values of x this line oscillates along a straight line parallel to the axis of abscissæ, and at a distance from it equal to the exact value of the series, *i.e.* to $\frac{1}{\sqrt{c}} \sin\left(\sqrt{c} \log (1+\sqrt{2})\right)$. Let us demonstrate this for a certain value of c, e.g. $c = 24$. We find

x	$\Sigma(x)$	x	$\Sigma(x)$
1	+1·00000	12	−2·43133
2	−3·16667	13	+1·88211
3	+3·70833	14	−2·10568
4	−4·31251	15	+1·59235
5	+3·81975	16	−1·84723
6	−3·94285	17	+1·36108
7	+3·27238	18	−1·63963
8	−3·35875	19	+1·17409
9	+2·71167	20	−1·47067
10	−2·84399	21	+1·02088
11	+2·24871	22	−1·33133

If now we take the arithmetical means of each two consecutive $\Sigma(x)$, which we may call $\Sigma_1(x)$, these may again be represented by a zigzag line, whose excursions from the horizontal line are obviously much smaller, and if we repeat this process of forming arithmetical means, the fourth operation of this kind will lead to the following set of values for $\Sigma_4(x)$:

x	$\Sigma_4(x)$	x	$\Sigma_4(x)$
13	−0·18840	17	−0·18841
14	−0·18842	18	−0·18844
15	−0·18840	19	−0·18842
16	−0·18844	20	−0·18844

Now we convince ourselves, by computing the higher terms of the series for $x > 22$, that however far we may extend the calculations, the values of $\Sigma_4(x)$ will always be found between the last two figures of the preceding table, and will more and more converge towards their arithmetical mean, viz., −0·18843. But the exact value of the series should be

$$\frac{1}{\sqrt{24}} \sin\left(\sqrt{24} \log (1+\sqrt{2})\right),$$

or, if the angle be expressed in degrees,

$$\frac{1}{\sqrt{24}} \sin\left(\sqrt{24} \times 50°\cdot 499\right) = -0\cdot 188438.$$

We see, then, that in this case the first twenty terms of the series are quite sufficient to find the value of our series with an error of less than one unit of the fifth

decimal place. The same number of terms is therefore also sufficient for the calculation of the more convergent Seiche-function

$$\mathfrak{S}(c, 1) = 1 - \frac{c}{2.3} + \frac{c(c+6)}{2.3.4.5} - \frac{c(c+6)(c+20)}{2.3.4.5.6.7} + \ldots.$$

for the same value of c. This method enables us to compute with sufficient accuracy the $\mathfrak{S}$- and $\mathfrak{C}$-functions, even for great values of the argument c, from a comparatively limited number of terms of the series. I subjoin the results of my computations in the following Table III. :

TABLE III.

c	$\mathfrak{C}(c, 1)$	$\mathfrak{S}(c, 1)$	$\Delta\mathfrak{C}$	$\Delta\mathfrak{S}$
0	+ 1·0000	+ 1·0000	− 0·0014	+ 0·0025
1	+ 0·5894	+ 0·8731	− 0·0010	+ 0·0024
2	+ 0·2328	+ 0·7558	− 0·0006	+ 0·0025
3	− 0·0748	+ 0·6479	− 0·0003	+ 0·0023
4	− 0·3368	+ 0·5490	− 0·0002	+ 0·0025
5	− 0·5568	+ 0·4580	0·0000	+ 0·0024
6	− 0·7388	+ 0·3748	+ 0·0001	+ 0·0025
8	− 1·0007	+ 0·2299	+ 0·0003	+ 0 0024
10	− 1·1468	+ 0·1108	+ 0·0003	+ 0·0024
12	− 1·1985	+ 0·0147	+ 0·0002	+ 0·0025
16	− 1·0908	− 0·1208	0·0000	+ 0·0022
20	− 0·8019	− 0·1956	− 0·0010	+ 0 0019
24	− 0·4235	− 0·2255	− 0·0021	+ 0·0017
28	− 0·0226	− 0·2234	− 0·0033	+ 0·0015

But for the construction of final tables, from which the $\mathfrak{C}$- and $\mathfrak{S}$-functions might be obtained for any value of c by interpolation, the preceding Table III. is not sufficient. In order to facilitate the further computations, I have searched for convenient formulæ of interpolation, so that the still troublesome direct calculations of the series may be avoided, and I have found that the following consideration leads to a convenient result.

It may be easily shown that the differential equation

$$(1+w^2)\frac{d^2y}{dw^2} + c\left[1 - \frac{\frac{3}{4}}{c(1+w^2)}\right]y = 0$$

is satisfied by the particular solutions

$$(1+w^2)^{\frac{1}{4}} \cos\left(\sqrt{c-\tfrac{1}{4}} \log\left(w + \sqrt{1+w^2}\right)\right) \quad \text{and} \quad \frac{(1+w^2)^{\frac{1}{4}}}{\sqrt{c-\frac{1}{4}}} \sin\left(\sqrt{c-\tfrac{1}{4}} \log\left(w + \sqrt{1+w^2}\right)\right).$$

Now, the close similarity between this differential equation and the hyperbolic Seiche-equation, into which it converges if c is increased indefinitely, makes it probable that the solutions of the latter, *i.e.* the $\mathfrak{C}$- and $\mathfrak{S}$-functions, are also closely related to the

two simple transcendents just now mentioned. It is, indeed, easily seen that the values of $\mathfrak{C}(c, 1)$ and $\mathfrak{S}(c, 1)$ are approximately represented by

$$2^{\frac{1}{4}}\cos[\sqrt{c+\gamma}\log(1+\sqrt{2})] \quad \text{and} \quad \frac{2^{\frac{1}{4}}}{\sqrt{c+\gamma}}\sin[\sqrt{c+\gamma}\log(1+\sqrt{2})]$$

where γ is a certain constant.

By slightly altering the constants, these two expressions can be made to represent the $\mathfrak{C}$- and $\mathfrak{S}$-functions even more closely. We have then approximately

$$\mathfrak{C}(c, 1) = 1{\cdot}19942 \times \cos[\sqrt{c+0{\cdot}438} \times 50°{\cdot}458]$$
$$\sqrt{c+0{\cdot}438}\,\mathfrak{S}(c, 1) = 1{\cdot}19942 \times \sin[\sqrt{c+0{\cdot}438} \times 50°{\cdot}458].$$

If we calculate the expressions on the right-hand side for the values of c given in Table III., by comparison of the calculated figures with the exact values of $\mathfrak{C}(c, 1)$ and $\mathfrak{S}(c, 1)$ shown in the same table, we find the corrections $\Delta\mathfrak{C}$ and $\Delta\mathfrak{S}$ which have to be added to the computed transcendents. These corrections are conveniently small to permit an easy interpolation, and we are thus in a position to ascertain the correct numerical values of $\mathfrak{C}(c, 1)$ and $\mathfrak{S}(c, 1)$ for values of c lying between those mentioned in Table III. by simply computing the corresponding values of the right-hand sides of the above approximate equations and by adding the corresponding corrections $\Delta\mathfrak{C}$ and $\Delta\mathfrak{S}$ interpolated from the figures of the same table. In this way the following Table IV. has been constructed, which shows, for conveniently small intervals, the $\mathfrak{C}(c, 1)$ and $\mathfrak{S}(c, 1)$ for the arguments c between zero aud 30·0. To exhibit at the same time, more concretely, the character of the functions $C(c, 1)$, $S(c, 1)$, $\mathfrak{C}(c, 1)$and $\mathfrak{S}(c, 1)$, their graphs are given in fig. 2.

TABLE IV.

c	$\mathfrak{C}(c, 1)$	$\mathfrak{S}(c, 1)$	c	$\mathfrak{C}(c, 1)$	$\mathfrak{S}(c, 1)$
0·0	+1·0000	+1·0000	13·0	−1·1946	−0·0259
0·5	+0·7877	+0·9353	14·0	−1·1741	−0·0619
1·0	+0·5894	+0·8731	15·0	−1·1389	−0·0932
1·5	+0·4047	+0·8133	16·0	−1·0908	−0·1208
2·0	+0·2328	+0·7557	17·0	−1·0314	−0·1446
2·5	+0·0730	+0·7008	18·0	−0·9626	−0·1648
3·0	−0·0748	+0·6480	19·0	−0·8854	−0·1817
3·5	−0·2111	+0·5974	20·0	−0·8019	−0·1956
4·0	−0·3368	+6·5489	21·0	−0·7126	−0·2067
4·5	−0·4519	+0·5024	22·0	−0·6192	−0·2153
5·0	−0·5568	+0·4580	23·0	−0·5224	−0·2214
6·0	−0·7388	+0·3747	24·0	−0·4235	−0·2255
7·0	−0·8856	+0·2989	25·0	−0·3229	−0·2275
8·0	−1·0007	+0·2299	26·0	−0·2225	−0·2277
9·0	−1·0868	+0·1673	27·0	−0·1219	−0·2264
10·0	−1·1468	+0·1108	28·0	−0·0226	−0·2234
11·0	−1·1831	+0·0600	29 0	+0·0755	−0·2192
12·0	−1·1984	+0·0146	30·0	+0·1707	−0·2137

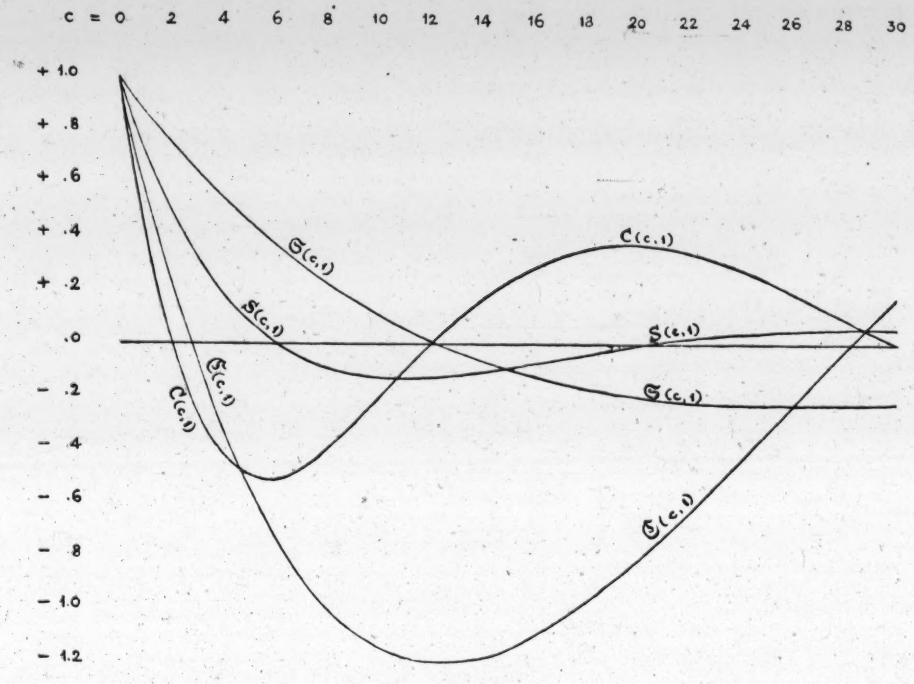


FIG. 2.

An extension of Table IV. beyond $c=30$ is probably unnecessary for practical purposes. That the approximate formulæ for $\mathfrak{C}(c\,,1)$ and $\mathfrak{S}(c\,,1)$ are still valid for $c>30$ may, however, be concluded from the fact that the higher roots of the equations

$$\cos[\sqrt{c+0{\cdot}438}\times 50^{\circ}{\cdot}458]=0 \qquad \text{and}$$
$$\sin[\sqrt{c+0{\cdot}438}\times 50^{\circ}{\cdot}458]=0$$

agree very closely with those of the corresponding Seiche-equations $\mathfrak{C}(c\,,1)=0$ and $\mathfrak{S}(c\,,1)=0$. We find for the former 79·099 and 50·465, whereas the corresponding roots of the latter were previously found to be 79·053 and 50·466. As regards the lower roots which are included in Table III. we have in the case of the hyperbolic Seiche Cosine

$$1{\cdot}19942\cos[\sqrt{c+0{\cdot}438}\times 50^{\circ}{\cdot}458]-0{\cdot}0004=0\,;\ c=\ 2{\cdot}742$$
$$1{\cdot}19942\cos[\sqrt{c+0{\cdot}438}\times 50^{\circ}{\cdot}458]-0{\cdot}0034=0\,;\ c=28{\cdot}230$$

and for the hyperbolic Seiche Sine

$$\frac{1{\cdot}19942}{\sqrt{c+0{\cdot}438}}\sin[\sqrt{c+0{\cdot}438}\times 50^{\circ}{\cdot}458]+0{\cdot}0024=0\,;\ c=12{\cdot}341\,.$$

These roots agree almost exactly with those previously found by an entirely different approximative method (see (35)).

In conclusion, I wish to express my great indebtedness to Professor CHRYSTAL for the interest he has taken in this investigation, and especially for having kindly permitted me to publish this mathematical discourse on the differential equations of his Seiche-problem along with his own physical and mathematical researches. The general type

of differential equations, of which the Seiche-equations are special cases, is well known to mathematicians through the important rôle it plays in the theory of the associated LEGENDRE-functions. But, as far as I am aware, its solutions have been investigated only in the special cases where n represents integers (see WHITTAKER, *Modern Analysis*, p. 235). In the theory of the $C_n^a(w)$ polynomials, which are particular integrals of the general equations here considered, the two synectic integrals, with which we were particularly concerned in this investigation, are of less importance than some of the other hypergeometric series. But their property as synectic solutions renders them particularly useful in the special physical problem before us. It is not improbable that in other problems—for instance, in such which are based on the differential equation of the LEGENDRE-functions ($a = \frac{1}{2}$)—the corresponding synectic integrals, the LEGENDRE Sine and Cosine, might also be of special importance. Since the latter, as well as their associated functions, may be derived from the Seiche-functions by differentiation, we may consider these as the typical representatives of the whole class; and this fact, doubtless, gives to Professor CHRYSTAL'S investigation a considerable importance from the mathematical point of view,—an importance still more enhanced, on the one hand, by the introduction of the corresponding hyperbolic functions, to which he has now for the first time directed the attention of mathematicians, and, on the other, by the remarkable relations between this general class of functions and another class represented by simple transcendents.

XXVII.—The Tardigrada of the Scottish Lochs. By James Murray. *Communicated by* Sir John Murray, K.C.B., etc. (With Four Plates.)

(MS. received April 26, 1905. Read June 5, 1905. Issued separately July 20, 1905.)

Introduction.

Although they are thoroughly aquatic animals, the Tardigrada are not very abundant in permanent waters. They are most thoroughly at home in situations where the supply of moisture is intermittent, and are therefore conspicuous members of that numerous community of animals known as moss-dwellers. They share with the Bdelloid Rotifera the power of withstanding dessication.

Although they have their headquarters in land mosses, many species are quite at home in ponds, rivers, and lakes.

As lacustrine animals they belong entirely to the littoral region, into which they no doubt continually migrate from the adjoining mosses. A favourite habitat is that strip of shore between the highest and lowest levels of the lake, the 'grève inondable' of Forel. Into this often mossy margin they may migrate in the ordinary way, when the loch is low. The next step may be involuntary—the loch rises during floods, and the bears, in common with many other animals, find themselves, willy-nilly, converted into lake-dwellers. It appears to be certain that of the water-bears introduced into lakes, by whatever means, some have found the conditions very congenial. Several species have hitherto been found nowhere but in lakes.

The condition which renders the margins of lakes favourable to many of the moss-haunting animals is, I believe, the thorough aeration of the water resulting from the perpetual lapping of the waves upon the shore; the water of the lake in this respect resembling running water; and there are many species of microscopic animals, so sensitive to impurity that they are never found in bogs or other stagnant waters, which abound in running streams and in the littoral region of large or pure lakes.

No Tardigrade is known to swim—they have no place in the pelagic region of the lakes—nor are any of them truly abyssal, though, like so many other animals in Scottish lochs, they may extend to considerable depths, and several species have been obtained at depths of about 300 feet in Loch Ness.

The observations of the Lake Survey upon Tardigrades have been chiefly made in Loch Ness and Loch Morar. A few collections were made in Loch Treig and one or two other lochs, and an examination of these confirms the belief that some of the water-bears are characteristic of lake margins.

In the preparation of this paper I have been greatly indebted to Mr D. G. Scourfield, and to Prof. Richters of Frankfort, who kindly consented to examine my drawings, and assisted me with advice, and with literature to which I had not myself access. Without this willing help the considerable material at the disposal of the Lake Survey could not have been utilised.

References to the bibliographical list are throughout the text made by figures enclosed in parentheses, thus (1), (15).

The animals are all drawn to the same scale, so that their relative sizes may be seen. The principal measurements are given both in fractions of an inch and in microns. For convenience of reduction the inch is taken as equal to the round 25,000 microns, which is a near enough approximation for practical purposes.

Structure.

The Tardigrada are articulated animals, regarded as having their nearest relatives in the Arachnida.

The structure is only treated here in so far as is necessary for systematic purposes.

Water-bears are segmented animals, having four pairs of jointed legs, the segmentation of both body and limbs very obscure and superficial. The blood consists simply of a body-fluid, filling the whole of the body cavity between the skin and the alimentary canal. In the body-fluid are usually numerous large nucleated cells, formerly regarded as blood-corpuscles, but now called fat-cells (13); small dark granules may also be present.

Skin.—This may be smooth and hyaline, pigmented, papillose, warted, or spiny, indistinctly segmented or thickened dorsally and formed into a series of protective plates, symmetrically arranged (*Echiniscus*).

Legs.—There is usually a distinct basal portion of each leg, the skin of which seems to be an extension of that of the body, forming a kind of sheath for the leg proper; two joints may sometimes be distinguished in this. There may be one or many claws on each leg, which may all be free, or united into groups of two or three.

Head.—Two joints, sometimes three, are distinguishable in the head. There may be many palps and setæ on the head, or none; eyes may be present or absent; there may be a rostral prolongation, or not.

Masticating Apparatus.—This is somewhat elaborate, the food being acted upon by two sets of organs which function as teeth. The *teeth* proper are a pair of straight or curved rods, looking very much like chop-sticks, tapering in front to very fine stiletti-form points which enter the *mouth*, a short, funnel-shaped expansion of the anterior end of the alimentary canal, or the *throat*, a narrowed portion of the tube, separating the mouth from the gullet. These stiletto-like points pierce the cells of the plants or animals which serve as food, which are then sucked. The teeth are much enlarged posteriorly, and usually forked; they may be connected with the gullet by supports

called the *bearers*. The food passes by the *gullet* to the *pharynx*. This is a strong muscular bulb, through the centre of which the alimentary canal passes. The function of this organ is supposed to be to force the food into the stomach, which it does by a pumping action. In several genera there are rows of hard rods or nuts round the tube passing through the pharynx; in these cases the rods seem to have the further function of pounding the food as it passes, thus acting as a second set of teeth. The pharynx, with its rows of hard rods, has some analogy with the mastax of rotifers. A short *œsophagus* leads from the pharynx to the stomach. The large cells forming the walls of the stomach have contents of a characteristic colour—brown, yellow, red, or blue.

Reproduction.—So far as known, all Tardigrades are oviparous. The eggs are spherical, oval, or elliptical, smooth, viscous, or spiny. They are either deposited free, in which case they are spiny or viscous; or they are laid several together in the skin as it is moulted, being then always smooth. In some genera, the larvæ do not differ in any important degree from the adults; in others, they differ considerably, and gradually acquire the adult form through a series of moults.

There is a marked uniformity of structure throughout the whole group, the main classification being founded on no more important characters than the texture of the skin, the number of claws, the form of teeth and pharynx, and the presence or absence of certain feelers on the head.

Echiniscus.

Generic Characters.—Skin of the back thickened and forming a number of plates or shields, symmetrically arranged singly or in pairs. Claws two or four, separate and independent. Two eyes. Teeth and gullet long, straight; no bearers. Four short setæ and two blunt palps near the mouth; two longer lateral setæ between the head and the next segment.

Plates.—The number of plates varies, ten being most common. When this normal number is present they have almost invariably the same arrangement:—(1) The *head plate* or frontal plate; (2) The *shoulder plate*, a larger plate, crossing the back and extending down the sides; (3) *First median plate*, a small triangular plate in the middle of the back, the apex pointing backward; (4) *First pair of plates*, two equal plates, meeting in the middle of the back, and extending down the sides; (5) *Second median plate*, triangular, with apex pointing backward, sometimes quadrangular; (6) *Second pair*, similar to the first pair; (7) *Third median*, triangular, apex directed forward; (8) *Lumbar plate*, a large plate, covering the whole posterior part of the body, and the fourth pair of legs, usually cut into a trefoil by two deep incisions. The middle portion of the trefoil I distinguish as the *tail-piece*. In many species described by Richters it is quite separated from the lumbar plate, and is then called the *anal plate*.

If the number of plates were constant, or if a greater or less number were due to subdivision or suppression, the homologous plates could be distinguished through all the species by their names or numbers. There are, however, some species—(*E. islandicus*)

(13)—in which there are extra plates, the homologues of which are difficult to trace. The third median is often lacking, or it may be united to the lumbar; the second median also is sometimes absent.

Owing to these variations, it is judged better to give under each species a formula setting forth the number and arrangement of the plates, and of the various processes (setæ, spines, knobs) which they bear. When the homologous plates are recognisable, they will be numbered as above.

Richters (9) divides the body into six principal segments:—I. (=*head plate*); II. (=*shoulder plate*); III. (=*first pair*); IV. (=*second pair*); V. (=*lumbar plate*); VI. (=*tail-piece* or *anal plate*).

Processes.—All the species have processes of some sort; many have numerous setæ, spines, or short knobs. Six setæ are invariably present, viz., the four near the mouth, and a pair behind the head. Besides these, there are usually some dorsal and lateral hairs or spines. The dorsal processes arise (with the single exception of *E. gladiator*) from the posterior margins of the plates which bear them. The lateral processes also spring from the posterior margin, at the ventral limit of the plates (postero-ventral angle). They are regarded by Richters as arising independently of the two plates between which they are found; but they seem to me to be always more intimately connected with the anterior of the two, and are often as rigidly joined to it as are the dorsal processes, remaining attached to it after the skin has been cast, and the softer integument between the plates decayed away.

Following the practice of Plate (5) and subsequent writers, the four short setæ of the face are disregarded (as being invariable) and only the longer head pair reckoned among the lateral setæ. It is understood that all processes are paired, and rise from the posterior edges of the plates (except the median spine of *E. gladiator*).

Dorsal processes are rarely found on any but the paired plates; lateral processes may be on any or all of the plates which extend over the sides.

Richters (12) distinguishes the lateral processes by the letters a, b, c, d, e; a = the head seta, b springs from the shoulder plate, c from the first pair, d from the second pair, e from the cut separating the tail-piece or anal plate from the lumbar plate; e might as readily be reckoned dorsal as lateral, as when it is a spine it often rises some distance up the back. The positions of the various processes have been usually indicated by their relations to the four legs; they can be more accurately located by reference to the plates. The lateral processes, b, c, d, e, are over the four legs respectively.

Texture of Skin.—The plates may be quite smooth (*E. islandicus*), but are usually covered with larger or smaller granules, which may be of equal size and uniformly distributed, or irregular both in size and spacing. Some appear to have perforations in place of granules, or show other peculiarities, which will be noticed in the detailed descriptions. The whole of the skin, as well as the plates, is sometimes finely granular, the proximal part of the legs in some species coarsely so.

There is some doubt as to the precise nature of the apparent perforations of the dorsal plates of certain species. I suspect that they may arise by decay of the granules, and for this reason have made no use of them as specific characters. They are found in living animals, but are commoner in empty skins. That they are of some specific value is shown by their constant occurrence in association with definite arrangements of plates and spines. Where they occur, the perforations are very distinct, and marked by clean sharp edges.

Legs.—The first leg has often a small sharp spine, the last leg a similar spine, or more commonly a blunt palp, near the base. Both of these have probably been generally overlooked, and may yet be found to be always present. The last leg has, in most species, a serrate fold of skin about the middle of its length, which I call the *fringe*. The claws are four in number in most species, probably in all, when fully grown and mature. Generally the inner claws of each four have a decurved spine, called the *barb*, near the base, or as high as half-way up the claw. The outer claws are devoid of barbs, except in a very few species (*E. blumi*, etc.). The barbs of the outer claws are straight, and point outwards or upwards. The barbs of the last legs are larger than the others, and often these alone have them; there may be as many as three barbs on each outer claw of the last legs.

Teeth and Pharynx.—The teeth are always very long and straight, enlarged and forked at the ends, which are often closely applied to the pharynx; the points enter the mouth. The pharynx is sometimes minute and round, sometimes pretty large and somewhat cordate. As a rule there are no rods, such as are found in *Macrobiotus*, two obscure curved lines which diverge from the end of the gullet probably representing them; but RICHTERS has seen rods in *E. islandicus*. I have never seen *simplex* forms in this genus.

Reproduction.—All lay the eggs in the moulted skin. It has been thought that the number of eggs is characteristic, as it is in many species of *Macrobiotus*. To a certain extent this is so; but two species have been observed to lay eggs when still very small (*E. mutabilis* and *E. reticulatus*), laying then only one egg, while larger examples laid two, three, and four respectively.

Development.—Comparatively few of the species have been seen to hatch out, all which have been so observed having only two claws, which leads RICHTERS to suppose that all hatch in this form. Against this must be placed the fact that some individuals, of species which lay large eggs, have been found, so small that they might easily be supposed to be newly-hatched larvæ, but with four claws and all the outward characters of the adult. These species were abundant in the collections where they occurred, and increased in them, yet two-clawed larvæ were never found (*e.g. E. reticulatus*).

The larvæ usually lack some of the setæ possessed by the adult, and those which they have are relatively shorter. I don't know that any example has ever been kept under observation from its hatching to maturity; but where a species is abundant, examples may be selected at all stages of growth. In examples which I have seen

moulting, there was a considerable elongation of all the dorsal and lateral processes, as well as development of the fringe and of the barbs of the outer claws. Many observations of a species, common in Loch Morar, and which I regard as belonging to *E. granulatus*, are instructive as bearing on the value of all those points as specific characters (Plate II. figs. 6*a* to 6*j*). The larva was not certainly seen, but many moults of large animals were observed. The youngest individuals seen had four claws, without barbs on the outer ones; the fringe consisting of short blunt processes standing far apart; the dorsal processes being a large spine on the first pair of plates, a short spine on the second pair, and a mere knob on the lumbar plate; the lateral setæ were *c* and *d* (RICHTERS).

In a single moult the large dorsal spine elongated till it might be called a seta, the short dorsal spine became a long one, the knob became a spine, the fringe acquired long teeth standing close together, straight barbs appeared on the outer claws of the last legs, and the lateral setæ elongated. In the last stage seen there were two pairs of dorsal setæ, the first very long; the lateral setæ were also very long; the outer claws of all the legs had straight barbs, and those of the last legs had three such barbs.

Although no new processes appeared during these moults, except the barbs of the outer claws, the changes are sufficiently great to render imperative extreme caution in separating species by any of those characters, even if sexually mature individuals are seen. We may fully expect that the working out of the life histories would lead to the union of several of the earlier described species, and perhaps of some of the later ones as well.

Species.—All observers have agreed in basing their species chiefly upon the number and position of the spines or other processes; but it now begins to be suspected that this may carry us too far, and give us a multitude of species founded upon larval forms. RICHTERS advises that no species be described unless the eggs have been seen, or there is some very marked peculiarity. Now that it is known that some species lay eggs when not nearly full grown, even this rule may not be a perfect safeguard. It is a safe rule that no form should be separated from a known species merely because of one pair of spines more or less, or a difference in size of these appendages, unless there are other characters, whether of texture, claws, fringe, or what not, to support it. Another rule, laid down by JENNINGS (7) in regard to Rotifera, might well be applied to Tardigrada, viz., that no species should be described without an accompanying figure. JENNINGS' remark, that in most instances the description could be better dispensed with than the figure, applies equally to all microscopic biology.

Echiniscus arctomys—EHR. (4), (5), (9).

Specific Characters.—Small; nine minutely punctate dorsal plates. No setæ or spines except the six invariably present on the head. Legs slender, no fringe on last pair; all claws without barbs.

In many respects this seems like the larva of some larger species, but the eggs have been frequently seen. It is distinguished mainly by the lack of characters (fringe, barbs, setæ), which most species have when full grown. No formula of the arrangement of the plates can be given, as in the examples observed they were obscurely separated, the median plates being especially uncertain, and the separation of the pairs indistinct. The number, nine, is that given by PLATE. The whole skin is punctate. There is a blunt palp on the fourth leg.

Loch Ness, frequent, 1904.

Echiniscus gladiator, n. sp. (Plate I. figs. 1*a* to 1*c*.)

Specific Characters.—Small, yellow or pale red, all minutely punctate. Plates obscure, the median slightly developed, the pairs hardly divided. Large median recurved spine on anterior edge of second pair of plates. Lumbar plate deeply trefoliate. Legs slender, no fringe on last; inner claws with decurved barb, very strong on those of last legs. Eyes clear, not pigmented.

This is related to *E. arctomys*, which it resembles in narrow form, slender legs, lack of fringe, and minute punctation, and like that it might be a larva. The eggs have not been observed in this instance; but it differs markedly from *E. arctomys*, the only species to which it has any resemblance, not only in the great median spine, but in the strong barbs of the last inner claws.

Length, up to about $\frac{1}{93}$ inch (= 269μ).

Among mosses and hepatics from the shores of Burlom Bay, Loch Ness, frequent; in Loch Ness, rare, February 1904.

Echiniscus mutabilis, n. sp. (Plate I. figs. 2*a* to 2*d*.)

Specific Characters.—Fairly large, narrow, yellow, all minutely punctate with pellucid dots. Plates many, scarcely of firmer texture than the rest of the integument —partly outlined by folds, partly indicated only by interruption of the dots. Median line on most plates caused by cessation of the dots. No fringe on last legs. Inner claws with small decurved barb.

Arrangement of Plates.—(1) Head, entire, with usual six setæ and two palps; (2) Shoulder, divided in pair or four; (3) Median, triangular, divided in three; (4) Pair, entire; (5) Median, triangular, divided in three; (6) Pair, entire; (7) Median, triangular, divided in two; (8) Lumbar, divided in five (an anterior pair, and the usual trefoil).

The usual ten plates are present, the additional ones arising from division of these. Two varieties are distinguished:—(*a*) Plates sharply outlined, the lumbar having its anterior portion separated as a distinct pair of plates, which partly overlap the posterior trefoil; dots comparatively large, regularly spaced. (Plate I. fig. 2*a*.) (*b*) Plates

rather more numerous (from further subdivision), some of them only faintly indicated, the lumbar with its anterior portion not forming separate plates—general arrangement the same; dots very minute, as in *E. arctomys*. (Plate I. fig. 2*b*.)

These two varieties can be distinguished among the smallest individuals, and appear to be constant, no intermediate states having been found. The differences are not sexual, both having been repeatedly found with eggs. A spine on the first leg, and palp on the fourth, have been frequently seen in both varieties.

Reproduction.—Eggs from one to four in number, laid in the moulted skin. An example measuring $\frac{1}{216}$ inch (about 116μ) laid a single narrow egg, which measured $\frac{1}{580}$ inch by $\frac{1}{945}$ inch (43μ by 26μ). Larger examples laid two, still larger three, and the largest observed four eggs, which are larger and relatively broader, those in one skin measuring $\frac{1}{375}$ inch by $\frac{1}{500}$ inch (66μ by 50μ). They are usually dull yellow, but sometimes pale red. It appears from the above measurements that the species lays eggs when far from full grown. The newly-hatched larva has not been seen, but young measuring no more than $\frac{1}{227}$ inch (110μ) had four claws, the inner barbed, and all other outward features of the adult.

Related to *E. arctomys*, which it resembles in narrow form, obscure plates, slender legs, lack of fringe, and in the finely punctate skin, it differs in the larger size, more numerous plates, and in having barbs on all the inner claws. Size, up to $\frac{1}{93}$ inch, exclusive of legs (= 269μ).

In Loch Ness, Loch Morar, and ponds at Fort Augustus, abundant—1903–4.

Echiniscus wendti—Richters. (Plate I. figs. 3*a* to 3*c*.) (10), (15).

Specific Characters.—No setæ except the usual six on the head, the lateral setæ at the back of the head twice as long as in *E. arctomys*. A fringe on the last legs. A strong decurved barb on inner claws of last legs. Granulation small and uniform. A spine on the first leg, and a blunt palp at the base of the fourth.

Arrangement of Plates.—(1) Head; (2) Shoulder; (3) Median triangular; (4) Pair; (5) Median triangular; (6) Pair; (7) lacking; (8) Lumbar, trefoliate.

Its discoverer distinguishes the species by the long head seta, the fringe, and the strong barb. Examples from Loch Morar agree in all those characters, but the granulation is rather coarse, and appears to be variable.

Loch Morar, 1904, frequent.

Echiniscus reticulatus, n. sp. (Plate I. figs. 4*a* to 4*c*.)

Specific Characters.—Stout, broad, bright red. Plates ten, arranged on the normal plan. Lateral setæ on head very long. Plates covered with pattern of large hexagons or circles, a slightly raised rim enclosing a flattish depressed surface. A long

sharp spine on front legs, a blunter spine on the last legs. Fringe on last legs. All inner claws with very small decurved barb near base.

Resembling *E. wendti* in having very long lateral setæ on the head and no other setæ on the body, it differs in the texture of the plates, the small barbs of the inner claws, and the presence of the third median plate. This plate is variable, and sometimes appears to be united to the lumbar plate, though at other times quite distinct. The hexagonal pattern on this plate is very faint or quite obsolete. The lumbar plate is rendered trefoliate by two deep cuts, and is besides divided into four facets, the two lateral and the posterior facets being bent at a sharp angle to the median facet. This gives the appearance of a separate anal plate; but there is no real separation, the pattern passing uninterrupted over the angle. The length of the lateral seta is equal to the diameter of the body at the shoulder, or much greater. The teeth of the fringe are often bifid.

Reproduction.—One to four eggs laid in the cast skin, the larger number laid by larger, and presumably older, individuals. A skin measuring $\frac{1}{117}$ inch (214μ) contained three bright red eggs of $\frac{1}{351}$ inch by $\frac{1}{422}$ inch (71μ by 59μ). The head seta in this measured $\frac{1}{176}$ inch (142μ). The newly-hatched larvæ have not been seen, but individuals so small that it might be supposed they had not moulted since hatching had four claws, with the inner barbed, and the reticulated plates quite distinct.

Loch Morar, very abundant, Loch Ness, rare—1903–4. It has not yet been found anywhere except in lakes.

Echiniscus oihonnæ—Richters. (Plate I. figs. 5*a*–5*b*.) (10), (15).

Specific Characters.—Small, plates ten, only two median triangular, anal plate separate. Five lateral setæ or spines, and two dorsal spines, on each side. Fringe on last legs. Barbs on the inner claws, those of the last claws very strong.

Arrangement of Plates.—(1) Head, with longish lateral seta; (2) Shoulder, long lateral spine and smaller one above it; (3) Median, triangular; (4) Pair, lateral seta and small spine, strong dorsal seta; (5) Median, triangular; (6) Pair, lateral curved spine and short spine, dorsal short spine; (7) lacking; (8) Lumbar, very long whip-like seta; (9) Anal. In Scottish examples the anal plate is not separate.

Only two examples seen; no eggs. Margin of Loch Ness, February 1904. Loch Earn.

Echiniscus granulatus—Doy. (Plate II. figs. 6*a* to 6*f*.) (3), (5).

Specific Characters.—Plates nine, arranged in the normal manner, coarsely granulate. Three long lateral setæ, and a short spine at the junction of the tail-piece with the lumbar plate; two dorsal setæ or spines on each side. Spine on front leg, and blunt palp on last leg. Fringe on last leg. Inner claws with decurved barbs.

Arrangement of Plates.—(1) Head, with moderate lateral setæ; (2) Shoulder;

(3) Median, triangular; (4) Pair, long lateral seta, and strong dorsal spine or seta; (5) Median, triangular; (6) Pair, long lateral seta, and strong dorsal spine or seta; (7) lacking; (8) Lumbar, trefoliate, with short spine.

Among forms agreeing with *E. granulatus* in having three dorsal processes, on the first and second paired plates and the lumbar plate respectively, diminishing in size from the first backwards, there is considerable variety in the size of the processes and in the number of lateral setæ. The dorsal processes may all be spines; or the first, or both first and second, may be setæ. There may be only two lateral setæ, *a* and *d*; or they may be three, *a*, *c*, and *d*. There is no justification for separating any of these as distinct species, unless after a full study of the development, as the eggs are unknown. Three varieties were found in Loch Morar.

First (figs. 6*a* and 6*b*). An elongate, large animal, with only two lateral setæ, *a* and *d*, one dorsal seta, and two spines. Straight barbs on the outer claws of last legs. Granules variable in different examples—very coarse or moderately fine; uniform. Pharynx large, cordate.

Though large, it has not been seen with eggs.

Size, $\frac{1}{85}$ inch = 294μ

Second. Two large dorsal spines and a small one; three lateral setæ, *c* and *d* very long. Barbs of outer claws not seen. Granules moderate. Size, $\frac{1}{90}$ inch = 277μ.

Third (figs. 6*c* and 6*d*), like the second, but first dorsal process a very long seta. Straight barbs on the outer claws, up to three on those of the last legs. Granules moderate. Size, up to $\frac{1}{80}$ inch = 312μ.

The species resembles *E. blumi* (15) in having barbs on the outer claws. That species has, however, more numerous lateral and fewer dorsal processes. The elongation of the last dorsal process in *E. granulatus* would produce an animal like *E. blumi*. The absence of the barbs of the outer claws from descriptions of species must not be regarded as of much importance, as they may have been overlooked, or they may only appear at a late stage in development. For fuller account of appearance of barbs in this species, see *ante*, under development of *Echiniscus*.

Habitat.—Loch Morar, abundant; Loch Ness, frequent.

A larva, probably of this species, is shown in fig. 6*e*.

The first variety described above differs from this larva in that the second lateral process is on the second paired plates, instead of the first; so it may prove to belong to another species.

Echiniscus spitzbergensis—Scourfield. (Plate II. figs. 7*a* to 7*c*.) (6).

Specific Characters.—Plates nine, arrangement normal. Four lateral setæ (one on the head, shoulder, and each of the paired plates); long dorsal seta on first pair, and shorter spine on second pair. Inner claws with small decurved barbs. Granules very large.

I identify as this species an animal rare in Loch Morar. Though differing in some details from Mr SCOURFIELD's species, I do not think we would be justified in separating it, considering that there were no eggs seen in either case, and in view of the great change in the size of the processes which takes place during development. Only empty skins of the Loch Morar animal were seen. In place of the granules the plates were covered by large quoit-like rings, the centres perforate, which I believe to originate in the decay of the granules. They further differed in the dorsal spines on the second pair of plates being long, and the small spines of the same plates lacking. The lateral setæ, *a*, *b*, *c*, *d* (RICHTERS), increase in size from *a* to *d*, which is very long. The lumbar plate is trefoliate; in SCOURFIELD's examples, entire.

Length, Loch Morar examples, $\frac{1}{100}$ inch (250μ).

DOUBTFUL SPECIES.

Many examples of *Echiniscus* have been found which, while differing more or less from the descriptions of all known species, could not, in the absence of eggs, be certainly identified, or regarded as distinct species. They are figured here, with short descriptions, as an assistance to other observers. All were sufficiently large to be regarded as probably nearly full grown, though size is not a quite safe criterion of age.

Echiniscus, sp.? (Plate II. figs. 8*a*–8*b*.)

Plates ten, normal. Lateral setæ five, increasing in length from front to back. Dorsal processes—a long seta on the first paired plates, a short knob on the second pair. Fringe on last legs. Mid claws barbed. Granules of moderate size. The section of the genus having five lateral processes contains about a dozen species. Some of them (*E. duboisi*, *E. conifer*, *E. spinulosus*, *E. oihonnæ*, etc.) have very distinct characters. If we bear in mind the elongation of the processes during development, many of the other species will appear less certain, and it is noteworthy that the eggs of most of these are unknown (**14**). Some of the forms having fewer lateral processes may be younger stages of the same species. This and the two following forms belong to this section, differing mainly in the proportions of the processes.

Loch Ness, at pier, 7th February 1904.

Echiniscus, sp.? (Plate II. figs. 9*a*–9*b*.)

Plates normal. Four of the lateral processes are long spines (? setæ) with bulbose bases. Dorsal processes—a long spine on the first paired plates, a very short broad spine on the second pair. Granular or perforate. This form, with small perforations, as shown in fig. 9*c*, was frequent in Loch Morar, and was regarded as a distinct species till another form was observed, identical with it in all else, but with fairly large

uniform granules. It is a curious fact that the perforations have a quite different size and arrangement from the granules, so that they could not be derived from them, as was supposed to be the case with a similar form of *E. spitzbergensis.* (Plate II. fig. 7*a*.)

Echiniscus, sp. ? (Plate II. fig. 10*a*.)

Two lateral setæ (one after the plates of the first pair). Dorsal seta on plates of first pair, short broad spine on plates of second pair. Spine on front leg. Fringe. Inner claws barbed. Nearest *E. aculeatus*, differs in lateral process not double. (5).

Loch Ness, November 1903.

Two-clawed Larvæ.

Three larval forms having two claws were seen. Two of these are referred to under the species to which they are supposed to belong (*E. wendti*, *E. granulatus*). The third could not be identified.

Echiniscus, sp. ?, larva. (Plate II. fig. 11.)

Plates ten, arrangement normal. Three lateral processes—a short curved spine on plate of second pair, a longer seta at junction of tail-piece and lumbar plate. No dorsal processes. Granules moderate. Fringe of longish blunt spines. Claws two, the barbs large. Blunt palp on last legs. Mouth palp appears to spring from elongate curved process which bears the anterior mouth seta. Size $\frac{1}{166}$ inch.

Shore of Loch Ness at Fort Augustus.

Macrobiotus.

Generic Characters.—Obscurely segmented, without hardened dorsal plates. Claws four, united in pairs, or one pair and two free claws. Teeth with bearers; gullet short, rigid. Pharynx with several rows of hard rods or balls.

The genus *Doyeria* cannot now be maintained, as it has been shown by Richters that most (and probably all) species of *Macrobiotus* may get into a condition in which the teeth are as in Plate's genus *Doyeria*. The distinction of the genus *Diphascon* is also a slender one, there being intermediate forms between it and *Macrobiotus*.

The species of *Macrobiotus* are distinguished by the form of the claws, the texture of the integument, and the number and arrangement of the pharyngeal thickenings.

The last character is most reliable, but in many individuals its value is lessened by a curious reduction of parts which takes place. The eyes are of little importance, as

they may be present in some individuals of a species and not in others. The fat-cells in the body-fluid have a characteristic colour; in most species they are clear and hyaline, but in a few they are golden yellow or dark brown.

The segments of the body are superficial, affecting the skin only. There appear to be usually two segments to the head, one to each pair of limbs, and intermediate segments—ten in all; but they are often increased in number by subdivision, and there are very commonly two between the third and fourth legs.

Simplex Forms (15).—Individuals of species of Macrobiotus are often found which exhibit a remarkable reduction of the masticating apparatus: the teeth are straight, without stays—they are not functional; the rods of the pharynx disappear; in extreme cases, the mouth and gullet are quite obliterated. This state can only be temporary, or the animals would die; and they often appear in good health, and may have the stomach filled with food. I can only suggest that it is temporary, and a preliminary to moulting; but if this is so, it is a remarkable parallel, among animals so high in the scale, to the disappearance of the mouth in ciliata during fission.

In exceptional cases, the pharynx and teeth entirely disappear.

Reproduction.—Two forms of eggs are laid, the one kind round and spiny, the other smooth and oval or elliptical. The spiny eggs are laid singly and free; the smooth eggs are laid in the moulted skin, which serves as a protective capsule for them. So far as known, the same species always lays the same kind of egg. The smooth and spiny eggs are not, as from analogy we would expect, the summer and winter eggs of the same species; but further observation on the point is needed. The laying of smooth eggs in the cast skin is the prevalent mode of reproduction in the genus as in the order. It is very difficult to trace which species lay spiny eggs, as, for some unexplained reason, animals containing such eggs are very rarely seen. When the young contained in the smooth eggs are ready to hatch, it is seen that the teeth and pharynx are very large and fully developed. The stiletto-like teeth are continually applied to one spot in the shell till they weaken and finally pierce it. At this stage the pharynx is not very greatly inferior in size to that of the mature animal, and the characteristic thickenings are all present.

Two groups of species are to be distinguished in the genus. The first, typified by *M. hufelandi*, have the two pairs of claws similar, the claws strong, the claws of each pair rigidly united and one of them slightly larger than the other, the larger claw of each pair with a strong supplementary point. The second group includes species having the two pairs of claws dissimilar, slightly united at the base only, the larger pair having one very long slender claw and a much shorter one, the other pair similar but smaller, or of two nearly equal claws—supplementary points none, or very fine.

I believe all species of the first group lay spiny or viscous eggs; those of the second, smooth eggs enclosed in the skin. The species having two single claws and a pair form an extension of the second group.

Macrobiotus hufelandi—C. Sch. (1), (14).

Specific Characters.—Large, dark-coloured, dark granules, in addition to the fat-cells, in the body-fluid. Pharynx large, shortly elliptical, with two narrow rods and a small nut in each row of thickenings. Teeth large, strong, curved, with strong bearers, entering the throat. Claws, two similar pairs, each pair of a longer and a shorter claw closely united, the larger claw of each pair with a double point (supplementary claw near apex).

This is the water-bear *par excellence*, though no doubt the early observers confused several species together under this name. It appears to be widely distributed over the world, though perhaps less so than was formerly supposed.

There is a group of species, all very closely related to *M. hufelandi*, some so closely that they can only be distinguished by the different forms of the egg spines. These occur all over the world, and have no doubt been often mistaken for *M. hufelandi*, in the absence of eggs.

One of the largest Tardigrada, attaining to $\frac{1}{40}$ inch (625μ), and perhaps upwards. Eyes are normally present—the blind condition having been described as a distinct species (*M. schulzei*, Greeff).

Habitat.—Common in the shallow waters of lakes; in Loch Ness it has been found at a depth of 300 feet. Loch Morar; Loch Treig.

Macrobiotus echinogenitus—Richters. (10), (14), (15).

Specific Characters.—Hardly distinguishable from *M. hufelandi* except by the eggs, which are covered with conical processes, having acute—often curved—tapering points. Those of *M. hufelandi* have the processes narrower cones, expanding at the apices into little discs.

I have seen only the *simplex* form of this. The eggs are, however, very abundant in Loch Morar.

Habitat.—Loch Ness, Loch Morar; common.

Macrobiotus islandicus—Richters. (Plate III. figs. 12*a* to 12*c*.) (13).

Specific Characters.—Hyaline, except stomach. Teeth strongly curved, with bearers; teeth enter the mouth. Pharynx round, two short rods in each row, each about twice as long as broad, besides a little round nut attached to the end of the gullet. Claws, two unequal pairs, the longer claw of each pair with a supplementary point. Stomach cells filled with dark blue granules.

The eggs were not seen, but Richters found them in Iceland.

Loch Ness, common, 1903–4. Not yet seen elsewhere.

Macrobiotus ornatus—Richters. (Plate III. figs. 13*a* to 13*c*.) (8).

Specific Characters.—Glabrous and spineless, or finely or coarsely papillose, covered with large granules on the back, or with many rows of long spines on the back and sides. With or without eyes. Teeth somewhat weak, slightly curved, with bearers. Pharynx circular, thickenings three in each row, round or nearly so. Claws, two similar pairs, one of each pair longer.

I follow Richters in making the verrucose and spiny forms mere varieties of one species, although I have seen no intermediate varieties, and would have regarded them as distinct. As a logical consequence, the glabrous form must also be united with them. Of the three varieties, the warted one is largest, and is the only one possessing eyes (in Scottish examples). The spiny form has only been found in ground moss and in ponds, not yet in lakes. The other two varieties are lacustrine, the glabrous one being very frequent at lake margins.

Glen Roy, 1902—pond at Fort Augustus (var. *spinosissimus*); Loch Ness (var. *verrucosus*); Loch Ness, Loch Morar, Loch Treig, smooth variety.

The eggs have not been observed in Scotland, but Richters found them in the cast-off skin.

Macrobiotus annulatus, n. sp. (Plate III. figs. 14*a* to 14*c*.)

Specific Characters.—Skin pale yellow, stomach brown. All papillose except face and distal portion of legs. Papillæ large, round, equal; on back and sides arranged in regular lines running round the body, but lost on the under surface. The usual apparent segments of *Macrobiotus* here divided into lesser segments, on each of which are two or three of the rows of tubercles. Two black eyes. Teeth strong, curved, with bearers. Pharynx nearly as broad as long, round or slightly cordate. Two narrow rods in each row, and a lesser round nut next the end of the gullet. Claws, two pairs slightly united, one claw of each pair longer than the other. Longer claws, with fine supplementary points.

Reproduction.—Three elliptical eggs are usually laid in the moulted skin; they measure about $\frac{1}{384}$ inch (67μ) long. A curious habit prevails, which I have not observed or heard of in any other Tardigrade. The skin is not completely moulted, but remains attached to the front of the head, and is carried about, with its contained eggs, for a long time, in some cases till the eggs hatch. As all my observations were made upon animals kept in captivity, and therefore under conditions different from those to which they would be subjected in their natural home, we cannot be sure that this habit is normal. It is noteworthy, however, that on every occasion when it was observed the eggs were thus carried, and other species kept in the same way did not do so. It was under almost continuous observation for more than a year, and many hundreds of examples were seen carrying the skinful of eggs, and the practice was repeated by successive generations.

It is difficult to imagine how the eggs can be deposited in the skin while it remains attached to the head, so that one is tempted to suppose that the moult is completed in the usual way, and the skin picked up again afterwards; but this has not been seen.

The egg measuring $\frac{1}{384}$ inch (67μ), produced a larva $\frac{1}{176}$ inch (142μ) long. The pharynx was $\frac{1}{1056}$ inch (24μ) long. There was no trace of eyes nor of papillæ on the skin or supplementary points to the longer claws, but otherwise the form was as the adult.

Size, about $\frac{1}{60}$ inch (417μ) or larger. Having some resemblance to *M. granulatus* —Richters (10), which has, however, claws of quite different structure. The supplementary points of the longer claws of each pair are much more distinct than is usual in species having smooth eggs.

Habitat.—Bog pool at Fort Augustus, very abundant; margin of Loch Morar, rare, 1904.

Macrobiotus papillifer, n. sp. (Plate III. figs. 15*a* to 15*c*.)

Specific Characters.—Hyaline, two black eyes. Back and sides covered with conical acuminate processes, arranged in transverse and longitudinal rows. Similar processes on the head, or lacking. Teeth strong, curved, with bearers. Pharynx nearly as broad as long, with three equal thickenings in each row, which are about twice as long as broad. Claws, two nearly equal pairs, one claw of each pair longer.

Length, up to $\frac{1}{100}$ inch (250 microns). Eggs laid in the cast-off skin. Five eggs were laid in one skin, the animal being seen to leave the old skin by the anterior end.

Habitat.—Loch Ness, common; Loch Morar, rare.

This species is comparable with *M. tuberculatus*—Plate (5). The processes are more numerous and of different form; but this would not justify its separation, if we had not a more reliable character in the relatively large pharynx, with three short rods in each row of thickenings. Plate says that there are only two rods in each row in *M. tuberculatus*, though his figure shows three.

Scourfield, who has seen *M. tuberculatus*, regards this as distinct.

Macrobiotus oberhäuseri—Doy. (3).

Specific Characters.—Dorsum, with nine transverse bands of a brown colour. Pharynx, small round, with three short oval thickenings in each row. Claws, one pair and two single claws. Eggs laid in the cast-off skin.

Various diverging if not conflicting diagnoses of this species are given by different authors, and it is probable that different species have been confused together. An animal having the transverse bands of colour and small pharynx was observed in Loch Ness, but it had not the two free independent claws which, according to Plate, this species should have. This probably indicates only different interpretations of the structure of the slightly united larger pair of claws.

Habitat.—Frequent in Loch Ness and Loch Morar; occurring at considerable depths in Loch Ness.

Macrobiotus macronyx—Doy. (Plate III. figs. 16*a* to 16*d*.) (2), (11), (12).

Specific Characters.—The claws of each pair of unequal sizes, independently movable (5). Teeth and pharynx as in *M. hufelandi*, but the thickenings are slender rods. Two eyes. Numerous eggs laid in the cast skin.

In a short stream draining Loch Geireann Mill, North Uist, abundant among *Fontinalis*. High tides reach to the spot where it was found, May 1904.

From a drawing sent to him Richters determined the Lake Survey examples to belong to this species, but the animal differs in some respects both from Plate's description and from Richters' own figure (11). The smaller claw of each pair is relatively much smaller than Richters draws it, and I saw no indication that they were separately movable. The lesser claws of all the legs, except the fourth, appeared so reduced and closely united to the larger claws as to resemble the barbs of the inner claws of *Echiniscus*. The lesser claws of the fourth pair of legs are considerably larger, but even these do not appear to be separately movable.

Macrobiotus intermedius—Plate. (5).

Specific Characters.—Claws, two similar pairs, strong, one of each pair longer and with double point. Pharynx small, round; no eyes.

Resembling *M. hufelandi* in the form of the claws, and *M. oberhäuseri* in the small round pharynx.

Habitat.—Margin of Loch Ness, frequent.

Macrobiotus, sp.? (Plate III. figs. 17*a* to 17*c*.)

Only cast skins enclosing eggs having been seen, the animal cannot be named or fully described. It is of moderate size, the whole of the skin covered by a hexagonal reticulation, very uniform and regular. Each hexagon is a slight concavity surrounded by a raised edge. There is a boss on each of the last legs. The claws are two pairs of nearly equal size, stout, closely united, with no supplementary points. The five eggs are very small.

Loch Morar, rare.

Eggs of Macrobiotus. (Plate IV. figs. 18 to 22.)

It has been already pointed out that several species related to *M. hufelandi* lay spiny eggs. Richters supposes that the spiny armature serves to secure dissemination through the action of rain. It appears that there are more species of *Macrobiotus* in

the Scottish Lakes than have been recognised, as several forms of eggs have been found which cannot be identified as those of any known species. Five forms are here figured, only two of which can be named. The others are not proven to be even Tardigrade eggs, but they are so similar to eggs of known species as to leave little doubt on the matter.

M. hufelandi—C. Sch. (Fig. 18.) (14).

Processes narrow, tapering, expanded at ends into discs. There is a good deal of variation in the size of the eggs, and in the size and spacing of the spines.

Loch Ness and Loch Morar, common.

M. echinogenitus—Richters. (Fig. 21.)

Processes conical, acuminate, papillose, the long points usually curved over (in Scottish examples). There appears to be even more variation than in *M. hufelandi*, according to Richters (14). Since, as he says (10), the two species can hardly be distinguished except by the eggs, it may well be contended that some of these varieties of eggs represent equally good species.

Loch Morar, very abundant.

Macrobiotus, sp.? (Fig. 19.)

Possibly a variety of *M. echinogenitus*, but more probably distinct. Loch Morar.

Macrobiotus, sp.? (Fig. 20.)

This beautiful egg is also possibly a variety of *M. echinogenitus*. It is most like Richters' variety with blunt processes (14), but they are here acuminate. Loch Morar.

Macrobiotus, sp.? (Fig. 22.)

Conical processes close together at the bases. Loch Ness.

Diphascon.

Generic Characters.—Gullet elongated between the teeth and the pharynx into a flexible tube; otherwise as *Macrobiotus*.

In those species which have the flexible gullet well developed this seems to be a sufficiently distinct genus, but it is connected with *Macrobiotus* by *D. angustatum*, in which the flexible portion is very short and hardly distinguishable, and the two genera may have to be united. All the species known to me are destitute of eyes. Teeth-bearers are usually present, but they are often weak, and in *D. angustatum* they are frequently absent.

Diphascon chilense—PLATE. (Plate IV. figs. 23*a*–23*b*.) (5).

Specific Characters.—Pharynx small, nearly circular, the thickenings short, roundish, four or five in each row. Throat and gullet slender. Claws of two nearly equal pairs, one claw of each pair somewhat longer.

This description is based on PLATE's figure, as he gives no specific characters. The Lake Survey specimens were hyaline, and measured about $\frac{1}{110}$ inch (227μ). The short diameter of the elliptical pharynx was fully $\frac{3}{4}$ of the long diameter. There were three short rods in each row, with a smaller round nut at each end of the row, the anterior one connected with the end of the gullet.

This species, on which PLATE founded the genus, is widely diffused. RICHTERS found it as far north as Tromsö, and it ranges into the antarctic circle. It shows some variation in different localities, but not enough to justify the separation of varieties.

Habitat.—Among moss at the margin of Loch Ness, not in the loch, February 1904.

Diphascon spitzbergense—RICHTERS. (Plate IV. figs. 24*a*–24*b*.) (10), (15).

Specific Characters.—Pharynx narrow, elongate, short diameter $\frac{2}{3}$ to $\frac{1}{2}$ of long diameter; thickenings of straight rods—(1st) at anterior end a short rod, (2nd) a longer rod, (3rd) a little round nut. Teeth longer and stronger than in *D. chilense*, nearly straight, with strong bearers. Gullet much thicker than in *D. chilense*, and shorter.

The specimens taken in Loch Ness, though differing in some small particulars, must be referred to this species. The pharynx is relatively broader, but the arrangement of rods is identical. The animal is hyaline; the claws two unequal pairs, the longer claw of one pair considerably longer than that of the other.

Habitat.—Loch Ness, March 1904.

Diphascon angustatum, n. sp. (Plate IV. fig. 25*a* to 25*c*.)

Specific Characters.—Large, hyaline, broadest about the third legs, thence tapering to the narrow snout-like head. Pharynx narrow, twice as long as broad; two slender rods in each row—first short, second twice as long. Gullet very short and wide, marked by annular rings, very slightly flexible. Mouth and throat wide. Teeth straight, only slightly divergent, with or without small weak bearers. Claws two unequal pairs, one pair with one claw long and slender.

Distinguished from *D. spitzbergense* by the general form, tapering anteriorly, the shorter and wider gullet, still narrower pharynx, and number of pharyngeal thickenings.

Habitat.—Loch Ness, February 1904. Common. RICHTERS has also observed this species in Germany.

Milnesium.

Generic Characters.—Mouth surrounded by circlet of six blunt palps; two similar palps further back, as in *Echiniscus*; pharynx narrow, without hard rods; each foot with four claws, of which two are long and slender, the others short and two- or three-branched.

Resembling *Echiniscus* in having a pair of blunt processes on the face, otherwise nearer *Macrobiotus*.

Milnesium tardigradum—Doy. (Plate IV. figs. 26*a* to 26*c*.) (3), (9).

Specific Characters.—Large, hyaline; fish-shaped, tapering to both ends. Mouth and gullet very wide and short; teeth short, with weak bearers. Pharynx about twice as long as broad. A pair of black eyes. The single claws very long and slender, with expanded base; the triple-hooked claws short, broad, two of the hooks nearly equal, a smaller one near base.

According to Richters, *M. alpigenum* (Ehr.) is not a distinct species.

This is a widely distributed form.

Habitat.—Loch Ness, among moss growing on pier, February 1904.

LIST OF WORKS REFERRED TO.

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(3) Doyère, *Ann. Sc. Nat.*, Paris. II. Sér. T. 14, p. 269. 1840.
(4) Ehrenberg, *Mikrogeologie*, 1854; Atlas, pl. 35B.
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(10) ,, ,, "Nordische Tardigraden," *Zool. Ang.*, Bd. xxvii., 1903, p. 168.
(11) ,, ,, "Der kleine Wasserbär," *Prometheus*, 1903, p. 44.
(12) ,, ,, "Verbreitung der Tardigraden," *Zool. Ang.*, Bd. xxviii., 1904, p. 347.
(13) ,, ,, "Islandische Tardigraden," *Zool. Ang.*, Bd. xxviii., 1904, p. 373.
(14) ,, ,, "Eier der Tardigraden," *Ber. Senckenbg. Natf. Ges.*, 1904, p. 59.
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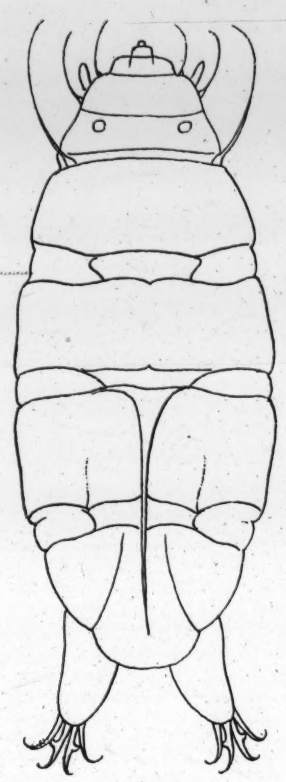
MURRAY: TARDIGRADA OF THE SCOTTISH LOCHS.—PLATE I.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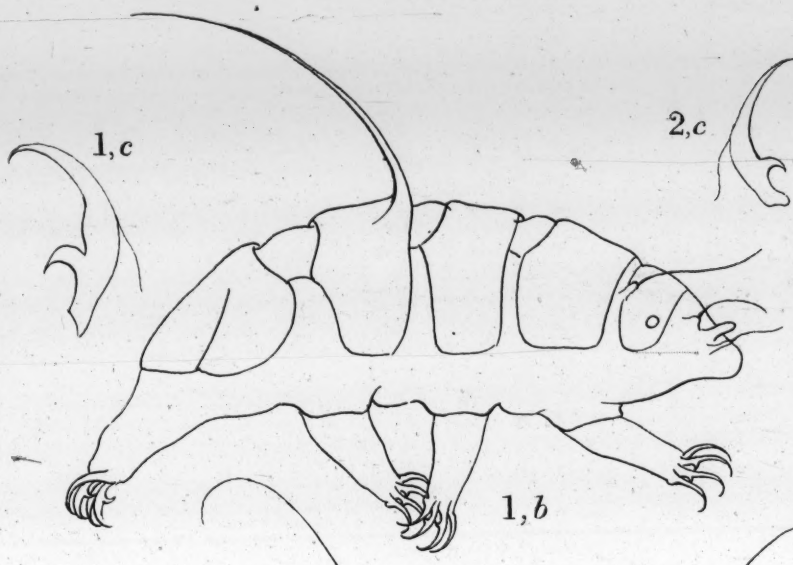
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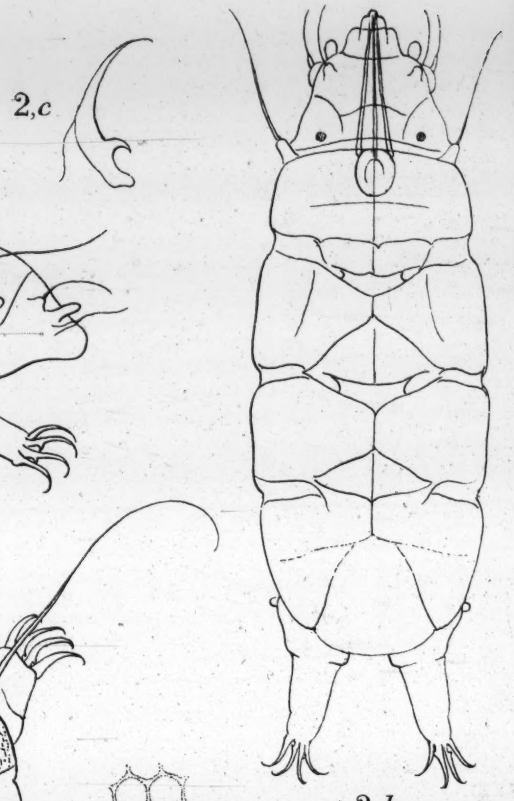
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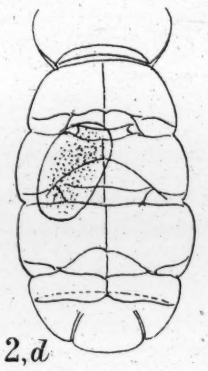
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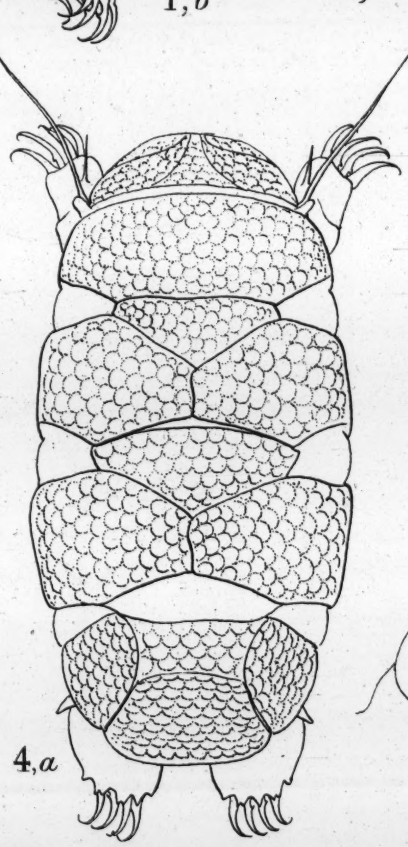
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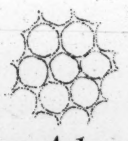
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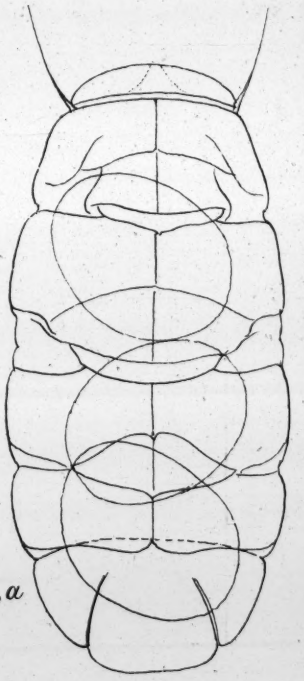
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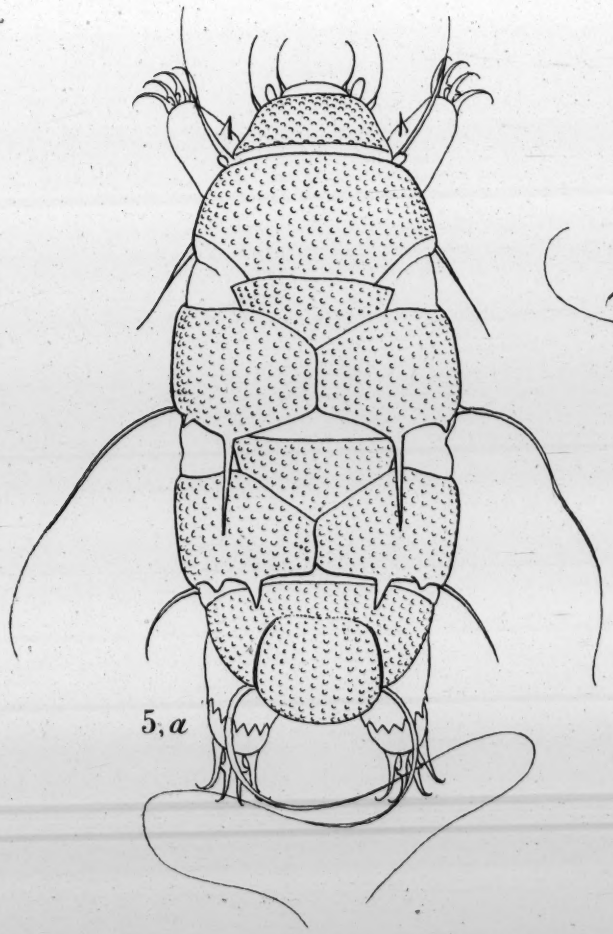
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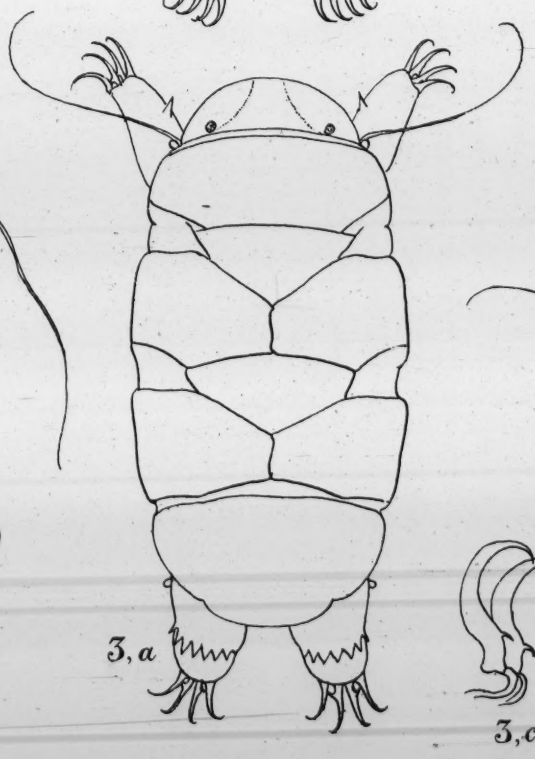
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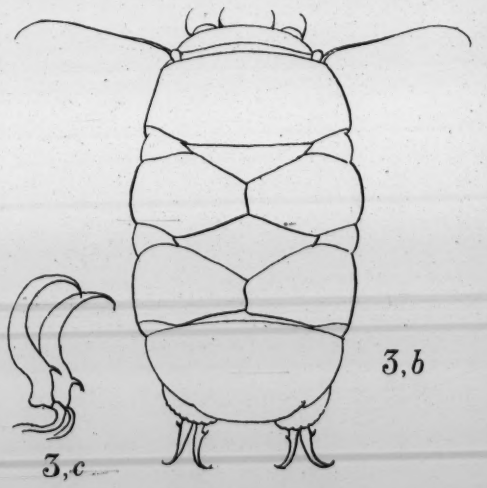
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3, c

McFarlane & Erskine, Lith. Edin.r

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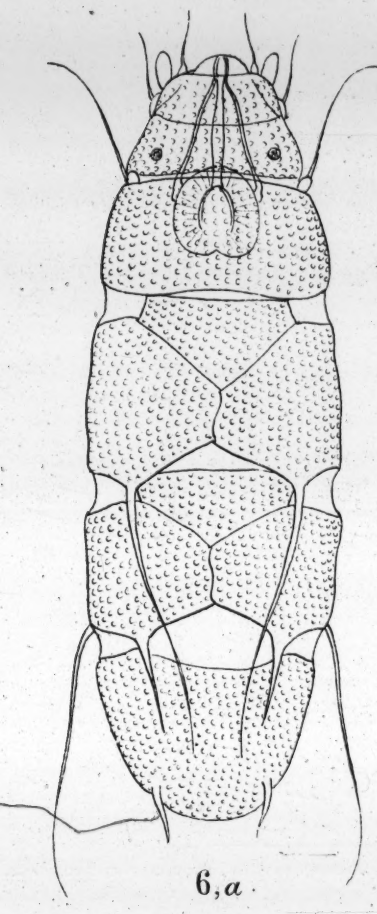
4, E. RETICULATUS, n. sp. 5, E. OIHONNÆ, Richters.

MURRAY: TARDIGRADA OF THE SCOTTISH LOCHS.—PLATE II.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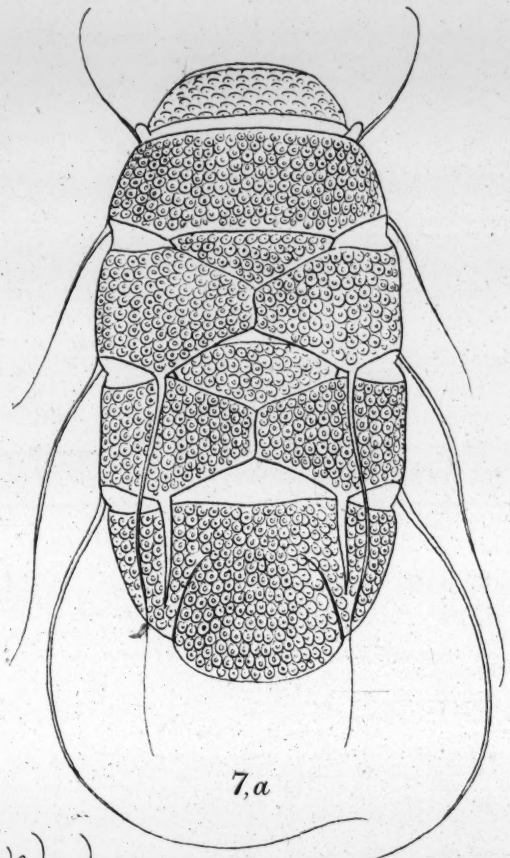
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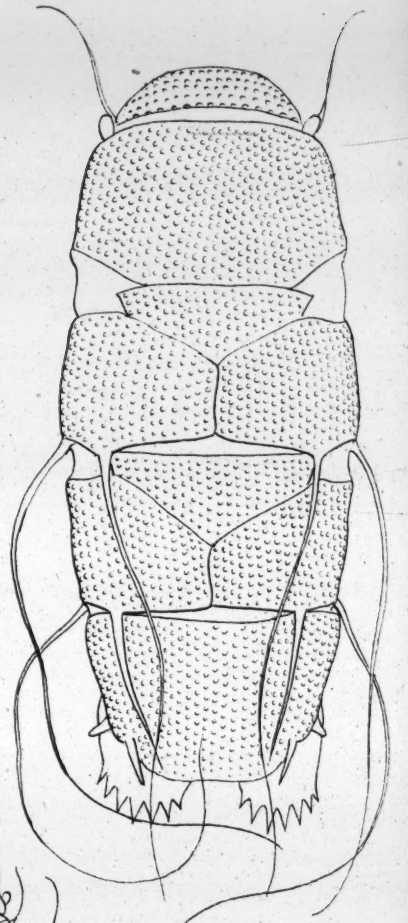
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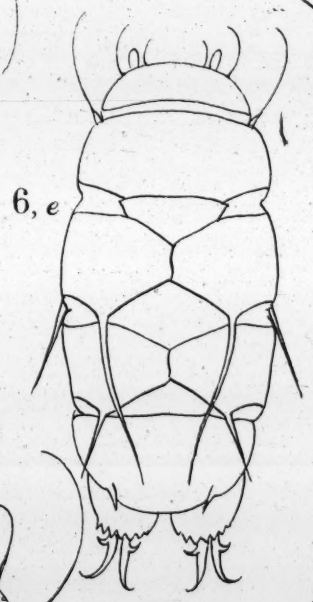
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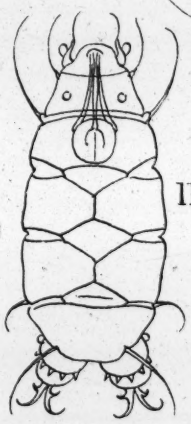
11, c

11, b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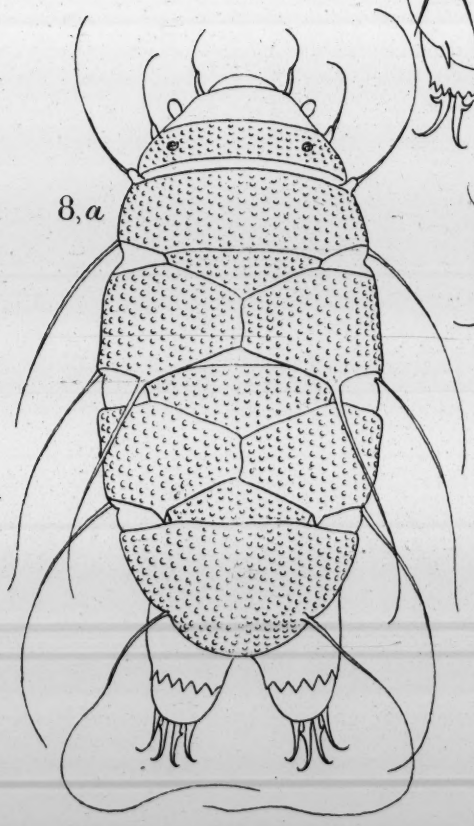
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6, f

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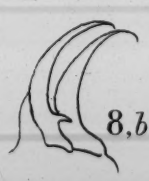
9, c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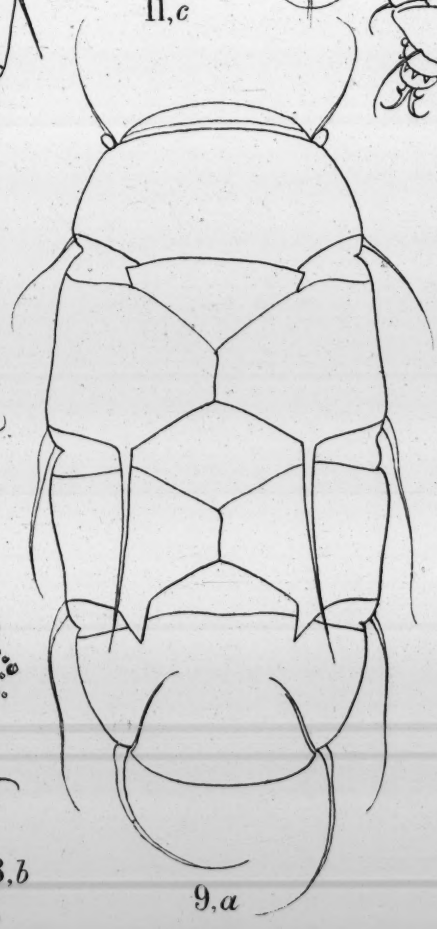
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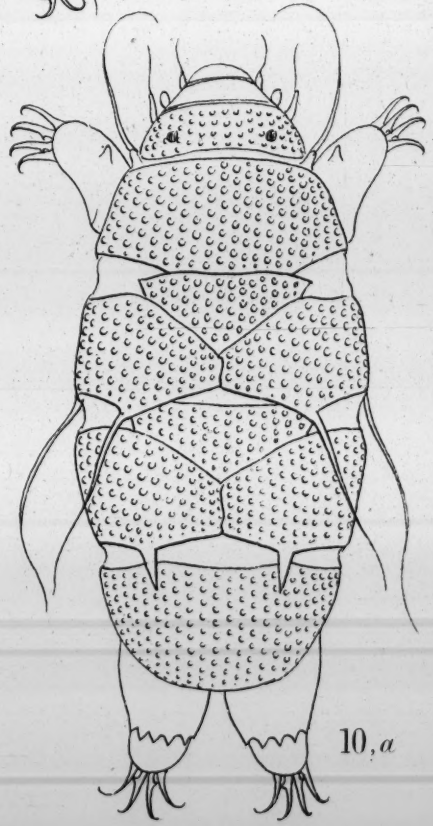

10, a

McFarlane & Erskine Lith Edin.r

6, ECHINISCUS GRANULATUS, Doy. 7, E. SPITZBERGENSIS SCOURFIELD. 8, ECHINISCUS. sp.?
9, ECHINISCUS. sp? 10, ECHINISCUS. sp.? 11, ECHINISCUS, sp.? LARVA.

EXPLANATION OF PLATES.

The figures of the complete animals are all drawn to the same scale, to enable comparisons to be made; the details are drawn larger, but to no uniform scale. Where the granulation of the plates is minute it is omitted from the drawings, as it could only be indicated in an exaggerated form. The coarser granulation or reticulation of some species is shown.

PLATE I.

1. *Echiniscus gladiator*, n. sp.

a, dorsal view.
b, lateral view.
c, inner claw of last legs.

2. *Echiniscus mutabilis*, n. sp.

a, dorsal view of typical example, three eggs.
b, dorsal view of variety.
c, inner claw of last legs.
d, small example of type, with one egg.

3. *Echiniscus wendti*—RICHTERS.

a, dorsal view.
b, larva with two claws.
c, claws of larva.

4. *Echiniscus reticulatus*, n. sp.

a, dorsal view.
b, reticulation, to larger scale.
c, inner and outer claws of fourth leg.

5. *Echiniscus oihonnæ*—RICHTERS.

a, dorsal view.
b, inner claw of last leg.

PLATE II.

6. *Echiniscus granulatus*—DOY.

a, variety, dorsal view.
b, outer and inner claws of last leg of same.
c, older and more typical example.
d, two claws of last leg, showing three barbs on outer claw.
e, two-clawed larva, probably of this species.
f, claws of the larva.

7. *Echiniscus spitzbergensis*—SCOURFIELD.

a, dorsal view.
b, the apparent rings on the plate.
c, outer and inner claws of last leg.

8. *Echiniscus*, sp.?

a, dorsal view.
b, outer and inner claws, last legs.

9. *Echiniscus*, sp.?

a, dorsal view.
b, granules seen on some examples.
c, irregular perforations seen on others.

10. *Echiniscus*, sp.?

a, dorsal view.

11. *Echiniscus*, sp.?, larva.

a, dorsal view.
b, teeth and pharynx.
c, claws of last leg.

Plate III.

12. *Macrobiotus islandicus*—Richters.

a, dorsal view.
b, claws.
c, teeth and pharynx.

13. *Macrobiotus ornatus*—Richters.

a, dorsal view, var. *spinosissimus*, Richters.
b, lateral view, var. *verrucosus*, Richters.
c, teeth and pharynx.

14. *Macrobiotus annulatus*, n. sp.

a, mature example, carrying skin with eggs.
b, teeth and pharynx.
c, claws, under pressure.

15. *Macrobiotus papillifer*, n. sp.

a, dorsal view.
b, teeth and pharynx.
c, claws.

16. *Macrobiotus macronyx*—Doy.

a, lateral view.
b, teeth and pharynx.
c, pair of claws of last leg.
d, pair of claws of first leg.

17. *Macrobiotus*, sp.?

a, empty skin with five small eggs.
b, part of reticulation, on larger scale.
c, claws.

Plate IV.

18. Egg of *Macrobiotus hufelandi*, C. Sch.
19. Egg of *Macrobiotus*, sp.?
20. Egg of *Macrobiotus*, sp.?
21. Egg of *Macrobiotus echinogenitus*, Richters.
22. Egg of *Macrobiotus*, sp.?

23. *Diphascon chilense*—Plate.

a, dorsal view.
b, teeth and pharynx.

24. *Diphascon spitzbergense*—Richters.

a, dorsal view.
b, teeth and pharynx.

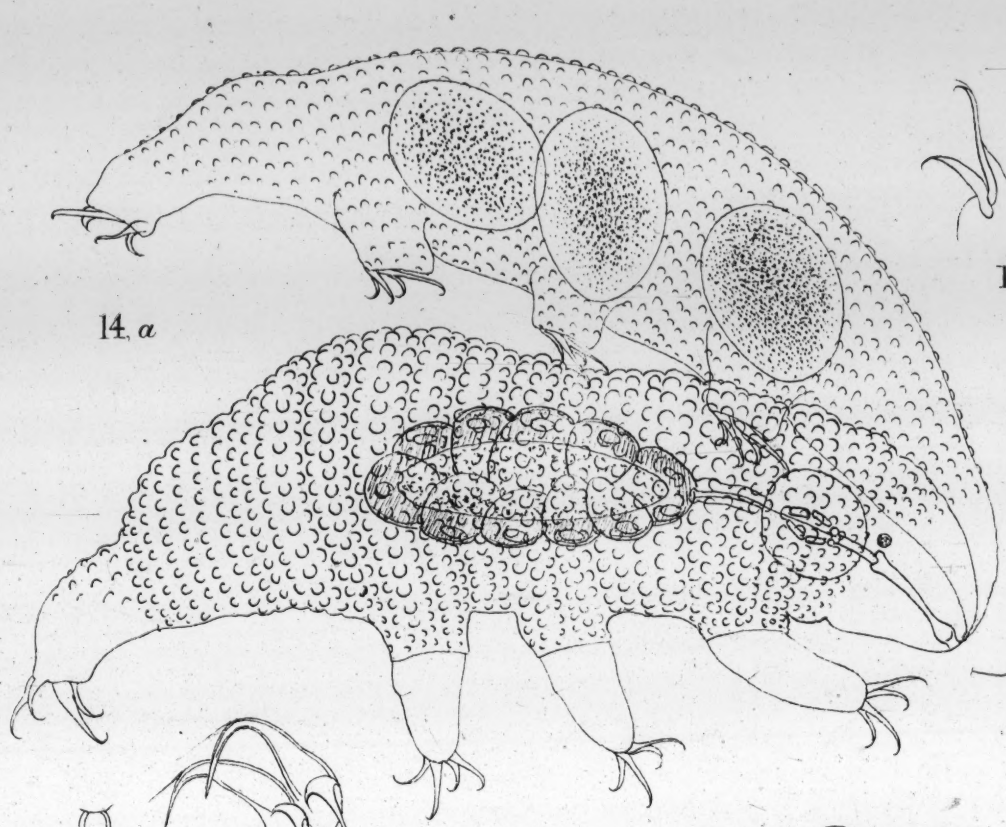
25. *Diphascon angustatum*, n. sp.

a, dorsal view.
b, teeth and pharynx.
c, claws.

26. *Milnesium tardigradum*—Doy

a, dorsal view.
b, teeth and gullet.
c, claws.

MURRAY: TARDIGRADA OF THE SCOTTISH LOCHS.—PLATE III.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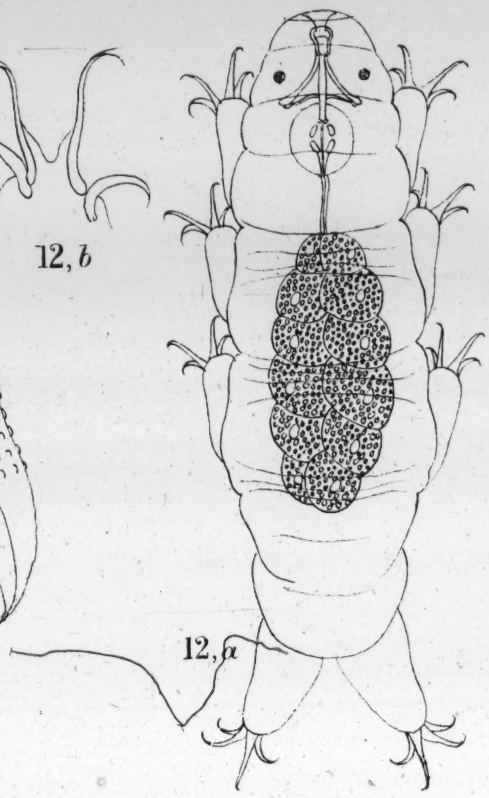
12, b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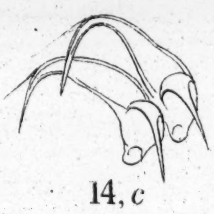
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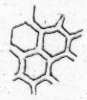
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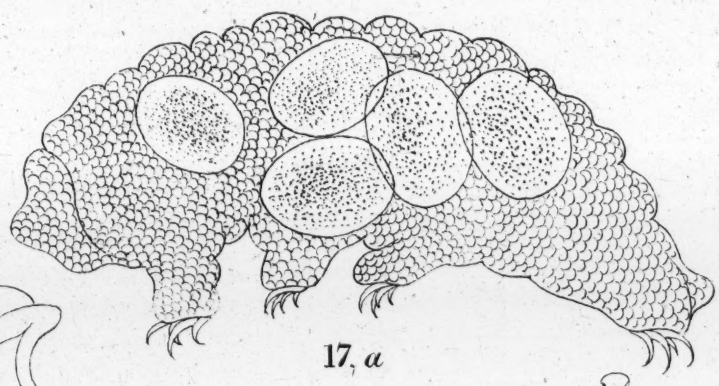
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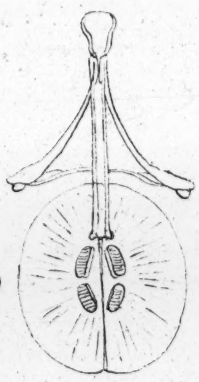
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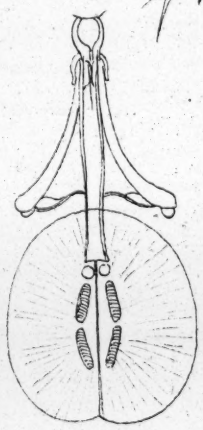
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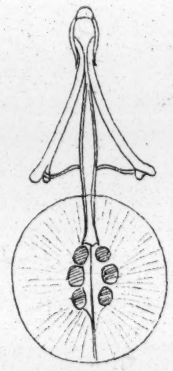
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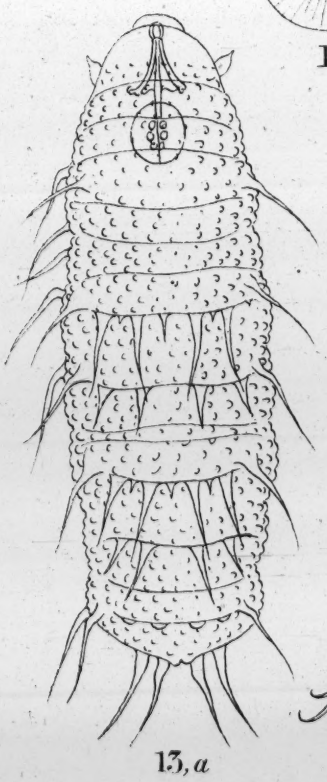
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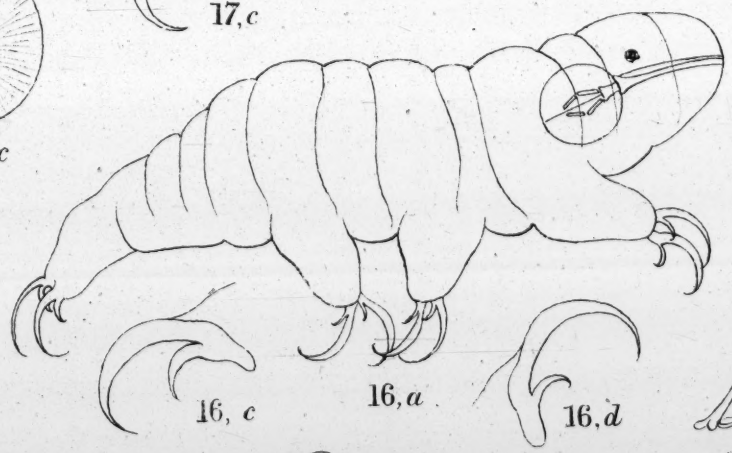
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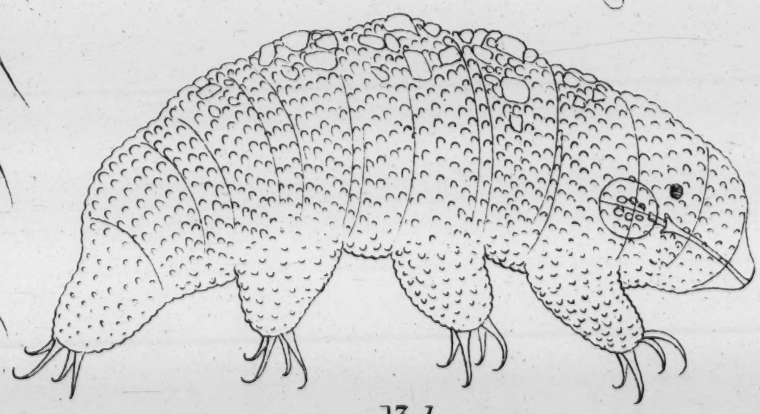
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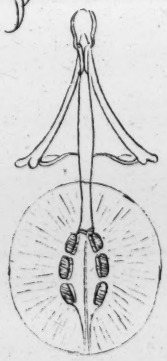
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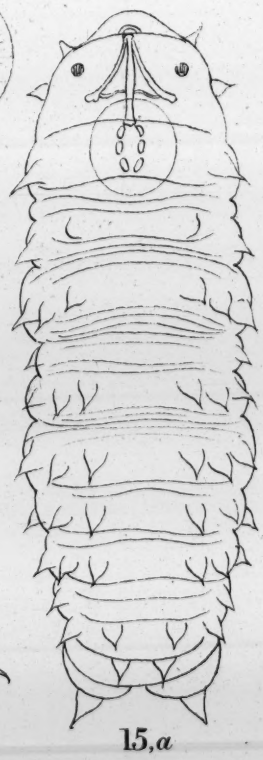
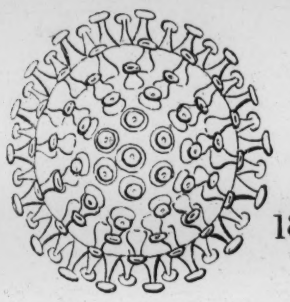

15, a

M'Farlane & Erskine Lith Edin.r

12. MACROBIOTUS ISLANDICUS, Richters 13. M. ORNATUS, Richters. 14. M. ANNULATUS, n. sp.
15. M. PAPILLIFER, n. sp. 16. M. MACRONYX, Doy. 17. MACROBIOTUS. sp.?

MURRAY: TARDIGRADA OF THE SCOTTISH LOCHS.—PLATE IV.



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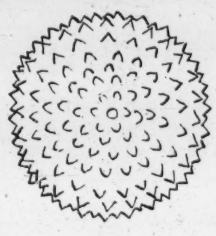

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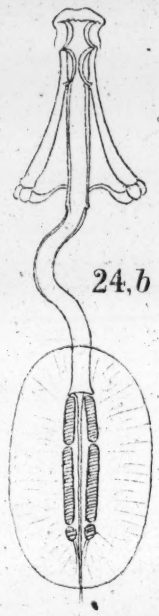

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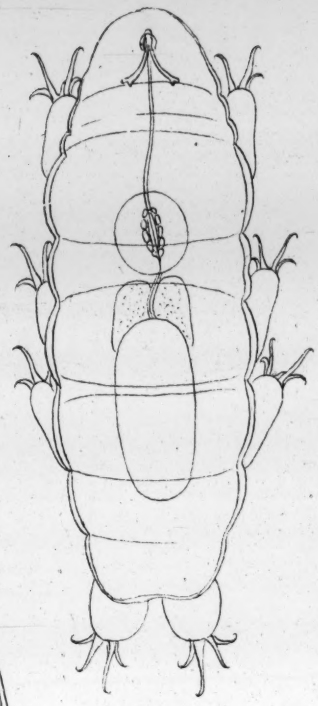

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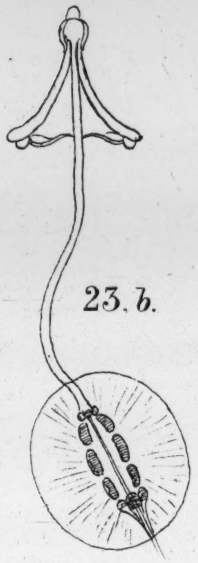

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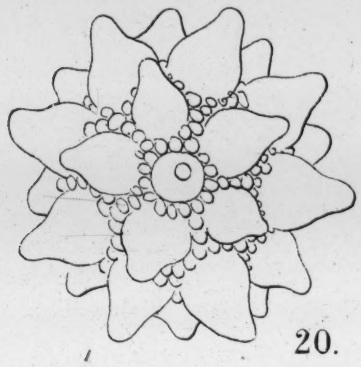

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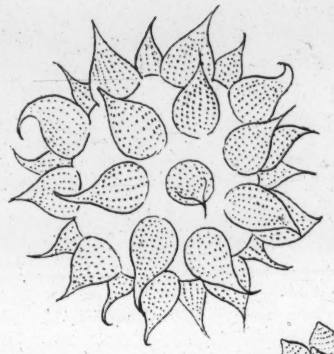

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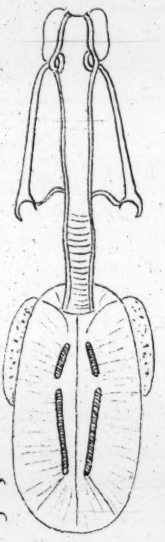

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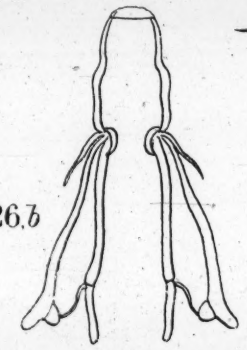
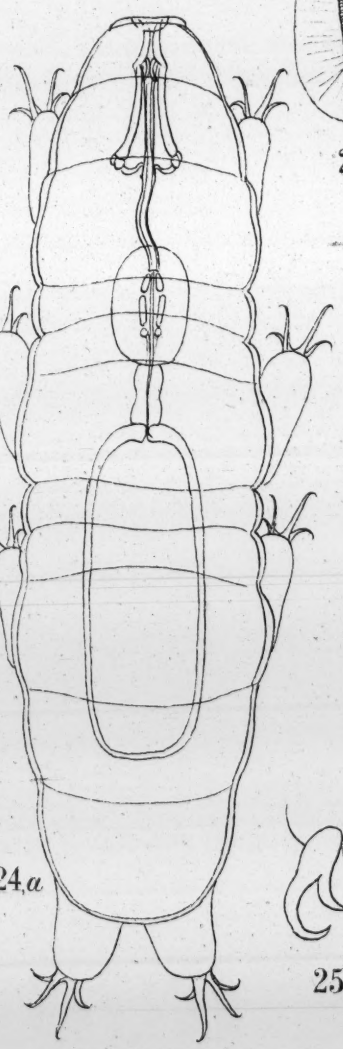
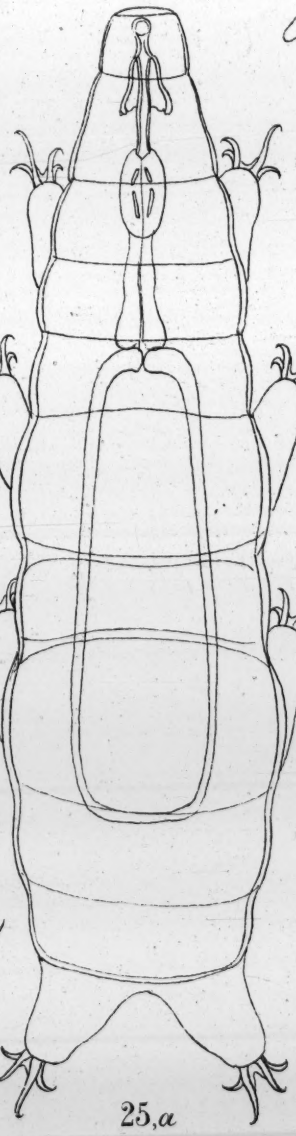


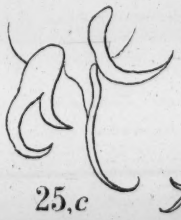
26. *b*



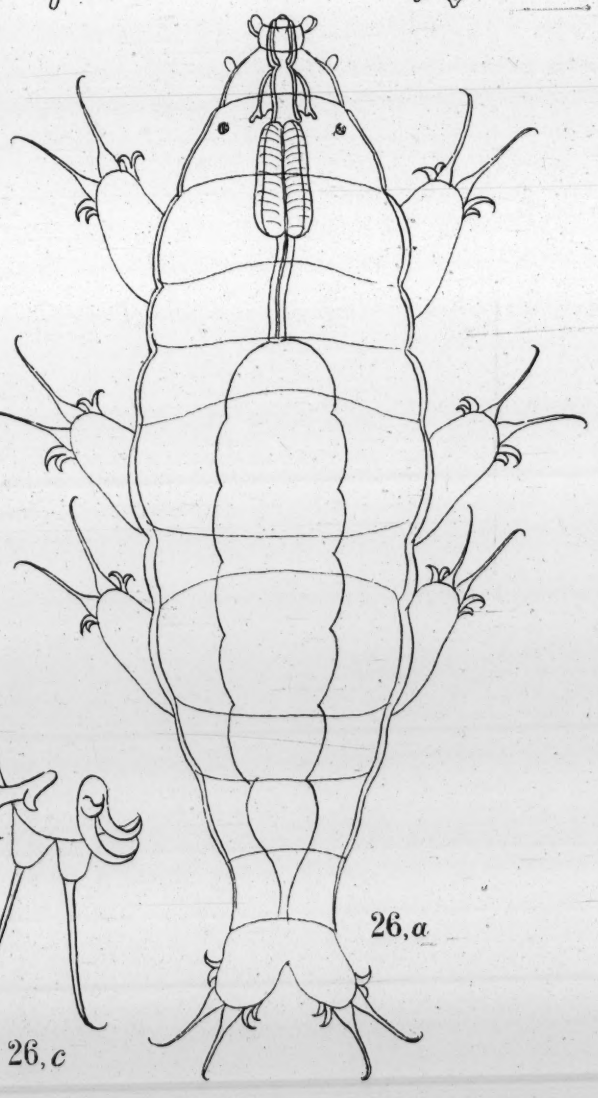
24, *a*



25, *a*



25, *c*



26, *a*

26, *c*

McFarlane & Erskine, Lith. Edin.r

18, MACROBIOTUS HUFELANDI, C, Sch., EGG. 19, 20, 22, EGGS OF MACROBIOTUS, sp.? 21, M. ECHINOGENITUS, Richters.
23, DIPHASCON CHILENSE, Plate. 24 D, SPITZBERGENSE, Richters. 25 D. ANGUSTATUM, n.sp.
26, MILNESIUM TARDIGRADUM, Doy.

XXVIII.—The Plant Remains in the Scottish Peat Mosses.

By **Francis J. Lewis**, F.L.S., Assistant Lecturer in Botany, University of Liverpool. *Communicated by* Professor Geikie, LL.D., F.R.S. (With Six Plates.)

PART I.

The Scottish Southern Uplands.

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The following paper deals with an investigation of the successive zones of plant remains contained in the deeper peat deposits covering areas in the Scottish Southern Uplands. The field work was carried on during the summer and early autumn of 1904, and the detailed examination of the peat in the laboratory during part of the winter. No attempt has been made to work out the detailed flora of the different zones, but attention has chiefly been directed to the dominant plant remains found at different horizons in the mosses. Whilst the list of plants from each zone is small, the general facies of the flora of any layer can be gauged from the abundant presence of a few characteristic plants such as *Salix reticulata* and Empetrum, or Sphagnum and Eriophorum. Thus, while the investigation is incomplete as regards any addition to the history of the British Flora, it will, I hope, throw some light upon the succession of vegetation over the older peat mosses since their origin.

Much work has already been done, chiefly by Clement Reid (1), on the plant remains from some of the interglacial and earliest post-glacial deposits in England. The remains have chiefly been taken from clay and sand beds, and for that reason would generally be more plentiful and better preserved than the remains contained in the peat; for the flora of peat mosses is comparatively small, and many square miles are often tenanted by a few dominant plants, such as Sphagnum or Eriophorum. Although the plant remains from the older peat mosses may not add much to our knowledge of the history of the British Flora, yet as they date from late glacial times, they will indicate the type of conditions which have prevailed both in the lowlands and highlands at each successive period down to the present.

Much has already been written by different observers on the subject of the Scottish peat mosses, and is summarised in Professor Geikie's *Prehistoric Europe* and need not be referred to in detail here.

The recording of a plant, or set of plants, from a peat moss is of little value without a description of the beds which lie above and below the plant remains, the character of the flora of the moss, and other features which would help to determine the age of the peat. The kind of observations that are needed have been described by Clement Reid (2), who lays stress on the point that observations on the succession of plant remains are

far more likely to throw light on the questions at issue, than the collecting of remains over large areas where no succession can be made out.

On the Continent the plants preserved at different horizons in peat mosses have been fully described by STEENSTRUP, BLYTT, GUNNAR ANDERSSON, NATHORST, SCHROETER, and many others, and I hope that correlation between these zones and those in the British peat may be possible when more areas have been investigated in this country.

Method of Survey.—The following results have been obtained from sections, made for the purpose of this survey, in the untouched portion of the mosses. The peat of many of the lowland and some of the upland districts is still dug for fuel, this being particularly the case in Wigtonshire, Ayrshire, and Selkirkshire. Even in these districts, however, the turbaries have trenched very little on the mosses, being confined to the drier margins. The surface peat, being unsuitable for fuel, is generally taken off the top of the turbaries and laid down upon the excavated area. This quickly becomes grown over with vegetation, so that it is frequently difficult to determine the exact boundary of some of the older turbaries. To avoid any error due to this source, independent sections have been made in all cases away from the turbaries.

The following results are based upon evidence obtained both by means of sections and borings. The borings were made with a $2\frac{1}{2}$-inch clay auger with rods to bore to a depth of 20 feet. Fairly good cores were obtained when dealing with dry peat; but the evidence from borings alone is not always to be trusted, as a layer of wiry stems imbedded in soft peat may easily be pushed aside by the boring tool without being represented in the core. Except in the few places where it was impossible to cut sections, the borings have only been used for verifying facts already ascertained by digging in other parts of the mosses, or for ascertaining the average depth of a large moss.

The sections were generally in the form of a pit 6 feet wide and 8 feet long where the depth of peat to be cut through did not exceed 8 or 10 feet; but in many cases the sections had to be carried down 16 or 17 feet before the basal layers of the moss were reached, and in such case the section would be enlarged to 12–16 feet in length with a series of steps or terraces at one end. After the underlying rock had been reached and cut into as far as possible, the sides of the section were carefully examined for evidence of stratification. Material would then be cut from each layer, placed in tins, labelled, and sent to the laboratory for detailed examination. Larger blocks of the peat, to show the sequence of beds, were also cut from many of the sections and sent to the laboratory for more detailed examination than was possible in the field. In cases where the peat rests upon sand, it is frequently traversed towards the base by cracks, and the inrush of water through these caused much delay. This was particularly the case in the Merrick mosses, where many sections had to be abandoned when only half finished owing to the rush of water.

In most of the British peat mosses the plant remains are not so perfectly preserved as in the Continental peat, and seeds are comparatively few in number. I have found that the isolation of the plant remains can most easily be effected by examination of

the peat under a dissecting microscope, though this is at best a tedious process. Gunnar Andersson (3) has described a special method of loosening the peat by treatment with a strong oxydising agent such as nitric acid. This partly bleaches the peat, which, placed in a metal net, is kept below water. The finer material, consisting of groups of cells and unicellular organisms, floats out in the water; the coarser remains, such as seeds, stems, etc., are retained by the net, and can then be spread out on a slide and examined under the microscope. This method yields good results when dealing with peat rich in plant remains. Specimens of the peat from different layers have also been embedded in paraffin wax, and serial sections cut with a microtome and mounted in Canada Balsam for microscopic examination.

The following areas were investigated during last year:—

1. Upland mosses in Kirkcudbrightshire and Ayrshire:
   The mosses lying between the Merrick and Kells range.
2. Upland mosses in Selkirkshire:
   Mosses in the Tweedsmuir and St Mary's Loch district.
3. Hill-top mosses in Peeblesshire and Edinburghshire:
   Peat of the Moorfoot Hills.
4. Lowland mosses in Wigtonshire:
   Flow of Dergoals, Dirskelpin Moss, Knock Moss, Anabaglish Moss.
5. Buried peat and clay beds in the Earn Valley.
6. Mosses resting on the 25-feet raised beach of the south coast:
   Moss of Cree in Wigtonshire. Priestside Flow in Dumfriesshire.

A preliminary examination was made of the peat of Lochar Moss in Dumfriesshire, some of the Scottish Midland Plain mosses, and the hill-top mosses of Cross Fell in Cumberland; but, owing to want of time, a complete investigation of these mosses had to be postponed.

The Upland Mosses of the Merrick and Kells District.

The mosses investigated in this district lie at elevations of 700–1000 feet above sea-level, and are situated in the valley running north and south between the Merrick and Kirriereoch range of hills on the west and the Kells range on the east (one inch Ordnance Survey—sheet 8). The valley is about 10 miles in length with an average width of 2–3 miles, and is drained to the north by the Gala Lane flowing into Loch Doon, and by Cooran Lane flowing south. The divide between the two drainage systems is situated 5 miles south of the head of Loch Doon, in the form of a low neck of land with an elevation of about 1000 feet running between Mullwarchar on the Merrick range to Corserine on the Kells range. The Merrick range, as a whole, is granitic in structure, whilst the Kells consists of Silurian rocks.

The marks of glacial action are evident everywhere in the district; the rocks are frequently ice scratched, perched blocks are numerous, and small moraines in a remarkably

good state of preservation are to be seen at the entrance to many of the small lateral valleys and along the foot hills of both the Merrick and Kells. These small moraines are contemporaneous with the numerous moraines in the Loch Skene and Tweedsmuir district at similar elevations, and belong to the "third" epoch of glaciation, or the period of local ice-sheets and valley glaciers of the Southern Uplands (4). The mosses occurring here are evidently younger than the moraines, as in many places they run up to the foot and actually rest upon these moraines.

The peat forms an irregular border on the sides of the Gala Lane and Cooran Lane, varying in width from $\frac{1}{2}$ a mile to $1\frac{1}{4}$ miles. The five miles of peat south of a line drawn from the foot of Craignaw to Elderholm is covered at present by Sphagnum. The five miles of peat lying to the north of this line is better drained, and tenanted by a much drier type of vegetation consisting of *Calluna vulgaris*, L.; * *Eriophorum vaginatum*, L.; *Myrica Gale*, L.; *Carices*, and *Juncus Squarrosus*, L.

Tree vegetation is entirely absent from the district, the first natural woodland on the east lying 9 miles away in the Ken Valley on the other side of the Kells range, and westward, 15 miles away on the other side of the Merrick Hills in the Barrhill district.

For purposes of description the peat can be divided into two districts by a line drawn across the valley from the foot of Craignaw to Elderholm, for the features presented by sections in the southern area are somewhat different from those in the northern area. The northern area will be described first.

The peat is undergoing denudation at the present day, being channelled into peat-hags. (Fig. 1.) The amount of denudation, however, is not so great as in other hill districts situated farther east both in Scotland and England. The first series of sections were made near the rising place of Cooran Lane, and the following plant-beds were exposed:—

1. Peat formed chiefly from *Scirpus* and *Eriophorum vaginatum*, L., 7 feet.
2. Layer of *Pinus sylvestris*, L.; trees with stools of large size and numerous cones, ... 3 ,,
3. Peat formed chiefly from *Sphagnum*, ... 2 ,,
4. *Eriophorum vaginatum*, L., peat, ... 6 inches.
5. A layer of the stems of *Empetrum nigrum*, L., sharply divided from the peat above and below, ... 5 ,,
6. *Eriophorum vaginatum*, L., peat, ... 14 ,,
7. Sphagnum peat, ... 4 ,,
8. A layer of Betula.
9. Structureless peat mixed in some places with much coarse granite sand.
10. Coarse granite sand. Bored through for 2 feet, but further boring stopped by the rush of water and the difficulty of obtaining a core.

* The nomenclature of HOOKER'S *Student's British Flora*, third edition, is followed throughout.

Horizontal exposures were then made of the successive beds, and the same general features were found to be present. The Empetrum bed can be traced for some distance without cutting into the peat, as the channels in the peat-hags are worn down to about a foot below its level. (Fig. 2.) Three sections were made between Elderholm and the watershed—a distance of about a mile—and the same succession of plant-beds were exposed in all cases.

Sections were then made on the western margin of the peat at the foot of Craignaw —one of the out-lying hills of the Merrick range. The ground rises here to a little over 1000 feet, and the peat is better drained than in the centre of the valley. The same sequence of beds was found, but the character of the Empetrum bed alters considerably. Empetrum is only occasionally present, and its place is taken by abundant *Salix herbacea* and *S. reticulata*. The Eriophorum and Sphagnum beds are also thinner above and below the Empetrum bed, and the growth of the peat has been slower on this sloping, well-drained ground than at the bottom of the valley.

Sections and borings were made north of the divide, towards Loch Doon on each side of Gala Lane, and the same plant beds were seen to extend here also. A section near Yellow Tomach, three miles north of the previously described section, exposed the following beds :—

1. Scirpus and Sphagnum peat, ... 3–4½ feet.
2. Layer of *Pinus sylvestris*, L.
3. Sphagnum peat, with traces of shrubby birch in the lower layers, 1 foot.
4. Layer of *Empetrum nigrum*, L., ... 3 inches.
5. Birch remains, with scanty Calluna and occasional patches of Sphagnum.
6. Below the last layer, but not sharply marked from it, stems and leaves of *Salix repens*, L.

Racomitrium ellipiticum, B. & S. *Epilobium palustre*, L.

Comparing this section with the one previously described, it will be seen that layers 1–4 agree, and that the difference lies in the absence of the underlying Eriophorum and Sphagnum layers in the northern section. In the case of the peat near Yellow Tomach, the growth of the birch appears to have been nearly continuous up to the occurrence of the Empetrum bed, although much Sphagnum is mixed with it in places. This variation at different spots does not, I think, impair the comparison between the different sections, as we should expect some variation in the contemporaneous vegetation at different places in the valley ; and the Empetrum bed always stands out as a kind of landmark — Empetrum dominant in some places, and *Salix herbacea*, L., and *S. reticulata*, L., dominant in others.

The occurrence of a compact layer of Empetrum, mixed with such northern forms as *Salix herbacea* and *S. reticulata*, undoubtedly marks a period in the growth of this peat when the conditions must have been very different to those under which the underlying peat and the overlying Eriophorum, Pine, and Scirpus-Sphagnum zones were

formed. *Salix herbacea* at the present time is confined to the summits of the highest English and Scotch mountains. South of the Tweed it occurs on a few of the highest Welsh mountains, and also on the Lake mountains, seldom occurring much below 2500 feet. In Scotland it is confined to the summits of the Highland mountains, and to a few of the summits in the Southern Uplands. *S. reticulata* is still more restricted in distribution, being confined to the Highlands between 2000–3200 feet. Northwards, both plants reach the limits of Arctic vegetation. Extreme northern types are absent from the present vegetation of the western part of the Scottish Southern Uplands.

The basal peat of Section 1 has not yielded any determinable plant remains. Microtome sections have been cut and examined under the microscope, when the peat is seen to be formed of structureless plant remains, except traces of Sphagnum and some isolated pollen grains agreeing closely with those of *Alnus glutinosa*. The basal layers of the peat near Yellow Tomach, however, yield evidence that the conditions about this time were not greatly different to those prevailing at the present day. These mosses, then, must have originated some considerable time after the disappearance of the local glaciers which deposited their moraines between the Merrick and Kells Hills, and at a period late enough for the climate to have become not less mild than at the present day. At a later date the whole district became clothed with woodland of a fairly northern type, mixed with Calluna moor, with a small growth of Sphagnum in the wetter places and bordering the moorland pools. After this period the vegetation undergoes a complete change. The woodland disappears, and the Calluna is replaced by Sphagnum, which in turn is again replaced by *Eriophorum vaginatum*, changes which indicate a steady increase in precipitation. The Eriophorum zone, later, gives place to an Arctic plant-bed consisting of *Salix herbacea*, *S. reticulata*, and *Empetrum nigrum*, and thus indicating a period when the conditions in the Galloway valleys must have been similar to those at present obtaining on the summits of our highest mountains. No other conclusion can, I think, be drawn from this zone, as it maintains its character so uniformly over an area many miles in extent, and corresponds closely with the type of vegetation covering large areas on tundras at the present time.

The presence of moraines on the 45–50 feet raised beaches in the Highlands, described by Hinxman (5), proves that the smaller moraines found in the Highland valleys do not merely represent the dying away of the ice-sheet which deposited its moraines in the valleys of the Southern Uplands, but that they belong to a much later return to glacial conditions—separated from the former by a period long enough to have enabled a temperate flora, represented here by the Betula zone, to have overspread the country. If that is so, the Arctic plant zone found in this peat must be contemporaneous with this later return to cold conditions—*i.e.* with the fourth glacial epoch, or the period of mountain valley glaciers when the snow-line stood at about 2500 feet. The plant-beds above the Arctic zone show a gradual return from cold conditions to somewhat more genial conditions, although at first characterised by great precipitation. As the conditions became drier, the whole of the valley became covered with pine forest. (Fig. 3.)

According to the geological evidence, after the warm period following the formation of mountain valley glaciers there was again a partial return to cold conditions, evidenced by the corrie moraines of the Highlands; but this moderate degree of refrigeration might cause but little change in the vegetation of the valleys in the Southern Uplands. Be that as it may, the pine forest in this district vanished, and its place was taken by a much wetter type of moorland vegetation, as the 7 feet of peat above the pine zone to the present surface is chiefly formed from such plants as Sphagnum, *Scirpus cæspitosus*, L., and Carices.

As I have read the evidence, these mosses began to form some time during the inter-glacial period between the district glaciers and the return of glacial conditions marked by the mountain valley glaciers; and if that is so, these mosses reproduce the general aspect of the vegetation at each succeeding period down to our own.

At the present time the peat is not growing to any appreciable extent, and it is difficult to say how much peat has actually been denuded away since the cessation of growth. It is interesting to note that the present vegetation is chiefly made up of the following plants: *Calluna vulgaris*, Salisb.; *Molinia cærulea*, Moench.; *Myrica Gale*, L.; *Juncus Squarrosus*, L.; *Erica Tetralix*, L.; and a small quantity of *Eriophorum vaginatum*, L., and *Scirpus cæspitosus*, L.; but immediately below the surface of the peat the remains of Sphagnum and Scirpus become dominant—evidence of the prevalence of an altogether wetter type of vegetation. The question of the denudation of the peat will be dealt with later.

The Southern Area of the Merrick-Kells Mosses.

A series of borings were made through the peat lying immediately south of the area just described. The average depth of the peat is about 15 feet. The floor of the moss is formed of coarse sand, through which borings were carried for 18 inches; but no change in the character of the sand was observed at this depth. The basal peat immediately resting upon the sand shows no recognisable plant remains, but consists of a fairly dry compact mass. Small blocks have been embedded in paraffin wax, from which microtome sections have been made, which on examination with the microscope showed no structures which could be identified with certainty. Traces of vascular tissue in the shape of a few spiral vessels were found in one place, and pollen grains resembling those of the alder, and fragments of birch twigs, but nothing that would help to determine the conditions under which this layer was formed. Either this basal peat is disintegrated drifted material from the higher peat of the north, or the earliest plant remains have been completely disorganised. A considerable quantity of coarse grit occurring amongst this basal layer suggests that the peat is really drift.

As far as the borings show, the whole of the upper peat is formed from Sphagnum, *Scirpus* sp., and Carices, with here and there traces of Eriophorum. More satisfactory evidence of the history of this area might be obtained by cutting sections; but the

amount of water present in the upper layers prevented this, as the sections became filled before they could be carried down farther than 3 or 4 feet. So far as the evidence collected goes, this peat has had a different history from the 7 or 8 miles of peat lying immediately to the north, as here all the beds represented in the northern area from the basal birch to the pine zone are wanting, and the peat appears to have had an uninterrupted swamp history.

The Upland Mosses of the Tweedsmuir and St Mary's Loch District.

(One inch Ordnance Survey—sheet 16.)—Peat occurs abundantly in the hill district lying between the head-waters of the Tweed and St Mary's Loch, both as upland peat covering the slopes and floor of many of the valleys, and as hill-top peat covering the summits of the hills up to 2500 feet, and is developed to a greater extent on the Hartfell and White Coombe hills than on the Broad Law group. The average depth of the hill-top peat is about 6–8 feet, and the depth in the valleys about 10–14 feet.

Work in this district was chiefly directed to the Megget Valley, and particularly to some of its tributary valleys. Megget Water drains the eastern slopes of the Broad Law group on the north, and the Talla Side and Lochcraig Head on the south, and some of its tributary valleys run far up into the upland and hill-top peat districts. The peat about to be described lies in Winterhope, the main southern valley leading into Megget Water. The burn flowing in this hope rises on the peat-covered ground near Loch Skene, and flows northward for about 5 miles before joining Megget Water.

Evidences of glacial action are plentiful over the whole of the district, the terminal and lateral moraines being particularly distinct. They are to be seen in many of the northern tributary burns of the Megget Water, in the main valley itself, and are beautifully shown at the junction of Winterhope Burn with Garley Burn. (Fig. 4.) The moraines are found at altitudes of from 900 feet–1500 feet, and are contemporaneous with the third period of glaciation or the district ice-sheets and valley glaciers of the Southern Uplands.

In some places the moraines rise out of the peat-covered districts, as at the head of Loch Skene (fig. 5), and in other places the moraines are themselves covered thickly with peat (fig. 9). The peat then in such positions is clearly younger than the moraines upon which it rests, and cannot be older than the peat described in similar positions in the Merrick-Kells district, and it contains evidence by which it can be directly compared with the peat layers in that district.

The sections were made at the junction of Winterhope Burn and Garley Burn, where a thick covering of peat is developed in the hollows between the moraines. At the present time it shows the usual "peat-hag" formation of high mounds and ridges of peat with deep channels between, the summits of the mounds being covered with a vegetation consisting of *Calluna vulgaris*, Salisb.; *Erica Tetralix*, L.; *Scirpus cæspitosus*, L. Farther up Garley Burn and beyond the area covered by peat, clay beds

alternate with beds of peat. These will be described later. Sections cut through the deep surface peat showed the following sequence of layers with plant remains. Layers 6, 7, 8, 9, and 10 are shown in fig. 10.

1. Peat containing Calluna stems, with traces of shrubby birch.
2. Peat containing much *Eriophorum vaginatum* and *Polytrichum*, 2 ft.
3. Upper forest of *Betula alba*, L. . . . . $1\frac{1}{2}$ „
   Menyanthes trifoliata, L.; *Epilobium palustre*, L. (Fig. 6.)
4. Sphagnum peat, ... $1\frac{1}{2}$ „
5. A zone formed mainly of the stems of *Empetrum nigrum*, L., with *Loiseleuria procumbens*, Desv., ... 4–5 in.
   This layer is characterised by a thin band of Eriophorum above and below.
6. Sphagnum peat. Traces of Calluna, ... 1 ft. 3 in.
7. Lower forest of *Betula alba*, L.; *Menyanthes trifoliata*, L.; *Potentilla Comarum*, Nestl., ... 1 „
8. Mossy layer, ... 3–6 in.
9. Brown sandy peat, ... 16 in.
   Containing *Ranunculus lingua*, L.; *Viola*, sp.; *Epilobium palustre*, L.—very abundant; *Menyanthes trifoliata*, L.; *Ajuga reptans*, L.; *Alnus glutinosa*, Gaertn.; *Corylus Avellana*, L.; *Salix purpurea*, L.; Fragments of coniferous wood, water bourne; *Potamogeton*, sp.; *Equisetum*, sp.; *Hypnum cordifolium*, Hedw.; *Tortula angustata*, Wils.
10. Light gray fine sand, ... 10 in.
11. Moraine material.

Other sections proved that these layers are continuous through the peat at this place. Comparison of these beds with those found in the Merrick-Kells mosses will be made later.

Mention has already been made of the clay beds interstratified with peat layers. The banks of some of the lateral burns draining into Garley Burn were cut back, and the following strata exposed (fig. 8):—

1. Vegetable soil, ... 1–3 in.
2. Sandy clay, ... 1 ft.
3. Peat containing birch, ... 5 in.
4. Peaty clay.
5. Peat containing birch.
6. Peaty clay.
7. Fibrous peat containing *Betula alba*, L.; *Menyanthes trifoliata*, L.; *Epilobium angustifolium*, L.; *Sphagnum*, sp.; *Hypnum*, sp.; *Alnus glutinosa*, Gaertn.; *Vaccinium Myrtillus*, L.; *Equisetum*, sp.

8. Brown clay, peaty above, and containing scanty remains of Equisetum; becoming pebbly towards the base, and resting on moraine material.

The clays of this section contain no stones, and show every evidence of having been deposited by water; and this alternation of clay beds and peat layers has evidently been caused by flooding, which has continued uninterruptedly for a period long enough for about a foot of sandy clay to be deposited. The regular alternation of clay beds with wood peat is at first very striking when seen in section, but it is difficult to correlate these beds with those described in the first section unless the clay beds correspond with the wet-condition peat of layers 8, 6, and 4 of Section 1. There is nothing in the relative positions of the two sections which would preclude this explanation, for local flooding affecting Section 1 during a wet period might not have spread to Section 2.

The peat covering the level ground and slopes on the N.E. side of Loch Skene, and forming the gathering ground of Winterhope Burn, was next examined. The same evidence of present denudation is to be seen here, the peat in some cases being wasted away to within 1 or 2 feet of the underlying glacial deposits. Several sections and borings were made in this locality, and all agreed in showing the following changes in vegetation:—

1. Scirpus and Sphagnum peat, with occasional Calluna, . . 3 feet.
2. Empetrum zone, ... 2 ,,
3. Traces of Betula.
4. Dry hard peat, with traces of Calluna towards the base, . . 2 ,,
5. Pebbles and clay.

On comparing the sequence of beds in the northern area of the Merrick-Kells mosses and the Winterhope peat, a striking similarity is seen. In both cases two woodland beds are present, separated by layers showing a considerable increase in precipitation. Thus, in both districts the basal woodland bed is covered with Sphagnum peat, which at a later date was replaced by a vegetation in which *Eriophorum vaginatum* was the dominant plant. Such plant associations cover the wettest areas on our moorlands at the present time, having been mapped in S. Yorks. by Dr Smith (6), and on the Northern Pennines by myself (7). The most interesting point, however, is the fact that these wet-condition beds are overlaid in both districts by a thin seam of Arctic plants—Empetrum with *Salix herbacea* and *S. reticulata* in the Merrick-Kells mosses, and Empetrum and *Loiseleuria procumbens* in the Tweedsmuir district; and the evidence can, I think, be hardly interpreted in any other way except that a considerable decrease in temperature took place at the time this bed was forming. The interest of these two districts is further increased by the fact that a gradual change takes place above these Arctic plant-beds. Immediately over them lies *Eriophorum vaginatum* peat, which again is covered with Sphagnum peat. The line of division between the lower surface of the Arctic plant-bed and the underlying Eriophorum peat

is very sharply marked; there is no gradual transition from Eriophorum conditions to Empetrum and Arctic willow conditions, and this may indicate that some time elapsed between the close of the Eriophorum formation and the beginning of the Empetrum, Salix, and Loiseleuria formation, during which time no peat was formed. The Arctic beds, although only 4–6 inches thick, may have required a length of time for their formation altogether incommensurate with their thickness. Such plant associations having Empetrum as the dominant plant, with Arctic willows and trailing plants such as Loiseleuria, cover large areas in Central and Southern Greenland and have been described by Warming (8), and this goes to show that the conditions then obtaining in the upland valleys of Galloway and Tweedsmuir were essentially similar to present-day shrub-tundra conditions in Greenland, and very similar to what might have been expected to prevail during the period of mountain valley glaciers when the snow-line lay at about 2500 feet, thus giving a permanent snow-cap to the highest hills of the Southern Uplands. The evidence thus collected from upland peat occupying the same position with regard to the valley moraines both in Galloway and Tweedsmuir, agrees in almost every detail. The woodland at the base of the Tweedsmuir peat throws some light upon the character of the structureless peat met with in some of the sections of the Merrick-Kells district. During the interglacial period following the district ice-sheets and valley glaciers, two types of woodland flourished in the Tweedsmuir district: the first consisted of such trees as Corylus, Alnus, and temperate willows; the second of more northern types, such as birch. The sections near Yellow Tomach, in the Merrick-Kells district, also show the same feature—temperate willows at the base merging above into birch, and patches of Calluna. In other sections in the Merrick district the birch rested upon peat in which no plant remains could be recognised; but the original vegetation here was probably of the same character as that found farther north in the same district, and also in Tweedsmuir. The latest forest in the Merrick-Kells mosses is pine, and in the Tweedsmuir district, birch; but this hardly prevents comparison between these beds, as it might be expected that diversity of tree vegetation would occur in different parts of the country during a forest period—even as tree distribution differs at the present day.

The Hill-top Peat of the Moorfoot Hills.

(One inch Ordnance Survey—sheet 24.)—The lower boundary of the peat in the Moorfoot Hills closely follows the 1750-feet contour line, seldom occurring below, and generally running somewhat above. From here the peat stretches upwards, and covers the summits of the highest hills up to 2136 feet. The peat-covered ground is tenanted by an association dominated by *Eriophorum vaginatum*, L.; *Calluna vulgaris*, L.; and *Vaccinium Myrtillus*, L., with *Rubus Chamaemorus*, L., on the higher ground; whilst the lower slopes below the peat covering are dominated by *Nardus stricta*, L. The distribution of the different plant associations covering this ground has been described in detail by Robert Smith (9). Nowhere in the districts investigated can the present

denudation of the peat be seen in a more striking form, the whole of the peat area being intersected by deep channels, whilst the larger burns have cut through the peat some distance into the underlying clay. (Fig. 11.) The boundary of the peat and Nardus vegetation forms an irregular line along the hill-sides, presenting the general features found on the hills of the Weardale watersheds (10). Long tongues of dark heath-covered peat frequently stretch into the surrounding Nardus areas, and outliers of peat occur quite low down on the hill-sides. These are islands of peat which have been left behind by the main peat mass as it retreated up the hill-sides, owing to denudation. Many of the h l-sides which are still clothed with peat are very steep, and it is difficult to see how the peat could have accumulated there under present conditions. The peat attains a depth of 16 feet in some places, but is more usually about 7 or 8 feet in thickness. Owing to its channelled condition and situation on steep hill-sides it is generally dry, and the plant remains are better preserved than elsewhere.

The mosses rest upon glacial deposits consisting of stiff gray clay, containing many stones and occasionally small nests of sand. All the sections agree in showing an absence of any great development of woodland, but a little shrubby birch appears in some places in the peat.

The plant-beds exposed at the same horizons in different sections agree with one another fairly well, but there is rather more variation than was found to be the case in the flat mosses previously described. This was perhaps to be expected, as the factors of aspect, degree of inclination of the ground, and altitude would come in on these hill-top mosses and tend to produce a vegetation which would vary much more at any one time, and prevent the growth of a uniform plant association such as we find developed over a flat low level moss. The general sequence of vegetation appears to have been the same all over the Moorfoot Hills, but some of the beds found to be present in one place are absent in others.

The lower edge of the mantle of peat in most places contains a basal layer of *Betula alba* of shrubby size mixed with Calluna, whilst in other places this growth is replaced by small Salices. The layers of peat immediately above this are formed chiefly of *Eriophorum vaginatum*, which, in the peat situated at higher elevations, rests directly upon the clay. The peat resting upon the Eriophorum layer is formed of *Empetrum nigrum*, and this layer has been found to be well developed in all the sections. This again is overlaid by *Eriophorum vaginatum* peat, on the higher lying ground. Later, this is replaced by peat containing much *Calluna vulgaris*, which above yields place to Scirpus and Sphagnum mixed with scanty *Eriophorum vaginatum*.

The four following sections show the variations met with in different positions, together with the plant remains recognised from each layer :—

Half-mile N. of Bowbeat at 1500 feet.	N.W. of Bowbeat at 1900 feet.	Emly Bank at 1900 feet.	Cleave Burn at 2000 feet.
1. *Sphagnum*, sp. *Scirpus pauciflorus*, Lightf.	*Sphagnum*, sp.	*Sphagnum*, sp.	*Sphagnum*, sp.
S. cæspitosus, L. (abundant).	*Scirpus pauciflorus*, Lightf.	*Calluna vulgaris*, L. (scanty).	*Eriophorum vaginatum*, L. *Scirpus pauciflorus.*
Eriophorum vaginatum, L.	*Eriophorum vaginatum*, L.	*Scirpus pauciflorus*, Lightf.	*Calluna vulgaris*, Salisb.
2. *Calluna vulgaris*, Salisb.	*Calluna vulgaris*, Salisb.		*Calluna vulgaris*, Salisb.
3.	*Eriophorum angustifolium*, L. *E. vaginatum*, L. *Narthecium ossifragum*, Huds. *Alisma Plantago*, L.	*Eriophorum vaginatum*, L. *Narthecium ossifragum*, Huds.	*Eriophorum angustifolium*, L. *E. vaginatum*, L. *Alisma Plantago*, L.
4. *Empetrum nigrum*, L. *Vaccinium*, sp.	*Empetrum nigrum*, L. *Arctostaphylos Uva-ursi*, Spreng.	*Empetrum nigrum*, L.	*Empetrum nigrum*, L.
5.		*Eriophorum vaginatum*, L.	*Eriophorum angustifolium*, L. *Calluna vulgaris*, Salisb. *Molinia cærulea*, Moench. *Polytrichum Commune*, L. *Vaccinium Vitis Idœa*, L. *Scirpus cæspitosus*, L.
6. *Betula. Calluna.* *Carices*, sp. *Epilobium palustre*, L. *Salix*, sp. *Ranunculus repens*, L. *Lychnis diurna*, Sibth.? *Ajuga reptans*, L. *Viola palustris*, L.?	*Betula. Calluna.*		

In the section by Cleave Burn there is no sign of birch at the base of the peat, and the peat dominated by Eriophorum rests directly upon the clay. There is a sharp division between the two, the upper surface being smooth and indented by the weight of the overlying peat, and the dry compressed remains of Eriophorum and Molinia peel away from this surface, leaving it quite clean. The appearance suggests that some considerable time elapsed between the deposition of the clay and the growth of the vegetation which now rests upon it, during which time the clay was consolidated and denuded to some extent by water, after which it became covered with a growth of *Molinia cærulea*, Moench., Polytrichum, and Eriophorum. At the same time, or possibly sooner, the lower slopes of these hills became covered with a shrubby growth of birch

mixed with much Calluna. The zone of Empetrum running through the peat of the whole district suggests that, at some period, colder conditions prevailed; but the evidence afforded by Empetrum alone is scarcely conclusive, for, although covering large areas within the Arctic Circle, it also occurs fairly abundantly at the present day on many hills of about the same altitude as the Moorfoots, although never forming a pure association on the North of England or Southern Upland Hills. The presence of *Arctostaphylos Uva-ursi* also suggests more northern conditions, as it is not at present found on the Southern Upland Hills.

If the Empetrum zone represents cold conditions during the formation of these mosses, it must either be contemporaneous with the same period which produced the *Salix reticulata* in the Merrick-Kells peat and the Loiseleuria in Tweedsmuir, or it must represent the later return to glacial conditions, when the Highland corries were tenanted by small glaciers whose trace can still be seen in the high level corrie moraines of the present day. The sequence of the beds above and below the Empetrum agrees so closely with that in the Merrick-Kells area and in Tweedsmuir, that I think the evidence is in favour of much the same—or, possibly, somewhat milder—conditions having caused its growth. If the Empetrum zone here is contemporaneous with the Arctic zone in the other districts, it might have been expected to contain plants still more Arctic in character, as the ground lies about 2000 feet instead of 800–1200 feet, and is not sheltered like the Tweedsmuir and Galloway valleys; and for this reason I suggest that the Empetrum zone in the Moorfoot peat is contemporaneous with the period of Highland corrie glaciers.

The Lowland Mosses of Wigtonshire.

(One inch Ordnance Survey—sheet 4.)—Between the towns of Glenluce and Newton Stewart, in Wigtonshire, lies an extensive tract of peat mosses which northward stretch as far as the Merrick district. The general level of the peat-covered ground lies at about 200–300 feet above Ordnance datum. The whole of the district is flat in character, broken by a few ridges of Silurian rocks with their longer axes pointing N.N.E. and S.S.W. in the direction of the great centre of ice dispersal of the glacial period in the Merrick and Kells range. (Fig. 12.) The mosses here occupy great hollows in the till between the outcrops of rock. The present vegetation covering the wetter mosses consists of Sphagnum; *Erica Tetralix*, L.; *Myrica Gale*, L.; *Eriophorum vaginatum*, L. —with a little *Calluna vulgaris* with tufts of *Cladonia rangiferina* amongst the scattered Calluna patches. On the drier mosses Calluna is better represented, with *Scirpus* sp. and *Eriophorum vaginatum* as subdominant plants.

The peat covering the district shows no sign of denudation; the surface is even, and closely covered with vegetation. Drainage channels have been cut in many places, but owing to the low-lying level character of the ground they are not very effective. Peat is chiefly used as a fuel in the district, but owing to the sparsely inhabited nature of the country the mosses have not been trenched upon by turbaries to any great extent,

as the peat has only been dug in a few of the drier places near the edges of the mosses. The general character of the country is illustrated in fig. 12, where the long whale-backed hillocks can be seen rising about 30–40 feet above the general level of the moss.

Many of the mosses are of the nature of flow mosses, merely consisting of a crust of peat firmly bound together by the wiry stems of *Myrica Gale*, Calluna, and Scirpus, underlaid by many feet of semi-liquid peat, and I found that it was impossible to cut sections in such cases, and had to fall back on borings in order to obtain specimens of the basal peat layers and underlying glacial deposits.

An investigation by means of sections and borings was made of the following mosses lying in this district: Flow of Dergoals, Dirskelpin Moss, Knock Moss, Anabaglish Moss.

The Flow of Dergoals represents the wettest type of moss found in the district, being covered with an association of *Sphagnum* sp. (dominant), *Erica Tetralix*, and *Myrica Gale*. As a result of twelve borings the moss was found to have an average depth of 18–20 feet, and in one place close to the eastern boundary no bottom was reached in 30 feet. An endeavour was made to section the central part of the moss, but the peat proved to be semi-liquid in character at a few feet below the surface. A section was made close to the eastern margin through 17 feet of peat. Much *Eriophorum vaginatum* occurred in the upper 12 feet of peat, with *Polytrichum Commune* in places. Below this, woodland began to appear, the peat containing abundance of Corylus wood and nuts. Lower still the Corylus became more scanty and yielded place to Betula, which continued until the floor of the moss was reached.

The birch zone contained in some places considerable quantities of *Polytrichum Commune* and *Equisetum*, sp. This general succession was fully borne out by the borings, Betula being everywhere met with at the base of the peat, with much Corylus mixed with its upper layers. The floor of the moss consists of stiff gray clay packed full of stones of all sizes. The succession of events over the area covered by this moss appears to be as follows:—At a period subsequent to the deposition of the till upon which it rests, the ground became covered with a growth of birch and Calluna. This woodland was gradually replaced by hazel and alder, which, however, did not stretch to the centre of the moss, but formed a fringe round the sides of the basin. There is some evidence that the conditions became wetter as the birch and Calluna died away, for the peat above this shows a fairly abundant development of Polytrichum and Equisetum. This may explain the general absence of hazel and alder from the deeper parts of the moss. Later still the conditions become favourable to the growth of the wettest types of moorland plants; all sign of woodland vanished, and a close carpet of such plants as Eriophorum, Polytrichum, Sphagnum, and Carices covered the ground. Comparatively recently this vegetation has given place to the present Sphagnum, *Erica Tetralix*, and *Myrica Gale* association. The general history of the neighbouring mosses appears to have been the same, but there are, as might be expected, considerable local variations.

Dirskelpin Moss, Knock Moss, and Anabaglish Moss may be described together, being all of the same type and continuous with one another. The depth of these mosses varies from 10 feet to over 20 feet, an average depth in the centre being about 14 feet. In some cases borings were made at short distances from one another from one side of the moss to the other, and these showed that the peat occupies large hollows in the till. In several places the surface of the moss is broken by long whale-backed ridges of Silurian rock. Borings were made round some of these ridges, and the ground was found to shelve down steeply at the N.N.E. end, as a depth of more than 20 feet of peat was recorded only 70 feet from the edge of the moss; whilst at the S.S.W. end of the ridge the ground shelved much more gradually, as the peat only reached a depth of 3–4 feet 300 feet from the margin. All these whale-backed hills showed the same feature: a deep excavation on the N.N.E. filled with a growth of peat, and, at the S.S.W. end, a great accumulation of till covered by very shallow peat.

The features presented by the borings, and confirmed by sections made along the drier margins of the mosses, were as follows:—The mosses everywhere rest upon a stiff clay containing in some places numerous nests of sand, the clay being filled with stones of all sizes. This forms a fairly level floor, rising steeply towards the margins and round the Silurian outcrops. The upper layers of the till contain many rootlets from the overlying peat, but otherwise are free from vegetable remains. The peat immediately overlying the till (for the first 2 or 3 inches) contains no recognisable plant remains, and is frequently banded with thin layers of coarse sand or grit. Above this occurs an uninterrupted layer of *Betula alba*, L., of small size, the largest diameter of wood met with being only 10 inches. The wood is much decayed, but pieces of bark are well preserved and mixed with decayed leaves. *Corylus Avellana*, L., and *Alnus glutinosa*, Gaertn., occur fairly abundantly in many places towards the top of the birch layer, but are not found in the very centre of the mosses; here the only woodland zone present is birch. At the same time there is no sign of any sharp separation between the birch zone and the hazel and alder, for they appear to merge gradually into one another; but the evidence for this must be received with caution, as a zone of wood might easily sink in such soft peat without leaving any trace of the operation. Above the birch and hazel layers the peat contains much *Equisetum*, sp., with *Phragmites communis* occasionally reaching a thickness of 4–5 feet. This layer is particularly noticeable in the peat round some of the lochans which occur here, and shows that after the passing away of the birch, hazel, and alder vegetation, very wet conditions prevailed when most of these mosses were covered with swamp, or by a series of shallow lakes. Later still the conditions evidently became drier again, for a zone of *Pinus sylvestris* appears immediately above the Equisetum and Phragmites peat. The trees here, unlike the basal birch, are large in size, the stools are *in situ*, and the wood is well preserved and still resinous in smell on being broken up The peat round the stools contains abundance of pine cones, twigs, and apparently the remains of leaves. The trees stand at a distance of 9–12 feet apart, but do not occur in the centre of the mosses but as a fringe round the margins, in much the

same way as GUNNAR ANDERSSON found in the Swedish peat. After the pine zone had flourished for some time, its place was taken by swamp plants showing much wetter conditions; for the peat above the pine zone is almost wholly formed from the remains of Sphagnum, *Scirpus* sp., and *Eriophorum vaginatum*.

I found no evidence of Arctic plants at the base of these mosses, the first recognisable remains above the till being birch and Calluna. The thin seam of peat generally underlying this layer contains no recognisable plant remains. It would seem, then, that these mosses did not originate until genial conditions had replaced the cold under which the till was deposited, and the frequent presence of sand in the first few inches of peat suggests that the basal layers are wash peat deposited when the hollows in the till were partly under running water.

Here, as in the previously described districts, there are two woodland beds present, separated by peat showing very wet conditions, but, unlike the peat in Galloway and Tweedsmuir, there is no layer between the woodland beds of a distinctly Arctic type. It is not unreasonable to suppose, however, that the conditions which would favour the growth of Arctic plants on well-drained mountain valley peat at an altitude of 800–1200 feet might not produce the same results on wet peat lying at only 200 feet and close to the southern coast. If the basal birch in these mosses is contemporaneous with the basal birch in Galloway and Tweedsmuir, then the whole of the intervening beds in those districts are represented by the Equisetum and Phragmites peat here. If the record contained in these mosses was complete, it should carry the story of moorland history a stage further back, as the ground has not been glaciated since the second *mer de glace*, the district ice-sheets and valley glaciers not having encroached so far upon the low ground away from the hills.

Lochar Moss (one inch Ordnance Survey—sheets 10 and 6) is the largest tract of peat in the south of Scotland, and is situated to the S.E. of Dumfries. The southern part of the moss lies on the 25-feet raised beach, whilst the northern portion lies at about 45 feet. A complete investigation of this moss had to be postponed, owing to want of time; but some borings made near Racks, at 40 feet above Ordnance datum, showed the peat to be 15 feet in depth. The sections all filled with water before the base of the moss was reached, but the borings showed a well-marked basal layer of birch embedded in dry black peat, overlaid by 12 feet of Scirpus, Sphagnum, and Calluna peat. The peat immediately overlying the birch zone is almost entirely formed of Sphagnum, but there is a gradual increase in the amount of Calluna towards the surface of the peat. The chief point of interest is the presence of the basal birch layer, thus agreeing with the mosses previously described in Wigtonshire as well as with the 25-feet raised beach mosses of Cree and Priestside Flow.

THE BURIED PEAT OF THE EARN VALLEY.

(One inch Ordnance Survey—sheet 48.)—An apparently continuous bed of peat underlies the Carse clays of the Earn and Tay valleys. Numerous exposures

of the peat can be seen from near Forteviot, in the Earn Valley, to as far east as Stannergate near Dundee. The character of the peat and the plant remains contained in it has been described, amongst other observers, by JAMIESON (11) and by GEIKIE (12).

I was enabled to examine the peat at several places between Forteviot and Forgandenny through the assistance of Prof. GEIKIE, who kindly supplied me with maps showing the position of some of the best exposures. The peat in these places is about 3 feet in thickness and forms a brown, dry, hard layer resting upon the valley gravels, silts, and clays, and overlaid by the Carse clays. The peaty material is crowded with wood of all sizes, flattened and very brittle, and overlaid by a seam of sand and silt crowded with leaves often in the most perfect state of preservation. In the peat which I examined, remains of the following plants occurred: *Quercus*; *Corylus Avellana*, L. — wood and numerous nuts; *Betula alba*, L.; *Alnus glutinosa*, Gaertn.; *Salix*, sp.; *Menyanthes trifoliata*, L. — several seeds; *Carex*, sp.; *Phragmites communis*, Trin.;—a list which adds only one fresh plant to those already described by the authors mentioned. The overlying sandy clay contains numerous leaves of *Salix*, sp., and fragments of oak, birch, and hazel leaves.

The plants contained in the peat evidently grew where they are now found, as numerous rootlets penetrate the underlying deposits. The upper leaf-bed, on the other hand, is clearly drifted material, as the individual leaves are separated by thin layers of fine sand or silt. The peat, occurring as it does immediately below the Carse clays, should be contemporaneous with the oldest peat in the Galloway and Tweedsmuir districts.

THE 25-FEET RAISED BEACH MOSSES OF WIGTONSHIRE AND DUMFRIESSHIRE.

(One inch Ordnance Survey—sheets 4 and 6.)

Moss of Cree.—This extensive moss lies on the west side of the River Cree, between Newton Stewart and Wigtown. Less than a mile to the southward the smaller mosses of Barrow and Carsegowan are met with, both being similar in character and situation to the Moss of Cree. On the westward and landward side the moss is bounded by a series of low hills about 100 feet in elevation, from the bases of which the beach slopes to the S.E., the height of the beach at the eastern boundary being about 12–15 feet above Ordnance datum. Viewed from the margin, the moss presents a fairly smooth surface with a gradual rise to the centre, which lies about 15 feet higher than the margins.

The present vegetation over most of the moss consists of *Erica Tetralix*, L.; *Calluna vulgaris*, L.—not abundant; *Myrica Gale*, L.—abundant near the eastward margin; *Salix*, sp.; *Carex*, sp.; *Narthecium ossifragrum*, Huds.; *Vaccinium oxycoccos*, L.; *Sphagnum*; *Drosera rotundifolia*, L.; *D. intermedia*, Hayne. The ground on each

side of the small stream, crossing the moss from east to west, is covered chiefly with Sphagnum and Molinia.

Eight borings were made across the centre of the moss from north to south, and five from east to west, and these showed that the floor of the moss was very level with a general dip to the S.E. The total depth of the peat varied from about 6–8 feet near the edge of the moss to about 20 feet near the centre. Sections made on the eastern margin of the moss, near Palwhilly, showed the following series of strata :—

1. Sphagnum peat mixed with some *Scirpus cæspitosus*, . . 3–4 feet.
2. Peat formed chiefly from Eriophorum, with traces of Sphagnum and Calluna, ... 3 „
3. Moss peat containing quantities of *Hypnum*, sp., and *Polytrichum Commune*, L.
4. A layer of *Betula alba* mixed with leaves of the same tree. The wood is of all sizes up to 10 or 12 inches in diameter, and several stools were found with roots penetrating the surrounding peat.
5. Stiff blue stoneless clay, the first few inches containing fairly numerous pieces of small wood of Quercus, Alnus, and Pinus. This wood is apparently drifted, the bark in most cases being worn off and the ends rounded.

Borings were taken down through the clay for 6 feet, but no change in character was observed at this depth. The wood fragments cease a few inches below the surface of the clay. This general sequence of beds was repeated in all the sections made from different places in the moss. Some of the sections laid bare large fragments of pine and oak wood lying at the base of the moss, but these all bore traces of long drifting. The clay on the banks of the river at the top of the moss contains many much larger fragments of these trees, all bearing signs of drifting, the only stools *in situ* being birch in the basal layers of the peat.

Priestside Flow.—This moss lies on the Solway coast between Annan and Dumfries, the inland edge of the moss being 40 feet above Ordnance datum with the coastward margin 25 feet above O.D. The vegetation is of a much drier type than that covering the Moss of Cree; *Calluna vulgaris* being dominant, mixed with some *Eriophorum vaginatum* and *E. angustifolium*. *Myrica Gale* is absent, and there is but little Sphagnum. A series of borings showed the peat to have an average depth of about 14 feet, with the same level floor gently sloping seaward as was found in the Moss of Cree.

The first section was made on the inland side about 40 feet above Ordnance datum, and the following beds exposed :—

1. Sphagnum peat containing *Carex*, sp., and *Scirpus cæspitosus*, . 7 feet.
2. Eriophorum peat, ... 4–6 in.
3. Peat containing shrubby birch in places, . . . 1 foot.
4. Peat consisting chiefly of the remains of *Phragmites communis*, 5 feet.

5. Hard dry peat of a dark red colour, containing remains of *Corylus Avellana*, *Alnus glutinosa*, *Quercus*—the latter apparently drifted.
6. Coarse gray sand devoid of plant remains. The Corylus in the basal layers is particularly abundant.

Two other sections taken on the landward side confirmed these results, and the sides of a large turbary near by showed the same features. Sections were then made on the seaward side, and the following beds exposed :—

1. Sphagnum peat containing Calluna and a small quantity of *Myrica Gale*, ... 7–8 feet.
2. Peat consisting chiefly of Phragmites, ... $3\frac{1}{2}$,,
3. Stiff gray clay containing remains of *Phragmites communis*, . 4 ,,

   This layer is interstratified with seams of peaty material containing remains of the same plant. Sandy layers also run through the clay, and these contain numerous remains of the rhizomes of *Holcus mollis*, L. At the bottom of this layer the clay becomes more sandy and black in colour, without any determinable plant remains. This seam is about 5 inches in thickness.
4. Fine sand.

Layer 3 has evidently been formed by constant flooding, which continued for a period long enough to deposit the clay layers between the peaty material. At the margin of this raised beach an abundant growth of Phragmites sprang up, only to be destroyed by flooding, which at the same time deposited a layer of sand and mud over the plant-bed. This growth of Phragmites may have occurred whilst the beach was still being raised, and the flooding caused, not by changes in the level of the land, but rather by shifting of sand-banks close in to the shore. The height of the moss at the place where these sections were made is about 25 feet above Ordnance datum. It is interesting to note that the basal peat with Corylus, represented in the higher part of the beach, is absent from the seaward side. It would appear that the higher parts of the beach actually became clothed with woodland before the seaward side had entirely emerged from the sea, and that towards the close of the period of upheaval the woodland died out over the upper ground and the whole beach became covered with a vegetation indicating wet conditions.

The two mosses first described may fairly be taken as representative of the mosses lying on the 25-feet raised beach of the south coast. The absence of any remains of Arctic plants at the base, and still more the presence of such temperate forms as Corylus and Alnus, with the abundance of nuts of the former, is of interest as showing that these mosses began to form under climatic conditions which were certainly not less favourable than those at the present day. Both the birch at the base of the Moss of Cree, and the Corylus of Priestside Flow, grew *in situ*, as none of the material shows any sign of water

action: the bark on the stems and twigs being quite uninjured, and the nuts in an excellent state of preservation, and, further, numerous roots of the same trees are to be found running through the peat. There is, then, every indication that the peat immediately overlying the clay represents the primitive vegetation covering the surface of these raised beaches. The drifted pine and oak in the clay underlying the Moss of Cree also shows that the climate inland was not less favourable to the tree growth than that at the coast, and much of the drifted timber must be débris from woodland which existed inland during the period of land elevation,

The Denudation of the Peat.—The sections already described show that considerable changes have taken place in the distribution of the vegetation during the growth of the mosses. The youngest peat in each district hitherto examined is formed of plants indicating extremely wet conditions, such as Sphagnum and Scirpus; but these plants, although still represented in the vegetation, do not occur in such profusion as they are found in the peat, but occupy isolated patches in the wettest spots, or occur mixed with other plants indicating somewhat drier conditions. Of the areas examined, the lowland mosses of Wigtonshire appear to be the only mosses in which peat is forming at the present day. These mosses are flat, badly drained, and are still dominated by large areas of Sphagnum, Scirpus, and Eriophorum. These features are also reproduced on the peat-covered ground of the flat-topped hills, plateaus, and gently-sloping moorlands of such districts as the Northern Pennines and Stainmore in Westmorland. The peat in these latter districts, however, nearly always shows traces of wasting, the greater rainfall and freedom of drainage favouring denudation. The peat of the hill-sides, although containing thick Sphagnum, Scirpus, and Eriophorum beds, is no longer clothed with these plants but with a much drier type of vegetation, and denudation has evidently gone on here for a long period. In the Moorfoot Hills and Tweedsmuir, many of the steepest hill-sides are thickly covered with peat; but this is only the remnant of what was once a much greater covering, both thicker and larger in extent. Geikie (13) has discussed the general features of denudation to be met with in Western Europe, and there is little to add to the account given in his paper.

The phenomena are too universal to be entirely accounted for by drainage operations; these may, indeed, accelerate the wasting of the peat in some districts, but cannot account for it in all. Furthermore, the peat on the eastern side of England and S. Scotland is denuded to a much greater extent than that in the western districts—*i.e.*, the wasting began earlier, and is more rapid in those districts having a smaller rainfall, other factors, such as the slope of the ground and elevation above sea-level, being equal. Comparing the peat of the Moorfoot Hills with that in the Galloway district, the amount of denudation is much greater in the former; for, although the peat is wasting away over most of the Galloway mosses, the shrinkage is not nearly so marked as it is on the Moorfoots. The same difference in the amount of denudation can also be seen in England on comparing the hills of the Lake district with the eastern slopes of the Pennines. Although the topography of Western Cumberland and Westmorland is not favourable

to any great growth of peat, yet, where it does occur, the amount of denudation to which it has been subjected is slight compared with the wasting away of the thick mantle of peat covering all the watersheds of the rivers Tees, Tyne, and Wear.

An examination of the successive beds of vegetation contained in the older mosses shows that the rate of peat formation has not been uniform, for the length of time required to form a layer of closely-compressed stems of Empetrum and Arctic willows only a few inches in thickness, might possibly be sufficient to form several feet of Sphagnum peat. It would seem, then, that peat formation has been almost arrested at some stages in the history of the mosses; and I have met with features in the Merrick-Kells mosses which suggest that the peat has been subjected to denudation about the time of the formation of the Arctic plant-bed.

Summary and General Conclusions.

The peat in all the districts examined shows a definite stratification of plant remains, indicating a swing from woodland to heath and moss, and again to woodland. In some districts, an Arctic plant-bed is interposed between the lower and upper woodland beds.

Gunnar Andersson has described alternations of woodland beds, with Sphagnum, and with heather layers, as occurring over large areas in Central and Southern Sweden, and he attributes such alternation to changes in drainage caused by the throwing up of a clay or sand bank by natural or artificial causes near a moss territory, thus causing flooding, and consequently favouring the growth of Sphagnum at the expense of woodlands. This may quite possibly have produced like results in similar districts in Britain; but in the hill districts described in the course of this paper such causes cannot have operated, as the mosses are situated on steeply-sloping ground on which no natural or artificial dam could be created. The regularity of the sequence of the beds, and their general agreement on similar although widely separated areas, tend to show that these beds represent successive changes in the vegetation which have been brought about by climatic changes at the passing away of the glacial period.

None of the Scottish districts investigated by the author show any remains of Arctic plants at the base of the peat; but, on the contrary, some of them, such as the 25-feet raised beach mosses, contain remains of hazel in the basal layer. From the discovery by the late Mr James Bennie, of Arctic plants in the old alluvia of the Edinburgh district (14), the same features were expected at the base of some of the deeper lowland mosses, such as those in Wigtonshire. The presence of woodland at the base of these mosses, however, suggests that they did not originate until a temperate climate had replaced the Arctic conditions of the *mer de glace* period. In the Cross Fell peat, at 2500 feet, a bed of Arctic willows and Empetrum has been met with lying on the clay at the base of the peat (15), and, on recently re-examining this bed, I was struck by its great similarity to the Arctic bed lying between the two woodland zones in the Merrick-Kells mosses.

A summary has been given by Prof. Geikie (16) of many of the more important

papers dealing with Arctic zones in the late glacial deposits of the Continent, and I hope that correlation between these zones and those in the British peat may be possible when areas have been investigated in the north of Scotland.

The general sequence of vegetation observed in the peat of the several districts may be summarised as follows :—

The Merrick and Kells mosses, and the mosses in the Tweedsmuir district, occur above and upon the moraines of the local glaciers of the Southern Uplands, and must, therefore, be of later date than these.

That these mosses began to grow at some period between the disappearance of the local glaciers and the reappearance of glacial conditions, is shown by the presence in both districts of an Arctic plant-bed running between the lower and upper woodland bed. The conditions which would favour the growth of such a vegetation in the S.W. of Scotland at only 800–1200 feet, would be severe enough to cause considerable glaciation in the Highlands. The plant-beds below and above the Arctic bed also tend to show that this layer indicates one of the smaller and later returns to glacial conditions; for the beds below show a gradual increase, and above, a gradual decrease, in precipitation. If this reading is correct, interest would attach to an examination of any deep peat deposits resting on the 50-feet raised beach, as we might expect to find, in that case, the representative of the Arctic zone of the Merrick mosses resting upon the surface of the beach.

The peat of the Moorfoots contains no widespread forest beds, basal birch only being found low down on some of the hill-sides. Eriophorum and Molinia have been found at the base of the peat on the steepest hill-sides, thus showing that these mosses began to form under extremely wet conditions, the higher ground being covered with Eriophorum bog whilst the lower slopes supported copses of birch and willows. There is no sign of Arctic vegetation at the base of this peat, but the basal swamp vegetation gives place above to a formation indicating much drier and probably colder conditions, represented by a zone of Empetrum with Arctostaphylos Uva-ursi.

The question arises whether this Empetrum bed can be correlated with the Arctic zone of the Merrick and Kells mosses and the Tweedsmuir peat. If it is contemporaneous, then the later return to cold conditions represented by the high level corrie moraines of the Highlands produced little effect upon the vegetation so far south as the Moorfoots, for there are no beds above the Empetrum zone in this peat which show any return to cold conditions.

The lowland mosses of Wigtonshire occupy large hollows in the till between the outcrops of Silurian rocks, and reach a depth of about 20 feet. No Arctic plants have been found at the base, the basal vegetation consisting of shrubby birch, which is continuous over the area. The beds above this represent lake or swamp conditions; but a return to forest conditions took place later, when the mosses became fringed with pine trees of large size. The peat above the pine zone is formed of wet-condition moorland plants.

The mosses lying on the 25-feet raised beaches contain no Arctic plants, and the general facies of the vegetation agrees with that in the upper layers of the older mosses inland. The basal layers consist of birch, hazel, and alder, which give place above to wet-condition plants such as Sphagnum, Eriophorum, and Phragmites.

Birch is represented in the basal layers of all the Scottish mosses described in this paper, and birch has also been found in the lower layers of some of the Highland peat, as described by Mr CLOUGH in the East Ross district (17).

In conclusion, I wish to express my thanks to Professor J. GEIKIE, LL.D., F.R.S., for much kindly advice and help during the progress of the work; and to Dr HORNE, F.R.S., for kindly lending me Geological Survey maps and lists of all the Scottish peat mosses, and for his valuable advice during the progress of the field work.

I am also much indebted to Mr CLEMENT REID, F.R.S., for kindly identifying several of the seeds from the different layers.

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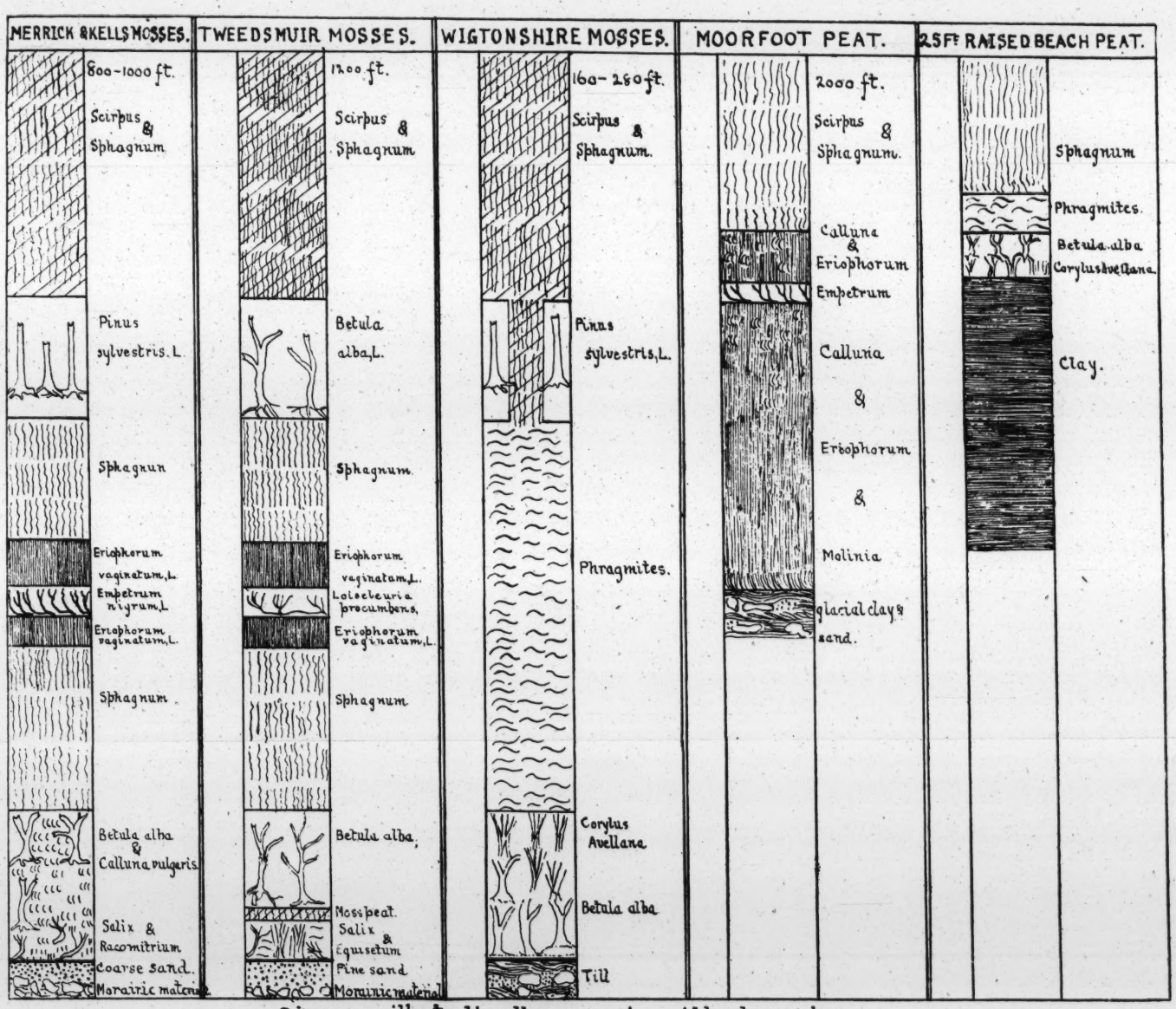

Diagram illustrating the succession of plant remains.

LEWIS: Plant Remains in the Scottish Peat Mosses.—PLATE I.

FIG. 1.—Denudation of the peat at the eastern margin of the Merrick-Kells Mosses. Kells Range in the background.

FIG. 2.—Pine zone and Empetrum nigrum zone separated by Sphagnum and Eriophorum, in the Merrick-Kells peat.

LEWIS: Plant Remains in the Scottish Peat Mosses.—PLATE II.

FIG. 3.—Pine zone in the Merrick-Kells peat.

FIG. 4.—Valley moraines in Winterhope.

LEWIS: Plant Remains in the Scottish Peat Mosses.—PLATE III.

FIG. 5.—Loch Skene, showing the peat running up to the base of the moraines at the foot of Lochcraig Head.

FIG. 6.—Upper Birch zone exposed on the banks of Winterhope Burn.

LEWIS: Plant Remains in the Scottish Peat Mosses.—PLATE IV.

FIG. 7.—Upper Layers of peat in Winterhope.

FIG. 8.—Alternation of peat and clay-beds in Winterhope.

LEWIS: Plant Remains in the Scottish Peat Mosses.—PLATE V.

FIG. 9.—Peat resting on the moraines at the foot of Loch Skene.

FIG. 10.—Basal layers of peat resting upon fine grey sand in Winterhope.

LEWIS: Plant Remains in the Scottish Peat Mosses.—PLATE VI.

FIG. 11.—Moorfoot Hills. Denudation of the peat on Emly Bank at 1900 ft.

FIG. 12.—Lowland Mosses in Wigtonshire. Dirskelpin Moss.

XXIX.—Semi-regular Networks of the Plane in Absolute Geometry.

By Duncan M. Y. Sommerville, M.A., B.Sc., University of St Andrews. *Communicated by* Professor P. R. Scott Lang. (With Twelve Plates.)

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§ 1. The networks considered in the following paper are those networks of the plane whose meshes are regular polygons with the same length of side.

When the polygons are all of the same kind the network is called *regular*, otherwise it is *semi-regular*.

The regular networks have been investigated for the three geometries from various standpoints, the chief of which may be noted.

1. The three geometries can be treated separately. For Euclidean geometry we have then to find what regular polygons will exactly fill up the space round a point. For elliptic geometry we have to find the regular divisions of the sphere, or, what is the same thing, the regular polyhedra in ordinary space. The regular networks which do not belong to either of these classes are then those of the hyperbolic plane.

2. The problem is identical with that of finding the partitions of a polygon into polygons of the same kind, with the same number of polygons at each point.* The boundary polygon is one of the meshes of the network. For elliptic networks the boundary is finite, for Euclidean networks it is wholly at infinity, and for hyperbolic networks it is wholly ideal.

This method gives a convenient mode of representing the networks, viz., by their stereographic projections upon the Euclidean plane. This representation will be employed throughout.

3. The problem corresponds to a particular case of the problem of determining all discontinuous groups of motions in the plane.†

It will be convenient here to collect the results. If n is the number of sides of each polygon, p the number of lines or polygons meeting at each point, N_2, N_1, N_0 the number of meshes, lines, and nodes respectively, the results may be summarised as follows:—

1. On the Elliptic plane there are five regular networks, corresponding to the five

* See V. Schlegel, "Theorie der homogenen zusammengesetzten Raumgebilde," *Nova Acta*, Bd. xliv., Nr. 4, 1883.

† W. Dyck, "Gruppentheoretische Studien," *Math. Annalen*, xx. 1–44 (1882), and W. Burnside, "Theory of Groups," ch. xii., xiii. Also Klein and Fricke, "Theorie der elliptischen Modulfunctionen." (For these references I am indebted to the referee.)

regular polyhedra. They are arranged in conjugate pairs, the number of meshes in one being equal to the number of nodes in the conjugate network. One is self-conjugate.

	n	p	N_2	N_1	N_0
Tetrahedral, . . .	3	3	4	6	4
{ Hexahedral, . . .	4	3	6	12	8
{ Octahedral, . . .	3	4	8	12	6
{ Dodecahedral, . . .	5	3	12	30	20
{ Icosahedral, . . .	3	5	20	30	12

2. On the Euclidean plane there are three regular networks, all infinite.

	n	p	N_2 :	N_1 :	N_0
Square, ...	4	4	1	2	1
{ Triangular, . . .	3	6	2	3	1
{ Hexagonal, . . .	6	3	1	3	2

3. On the Hyperbolic plane there are an infinite number, all infinite.

$$p = 3, 4, 5, 6, >6$$
$$n > 6, 4, 3, 3, \text{any value}$$
$$N_2 : N_1 : N_0 = 2p : np : 2n$$

§ 2. We proceed to investigate the semi-regular networks, and we shall take the three geometries separately.

I. The Euclidean Plane.

We shall consider, first, how the space about a point can be exactly filled with regular polygons. Each combination of polygons satisfying this condition determines a *species of node*, and all the semi-regular networks must be built up out of the various possible species of nodes. Two networks will be considered to be of the same *type* when they contain only nodes of the same species. It is obvious that there may be varieties of the same type. The types will be divided into Groups according to the kinds of polygons involved, and the groups into Classes according to the number of kinds of polygons. Class A. consists of the regular networks, and contains three groups with one unvaried type in each. The simplest type in any group is that which contains only one species of node. I call this the *simple type*; other types I call *composite*. A group does not necessarily contain the simple type.

The Species of Nodes.

§ 3. The angle of a regular n-gon is given by the formula

$$\alpha_n = \left(1 - \frac{2}{n}\right)180^\circ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The following table of values of α_n will be immediately useful :

n	3	4	5	6	7	8	12
a_n	60	90	108	120	$128\frac{4}{7}$	135	150

Taking the four simplest polygons, we find that the sum of the angles is 378°, *i.e.* >360°. Hence *there cannot be more than three different kinds of polygons at a point.* The species of nodes therefore fall under three classes.

CLASS A. contains the *homogeneous* nodes. Denoting the regular polygons by their initial letters, the nodes of Class A. can be denoted by

1. T_6. 2. S_4. 3. H_3.

§ 4. CLASS B. Let there be at a point p_1 n_1-gons and p_2 n_2-gons. Then we have

$$p_1a_1 + p_2a_2 = 2\pi,$$

hence from (1)

$$p_1\left(1-\frac{2}{n_1}\right) + p_2\left(1-\frac{2}{n_2}\right) = 2.$$

Integralising, we obtain on the left the function

$$n_1n_2(2-p) + 2(p_1n_2 + p_2n_1)$$

where $p = p_1 + p_2$. We shall denote this by Δ. It is easily seen that the sign of Δ characterises the network as elliptic, hyperbolic, or Euclidean. For Euclidean networks Δ is always zero; for elliptic and hyperbolic networks $\Delta \gtrless 0$ respectively.

For the regular networks there is a corresponding function $n(2-p)+2p$, and for three kinds of polygons we shall find a similar function. Where there is no risk of confusion we shall call each of these Δ. The values of Δ for the regular networks are as follows: Tetrahedral 3, Hexahedral and Octahedral 2, Dodecahedral and Icosahedral 1.

Solving now for n_2, we have

$$n_2 = \frac{2p_2n_1}{n_1(p-2) - 2p_1}.$$

We have to find the integral solutions of this equation under the following conditions: $n_1 \neq n_2 \geq 3$, $p_1, p_2 > 0$, $p \not< 3$ and $p \not> 5$, therefore $p_1, p_2 \not> 4$.

The only possible sets of values of p_1, p_2 are then

$$p_1 = 1, 1, 1, 2, 2.$$
$$p_2 = 2, 3, 4, 2, 3.$$

We shall take these cases in succession.

$p_1 = 1$, $p_2 = 2$: $\quad n_2 = \dfrac{4n_1}{n_1-2} = 4 + \dfrac{8}{n_1-2}$,

whence $\quad n_1 = 3, 4, 10$,
$\quad n_2 = 12, 8, 5$.

$p_1 = 1$, $p_2 = 3$: $\quad n_2 = \dfrac{3n_1}{n_1-1} = 3 + \dfrac{3}{n_1-1}$.

There are no unequal values of n_1 and n_2 satisfying this equation.

$$\left.\begin{matrix} p_1=1 \\ p_2=4 \end{matrix}\right\} \qquad n_2=\frac{8n_1}{3n_1-2} \quad \text{or} \quad 3n_2=8+\frac{16}{3n_1-2}$$

whence $n_1=6$, $n_2=3$.

$$\left.\begin{matrix} p_1=2 \\ p_2=2 \end{matrix}\right\} \qquad n_2=\frac{2n_1}{n_1-2}=2+\frac{4}{n_1-2}$$

whence $n_1=3$, $n_2=6$.

$$\left.\begin{matrix} p_1=2 \\ p_2=3 \end{matrix}\right\} \qquad n_2=\frac{6n_1}{3n_1-4}=2+\frac{8}{3n_1-4}$$

whence $n_1=4$, $n_2=3$.

There are thus six species of nodes in this class. They may be denoted by TD_2, SO_2, $\mathrm{Dec}\,P_2$, T_4H, T_2H_2, T_3S_2.

§ 5. Class C. Here we have

$$p_1\left(1-\frac{2}{n_1}\right)+p_2\left(1-\frac{2}{n_2}\right)+p_3\left(1-\frac{2}{n_3}\right)=2$$

or $$\Delta \equiv n_1n_2n_3(2-p)+2(p_1n_2n_3+p_2n_3n_1+p_3n_1n_2)=0.$$

Solving for n_3, $$n_3=\frac{2p_3n_1n_2}{n_2(p-2\,.\,n_1-2p_1)-2p_2n_1}.$$

We have to solve this equation in integers under the following conditions : $n_1 \neq n_2 \neq n_3 \geqslant 3$, $p_1, p_2, p_3 > 0$, $p \not< 3$, also $p \not> 4$ (for $3\alpha_3+\alpha_4+\alpha_5 > 360°$), therefore $p_1, p_2, p_3 \not> 2$. Further, if $p_1=2$, n_1 must be either 3 or 4, for $2\alpha_5+\alpha_3+\alpha_4 > 360°$. Again, we cannot have $n_1=5$, $n_2=6$, $n_3=7$ together, for $\alpha_5+\alpha_6+\alpha_7 > 360°$.

The following are therefore the only possible sets of values of p_1, p_2, p_3, n_1 :

$$p_1=1, \text{ or } 2\,;\ p_2=p_3=1\,;\ n_1=3, \text{ or } 4.$$

We shall take these cases in succession.

$$\left.\begin{matrix} p_1=p_2=p_3=1 \\ n_1=3 \end{matrix}\right\} \qquad n_3=\frac{6n_2}{n_2-6}=6+\frac{36}{n_2-6}$$

whence $n_2=7,\ 8,\ 9,\ 10,$
$n_3=42,\ 24,\ 18,\ 15.$

$$\left.\begin{matrix} p_1=p_2=p_3=1 \\ n_1=4 \end{matrix}\right\} \qquad n_3=\frac{4n_2}{n_2-4}=4+\frac{16}{n_2-4}$$

whence $n_2=5,\ 6,$
$n_3=20,\ 12.$

$$\left.\begin{matrix} p_1=2,\ p_2=p_3=1 \\ n_1=3 \end{matrix}\right\} \qquad n_3=\frac{3n_2}{n_2-3}=3+\frac{9}{n_2-3}$$

whence $n_2=4,$
$n_3=12.$

$$\left.\begin{matrix} p_1=2,\ p_2=p_3=1 \\ n_1=4 \end{matrix}\right\} \qquad n_3=\frac{2n_2}{n_2-2}=2+\frac{4}{n_2-2}$$

whence $n_2=3,$
$n_3=6.$

There are thus eight species of nodes in this class.

§ 6. Collecting all the species of nodes, we can arrange them in the following scheme:

Class A.	1. T_6.	2. S_4.	3. H_3.					
Class B.	4. T_3S_2.	5. T_2H_2.	6. T_4H.	7. TD_2.	8. SO_2.	[9.] P_2Dec.		
Class C.	10. TS_2H.	11. T_2SD.	12. SHD.					

[13–17.]	n_1	4,	3,	3,	3,	3,
	n_2	5,	10,	9,	8,	7,
	n_3	20,	15,	18,	24,	42.

In future we shall refer to these nodes by their numbers in this scheme.

Of the seventeen species of nodes, only eleven are capable of development to form networks.

In No. 9 the pentagon must be surrounded alternately with pentagons and decagons, which is impossible since 5 is odd.

In the same way, when $p_1 = p_2 = p_3 = 1$, n_1, n_2 and n_3 must all be even, for the n_1-gon must be surrounded alternately with n_2-gons and n_3-gons. Hence 13–17 cannot be developed by themselves; nor can they be developed in combination, for each contains a polygon which is not contained in any other node. Similarly, 9 cannot be developed in combination, hence these six species are excluded from all the networks.

Again, T_2SD cannot be developed by itself, for, taking the square (fig. 1, Plate IV.), we must have a dodecagon on one side and on the adjacent sides double triangles. At the free corners of the square we must now have dodecagons, but this brings two dodecagons at a point and introduces 7; excluding this, we must introduce 4.

§ 7. We can now divide the types of networks into groups and classes. Five kinds of polygons are at our disposal, but octagons only occur in the combination SO_2, hence there are only four classes.

Class A. Regular networks.
- Group I. Triangles (1).
- „ II. Squares (2).
- „ III. Hexagons (3).

Class B. Two kinds of polygons.
- Group I. Triangles and Squares (1, 2, 4).
- „ II. Triangles and Hexagons (1, 3, 5, 6).
- „ III. Triangles and Dodecagons (1, 7).
- „ IV. Squares and Octagons (2, 8).

Class C. Three kinds of polygons.
- Group I. Triangles, Squares, and Hexagons (1, 2, 3, 4, 5, 6, 10).
- „ II. Triangles, Squares, and Dodecagons (1, 2, 4, 7, 11).
- „ III. Squares, Hexagons, and Dodecagons (2, 3, 12).

Class D. Four kinds of polygons.

Group. I. Triangles, Squares, Hexagons, and Dodecagons (1, 2, 3, 4, 5, 6, 7, 10, 11, 12).

The numbers within the brackets denote the species of nodes which the group admits.

The Simple Types.

§ 8. Now let us consider the simple types. I observe, in the first place, that when the species of node admits of no variation, the simple type is *in general* unvaried.

The unique nodes are the following:

1°. Class A,
2°. Those in which $p = 3$,
3°. $p_1 = 1$, $p_2 = 4$,

while the following are varied:

1°. $p_1 = p_2 = 2$. Two forms, M_2N_2 and $(MN)_2$.
2°. $p_1 = p_2 = 1$, $p_3 = 2$. Two forms, LMN_2 and LNMN.

We have then the *unique simple types.*

Class A. T_6, S_4, H_3.
,, B. T_4H, TD_2, SO_2 (figs. 4, 2, 3).
,, C. SHD (fig. 5).

The type T_4H is one exception to the rule stated above, for it does admit of a variation. The network is asymmetrical, its mirror image being different from itself. It exists, therefore, in two enantiomorphic forms. The one can be obtained from the other by turning the plane over.

Of the other groups, C. II. and D. I. do not possess simple types, and there remain the three simple types T_3S_2, T_2H_2, and TS_2H, each of which is capable of infinite variation.

T_2H_2 has a variety in which there are no two triangles and no two hexagons together. We shall call this the *fundamental* variety. The opposite sides of any hexagon, when produced, define a strip which is capable of displacement without affecting the rest of the network. All the varieties can then be obtained by displacing any number of such parallel strips through a distance equal to the length of the side (fig. 6).

TS_2H has the fundamental variety in which there are no two squares together. Each hexagon is surrounded by squares and triangles, forming a group whose boundary is a regular dodecagon. All the varieties can then be obtained by turning any number of such groups through $\frac{\pi}{6}$. This operation, performed upon a single group, brings two squares together; performed upon two adjacent but not overlapping groups, it brings three squares together (fig. 7).

§ 9. The unique types and the fundamental varieties of T_2H_2 and TS_2H can be obtained from the regular networks by fairly obvious dissections. Thus, SHD (fig. 5) is obtained from either the triangular or the hexagonal network; for the squares, hexagons, and dodecagons have (1, 1) correspondence with the lines, meshes, and nodes respectively of the triangular network. In a similar way T_4H (fig. 4) is obtained from the same network; to each mesh there corresponds a triangle, to each node a hexagon, and to each line two triangles. And so for the others: it is only necessary to compare the figures, given below (§ 11), which represent the relative numbers of the various polygons, with the numbers of nodes, lines, and meshes in the regular networks. In the diagrams given for the unique types the regular network is indicated by shading.

T_4H and all the varieties of T_2H_2 can also be obtained from the regular triangular network by replacing all the groups of six covertical triangles by hexagons; and TS_2H can be obtained from SHD by bordering every dodecagon internally with squares and triangles.

§ 10. The type T_3S_2 forms an exception to what has been said regarding the way in which the network may be obtained. One of its varieties, that in which no two squares are together, can be obtained in a simple way from the regular square network; to each mesh corresponds a square, to each node a square, and to each line two triangles. But the other varieties cannot be obtained from this, nor, in general, in any simple way from the square network. The following forms may be enumerated, though the list is not exhaustive:—

(1) 1, 2, 3, ... squares always together. Each of these is unique, and the series forms a general type of variety, admitting of an infinite number of mixtures (fig. 8).

(2) 2, 3, ... squares or fewer together. Here we can distinguish

(i.) Two similar types, in which there occurs once only (*a*) a single triangle surrounded by three squares, (*b*) a triangular group of four triangles surrounded by three double squares. The network radiates from this figure as centre (fig. 10).

(ii.) A general type, obtainable by a dissection of the square network, in which (*a*) and (*b*) are excluded (fig. 9).

(iii.) Further, if an unlimited number of squares may be together, the groups (*a*) and (*b*) may occur more than once, or together.

§ 11. From what has been said it is evident that for any of the simple types, with the possible exception of T_3S_2, the relative number of the several kinds of polygons is definite, and the same for all the varieties. These numbers can be found by inspection and a knowledge of the number of meshes, lines, and nodes in the regular networks. General expressions for the ratios may be found as follows. The results show that T_3S_2 is not an exception in this respect.

Let p', p'', p''' be the number of n'-, n''-, n'''-gons meeting at each point; N_0 the number of nodes, N_2', N_2'', N_2''' the number of n'-, n''-, n'''-gons in the whole network.

Then each n'-gon has n' angles; but if we count up the whole number of angles contributed by all the n'-gons, each is counted p' times.

$$\begin{aligned} &\text{Hence} && n' N_2' &&= p' N_0 . \\ &\text{Similarly} && n'' N_2'' &&= p'' N_0 , \\ & && n''' N_2''' &&= p''' N_0 . \\ &\text{Therefore} && N_2' : N_2'' : N_2''' &&= \frac{p'}{n'} : \frac{p''}{n''} : \frac{p'''}{n'''} . \end{aligned}$$

So that the relative number of polygons is the same for all varieties of the same type.

For T_3S_2,	$T : S = 2 : 1$	For SO_2,	$S : O = 1 : 1$
T_2H_2,	$T : H = 2 : 1$	TS_2H,	$T : S : H = 2 : 3 : 1$
T_4H,	$T : H = 8 : 1$	SHD,	$S : H : D = 3 : 2 : 1$
TD_2,	$T : D = 2 : 1$		

§ 12. Let us investigate the analogous formulæ for composite types. Let ${}_1N_0$ be the number of nodes at which there are p_1' n'-gons, p_1'' n''-gons, p_1''' n'''-gons, and p_1^{iv} n^{iv}-gons, where one at least of the quantities p_1 is zero, and let

$${}_1N_0 : {}_2N_0 : \ldots : {}_rN_0 = 1 : k_1 : \ldots : k_{r-1} .$$

$$\begin{aligned} &\text{Then} && n' N_2' &&= p_1' {}_1N_0 + p_2' {}_2N_0 + \ldots + p_r' {}_rN_0 \\ & && &&= (p_1' + k_1 p_2' + \ldots + k_{r-1} p_r') {}_1N_0 \\ &\text{Therefore} && N_2' &&= \frac{1}{n'} (p_1' + k_1 p_2' + \ldots + k_{r-1} p_r') {}_1N_0 . \end{aligned}$$

Let N_2, N_1, N_0 have their usual meanings, then

$$\begin{aligned} N_0 &= {}_1N_0 + \ldots + {}_rN_0 = (1 + k_1 + \ldots + k_{r-1}) {}_1N_0 \\ N_2 &= N_2' + N_2'' + N_2''' + N_2^{iv} . \end{aligned}$$

Also, by the analogue of EULER's polyhedral formula,

$$N_2 - N_1 + N_0 = 1 .$$

N_1 can also be expressed in terms of ${}_1N_0, \ldots, {}_rN_0$ thus: at each of the ${}_1N_0$ points there are p_1 lines, and each line joins two points, hence

$$2N_1 = p_1 {}_1N_0 + \ldots + p_r {}_rN_0 ,$$

whence we get

$$\sum_{i=1}^{i=4} \left(\frac{1}{n^i} \sum_{\lambda=1}^{\lambda=r} k_{\lambda-1} p_\lambda^i \right) - \tfrac{1}{2} \sum_{\lambda=1}^{\lambda=r} k_{\lambda-1} p_\lambda + \sum_{\lambda=1}^{\lambda=r} k_{\lambda-1} = \frac{1}{{}_1N_0} .$$

Now, since the number of species of nodes in the whole network is finite, one at least of the quantities ${}_1N_0, \ldots, {}_rN_0$ must be infinite. Hence we may put ${}_1N_0 = \infty$. The equation then becomes

$$\sum_{\lambda=1}^{\lambda=r} \left(\frac{p_\lambda'}{n'} + \frac{p_\lambda''}{n''} + \frac{p_\lambda'''}{n'''} + \frac{p_\lambda^{iv}}{n^{iv}} - \frac{p_\lambda}{2} + 1 \right) k_{\lambda-1} = 0 .$$

But this is an identity on account of the fundamental relation $\Delta = 0$. Without further

conditions, then, we cannot find any relation between the k's, which are therefore indeterminate. This shows that in general a composite type admits of infinite variation, the ratios of the number of polygons being indefinite.

Composite Types.

§ 13. We have defined two networks to be of the same type when they involve only nodes of the same species, whatever may be their form and structure. The determination of the composite types in any group thus reduces to finding all the possible combinations of nodes admitted by the group. In order to eliminate the impossible combinations, we can find what combinations must occur. We shall take each group in turn.

Class B. I. Triangles and Squares (1, 2, 4).

(a) 2 must be accompanied by 4 (fig. 11).

Hence the only combinations are 1, 4; 2, 4; 1, 2, 4, each of which gives a composite type. Examples of each are given by joining together infinite parallel strips of the triangular and square networks.

II. Triangles and Hexagons (1, 3, 5, 6).

(b) 3 must be accompanied by 5 (fig. 12).

Hence the only combinations are 1, 5; 1, 6; 3, 5; 5, 6; 1, 3, 5; 1, 5, 6; 3, 5, 6; 1, 3, 5, 6. There are thus eight composite types. Every variety of them can be obtained from the triangular network by replacing groups of six triangles by hexagons.

III. Triangles and Dodecagons (1, 7).

IV. Squares and Octagons (2, 8).

Each of these is unique and admits of no composite types.

§ 14. Class C. I. Triangles, Squares, and Hexagons (1, 2, 3, 4, 5, 6, 10).

The number of combinations of these numbers, two or more together, so as to include the three polygons, is 105. We proceed to establish rules for the rejection of impossible combinations.

(a) 2 must be accompanied by 4 or 10 (fig. 11).

(b) 3 ,, ,, ,, 5; and either 6, or 10 and 4 (fig. 12).

Exclude 6; then we must have 10, since squares and hexagons come together; and since there are always two triangles together, we must also have 4.

Exclude 10; then squares cannot be introduced till the hexagons have been surrounded by triangles, and we have 6.

(c) 5 must be accompanied by either 10, or 4 and 6 (fig. 12).

Exclude 10; before squares can be added we get 6, and the further addition of squares gives 4.

Hence if 10 is excluded, we must have 4 and 6.

For 4 is necessary in order to give squares, and if we exclude 6 we also exclude 5, and therefore 3. We are then left with only 1, 2, 4, which do not involve hexagons.

(d) 6 must be accompanied by 4 or 5.

Excluding 4 and 5, the hexagon must be surrounded by triangles, and squares can never be introduced without producing 4.

(e) 1 must be accompanied by either 4, or 5 and 6.

Exclude 5 and 6; then T_6 must be surrounded by either triangles, or triangles and squares. Since hexagons are excluded at this stage, either of these introduces 4.

Exclude 4 and 6; then T_6 must be surrounded by hexagons. Now squares can only be introduced after the concavities have been filled up. If we fill them with hexagons fresh gaps are produced, and if we fill them with triangles there are always two triangles together, and the addition of squares is impossible without giving 4. Hence we cannot exclude both 4 and 6.

Exclude 4 and 5; then 6 is also excluded by (d), and T_6 can only be surrounded by triangles.

Also 1 can only be continued by 4, 5, or 6; hence if we exclude 4, we must have both 5 and 6.

Rejecting according to these rules, we are left with forty-seven combinations, each of which gives a composite type. The combinations may be represented by the following notation. Let $C_r(a_1, \ldots, a_n)$ stand for any one combination of r or more of the a's, then the forty-seven combinations are

$$5 + C_2(4, 6, 10) + C_0(1, 2, 3)$$
$$4 + C_1(6, 10) + C_0(1, 2)$$
$$10 + C_1(2, 5).$$

§ 15. Class C. II. Triangles, Squares, and Dodecagons (1, 2, 4, 7, 11).

(a) 2 must be accompanied by 4 (fig. 11).

(f) 7 „ „ „ 11 (fig. 1).

After rejection there are left eleven combinations, all of which give composite types except 1, 7, 11. There are therefore the following ten composite types in this group:

$$4, 11 + C_0(1, 2, 7); \quad 1, 11; \quad 7, 11.$$

III. Squares, Hexagons, and Dodecagons (2, 3, 12).

Excluding triangles, 12 can only be continued in one way, hence there are no composite types in this group.

§ 16. Class D. Triangles, Squares, Hexagons, and Dodecagons (1, 2, 3, 4, 5, 6, 7, 10, 11, 12).

The number of combinations of two or more together of these numbers which involve all four polygons is 856, but we have the following rules for the rejection of impossible combinations:

(a) 2 must be accompanied by 4 or 10.
(b) 3 „ „ „ 5; and either 6, or 10 and 4.
(c) 5 „ „ „ either 10, or 4 and 6.
(d) 6 „ „ „ 4 or 5.
(f) 7 „ „ „ 11.

The previous proofs of these still hold.

(e) 1 must be accompanied by either 4, 5 and 6, or 11.

If we exclude 11, triangles or hexagons must be in combination with squares, and we have seen that squares can never be introduced if we exclude 4 and either 5 or 6. But 1 may be continued by 11 (fig. 16).

(g) We must have either 11, or 10 and 12.

For, excluding 11, we must have 12 at least, for dodecagons are only given by 7, 11, and 12, and 7 is excluded by (f).

Starting therefore with 12, we must have fig. 13. The concavities can now be filled either with dodecagons, or with squares and triangles. The latter gives 10, the former never introduces triangles.

(h) If 4 and 12 be excluded, the only combination is 7, 10, 11.

Excluding 4 and 12, we must have 11. Let us start therefore with fig. 14 (the heavy lines). a cannot be a square, for that gives a hexagon at b; nor a hexagon; nor a dodecagon, since 12 is excluded. Hence a must be a triangle, and we get the figure with dotted lines.

Again, if we start with fig. 15 (the heavy lines), it must be continued as in the dotted lines.

Lastly, let us start with fig. 16. A dodecagon at a gives us a variation of fig. 14 with the hexagon turned through 60°, a square gives fig. 15, and a triangle fig. 16 with the dotted lines.

The continuation of any of these figures (under the given conditions) will introduce no angles other than 1, 7, 10, 11; and fig. 16 must be excluded, since it does not contain hexagons. Hence the only possible combination is 7, 10, 11.

(i) If 4 and 10 be excluded, the only combination is 1, 11, 12.

We must have 12, by (h); and 11, by (g).

Starting with 12 we get fig. 17. At a we must put either a hexagon or six triangles, hence the figure can only be continued as in the dotted lines, where some of the hexagons, but not all, must be filled up with triangles. This is the combination 1, 11, 12.

Rejecting according to these rules, there are left 222 combinations. I have tested

these and found composite types corresponding to 176 of them. Of the remainder it is probable that a considerable proportion do not give types. Thus it seems probable that the only types which involve 11 without 4 are 1, 10, 11, 12; 1, 11, 12; 7, 10, 11; and 7, 10, 11, 12.

The combinations are all included in the following lists. A. contains the 222 left after rejection, B. those which I have not verified.

A. 4, 5, 11 + C_0(1, 2, 3, 7, 12) + C_1(6, 10).
4, 5, 10, 12 + C_0(1, 2, 3, 6).
4, 11 + C_0(1, 2, 7) + C_1(6, 10, 12).
4, 10, 12 + C_0(1, 2, 6).

5, 6, 10, 11, 12 + C_0(1, 2, 3, 7).
10, 11, 12 + C_0(1, 2, 5, 7).
5, 6, 10, 12 + C_0(1, 2, 3).
10, 12 + C_0(2, 5); 1, 11, 12; 7, 10, 11.

B. 3, 4, 5, 6, 7, 11, 12 + C_0(1, 2).
2, 4, 6, 11 + C_1(7, 12) + C_0(5).
2, 4, 5, 10, 11, 12 + C_0(7). 4, 11, 12.
3, 4, 5, 6, 11, 12. 4, 5, 6, 11.

5, 6, 10, 11, 12 + C_0(1, 2, 3, 7).
10, 11, 12 + C_0(1, 2, 5, 7) [except 1, 10, 11, 12 and 7, 10, 11, 12.]
5, 6, 10, 12.

§ 17. Many of these composite types can be obtained from the simple types by filling up the hexagons and dodecagons. Thus, as we have seen, the type 1, 11, 12 can be obtained from the simple type 12 by filling up some of the hexagons with triangles. From the same simple type can be obtained nine other composite types involving the nodes 1, 4, 10, 11, 12. In the same way, having obtained one example of one type, it is generally possible to obtain a number of other types from it by some simple substitutions or displacements. A classification of the composite types might thus be attempted, based upon their structure. In this way types which are widely separated in the present classification would be brought together, and *vice versa*. It is to be noted, however, that the general variety of a type may fall on the lines of no simple network, so that a classification such as that suggested would be difficult to apply in the general case.

II. The Elliptic Plane.

§ 18. We proceed to investigate the semi-regular networks upon the elliptic plane, or, what is the same thing, upon the sphere, or in general upon a closed surface of constant positive curvature.

We shall first find what species of nodes are possible. Since the angle of a regular polygon here depends upon the area of the figure, it is obvious that the number of species of nodes is infinite. Whatever holds on the Euclidean plane regarding the number of polygons which can meet at a point will hold *a fortiori* for the sphere. Hence we have only two cases to consider: viz., at a point there may be 1° two, 2° three

different kinds of polygons, but not more. The species of nodes thus fall under three classes, Class A. consisting of the homogeneous nodes. We shall take the other two classes separately, and find those nodes which give simple types.

§ 19. Class B. Let there be p_1 n_1-gons and p_2 n_2-gons at a point; then, if a_1, a_2 are the angles of the n_1- and n_2-gon,

$$n_1 a_1 > (n_1 - 2)\pi .$$

Also
$$p_1 a_1 + p_2 a_2 = 2\pi .$$

Therefore
$$p_1\left(1 - \frac{2}{n_1}\right) + p_2\left(1 - \frac{2}{n_2}\right) < 2 ,$$

or,
$$n_1 n_2 (2 - p) + 2(p_1 n_2 + p_2 n_1) \equiv \Delta > 0 .$$

Giving Δ positive integral values we get a series of equations to solve under the following conditions: $n_1 \neq n_2 \geqslant 3$, $3 \not> p \not> 5$, therefore p_1, $p_2 \not> 4$. Also, if $p_1 = 2$, $p_2 = 3$, the smallest values of n_1 and n_2 which are possible are $n_1 = 4$, $n_2 = 3$, but these make $\Delta = 0$, hence the only possible sets of values of p_1 and p_2 are

$$\begin{aligned} p_1 &= 1, 1, 1, 2, \\ p_2 &= 2, 3, 4, 2. \end{aligned}$$

Further, if $p_1 = 1$, $p_2 = 2$, in order that the node may give a simple type, n_2 must be even, for the n_2-gon must be surrounded alternately with n_1-gons and n_2-gons.

We have then

$$n_1 = \frac{2n_2 p_1 - \Delta}{n_2(p-2) - 2p_2} .$$

We shall take each set of values of p_1, p_2 in turn.

$p_1 = 1$, $p_2 = 2$:
$$n_1 = \frac{2n_2 - \Delta}{n_2 - 4} = 2 + \frac{8 - \Delta}{n_2 - 4} .$$

Since n_2 is even, Δ must be even.

$\Delta = 2$.	$n_2 = 6, 10$	$\Delta = 6$.	$n_2 = 6$
	$n_1 = 5, 3$		$n_1 = 3$
$\Delta = 4$.	$n_2 = 6, 8$	$\Delta = 8$.	$n_2 = 4$
	$n_1 = 4, 3$		$n_1 =$ any integer.

$p_1 = 1$, $p_2 = 3$:
$$n_1 = \frac{2n_2 - \Delta}{2n_2 - 6} = 1 + \frac{3 - \frac{1}{2}\Delta}{n_2 - 3}$$

Δ must be even.

$\Delta = 2$.	$n_2 = 4$	$\Delta = 6$.	$n_2 = 3$
	$n_1 = 3$		$n_1 =$ any integer.
$\Delta = 4$	impossible		

$p_1 = 1$, $p_2 = 4$:
$$n_1 = \frac{2n_2 - \Delta}{3n_2 - 8}$$

or
$$3n_1 = 2 + \frac{16 - 3\Delta}{3n_2 - 8} .$$

$\Delta = 1$.	$n_2 = 3$	$[\Delta = 3$. $n_2 = 3 = n_1$, excluded.]
	$n_1 = 5$	
$\Delta = 2$.	$n_2 = 3$	$\Delta \not> 3$.
	$n_1 = 4$	

$p_1=2$, $p_2=2$ $\Big\}$ $\qquad n_1=\dfrac{4n_2-\Delta}{2n_2-4}=2+\dfrac{4-\frac{1}{2}\Delta}{n_2-2}$

$\Delta=2$. $n_2=3$, $n_1=5$ $\qquad$ [$\Delta=6$. $n_2=3=n_1$, excluded.]

$\Delta=4$. $n_2=3$, $n_1=4$ $\qquad \Delta \not> 6$.

§ 20. Class C. We have here the equation

$$p_1\left(1-\frac{2}{n_1}\right)+p_2\left(1-\frac{2}{n_2}\right)+p_3\left(1-\frac{2}{n_3}\right)<2,$$

or $$n_1n_2n_3(2-p)+2(p_1n_2n_3+p_2n_3n_1+p_3n_1n_2)\equiv\Delta>0.$$

We have to solve this equation under the following conditions: $n_1 \neq n_2 \neq n_3 \geqslant 3$, $3 \not> p \not> 4$, therefore p_1, p_2, $p_3 \not> 2$, and we cannot have $n_1=5$, $n_2=6$, $n_3=7$ together.

Further, if $p_1=p_2=p_3=1$, n_1, n_2, and n_3 must all be even, for the n_1-gon must be surrounded alternately with n_2-gons and n_3-gons. Again, if $p=4$, $n_3=3$, $p_3=2$, then starting with an n_1-gon we must have on successive sides an n_2-gon, a double triangle, a double triangle, an n_2-gon, and so on (fig. 18). Hence n_1, and similarly n_2, must be a multiple of 3.

The only possible sets of values of p_1, p_2, p_3, n_3 are therefore

$p_1=p_2=p_3=1$, $n_3=4$;
$p_3=2$, $p_1=p_2=1$, $n_3=3$ or 4.

We have then

$$n_1=\frac{2n_2n_3p_1-\Delta}{n_2(p-2.n_3-2p_3)-2p_2n_3}.$$

$p_1=p_2=p_3=1$, $n_3=4$ $\Big\}$ $\qquad n_1=\dfrac{8n_2-\Delta}{2n_2-8}=4+\dfrac{16-\frac{1}{2}\Delta}{n_2-4}$.

Since n_2 and n_1 are both even, Δ must be a multiple of 8.

$\Delta=8$. $n_2=6$, $n_1=10$ $\qquad$ [$\Delta=24$. $n_2=6=n_1$, excluded.]

$\Delta=16$. $n_2=6$, $n_1=8$ $\qquad$ [$\Delta=32$. $n_2=4$, excluded.]

$p_1=p_2=1$, $p_3=2$, $n_3=3$ $\Big\}$ $\qquad n_1=\dfrac{6n_2-\Delta}{2n_2-6}=3+\dfrac{9-\frac{1}{2}\Delta}{n_2-3}$.

Since n_2 and n_1 are both multiples of 3, Δ must be a multiple of 18, but $\Delta=18$ gives $n_2=3$ and any higher multiple is excluded, hence there are no developable nodes of this type.

$p_1=p_2=1$, $p_3=2$, $n_3=4$ $\Big\}$ $\qquad n_1=\dfrac{8n_2-\Delta}{4n_2-8}=2+\dfrac{4-\frac{1}{4}\Delta}{n_2-2}$.

$\Delta=4$. $n_2=3$, $n_1=5$ $\qquad$ [$\Delta=8$. $n_2=3$, $n_1=4$ $\Big\}$ excluded.]

[$\Delta=12$. $n_2=3=n_1$, excluded.]

§ 21. We have found, then, the following developable species of nodes:

Class B.
$$\left.\begin{matrix} p_1=1 \\ p_2=2 \end{matrix}\right\} \quad \begin{matrix} n_1= 3,5,3,4,3;\ 3,5,6,\ldots. \\ n_2=10,6,8,6,6;\ 4,4,4,\ldots. \end{matrix}$$
$$\left.\begin{matrix} p_1=1 \\ p_2=3 \end{matrix}\right\} \quad \begin{matrix} n_1=3;\ 4,5,6,\ldots. \\ n_2=4;\ 3,3,3,\ldots. \end{matrix}$$
$$\left.\begin{matrix} p_1=1 \\ p_2=4 \end{matrix}\right\} \quad \begin{matrix} n_1=5,4. \\ n_2=3,3. \end{matrix}$$
$$\left.\begin{matrix} p_1=2 \\ p_2=2 \end{matrix}\right\} \quad \begin{matrix} n_1=3,3. \\ n_2=5,4. \end{matrix}$$

Class C.
$$\left.\begin{matrix} p_1=1 \\ p_2=1 \\ p_3=1 \end{matrix}\right\} \quad \begin{matrix} n_1= 4,4. \\ n_2= 6,6. \\ n_3=10,8. \end{matrix}$$
$$\left.\begin{matrix} p_1=1 \\ p_2=1 \\ p_3=2 \end{matrix}\right\} \quad \begin{matrix} n_1=3. \\ n_2=5. \\ n_3=4. \end{matrix}$$

Each of these gives a simple type of network.

§ 22. We shall find the number of polygons, lines, and points in the complete network.

Let N_2', N_2'', N_2''' be the number of n'-, n''-, and n'''-gons, N_2, N_1, N_0 the number of polygons, lines, and points respectively.

Then (§§ 11, 12)

$$N_2 = N_2' + N_2'' + N_2''' \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$
$$\left.\begin{matrix} n'N_2' = p'N_0 \\ n''N_2'' = p''N_0 \\ n'''N_2''' = p'''N_0 \end{matrix}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$
$$pN_0 = 2N_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

where $p = p' + p'' + p'''$.

Finally, EULER's polyhedral formula is

$$N_2 - N_1 + N_0 = 2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

From (2) and (1) we get

$$N_2 = \left(\frac{p'}{n'} + \frac{p''}{n''} + \frac{p'''}{n'''}\right) N_0,$$

and from (3)

$$N_1 = \frac{p}{2} N_0.$$

Substituting in (4),

$$\left(\frac{p'}{n'} + \frac{p''}{n''} + \frac{p'''}{n'''} - \frac{p}{2} + 1\right) N_0 = 2;$$

whence

$$N_0 = \frac{4n'n''n'''}{\Delta}.$$

Hence

$$N_1 = \frac{2n'n''n'''p}{\Delta}.$$

$$N_2' = \frac{4p'n''n'''}{\Delta}, \quad N_2'' = \frac{4p''n'''n'}{\Delta}, \quad N_2''' = \frac{4p'''n'n''}{\Delta}.$$

For Class B. the corresponding formulæ are

$$N_0 = \frac{4n'n''}{\Delta}, \quad N_1 = \frac{2n'n''p}{\Delta};$$

$$N_2' = \frac{4p'n''}{\Delta}, \quad N_2'' = \frac{4p''n'}{\Delta},$$

where $\Delta = n'n''(2-p) + 2(p'n'' + p''n')$.

The Simple Types.

§ 23. We proceed now to classify the simple types and investigate their varieties. The division into classes according to the number of kinds of polygons can still be made, but the subdivision into groups according to the kinds of polygons involved is useless, as there are an infinite number of kinds of polygons. We shall therefore, for the present, classify them according to the *types* of nodes. I consider two nodes to be of the same type when the values of p_1, p_2, p_3 are the same for both. A network will not in general admit of variation unless its node does so. But this rule is not always true; *e.g.* the type of angle PQ_3 is unvaried, but, as we shall see, one of the networks corresponding to this type admits of two distinct varieties. We shall give for each network the values of N_0, N_1, N_2, etc. Unless otherwise specified, the network is unique.

Class B. I. $p' = 1$, $p'' = 2$.

(1) $\Delta = 2$. (*a*) $n' = 3$, $N_2' = 20$
$n'' = 10$, $N_2'' = 12$
$N_0 = 60$, $N_1 = 90$, $N_2 = 32$ (fig. 19).

(*b*) $n' = 5$, $N_2' = 12$
$n'' = 6$, $N_2'' = 20$
$N_0 = 60$, $N_1 = 90$, $N_2 = 32$ (fig. 21).

(2) $\Delta = 4$. (*a*) $n' = 3$, $N_2' = 8$
$n'' = 8$, $N_2'' = 6$
$N_0 = 24$, $N_1 = 36$, $N_2 = 14$ (fig. 20).

(*b*) $n' = 4$, $N_2' = 6$
$n'' = 6$, $N_2'' = 8$
$N_0 = 24$, $N_1 = 36$, $N_2 = 14$ (fig. 22).

(3) $\Delta = 6$. $n' = 3$, $N_2' = 4$
$n'' = 6$, $N_2'' = 4$
$N_0 = 12$, $N_1 = 18$, $N_2 = 8$ (fig. 23).

(4) $\Delta = 8$. $n'' = 4$, $N_2'' = n$.
$n' = n$, $N_2' = 2$.
$N_0 = 2n$, $N_1 = 3n$, $N_2 = n + 2$ (fig. 24).

II. $p' = 1$, $p'' = 3$.

(1) $\Delta = 2$. $n' = 3$, $N_2' = 8$.
$n'' = 4$, $N_2'' = 18$.
$N_0 = 24$, $N_1 = 48$, $N_2 = 26$.

This type has two varieties. It contains a group formed by a quadrilateral surrounded

alternately with quadrilaterals and triangles. The boundary of this group is a regular octagon, and by turning it round through $\frac{\pi}{4}$ we get the other variety (figs. 26, 27).

(2) $\Delta = 6$. $n'' = 3$, $N_2'' = 2n$
$n' = n$, $N_2' = 2$
$N_0 = 2n$, $N_1 = 4n$, $N_2 = 2(n+1)$ (fig. 25).

III. $p' = 2$, $p'' = 2$.

(1) $\Delta = 2$. $n' = 3$, $N_2' = 20$
$n'' = 5$, $N_2'' = 12$
$N_0 = 30$, $N_1 = 60$, $N_2 = 32$.

A certain great circle divides this network into two equal groups. By turning one of these through $\frac{\pi}{5}$ a second variety is obtained (figs. 28, 29).

(2) $\Delta = 4$. $n' = 3$, $N_2' = 8$
$n'' = 4$, $N_2'' = 6$
$N_0 = 12$, $N_1 = 24$, $N_2 = 14$.

Like the preceding, this network has two varieties which may be obtained in a similar way, viz., by turning one of the groups through $\frac{\pi}{3}$ (figs. 30, 31).

IV. $p' = 1$, $p'' = 4$.

(1) $\Delta = 1$. $n' = 5$, $N_2' = 12$
$n'' = 3$, $N_2'' = 80$
$N_0 = 60$, $N_1 = 150$, $N_2 = 92$ (fig. 32).

(2) $\Delta = 2$. $n' = 4$, $N_2' = 6$
$n'' = 3$, $N_2'' = 32$
$N_0 = 24$, $N_1 = 60$, $N_2 = 38$ (fig. 33).

These two networks are asymmetrical. Each exists in two forms which are enantiomorph. The one could be obtained from the other by turning the sphere inside out, supposing this to be possible, as it would be in space of four dimensions.

§ 24. Class C. I. $p' = 1$, $p'' = 1$, $p''' = 1$.

(1) $\Delta = 8$. $n' = 4$, $N_2' = 30$
$n'' = 6$, $N_2'' = 20$
$n''' = 10$, $N_2''' = 12$
$N_0 = 120$, $N_1 = 180$, $N_2 = 62$ (fig. 34).

(2) $\Delta = 16$. $n' = 4$, $N_2' = 12$
$n'' = 6$, $N_2'' = 8$
$n''' = 8$, $N_2''' = 6$
$N_0 = 48$, $N_1 = 72$, $N_2 = 26$ (fig. 35).

II. $p' = 1$, $p'' = 1$, $p''' = 2$.

$\Delta = 4$. $n' = 3$, $N_2' = 20$
$n'' = 5$, $N_2'' = 12$
$n''' = 4$, $N_2''' = 30$
$N_0 = 60$, $N_1 = 120$, $N_2 = 62$.

Of this type there are five varieties, which may be obtained as follows:—In one of the varieties (fig. 36) there are no two quadrilaterals adjacent. Each pentagon has a quadrilateral on each of its sides and forms the centre of a group with a regular decagon as boundary. Let us call

this the fundamental variety. Then all the other varieties can be obtained from it by turning some of the groups through $\frac{\pi}{5}$. Let us denote this operation by R. In the fundamental variety the twelve pentagons occupy relatively the same positions as the meshes of the dodecahedral network, so that with respect to one of the groups the others can be divided into three sets: 5 *adjacent*, 5 *circumjacent*, and 1 *opposite*. Now suppose the operation R to be performed upon one of the groups. This gives a variety β_1 (fig. 37). Next suppose a second group to be operated upon. The adjacent ones cannot be moved, for the first operation has destroyed their symmetry. Operating upon the opposite one we get a variety β_2 (fig. 38), while operating upon one of the circumjacent groups we get a fourth variety γ_1 (fig. 39). From β_2 we cannot obtain any further variety, for each of the remaining groups is adjacent to one of those already operated on. From γ_1 we can obtain a fifth variety, γ_2 (fig. 40), by turning either of the two groups which are circumjacent to both. In β_1 and β_2 pairs of quadrilaterals occur, 5 in the former, 10 in the latter. In γ_1 and γ_2 there occur respectively 1 and 3 groups of three quadrilaterals.

§ 25. To every spherical network there corresponds a convex polyhedron whose vertices are the nodes of the network. The polyhedra which correspond to the semi-regular networks have for their faces regular plane polygons. These form only a class of convex polyhedra in general, but they are the only ones whose faces may be regular polygons, and which, at the same time, may be inscribed in a sphere.

If we examine the numbers of the several polygons in the various networks above we find that, with the exception of the two infinite series, they can all be connected with the regular networks. The series with two quadrilaterals and an n-gon at each point corresponds to a series of right prisms on a regular polygonal base, the altitude diminishing indefinitely as n increases.

The polyhedra corresponding to the other types can be obtained from the regular polyhedra by cutting off the corners in particular ways. Thus the octahedron (fig. 23) bounded by triangles and hexagons can be obtained from the regular tetrahedron by cutting off the corners, either triangles or hexagons corresponding to vertices, according to the depth of the section. When the numbers of the polygons are the numbers of faces, lines or vertices of a regular polyhedron, it is evident in what way they correspond. In some, however, the same kind of polygon may correspond to both edges and vertices, then its number has to be divided into two parts, each a multiple of the number of edges or vertices of the regular polyhedron.

This holds only for the unique types and the *fundamental* varieties of the other types, *i.e.* those in which no two polygons of the same kind are adjacent. The other varieties may or may not be obtainable from the corresponding regular polyhedron. Those of Class B. are not, while the four derived varieties of Class C. II. may still be obtained from the regular dodecahedron, since the positions of the pentagons are unchanged.

§ 26. We may therefore group the simple types in three divisions according to their morphology.* We shall use the notation $\mathbf{3}_1^4\mathbf{6}_2^4$ to denote a simple type consisting

* A correspondence between the regular polyhedra and certain general classes of polyhedra was considered by C. Jordan, "Recherches sur les polyèdres," *Comptes Rendus*, lx. 400–403, lxi. 205–208, lxii. 1339–1341, 1865–66.

of triangles and hexagons, where the subscript refers to the number of polygons at a point, and the index to the number of polygons in the whole network.

I. Tetrahedral.	$3_1^4 6_2^4$.		
II. Hexahedral.	$3_1^8 8_2^6$,	$3_1^8 4_3^{6+12}$,	$4\ ^6 3_4\ ^{+2(12)}$
	$4_1^6 6_2^8$,	$3_2^8 4_2^6$.	
	$4_1^{12} 6_1^8 8_1^6$.		
III. Dodecahedral.	$3_1^{20} 10_2^{12}$,		$5_1^{12} 3_4^{20+2(30}$.
	$5_1^{12} 6_2^{20}$,	$3_2^{20} 5_2^{12}$.	
	$4_1^{30} 6_1^{20} 10_1^{12}$,	$3_1^{20} 5_1^{12} 4_2^{30}$.	

With two exceptions, to each hexahedral network there corresponds a dodecahedral one, each pair being obtained in a similar way from the regular network. Thus $3_1^8 8_2^6$ and $3_1^{20} 10_2^{12}$ are obtained by shallow sections from the cube and the dodecahedron respectively; $3_2^8 4_2^6$ and $3_2^{20} 5_2^{12}$ by sections through the middle points of the sides, and so on. There is no dodecahedral network corresponding to $3_1^8 4_3^{18}$, nor is there a hexahedral network corresponding to $3_1^{20} 5_1^{12} 4_2^{30}$. It may be noticed that the values of Δ for corresponding networks bear the same ratio as the values of Δ for the regular networks, viz., 1 : 2.

In representing the networks upon the Euclidean plane the method of stereographic projection has been employed, though in some cases, in order to avoid undue crowding towards the centre of the figure, strict stereographic projection has been departed from. For simplicity the nodes are joined by straight lines instead of arcs of circles, so that the figures really represent the conical projections of the semi-regular polyhedra.

Composite Types.

§ 27. At first sight it might appear that a very large number of composite types could exist, for there are an infinite number of species of nodes, while on the Euclidean plane where there are a considerable number of composite types there are only a very few species of nodes. A little consideration will show, however, that it is probable that the number of composite types is extremely limited.

Let us take any species of node, p_1, p_2, p_3; n_1, n_2, n_3, and let a_1, a_2, a_3 be the angles of the different polygons, a the length of side, k the radius of the sphere.

Then

$$\cos\frac{a}{2k} = \frac{\cos\frac{\pi}{n_1}}{\sin\frac{a_1}{2}} = \frac{\cos\frac{\pi}{n_2}}{\sin\frac{a_2}{2}} = \frac{\cos\frac{\pi}{n_3}}{\sin\frac{a}{2}},$$

and

$$p_1 a_1 + p_2 a_2 + p_3 a_3 = 2\pi.$$

These four equations determine a_1, a_2, a_3 and a.

Now if an n_1-gon occurs in the same network in a different combination p_1', p_2', p_4; n_1, n_2, n_4, then a_4, the angle of the n_4-gon, must satisfy the two equations

$$\cos\frac{a}{2k} = \frac{\cos\frac{\pi}{n_4}}{\sin\frac{a_4}{2}}, \qquad p_1'a_1 + p_2'a_2 + p_4a_4 = 2\pi.$$

If we substitute in the first equation a value of a_4 corresponding to a possible set of values of p_1', p_2', p_4 obtained from the second, we must get an integral value for n_4. This will not in general happen.

The following negative results may also be obtained at once :

1. No composite types exist with only one kind of polygon, for the angles are all equal, and there must therefore be the same number of polygons at each point.

2. No composite types exist with $p=3$ at each point, for the angle and the side determine n.

§ 28. A certain number of composite types may be obtained by the following

THEOREM.—*Whenever a simple type contains a group of polygons bounded by a regular polygon, the replacement of that group by a single polygon will in general give a composite type.*

When such a group occurs it may be replaced by a single polygon, for in the corresponding polyhedron the boundary of the group lies in one plane. Further, the replacement of the group removes at least one line from the nodes at the boundary, and the resulting network contains some nodes with p lines and some with at most $p-1$, *i.e.* it contains at least two different species of nodes, and is therefore composite.

It follows that, in order that the simple type may give a composite type in this way, p must be >3. It may happen that the angle of the boundary polygon is $\geqslant 180°$. If we exclude this case we get the following composite types :

1. From $\mathbf{3}_5{}^{20}$ by replacing five covertical triangles by a pentagon.

This may be done in three ways, replacing 1, 2 or 3 sets by pentagons.*

The three networks are of different types. They may be denoted as follows, the symbol within brackets denoting the species of node and the coefficient the number of times it occurs in the network.

(*a*)	$6(\mathbf{3}_5)$	+	$5(\mathbf{3}_3\mathbf{5}_1)$			(fig. 41)
(*b*)	$2(\mathbf{3}_5)$	+	$6(\mathbf{3}_3\mathbf{5}_1)$	+	$2(\mathbf{3}_1\mathbf{5}_2)$	(fig. 42)
(*c*)			$3(\mathbf{3}_3\mathbf{5}_1)$	+	$6(\mathbf{3}_1\mathbf{5}_2)$	(fig. 43)

2. From $\mathbf{3}_1{}^8\mathbf{4}_3{}^{18}$ by replacing one of the octagonal groups (fig. 27 deleting the part within the heavy lines).†

$$12(\mathbf{3}_1\mathbf{4}_3) + 8(\mathbf{4}_2\mathbf{8}_1)$$

3. From $\mathbf{3}_1{}^{20}\mathbf{5}_1{}^{12}\mathbf{4}_2{}^{30}$ (fig. 36) by replacing the decagonal group.

* If two opposite groups of five triangles are replaced by pentagons we get the simple type $\mathbf{3}_3{}^{10}\mathbf{5}_1{}^2$ (like fig. 25).

† If both groups are replaced we get the simple type $\mathbf{4}_2{}^8\mathbf{8}_1{}^2$ (like fig. 24).

There are eight possible varieties. Let X denote the original group, R a group turned through $\frac{\pi}{5}$, O the decagon replacing a group, then, if XY denotes two *opposite* groups, $\overset{X}{YZ}$ three groups mutually *circumjacent*, the varieties can be expressed as follows:

$$(a)\ 45\ (\mathbf{3_1 5_1 4_2}) + 10\ (\mathbf{4_1 5_1 10_1})$$

$$(\alpha)\ O \qquad (\beta)\ OR$$

$$*\ (\gamma)\ \overset{O}{RX} \qquad (\delta)\ \overset{O}{RR}$$

$$(b)\ 30\ (\mathbf{3_1 5_1 4_2}) + 20\ (\mathbf{4_1 5_1 10_1})$$

$$(\alpha)\ OO \qquad (\beta)\ \overset{O}{OX} \qquad (\gamma)\ \overset{O}{OR}$$

$$(c)\ 15\ (\mathbf{3_1 5_1 4_2}) + 30\ (\mathbf{4_1 5_1 10_1})$$

$$\overset{O}{OO}$$

§ 29. Further, if we allow angles $\geqslant 180°$ we get the following:

1. From $\mathbf{3_4}^8$, replacing four covertical triangles by a quadrilateral (fig. 45).

   $(\mathbf{3_4}) + 4\ (\mathbf{3_2 4_1})$. Angle of quadrilateral 180°.

2. From $\mathbf{3_5}^{20}$ (fig. 44).

   $(\mathbf{3_5}) + 5\ (\mathbf{3_2 5_1})$. Angle of pentagon 216°.

3. From $\mathbf{3_1}^8\mathbf{4_3}^{18}$ (fig. 27 bounded by the heavy lines).

   $4\ (\mathbf{3_1 4_3}) + 8\ (\mathbf{3_1 4_1 8_1})$. Angle of octagon 196° 50′.

4. From $\mathbf{3_2}^8\mathbf{4_2}^6$, replacing the hexagonal group (fig. 31 bounded by the heavy lines).

   $3\ (\mathbf{3_2 4_2}) + 6\ (\mathbf{3_1 4_1 6_1})$. Angle of hexagon 180°.

5. From $\mathbf{3_2}^{20}\mathbf{5_2}^{12}$, replacing the decagonal group (fig. 29 bounded by the heavy lines).

   $10\ (\mathbf{3_2 5_2}) + 10\ (\mathbf{3_1 5_1 10_1})$. Angle of decagon 180°.

6. From $\mathbf{3_1}^{20}\mathbf{5_1}^{12}\mathbf{4_2}^{30}$ (fig. 37 deleting the part within the heavy lines).

   $5\ (\mathbf{3_1 5_1 4_2}) + 10\ (\mathbf{3_1 4_1 10_1})$. Angle of decagon 204° 6′.

These are all the composite types obtainable from the simple types. It seems probable that there are no others.

III. The Hyperbolic Plane.

§ 30. This case does not admit of exhaustive treatment. The number of types of networks is evidently infinite, for there is no limit to the number of lines at a point.

As a rule, the hyperbolic plane contains the types which cannot exist on the Euclidean or the elliptic plane. For example: one n-gon and two $2m$-gons at a point

* Asymmetrical. Two enantiomorphic forms, $\overset{O}{RX}$ and $\overset{O}{XR}$.

determine a simple hyperbolic network for all values of n and m for which the network is neither Euclidean nor elliptic. The networks are all infinite.

As regards composite types, we can apply the same remarks as were made in connection with the elliptic networks. The angle of a polygon is determined by the particular combination in which it occurs, and the multiplicity of composite types is thus limited. But, at the same time, it is infinite. For, consider the regular network $\mathbf{3}_p$ ($p>6$) (fig. 46). Any group of p covertical triangles can be replaced by a p-gon, so that from this network alone we obtain an infinite number of composite types.

Note added on July 29, 1905.—Since writing the above, I have come across some of the previous work on the subject. The semi-regular polyhedra have long been known. It appears, from the works of PAPPUS of Alexandria and KEPPLER, that they were described in a lost work of ARCHIMEDES.* PAPPUS † enumerates the series of thirteen (*i.e.* excluding the two infinite groups, figs. 24 and 25), with the numbers of their faces, edges, and vertices, for which he gives the general formulæ of § 22. KEPPLER ‡ establishes them by taking the different possible combinations, first binary and then ternary, containing triangles, squares, and pentagons successively. More recently, accounts of them have been given by MEIER HIRSCH § and R. BALTZER.‖ An elaborate article, containing numerous calculations relating to the radius of the circumscribed sphere, inclinations of the faces, etc., was presented by M. VALAT to the French Institute in 1854.¶ He refers to other writings, in particular to one by LIDONNE (1808), but gives no details of them. He shows also how the semi-regular polyhedra are obtained by truncating the Platonic solids. The connection between these polyhedra was also expressed by KEPPLER in an ingenious nomenclature which he employed to describe them. The following list of names corresponds to the table on p. 743; the numbers refer to the diagrams :—

I.	Tetrahedron truncum (23).	...	...
II.	Cubus truncus (20).	Rhombicuboctahedron (26).	Cubus simus (33).
	Octahedron truncum (22).	Cuboctahedron (30).	...
	Cuboctahedron truncum (35).	...	...
III.	Dodecahedron truncum (19).	...	Dodecahedron simum (32).
	Icosihedron truncum (21).	Icosidodecahedron (28).	...
	Icosidodecahedron truncum (34).	Rhombicosidodecahedron (36).	...

In none of these writings is any notice taken of possible varieties, the reason being probably that these varieties do not exhibit the same symmetry as the fundamental varieties. KEPPLER gives this as the reason for excluding the two infinite series.

* See also T. L. HEATH, *The Works of Archimedes* (Camb. 1897), p. xxxvi.
† *Collectio*, lib. v. pars 2.
‡ *Harmonices Mundi* (1619), lib. ii. pp. 61–65.
§ *Sammlung geometrischer Aufgaben* (Berlin, 1805–7), vol. ii. pp. 139–185.
‖ *Elemente der Mathematik* (1862), Bd. ii., Buch v. § 7.
¶ Published 1867, under the title "Des Polyèdres semi-réguliers, dits solides d'Archimède," *Mém. de la Soc. des Sciences phys. et nat. de Bordeaux*, v. 319–369.

KEPPLER has also gone into some detail regarding the Euclidean networks. He gives* all the developable species of nodes and some of the others, with examples of networks formed with them, and other patterns, containing star-polygons, which may be derived from them.

It remains to notice a class of polyhedra connected with the semi-regular polyhedra.† They are obtained by drawing tangent planes to the circum-sphere at the vertices. To a regular n-gon there corresponds then a regular n-hedral angle. A *regular* polyhedron treated in this way gives the *conjugate* regular polyhedron, but in a semi-regular polyhedron none of the polyhedral angles are regular, and so none of the faces of the "conjugate" polyhedron will be regular polygons. The regular polyhedra have both a circum- and an in-scribed sphere; the semi-regular polyhedra have only a circumscribed sphere, while the conjugate ones have only an inscribed sphere. The corresponding networks are constructed simply by taking as new nodes the centres of the old meshes. The polyhedra conjugate to the *fundamental* varieties have their faces all congruent. This does not hold for the other varieties (with the exception of that corresponding to $\mathbf{3}_1{}^8\mathbf{4}_3{}^{18}$, fig. 27). Two of this class are interesting as being the only ones which have all their *edges* equal, viz., the rhombohedra formed from the fundamental varieties of $\mathbf{3}_2{}^8\mathbf{4}_2{}^6$ and $\mathbf{3}_2{}^{20}\mathbf{5}_2{}^{12}$ (figs. 30 and 28). There is an analogous Euclidean network conjugate to T_2H_2, *i.e.* $\mathbf{3}_2\mathbf{6}_2$.

On p. 743 there occurs a misstatement. The hexahedral network $\mathbf{3}_1{}^8\mathbf{4}_3{}^{6+12}$ (Rhombicuboctahedron), though it contains only two kinds of polygons, really corresponds to the dodecahedral network $\mathbf{3}_1{}^{20}\mathbf{5}_1{}^{12}\mathbf{4}_2{}^{30}$ (Rhombicosidodecahedron), being obtained in a similar way from the corresponding regular network. Thus the correspondence between the hexahedral and the dodecahedral networks is complete.

* *Loc. cit.*, pp. 51–55.

† KEPPLER and BALTZER, *loc. cit.*; MEIER HIRSCH, *loc. cit.*, pp. 186-196; J. H. L. MÜLLER, *Trigonometrie* (1852), p. 345.

SOMMERVILLE—NETWORKS OF THE PLANE IN ABSOLUTE GEOMETRY.—PLATE I.

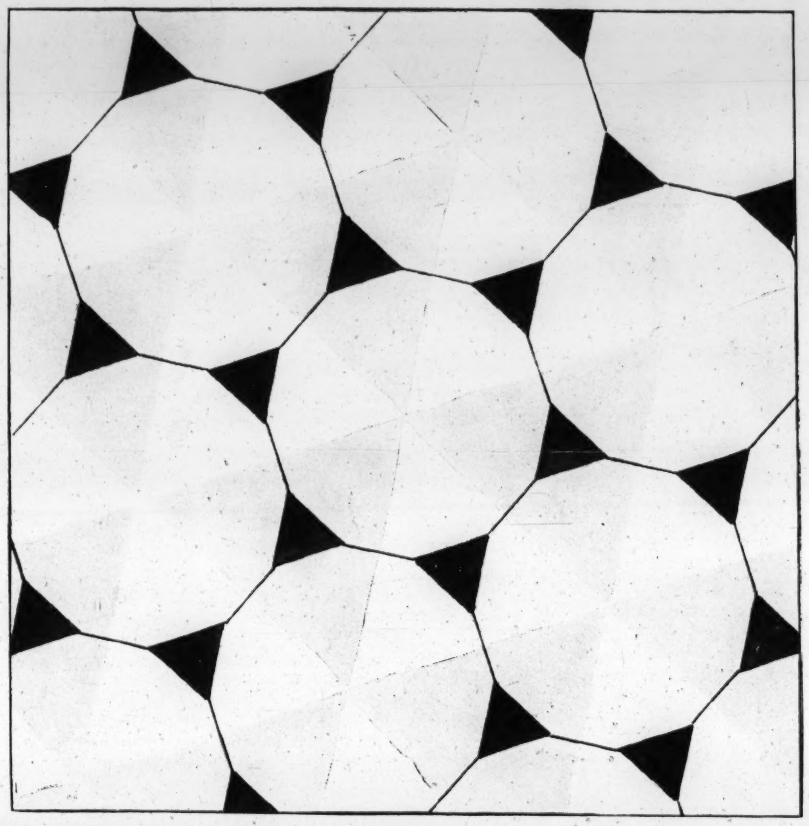


Fig. 2.

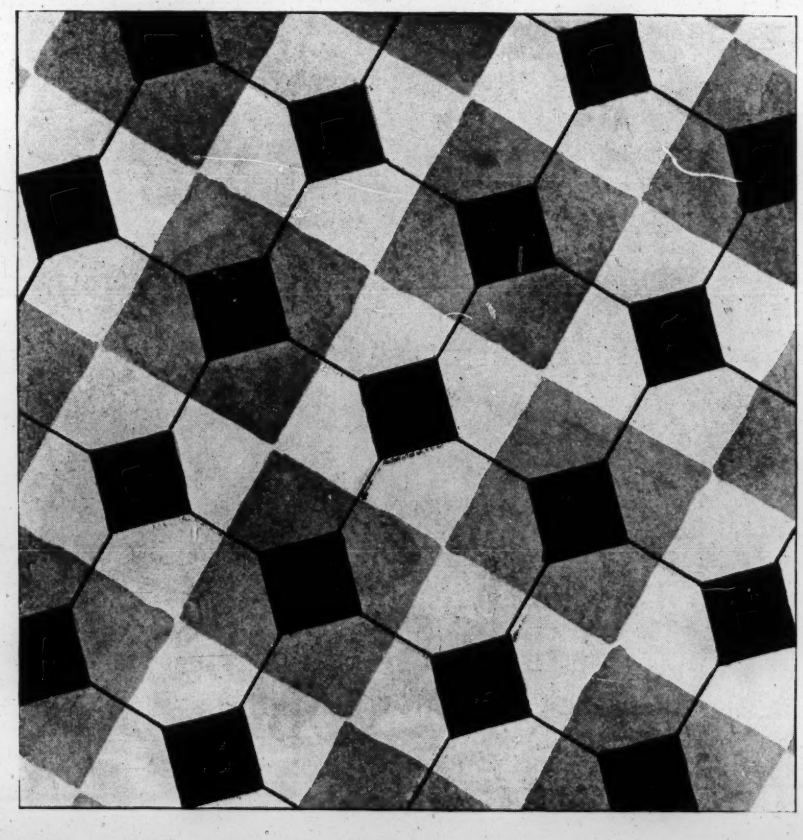


Fig. 3.

M'Farlane & Erskine, Edinburgh.

SOMMERVILLE—NETWORKS OF THE PLANE IN ABSOLUTE GEOMETRY.—PLATE II.

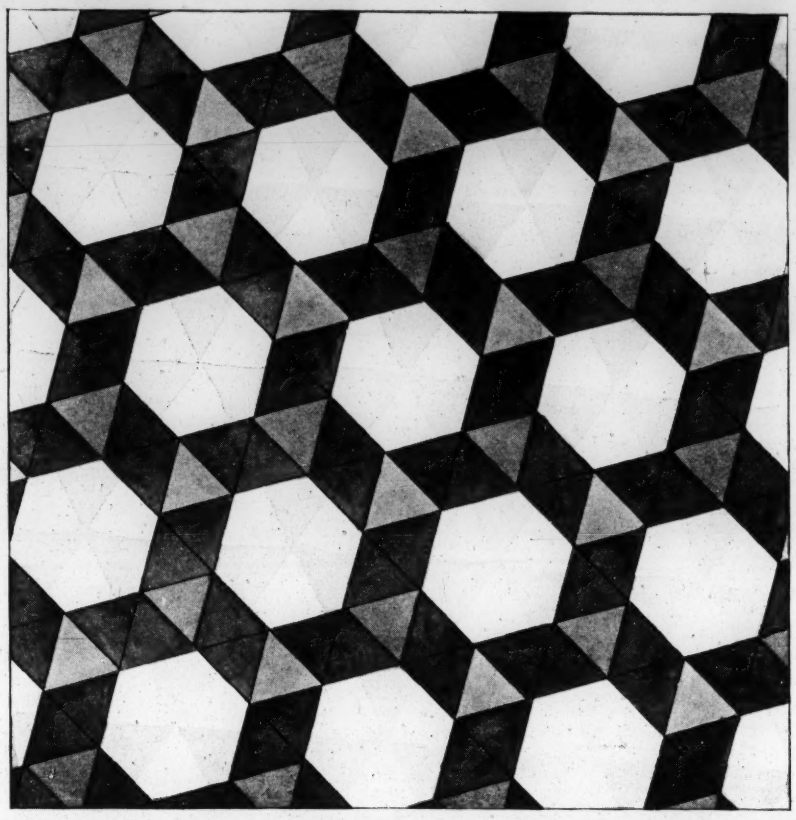


Fig. 4.

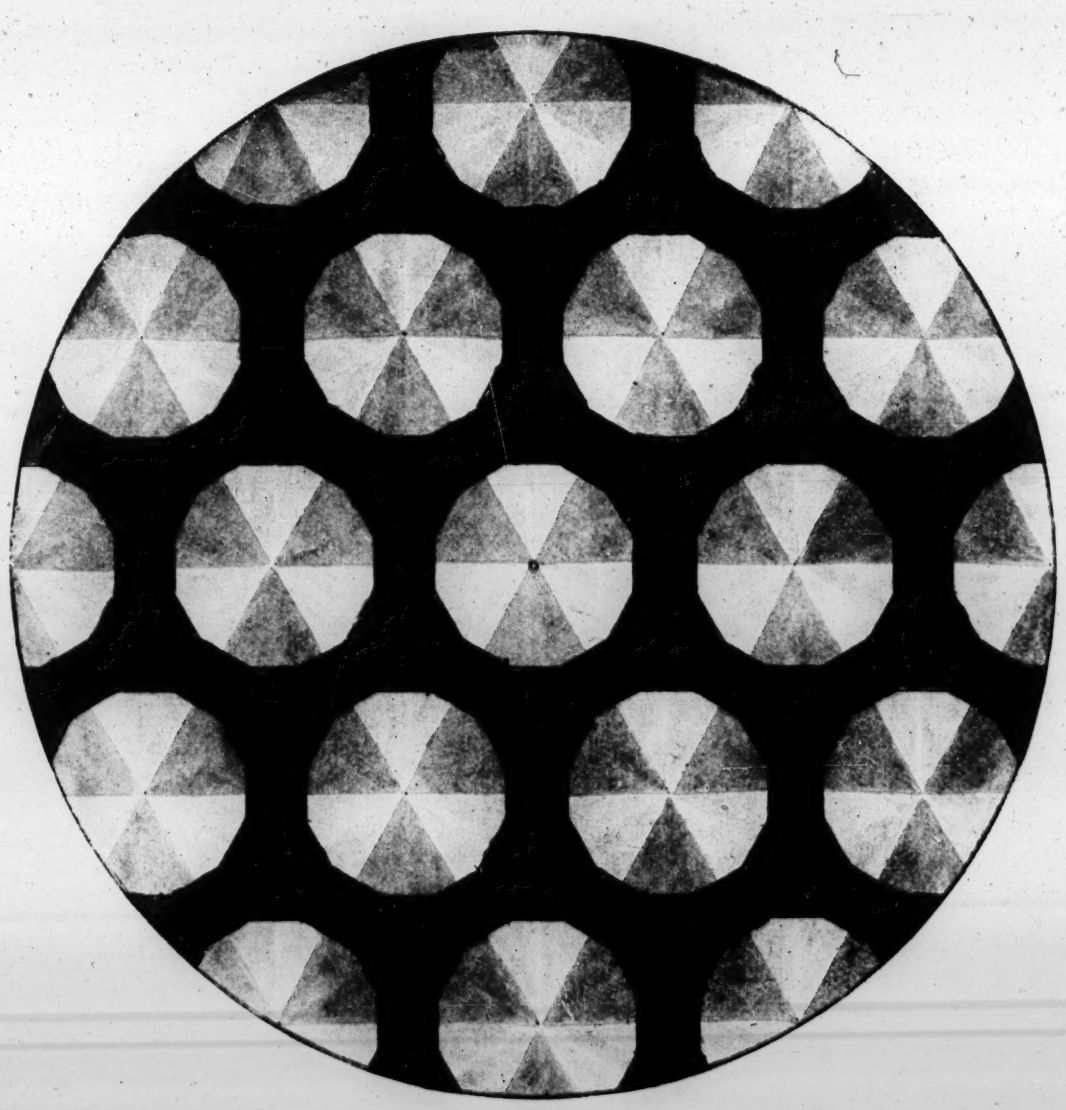


Fig. 5.

M'Farlane & Erskine, Edinburgh

SOMMERVILLE—NETWORKS OF THE PLANE IN ABSOLUTE GEOMETRY.—PLATE III.

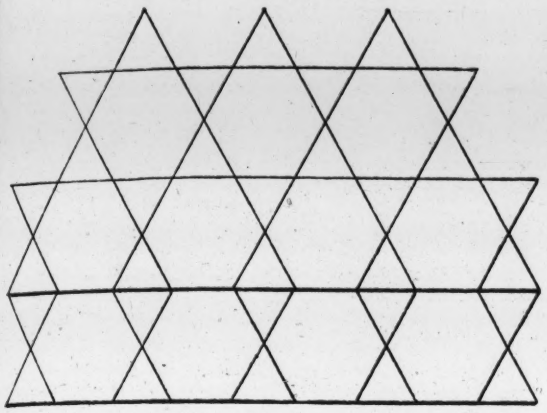


Fig. 6.

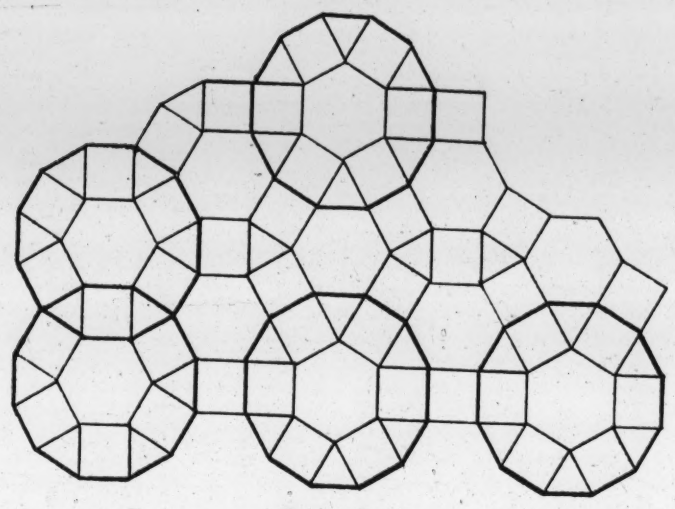


Fig. 7.

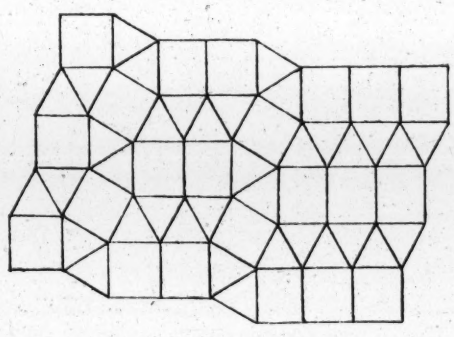


Fig. 8.

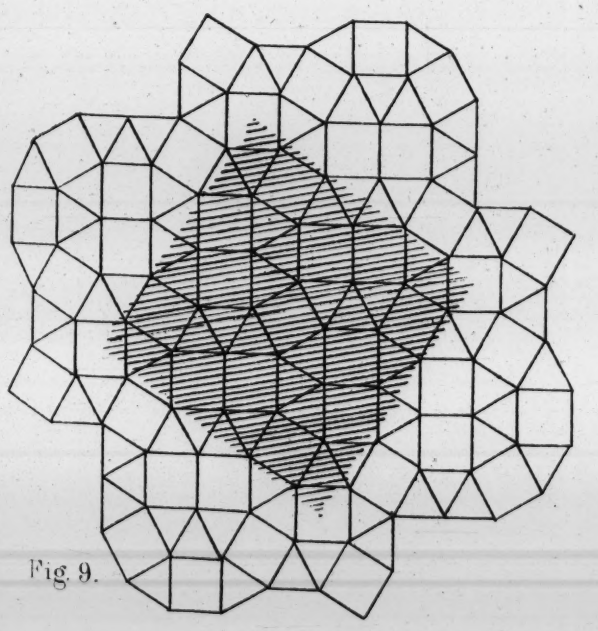


Fig. 9.

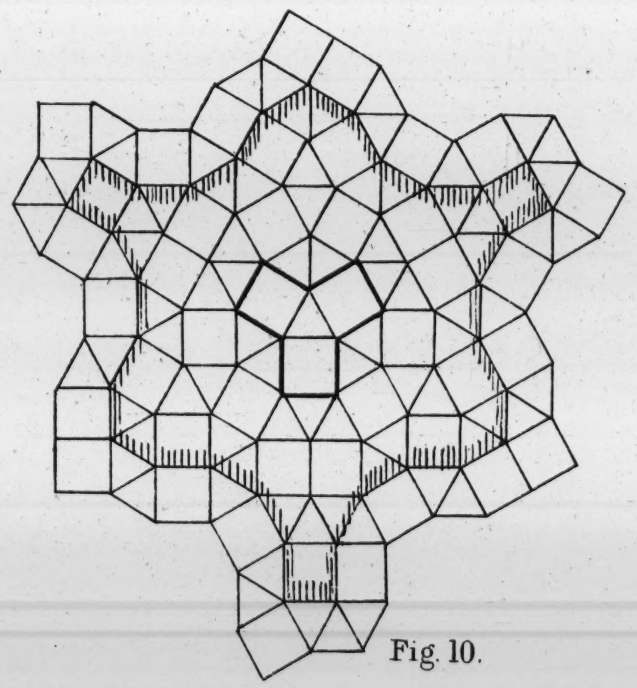


Fig. 10.

M^cFarlane & Erskine, Lith. Edin^r

SOMMERVILLE—NETWORKS OF THE PLANE IN ABSOLUTE GEOMETRY.—PLATE IV.

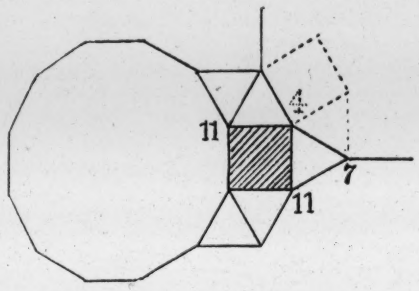
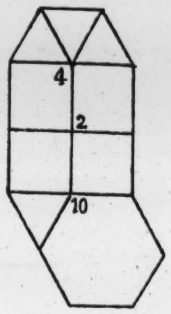


Fig. 1.



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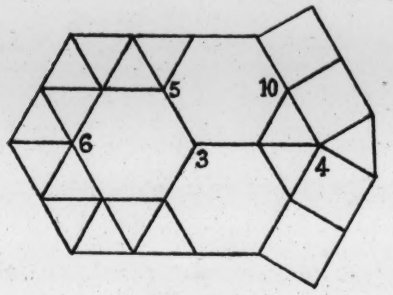

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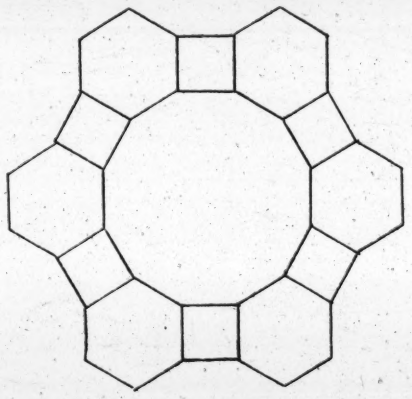

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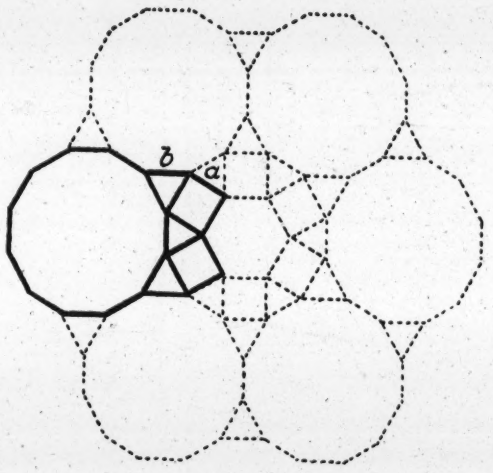

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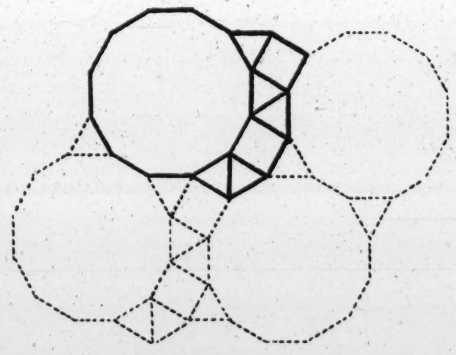

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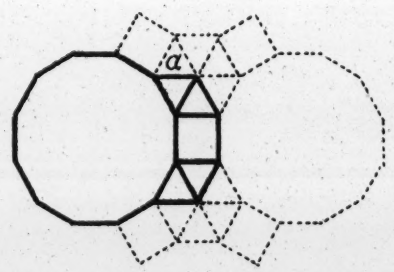


Fig. 16.

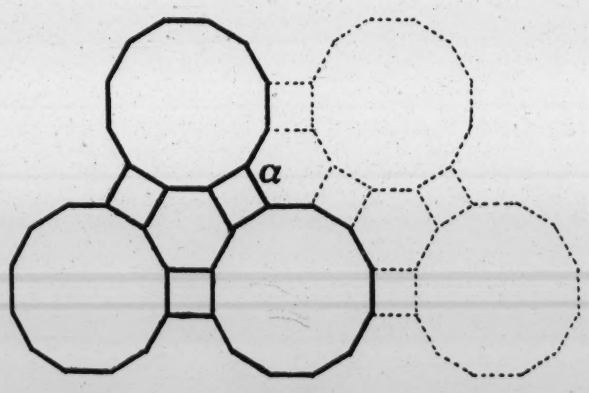


Fig. 17.

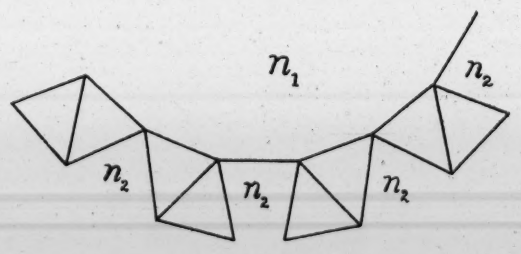


Fig. 18.

M^{c}Farlane & Erskine, Lith Edin^{r}

SOMMERVILLE—NETWORKS OF THE PLANE IN ABSOLUTE GEOMETRY.—PLATE V.

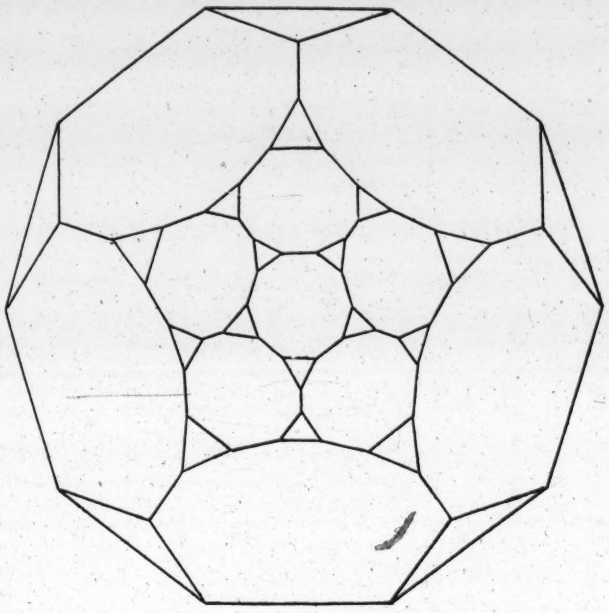
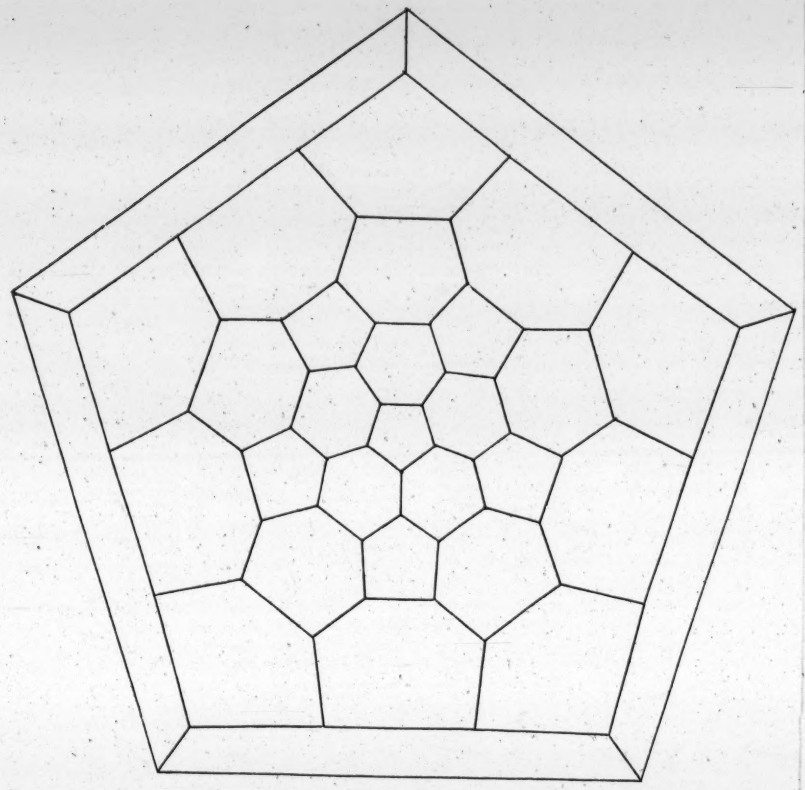


Fig. 19.



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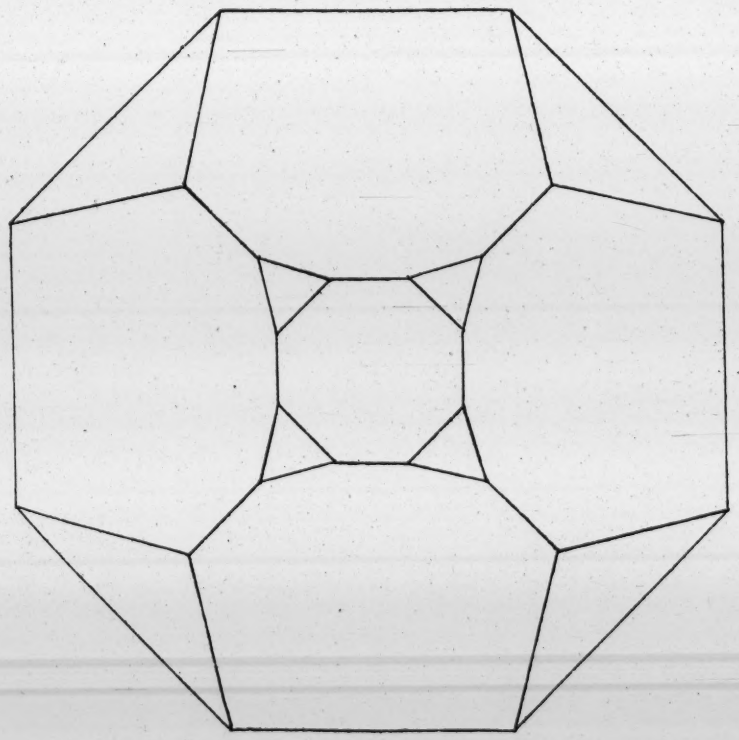

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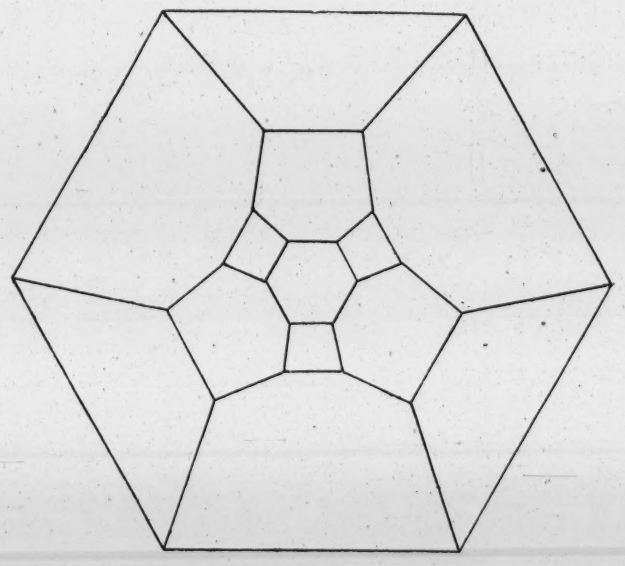
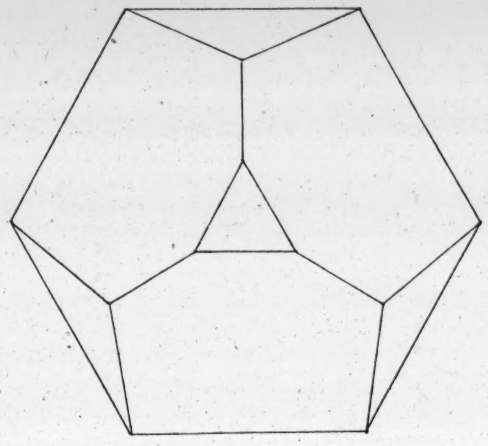


Fig. 22.

M^cFarlane & Erskine, Lith. Edin^r

SOMMERVILLE—NETWORKS OF THE PLANE IN ABSOLUTE GEOMETRY.—PLATE VI.



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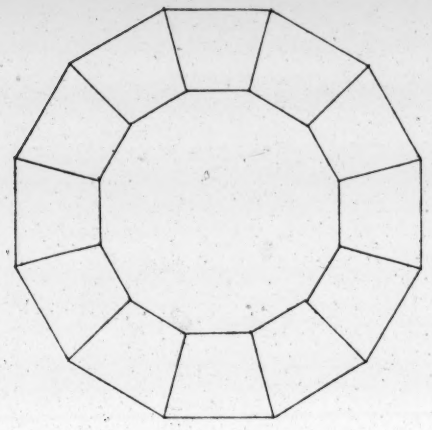

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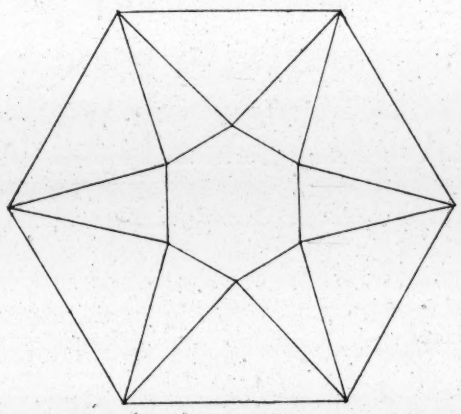

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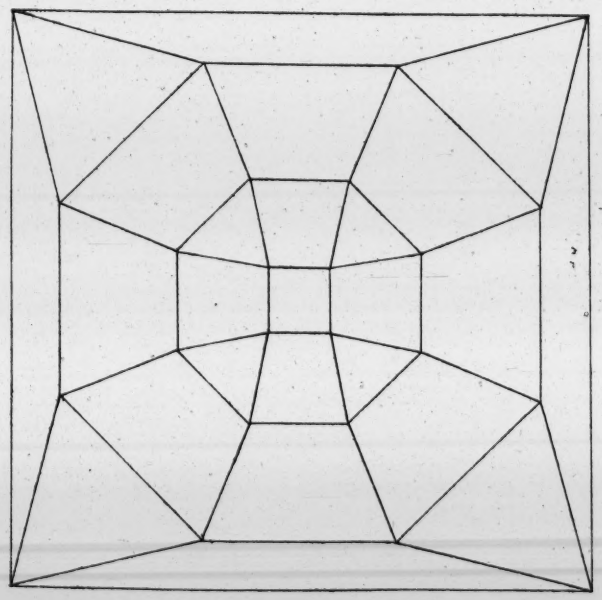

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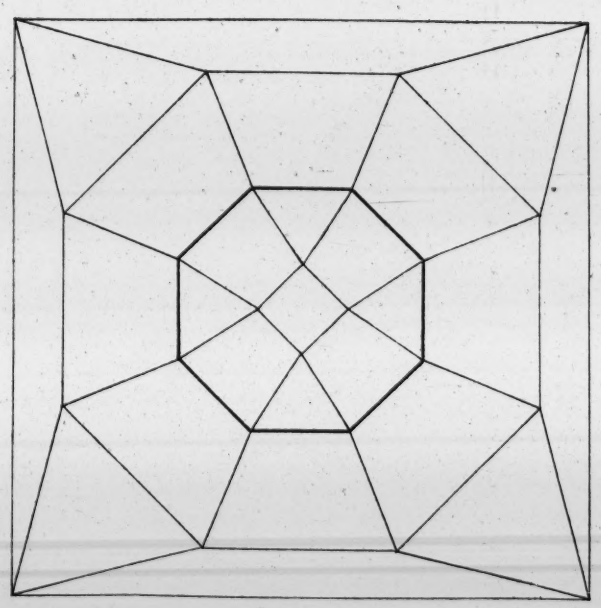
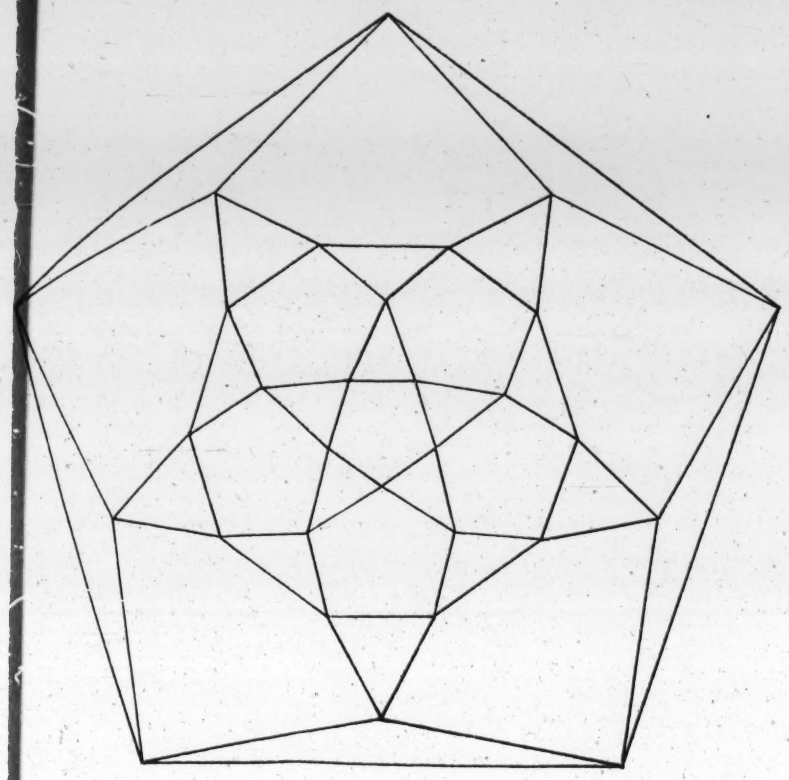


Fig. 27.

M^cFarlane & Erskine, Lith. Edin^r

SOMMERVILLE—NETWORKS OF THE PLANE IN ABSOLUTE GEOMETRY.—PLATE VII.



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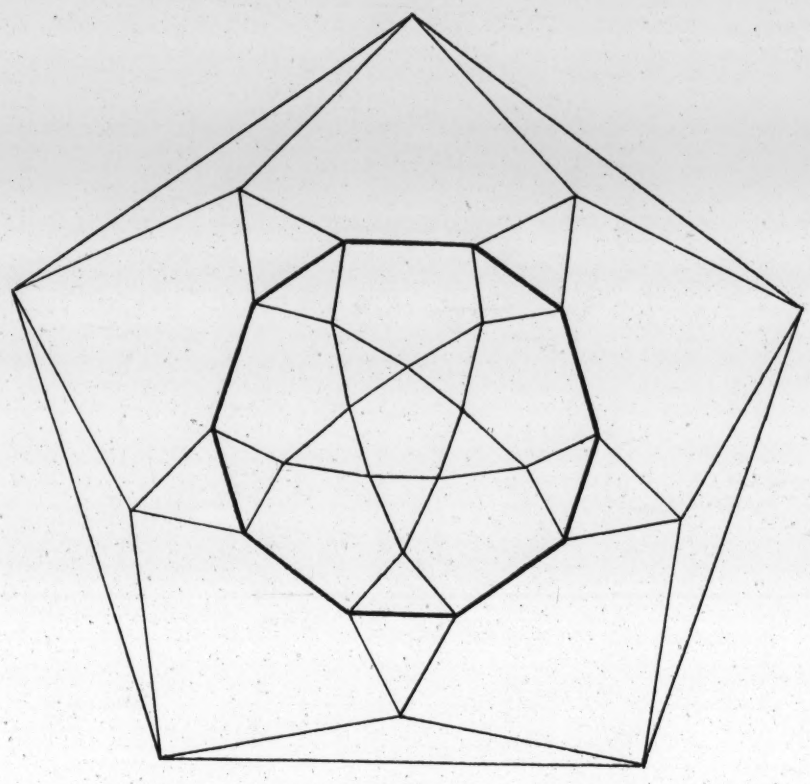

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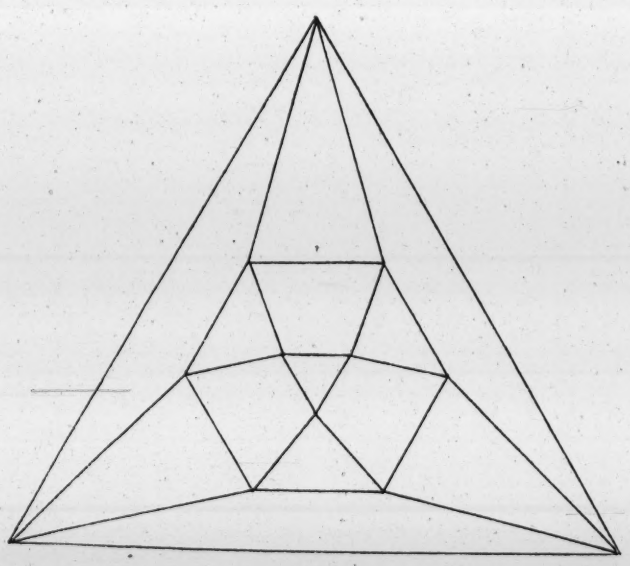

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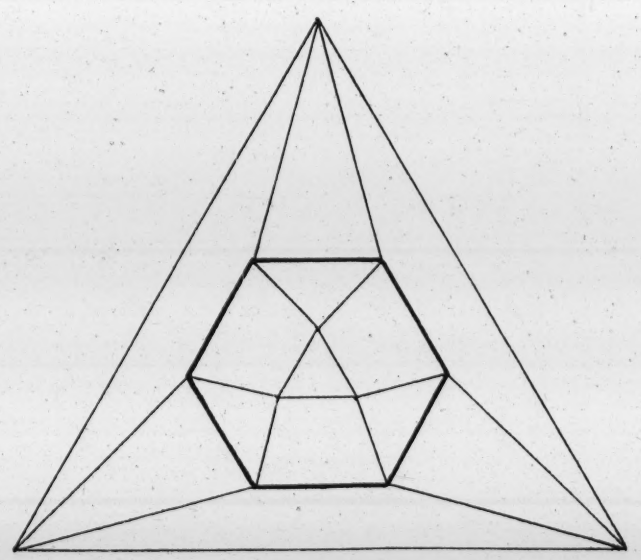
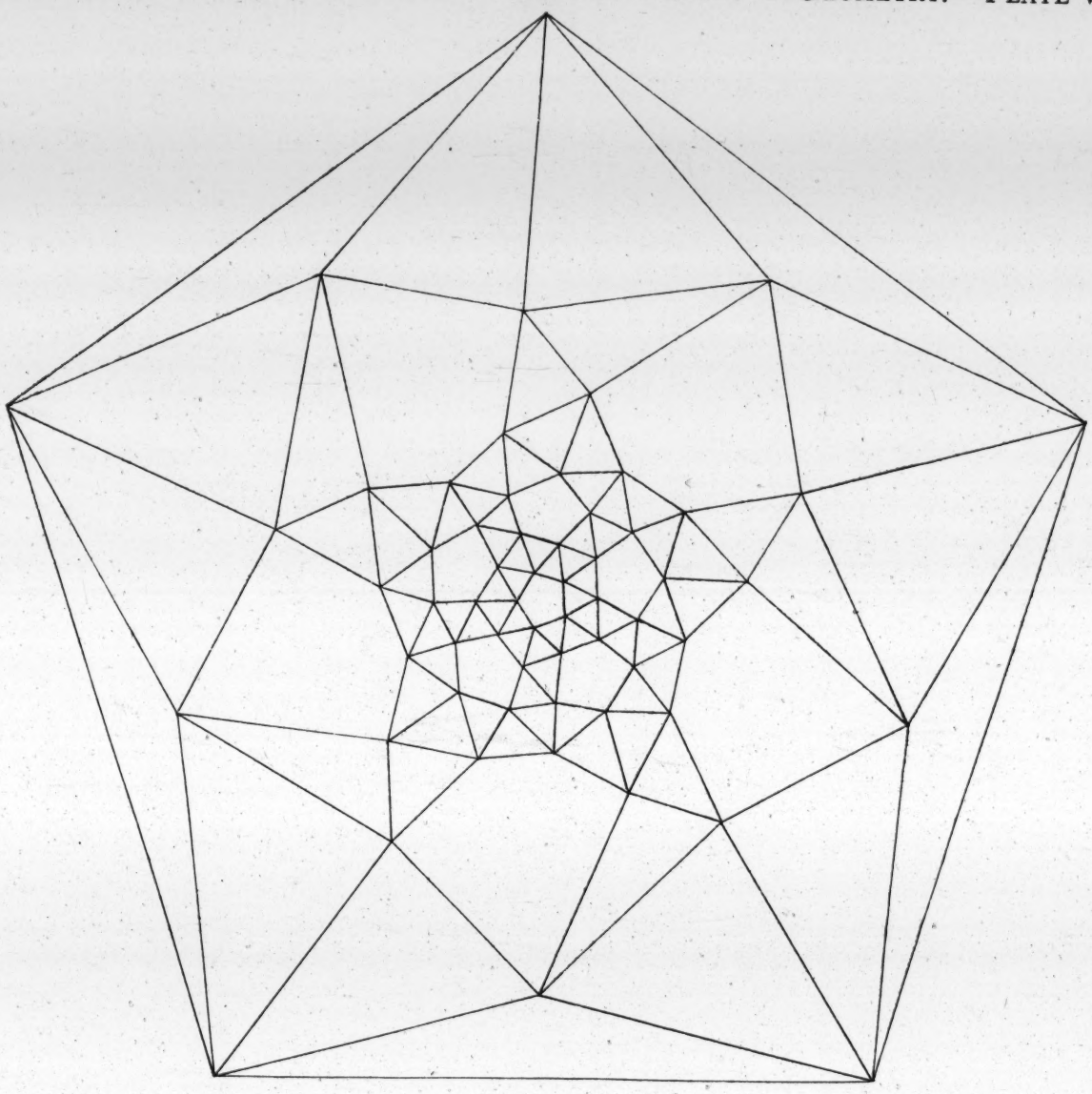


Fig. 31.

M^{c}Farlane & Erskine, Lith. Edin^{r}

SOMMERVILLE—NETWORKS OF THE PLANE IN ABSOLUTE GEOMETRY.—PLATE VIII.



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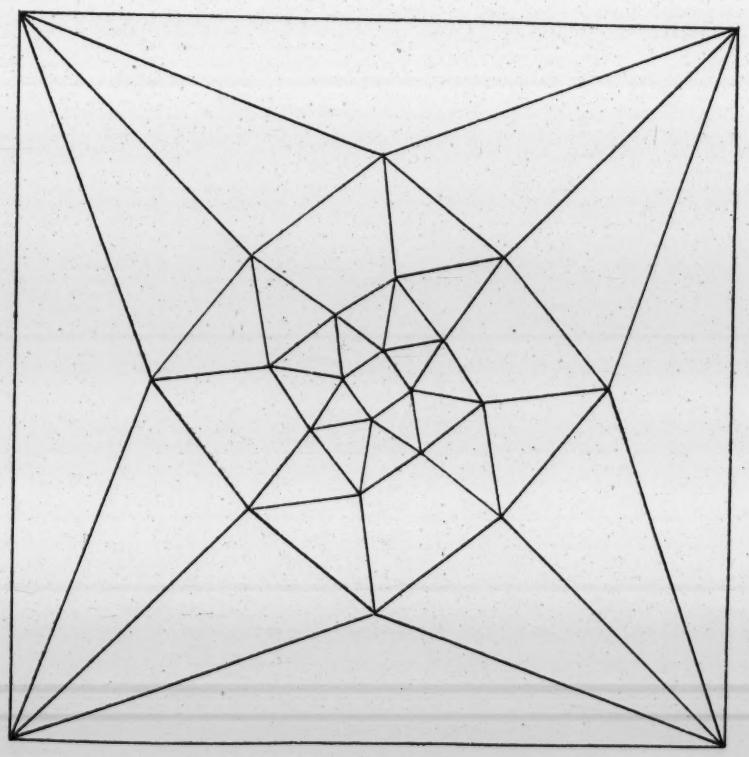
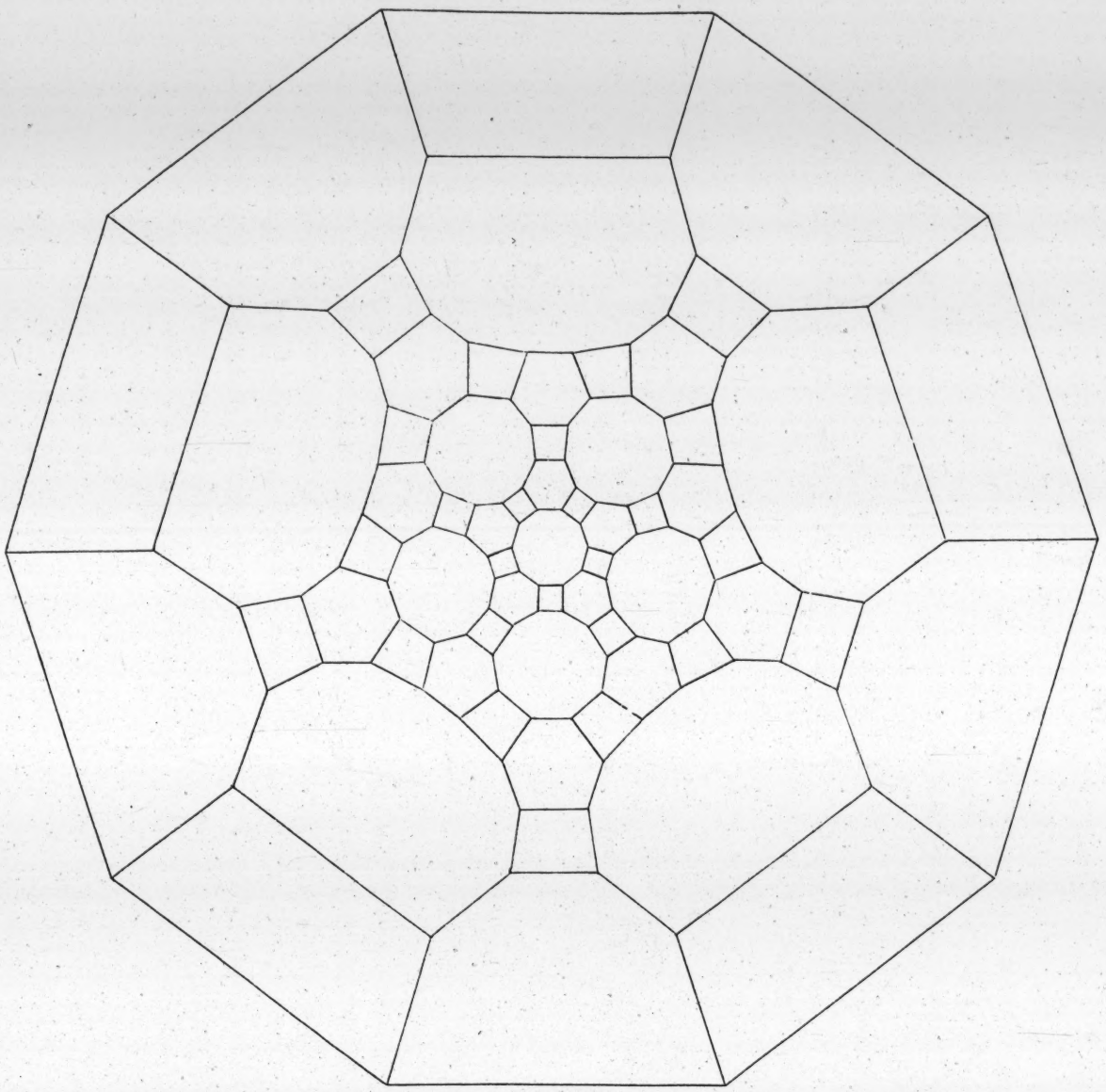


Fig. 33.

M^{c}Farlane & Erskine, Lith. Edin^{r}

SOMMERVILLE—NETWORKS OF THE PLANE IN ABSOLUTE GEOMETRY.—PLATE IX.



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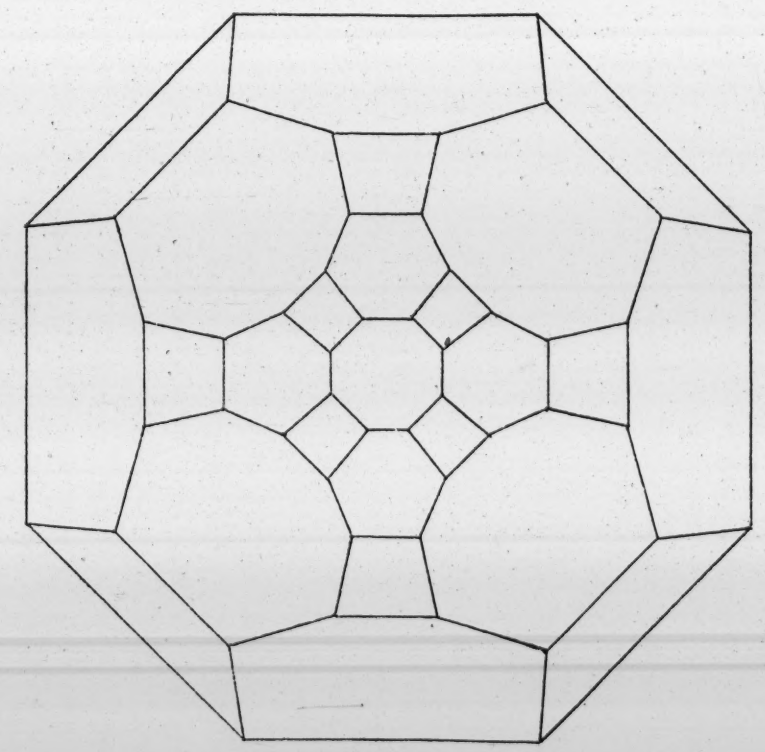


Fig. 35.

M^cFarlane & Erskine, Lith. Edin^r

SOMMERVILLE—NETWORKS OF THE PLANE IN ABSOLUTE GEOMETRY.—PLATE X.

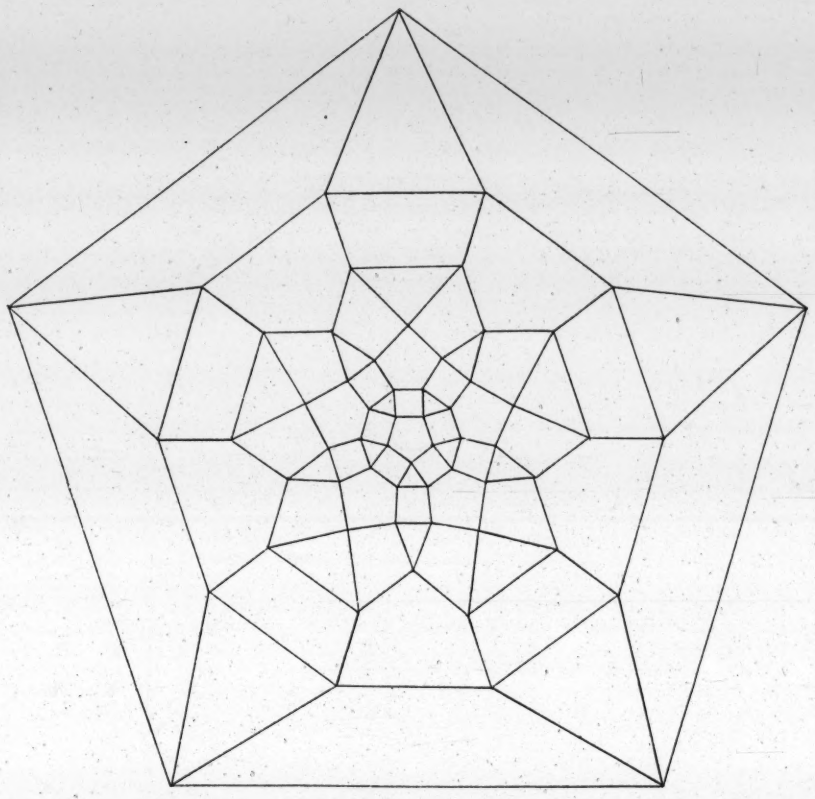


Fig. 36.

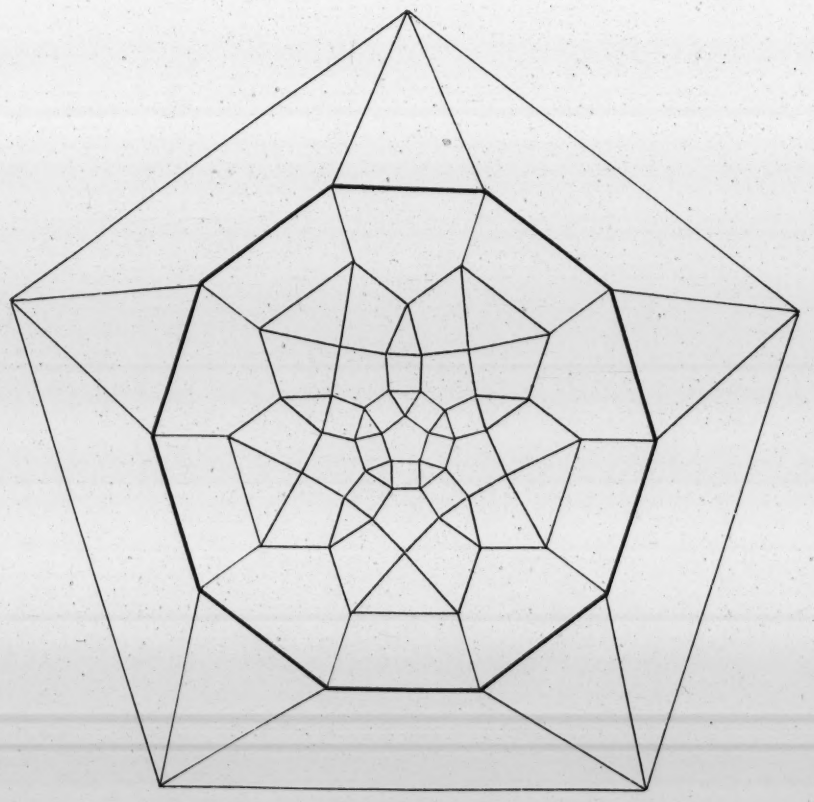


Fig. 37.

M^{c}Farlane & Erskine, Lith. Edin^{r}

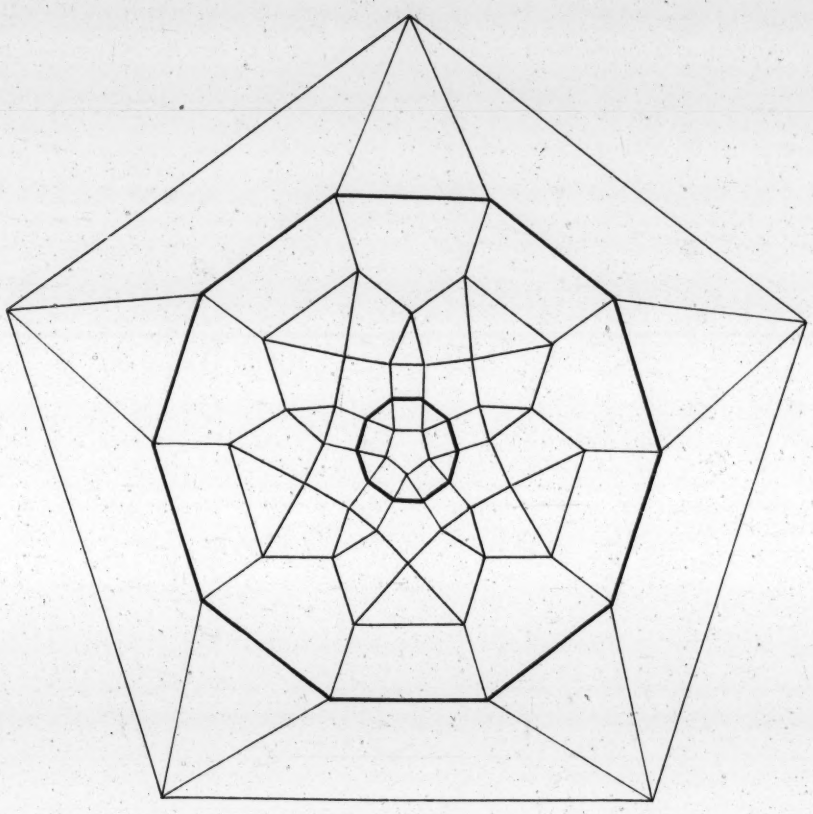


Fig. 38.

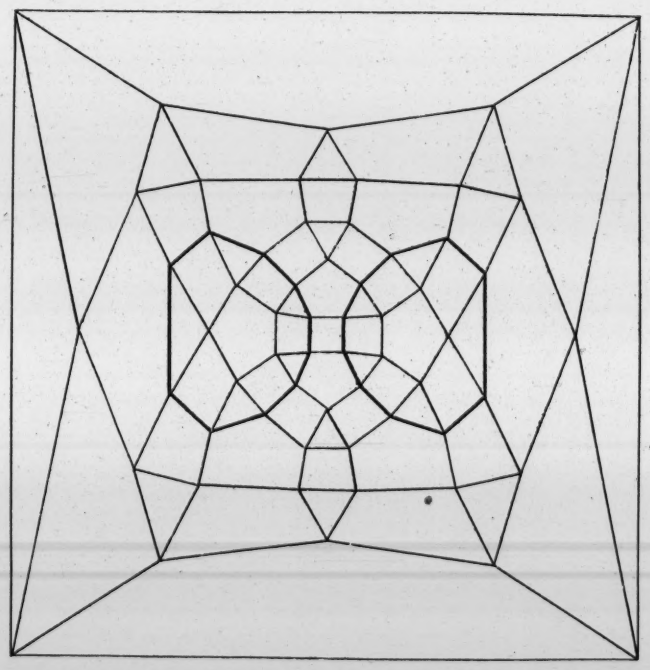
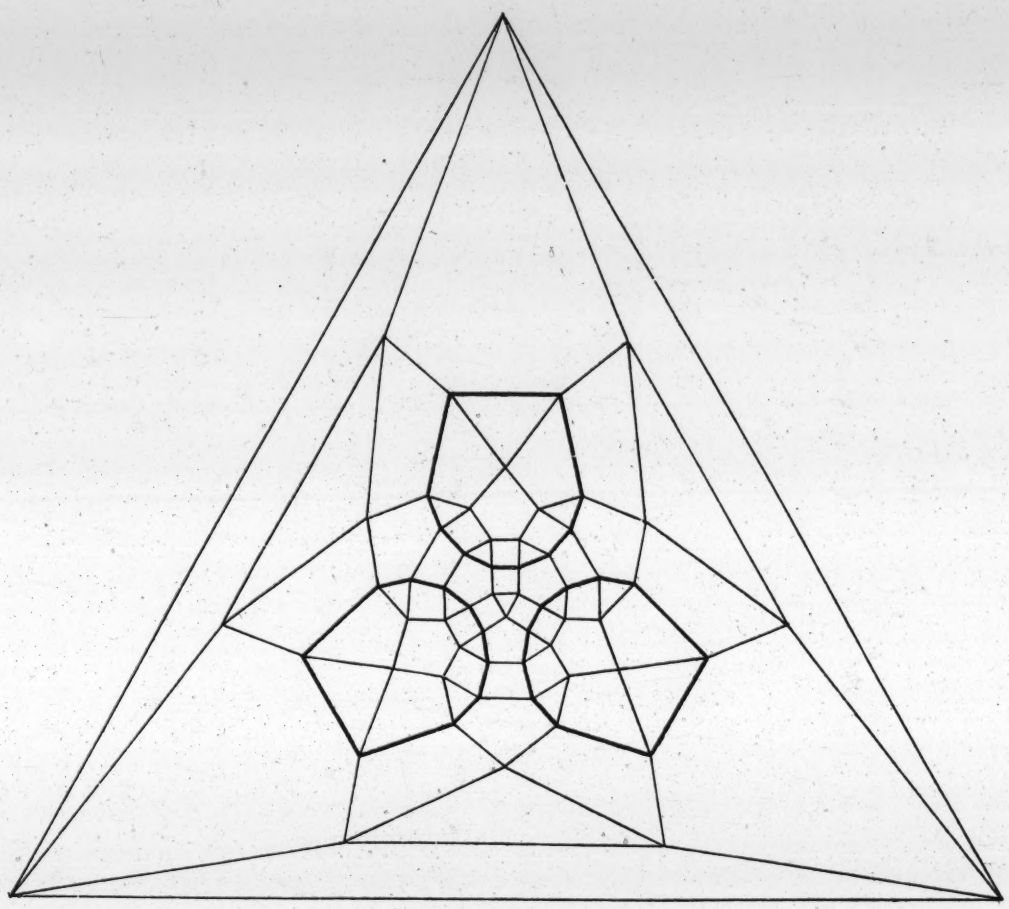


Fig. 39.

M^{c}Farlane & Erskine, Lith. Edin^{r}

SOMMERVILLE—NETWORKS OF THE PLANE IN ABSOLUTE GEOMETRY.—PLATE XII.



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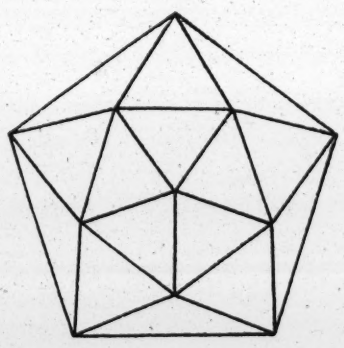
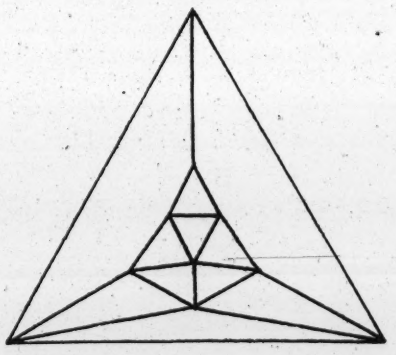


Fig. 41.



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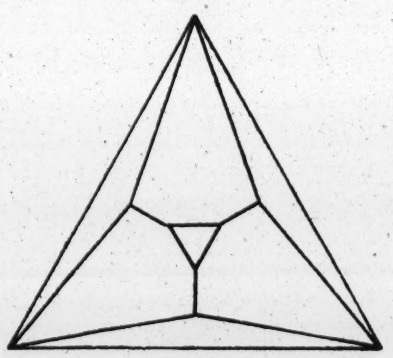

Fig. 43.

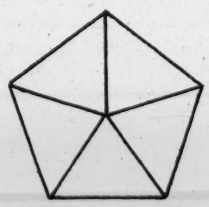

Fig. 44.

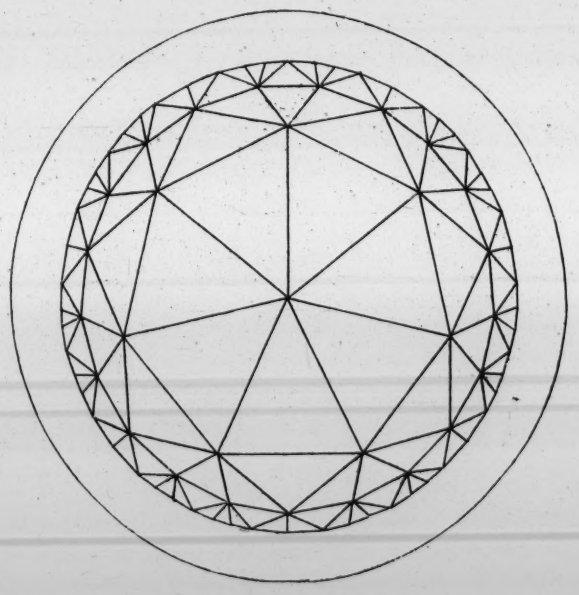

Fig. 46.

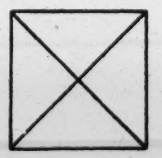

Fig. 45.

McFarlane & Erskine, Lith. Edin^{r}

XXX.—A Monograph on the general Morphology of the Myxinoid Fishes, based on a study of Myxine. Part I. The Anatomy of the Skeleton.

By Frank J. Cole, B.Sc. Oxon. *Communicated* by Dr R. H. Traquair, F.R.S. (With Three Plates.)

(MS. received June 3, 1905. Read June 19, 1905. Issued separately September 25, 1905.)

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A. Introduction.

The present work was commenced in the summer of 1902, with the object of writing a complete monograph of the morphology of *Myxine*. It had previously been undertaken by the late Professor G. B. Howes, F.R.S., but, owing to the pressure of other work, and the first indications of the illness which subsequently proved fatal, he handed the work over to me. During the earlier stages, however, he exhibited a characteristic interest in the research, and most generously placed his material and the late Dr Pollard's sections at my disposal. His death has removed a kindly and a stimulating figure from our midst; a man whose life was devoted to the service of his friends and the advancement of his favourite study, and whose generous and sympathetic nature was the wonder and the delight of all who knew him.

The work had not been long in progress before it became evident that it was much too extensive for publication as a single monograph. I therefore determined to issue it in parts, following the example of the illustrious founder of our knowledge of Myxinoid Anatomy—J. Müller. These parts will relate simply to the *anatomy* of *Myxine*, and will only take cognisance of such literature as contains *original* observations of Myxinoids. I have prepared an exhaustive Myxinoid bibliography, which will be published with my final part, so that in the meantime I need only direct attention to the papers on the skeleton mentioned below.* The anatomical parts will be followed by a separate con-

* The following works relate to the skeleton *generally*, those dealing with special points being referred to at the appropriate places:—A. A. Retzius, *Kgl. Vet. Akad. Stockholm*, 1824; J. Müller, *Abh. K. Akad. Wiss. Berlin*, 1834; P. Fürbringer, *Jena. Zeits.*, ix., 1875; W. K. Parker, *Phil. Trans.*, 1883; G. B. Howes, *Trans. Liv. Biol. Soc.*, vi., 1892; Neumayer, *Munchen. med. Abhand.* (Kupffer and Rüdinger), Hft. 74, 1898; Ayers and Jackson, *Jour. Morph.*, xvii., 1901, and *Bull. Cincinnati Univ.*, vol. i., 1900; Allis, *Anat. Anz.*, xxiii., 1903.

cluding section of a general character, in which the *morphology* of the Myxinoids will be treated in detail and the appropriate general literature discussed. The present part, therefore, like its successors, is not morphological, but is concerned with descriptive anatomy only. A very full description of the muscles will constitute Part II., and will be ready by the end of the year.

As regards the terminology of the skeleton, it is quite clear that the failure of PARKER and AYERS and JACKSON to correctly homologise the parts of the Myxinoid skeleton was due to insufficient data; and another attempt on my part, before the evidence of the muscles, blood-vessels, and nerves is available, and also the memoir on the development of the skeleton of *Bdellostoma* now being prepared by NEUMAYER, could only end in failure also. I have therefore adopted the terminology of AYERS and JACKSON *en bloc*, not because I approve of it—in fact, some of their terms have already been successfully challenged by ALLIS—but simply to avoid coining a set of terms which could only last a few years.* In my concluding section I shall, of course, discuss the morphology of the skeleton, and revise the terminology, with, let us hope, a reasonable prospect of arriving at results of some lasting value.

I have great pleasure in acknowledging the kindness of my friend Dr BEARD in lending me his sections of *Myxine*, and especially the series of the 6·5 cm. Hag, which has been of great use to me. Also the collection of my own material was made possible by a grant of £50 from the Government Grant Committee, with which I visited the marine laboratory at Cullercoats, under the charge of Mr ALEXANDER MEEK, and was entirely successful. Living *Myxine* may be collected at Cullercoats in great quantities, by methods which I shall describe in my Third Part. The laboratory at Cullercoats since my visit has unfortunately been entirely destroyed by fire, and it is to be hoped that the important work which Mr MEEK is doing there will not long be paralysed for the lack of a new well-equipped laboratory. Finally, I am greatly indebted to Professor W. F. R. WELDON, F.R.S., in whose laboratory at Oxford this work was done.

B. NOTOCHORD. (Fig. 18.)

The termination of the chorda at its cephalic and caudal extremities is described under the parachordal and caudal fin cartilages, and also below. As is well known, the notochord constitutes the only skeletal support of the back, and there is no appearance of cartilage, either in the chorda itself or in the neural tube, except in the region of the head and tail (*q.v.*). Still less are there any traces of bone or calcified tissue of any kind whatever, either here or in any other region of the body of *Myxine*. In this respect the Cyclostomes share with *Amphioxus* a unique position in the chordate series.

* For example, the structures named by AYERS and JACKSON "branchial arches" are, according to BASHFORD DEAN, developed *after* the gill pouches have disappeared from this region, and may therefore represent neomorphs developed in connection with the muscles of the "tongue." In fact, I am disposed to believe that much of the Myxinoid skeleton is recent and sesamoidal (as indicated by POLLARD), and therefore has no morphology at all!

The histology of the notochord of Myxinoids has been described by J. Müller, W. Müller,* G. Retzius,† v. Ebner,‡ and Ayers and Jackson. In transverse section, the chorda of *Myxine* is almost circular in the living condition, but is somewhat flattened dorsally in parts. The dorsal involution, however, so common in preserved material, and described by J. Müller, is an artifact produced by the fixing reagent. According to G. Retzius, the chorda contains no chondrin, glutin, or mucin, but albumen is always present. When the chordal tissue or "jelly" is removed the stout sheath does not collapse, as mentioned by J. Müller.

Skeletogenous Layer (*sk. l.*).—This layer consists of a few coarse fibres surrounding the elastica externa. As may be seen where the fibres are torn across (as shown in fig. 18), each fibre is formed of a large number of very fine fibrillæ. The fibres course almost straight round the chorda, and are usually closely packed together. External to these there may or may not be bundles of fibres coursing in various directions, and which often contain true elastic fibrils. With methyl-blue-eosin the skeletogenous layer stains a faint purplish blue. Arrived at the dorsal surface of the chorda, the skeletogenous layer forms the greater part of the pad of tissue filling in the angle formed by the meeting of the chorda and neural tube. At this place incipient patches of cartilage may be developed in *Bdellostoma*, according to Ayers and Jackson. The layer is then continued over the neural tube as the external sheath of the latter. In *Bdellostoma*, according to Ayers and Jackson, the skeletogenous layer splits at the top of the chorda so as to roof over the chorda as well as the neural tube. I have seen only very slight indications of this in *Myxine*; and, assuming of course that the *whole* of the neural tube is not formed by this layer, it is practically not represented between the chorda and the neural tube, and it thus forms *one* tube, enclosing both the chorda and the spinal cord. J. Müller's account differs both from mine and Ayers and Jackson's, and is clearly inaccurate. The dorsal vertical longitudinal septum separating the myotomes in the median plane is formed by the skeletogenous layer, and the latter is also continued into the septa intermuscularia and into the fascia superficialis interna (which latter is thus analogous to ribs), at these junctions the thickness of the layer being greatly increased. It thus also forms the connective tissue support of the myotomes. In the tail it further encloses the large blood-vessels, and is continued over the caudal cartilages. The skeletogenous layer is non-cellular (*i.e.* has no nuclei), but it is well infiltrated with blood-vessels. It is not, strictly speaking, part of the chorda, and internal to it no nerves or blood-vessels exist.

Elastica Externa (*el. ext.*).—This is a relatively thin cuticular-looking membrane, staining intensely with eosin, and which closely invests the notochordal sheath as its outermost covering. According to G. Retzius, it exhibits the chemical reactions of an elastic membrane. Its edges are doubtless by an optical effect sharply defined, and it may give off elastic fibrillæ into the skeletogenous layer and sparely into the external layer of the notochordal sheath (the latter according to v. Ebner). There are no

* *Jena. Zeits.*, vi., 1871. † *Arch. Anat. Phys.*, Anat. Abt., 1881. ‡ *Z. f. w. Z.*, 62, 1896 (complete paper).

nuclei in the elastica externa, and in *Bdellostoma* Ayers and Jackson describe parts of it as exhibiting a "distinctly fibrous structure." This is also the case in *Myxine* in the isolated elastica, as shown by v. Ebner, the fibres being circular and spindle-shaped and closely packed without any spaces. Ayers and Jackson state that no part of the chorda is a cuticular product—a conclusion previously emphasised by v. Ebner.

There is no *elastica interna*.

Notochordal Sheath (*nt. sh.*$^{1-3}$).—This consists typically, but not everywhere, of three perfectly distinct sheets—the *external*, *middle*, and *internal layers* of the notochordal sheath. According to G. Retzius, all three agree in structure and also exhibit the same micro-chemical reactions, and constitute the true chordal sheath traversed by perforating tubules (?), representing *one* sheath only, as first pointed out by Kölliker. As shown by v. Ebner, all three layers consist of non-cellular fibrillæ coursing transversely in large undulating curves round the chorda. The curves in the three layers do not correspond, or rather those of the middle layer do not correspond with the other two, thus emphasising the boundaries between the layers. The bends of the curves always corresponding, no matter how they may be directed, linear effects are produced in the isolated sheath, and in this way we may distinguish a dorsal, a ventral, and a paired lateral longitudinal line. Of the three layers, the external (*nt. sh.*^{1}) is usually as wide as the other two together, whilst the internal layer (*nt. sh.*^{3}) is always the weakest. v. Ebner has shown that in the tail, where one of the layers is suppressed, it is the middle one (*nt. sh.*^{2}). Stained with methyl-blue-eosin the external and internal layers stain a faint pale-blue, whilst the middle layer is sharply contrasted in pink.

Chordal Epithelium (*ch. ep.*).—This is very greatly reduced, and in this respect may be compared with *Bdellostoma* (Ayers and Jackson) and *Petromyzon* (v. Ebner). It consists of a very thin layer of granular protoplasm applied to the internal layer of the notochordal sheath, and which is raised up at intervals into small heaps, each lodging a nucleus. There is no observable division into cells, apart from the heaping arrangement, nor is there more than one layer of the nuclei, in which I confirm G. Retzius and v. Ebner. The chordal epithelium is connected with the chordal cells, or rather the walls of the latter are opposed to the epithelium. In the small 6·5 cm. and 10 cm. Hags the heaping arrangement is wanting, and the nuclei, as one would expect, are very close together—*i.e.* a typical epithelium exists. The chordal epithelium cannot, of course, be morphologically separated from the chordal cells; both form one coherent tissue. This is more evident in *Bdellostoma*, according to Ayers and Jackson, where the epithelium may be two or three layers deep, and transitional cells are found connecting it with the vacuolated chordal cells.

Chordal Cells (*ch. c.*).—The entire mass of the so-called chordal "jelly" consists of vacuolated nucleated cells. The size of these cells varies in different individuals and at different regions of the chorda, but generally there is a narrow zone of very small cells associated with the chordal epithelium, and they then rapidly increase in size

towards the middle of the chorda, where they are largest. The cells are, roughly, modifications of a spherical type—that is, they present the same appearance in transverse and longitudinal sections. Each cell is, I believe, quite independent—*i.e.* the fibrous-looking septa forming the network are always double, being formed by the walls of two adjacent cells, and there is no connection at the angles. This accords with the figures of G. Retzius and Ayers and Jackson. Between the walls of adjacent cells there is a very narrow space, occupied probably by a cement substance, and each wall is marked by very fine closely applied striations (as shown in the figures and as first pointed out by G. Retzius), indicating that the wall is fibrillar. Lying on the wall of the cell is the flattened oval nucleus, which is surrounded by a delicate membrane, is very coarsely granular, and contains one or more nucleolar spots each surrounded by a clear area. In *Bdellostoma*, according to Ayers and Jackson, the nucleus may lie in the *centre* of the cell, when it is connected with the walls by protoplasmic strands. According to v. Ebner and Ayers and Jackson, the walls of the cell bear a very thin, encrusting film of granular protoplasm. I find the nuclei in all the cells without exception, although they have so far only been recognised peripherally. The body of the cell is occupied by a clear, homogeneous fluid substance to such an extent that the chordal cells must be regarded as the most intensely vacuolated cells known, or possible. The fluid contains a granular substance in some forms, according to v. Ebner. Embedded in the centre of the chordal cells, or generally, perhaps, somewhat nearer the dorsal surface, is a condensed area known as the *fibrous core* (= the *chordastrang* of v. Ebner). Its extent varies very greatly in different individuals, and in some parts of the chorda may even be absent. In one specimen the fibrous core was cross-shaped in transverse section, but it is generally greatly flattened from above downwards and wide from side to side. As first suggested by Kölliker, and ascertained by v. Ebner, the fibrous core consists simply of chordal cells *elongated in a longitudinal direction*, and having relatively thick walls. That this explanation of the fibrous core is the correct one is obvious from an examination of thin longitudinal sections. Ayers and Jackson state that in *Bdellostoma* it is entirely *fibrous*, but they were evidently unaware that it had previously been correctly described.

As the chorda enters the parachordals its sheath may gradually thin down (except at one place, ventrally, where for a time it is even thicker), until at the extreme tip it is covered only by the now irregular elastica externa, and even this is wanting for a short space ventrally. The tapering of the chorda and the condition of its sheath is evidently very little disturbed by the growth round it of the parachordals. In the tail, as the caudal cartilages surround the chorda the fibrous sheath gradually disappears, leaving the elastica externa; but even after the chorda is largely invaded by soft cartilage the elastica externa and a portion of the fibrous sheath remain. Finally, however, first the fibrous sheath and then the elastica externa vanish, and there is a fusion, though never quite complete ventrally, between the now largely cartilaginous chorda and the median ventral bar of the skeleton of the tail.

C. The Cranium and Neural Tube.

The *neural tube* is a double-walled structure. It consists of an inner cylinder often selectively staining red with methyl-blue-eosin, and the base of which rests on the elastica externa of the somewhat flattened roof of the chorda, and an external layer, present only at its sides and roof, formed by the skeletogenous layer of the chorda. The former is comprised of fine transverse fibres very closely packed together, and, in fact, bears some resemblance to an elastic membrane. The cavity of the neural tube is very much larger than the spinal cord which it contains. There is sometimes seen wedged in between the fibres of the inner cylinder, in the mid-dorsal region, a wide conspicuous mass of *longitudinal* fibres. The neural tube is perforated laterally below at intervals by the roots of the spinal nerves.

The *cranium* is entirely membranous, and J. Müller is of course quite wrong in describing an infiltring cartilaginous substance in the cranium of *Bdellostoma*. It is perforated in front by the olfactory nerves, and laterally below by the cranial nerves. It is simply an expansion of the neural tube, and consists of the same two layers. Between the cranium and the olfactory capsule the skeletogenous layer is very extensive, as it also is ventrally and laterally at the termination of the chorda in the parachordals. The brain almost fills the cavity of the cranium, and in this respect may be contrasted with the spinal cord. The skeletogenous layer of the cranium is continued into the median dorsal longitudinal septum between the myotomes of the head, as in the spinal region. Anteriorly, the floor and roof of the cranium are joined up by a median vertical septum consisting of a double sheet of the inner wall of the cranium with the skeletogenous layer between, and which divides the anterior extremity of the cranial cavity into two chambers, each containing an olfactory lobe.

D. The Histology of the Skeleton. (Figs. 3 and 4.)

I do not propose to consider in any detail the finer structure of the myxinoid skeleton, which is a somewhat complex and contentious subject, but simply to discuss such facts as bear directly on the morphology of the skeleton. Consequently I leave over for the present my observations on the histogenesis of the cartilage.

The first observer to work at the histology of the myxinoid skeleton was J. Müller, who distinguished two kinds of cartilage,* which he refers to as "yellowish" or "brown" (hard) and "grey" (soft) cartilage, and which he describes as "cellular cartilage." Its peculiar structure, which he roughly worked out, differing apparently from any other variety of cartilage known, "greatly surprised" him. G. Retzius in 1881, in his work on the auditory organ, figures and briefly describes the cartilage of the auditory capsule of *Myxine*; and he states that it consists of a substance containing closely opposed oval or rounded cell cavities, and a weak intercellular substance which

* *i.e.* in *Bdellostoma*. In his concluding remark (p. 340) he wrongly excludes *Myxine*.

is arranged in *concentric layers* round each cell cavity. Here and there two such cavities are surrounded by the same concentric layers. This concentric structure seems to me to be due to the nature of RETZIUS' sections; but whatever its explanation may be, no subsequent observer has confirmed RETZIUS' description.

W. K. PARKER describes four varieties of skeletal tissue in myxinoids, which correspond to my hard and soft cartilage and pseudo-cartilage below, as far as I understand his description. The hard cartilage, he says, has a "greenish" colour. AYERS and JACKSON state that the *whole* of the cartilage (preserved in formol) assumes a "pink or reddish tinge," and this, together with their somewhat remarkable neglect of the literature of the subject, accounts for the failure of these authors to detect one of the most obvious and striking characteristics of the myxinoid skeleton. HOWES regards the distinction between the two kinds of cartilage as a "subtle" one; but the difference, as we shall see, is very real. POLLARD * describes the hard cartilage of *Myxine* as consisting of "only a hard yellow spongework of the intercellular matrix," the nuclei and protoplasm of the "procartilage" cells having disappeared. I have, fortunately, had an opportunity of examining the sections on which this statement was based, and find that as the material was stained in bulk with picro-carmine, the stain has reached the cartilage cells in the soft or more penetrable cartilage; but that the cells of the hard or denser cartilage are not stained, whilst the intercellular substance is coloured yellow. With the low power, therefore, POLLARD's description appears correct; but examination only with a Zeiss D at once reveals the cells in the hard cartilage, as shown in fig. 3, so that POLLARD's description is inaccurate.

The most reliable description of the cyclostome skeleton we owe to SCHAFFER,† most of whose points I had made out quite independently before I had an opportunity of consulting his work. The following abstracts, therefore, may be taken as including my own results also. *All the fresh cartilage is white and uncoloured*, but the hard cartilage is more opaque. The red colour assumed by the latter arises gradually in alcohol first on the surface and then penetrates inwards. I have, however, so far not found the red colour supervene on formol preserved material. The hard cartilage may be said to consist of a number of units each composed typically of one *cartilage cell* (*ct. c.*), a *cell capsule* (*c. ct. c.*), and a ring of *secondary ground substance* (*s. g. s.*), these units being held together by a *cement substance* (*c. sb.*), from which, however, they may be macerated out. I find that the independence of these units is more marked in some places than in others; for instance, in the palatine bar there is a complete ring of cement round each unit, as described by SCHAFFER, whilst in parts of the middle segment of the basal plate these rings are by no means complete, as shown in fig. 3. Hence maceration here has not been successful. In the lamprey, according to SCHAFFER, the intercellular substance ‡ of the *soft* cartilage forms a continuous network *which*

* 1894, p. 349; and 1895, p. 415.
† *Z. f. w. Z.*, 61, 1896, p. 606. *A. f. m. A.*, 50, 1897, p. 170.
‡ SCHAFFER'S term for the non-protoplasmic portion of the cartilage.

cannot be separated into units, and hence the intercellular substance here corresponds with the cement of the hard cartilage, so that we have at once a sharp *morphological* distinction between the hard and soft cartilages. This distinction, however, only partly applies to *Myxine*, for in places we find in the soft cartilage a deposit of secondary ground substance, and generally cell capsules are differentiated. The soft cartilage of *Myxine*, therefore, represents a stage further than that of the lamprey. There seems to be no question that the hard cartilage of cyclostomes, both as regards its minute structure and micro-chemical reactions, may be directly compared with the hyaline cartilage of other animals.

In addition to SCHAFFER'S memoir, the cartilage of cyclostomes has also been investigated by STUDNICKA.* This author finds the cartilage of the tail fin to represent an intermediate type, which I can confirm; but there must be added thereto the cartilage of the branchial skeleton. There is thus no *break* between the two kinds of cartilage, and this discontinuity is more in evidence in *Myxine* than in the lamprey. STUDNICKA doubts the presence (normally) of the cell capsule of SCHAFFER in *Myxine*; and whilst it is indeed true that there is a conversion of the capsule into ground substance in some cases, as in higher animals, my fig. 3, which was drawn before I had seen the papers of either author, amply confirms SCHAFFER in this respect. STUDNICKA was the first to examine what I have described below as the pseudo-cartilage of *Myxine*, which he terms "vorknorpel," and of which he correctly asserts, on account both of its morphology and micro-chemical reactions, that it cannot be regarded as true cartilage, but is a transition tissue. SCHAFFER, in his later paper, compares it with the tissue of the sesamoid nodule in the Tendo Achillis of the frog, and states that, whilst he does not regard it as true cartilage, it nevertheless exhibits considerable resemblances to the simplest form of true cartilage. He considers the posterior segment of the basal plate as a true sesamoid formation in the tendon of the M. retractor linguæ,† and its cells as peculiarly modified tendon cells. This has been independently stated by AYERS and JACKSON, evidently without knowledge of SCHAFFER'S work, and is clearly the correct view. In this connection I may mention that, in the cells of the frog's sesamoid above, MEVES was able to establish the presence of centrosomes, and this induced SCHAFFER to look for them in the similar tissue of *Myxine*. He succeeded in his quest,‡ and found that the cells of the posterior segment of the basal plate contained one or two centrosomes, each surrounded by a clear area. I can confirm this discovery, and am able to extend it to the cells of the hard cartilage, as shown in fig. 3.

Paraffin sections of the skeletal tissues of *Myxine* are invariably distorted and unreliable, and hence the histology of the skeleton is best studied by means of free-hand sections stained preferably with MANN'S methyl-blue-eosin. Provided care is taken to avoid being misled by certain deceptive appearances incidental to thick hand

* *A. f. m. A.*, 48, 1896, p. 606. Also 51, 1898, p. 452.

† This is surely a slip of the pen. The muscle should be the M. copulo-copularis, P. FÜRBRINGER (M. constrictor musculi mandibuli, AYERS and JACKSON).

‡ *Sitz. K. Akad. Wien*, Abt. iii., 105, 1896, p. 21.

sections, the micro-anatomy of the connective tissues may be satisfactorily worked out in this way.

Apart from the notochord, two kinds of skeletal tissue may be distinguished in *Myxine*—(*a*) *cartilage* and (*b*) *pseudo-cartilage*.* Further, there are at least *two varieties of each kind*, and all may be said to merge more or less perceptibly into each other. Of the cartilage, the two varieties are at once obvious. In the living condition, as already stated, the skeleton is uncoloured, but after it has been a long time in spirit a marked differentiation arises, the softer cartilage remaining white whilst the harder cartilage turns a deep reddish brown. This distinction is wonderfully emphasised by their staining reactions with methyl-blue-eosin, the soft cartilage staining blue and the hard red. There is also, of course, a considerable difference in consistency, as the terms soft and hard indicate. The combination of the morphological (already described) with the micro-chemical distinction makes the difference between *typical* hard and soft cartilage a very real one. The distribution of the two kinds of cartilage is illustrated in the figures by the two colours (representing the staining reactions with methyl-blue-eosin—soft cartilage, blue; hard cartilage, red), and hence there is no occasion to refer to it further here.

Hard Cartilage (fig. 3).—This consists of an intercellular substance or matrix in which very large cartilage corpuscles or cells are embedded. Each cell (*ct. c.*) is surrounded by a deeply staining thick capsule (*c. ct. c.*); but the matrix immediately around each capsule only stains slightly, and this, owing to the large size of the cells, accounts for the reticular appearance of the matrix emphasised by POLLARD. Wedged in between this secondary ground substance (*s. g. s.*), as it is called by SCHAFFER, is the staining portion of the matrix, or the cement substance (*c. sb.*), which more or less surrounds each ring of secondary ground substance. The cement is the most massive in older animals and in the larger cartilages. The cartilage cell itself consists of a very finely granular slightly staining reticulum, in which is embedded a round or oval nucleus (*n. ct. c.*) containing generally a single deeply staining nucleolar body surrounded by a clear space. In paraffin sections the nucleus appears clear and vesicular, with scattered globules of chromatin, whilst the sarcode is distinctly reticular and the centrosomes also visible. I have seen as many as four nuclei in one capsule, evidently prior to division, and the nucleolus itself may be multiplied. Various stages in the division of the cell, and the consequent formation of fresh intercellular substance, may be seen. With methyl-blue-eosin, the matrix stains red, the nucleus light blue, and the nucleolus a deep blue.

Soft Cartilage.—This is the pro-cartilage of PARKER and POLLARD, but not the pro-cartilage of STUDNICKA. It has a very strong affinity for methyl-blue, and in fact combines with this stain so intensely that it takes time to extract it. That there is, however, no *genetic* distinction between the hard and soft cartilage is shown by the fact

* A term applied in 1878 to similar tissue in the frog by STADELMANN. I adopt it in preference to SCHAFFER'S more cumbrous "vesicular supporting tissue."

that the soft cartilage, where it is connected with the hard—as, for example, the lateral labial cartilage with the external bar of the anterior segment of the basal plate—always passes imperceptibly into the hard cartilage without any demarcation or trace of suture. Hence we may regard the hard and soft cartilages as modifications of the same ancestral tissue. On the other hand the soft cartilage, for example, of the caudal fin and of the branchial skeleton undoubtedly approaches rather the structure of the hard pseudo-cartilage, thus connecting up the two kinds.* The essential difference between *typical* examples of the two varieties of the cartilage lies in the great reduction of the matrix, and of its character, in the soft cartilage. This in some places is a continuous, homogeneous, almost fibrous looking network; but, generally, cell capsules may be clearly distinguished, and secondary ground substance may even be added. Apart from this, the intercellular network is distinctly comparable to the cement substance of the hard cartilage, and is its characteristic feature. The cells and nuclei of the soft cartilage differ in no essential respect from those of the hard cartilage.

Pseudo-Cartilage (fig. 4).—The structure of the hard pseudo-cartilage, for we may distinguish hard and soft varieties here also, is best seen in the posterior segment of the basal plate and in the superior chondroidal bar. If a thin, free-hand, transverse section is made of the former it is seen to be U-shaped, and enclosed by a thick perichondrium of stout connective tissue fibres among which are interspersed groups of nuclei. Both the dorsal concave and the ventral convex borders are lined by a single palisade of vertical chambers, those at the latter border being much the larger. The central portion of the cartilage is occupied by similar chambers (but of much smaller size and irregular shape), and also by stout fibrous septa which usually pass more or less directly from one border to the other, branching as they go. All the peripheral chambers and many of the central ones are further divided by exceedingly fine partitions into a number of loculi, each loculus containing one cell formed of a glassy, transparent, unstaining sarcode, and one or more peculiar coarsely granular nuclei. These nuclei usually have one or more nucleolar bodies, each surrounded by a clear area. In some of the loculi a number of nuclei, each with an obvious nucleolus, were massed together, whilst the occurrence of centrosomes in these cells has been already mentioned. In the central loculi, the nuclei are generally much larger and of a very irregular shape. In spite of the fact that the matrix is here almost absent and its place taken by fibrous septa, and that the character of the cells is different, we may directly compare the pseudo-cartilage with the true cartilage in terms of the soft cartilage of the caudal fin and branchial skeleton. AYERS and JACKSON state that the pseudo-cartilage of the posterior segment of the basal plate resembles rather the structure of the notochord than the cartilage of the remainder of the skeleton; but this certainly cannot be accepted.

The soft pseudo-cartilage may be best studied by sections of the thick pad at the cephalic end of the basal plate (figs. 1 and 10). It has essentially the same structure

* *Cp.* the description of these two cartilages, and especially of the superior chondroidal bar.

as the hard pseudo-cartilage, except that (1) the intercellular features are generally much feebler, and hence the softer consistency; (2) the chambers are all smaller and of irregular shape—the regular peripheral chambers of the other variety being absent; (3) the fibrous septa are not only weaker, but ramify irregularly throughout the tissue; and (4) the nuclei are less coarsely granular (but otherwise similar). The soft pseudo-cartilage, in fact, bears somewhat the same relation to the hard variety as the soft cartilage does to the hard, whilst cartilage and pseudo-cartilage are connected up into a series by the hard pseudo-cartilage and the soft cartilage. Compare in this connection the histology of the superior and inferior chondroidal bars.

E. The Cranio-Pharyngeal Framework.

Under this generic title we may conveniently deal with that portion of the skeleton grouped around the mouth, pharynx, and central nervous system, and which exhibits *par excellence* that distinguishing feature of the mature myxinoid skeleton—the fusion of the parts into one continuous coherent whole. To what extent this *has* been formed by the fusion of independent elements, or whether it is more or less continuous from the first, we do not at present know. Our doubts on this point will, I hope, soon be set at rest by the speedy publication of the elaborate memoir which Dr L. Neumayer is now preparing on the development of the skull of *Bdellostoma*, based on material supplied by Prof. Bashford Dean. In the meantime, as far as *Myxine* is concerned, there is not a single independent cartilage in the entire skull, except a few of the nasal rings and the cartilage of the fourth tentacle. The division therefore into the following regions must, to a certain extent, be artificial in the present state of our knowledge.

"*Parachordal*" *Cartilages* (fig. 2, *p. c.*).—For a considerable distance behind the skull there is a very gradually diminishing deposit of soft cartilage at the mid-ventral line between the elastica externa and the skeletogenous layer of the notochord. In *Bdellostoma*, according to J. Müller, this is present as a *detached* ventral plate, and Neumayer also describes a detached ventral half-ring of cartilage behind the parachordals in *Myxine*. I have carefully searched for the latter in the sections of the 6·5 cm. and the 10 cm. Hags, but find no traces of the break. As the skull is approached this ventral deposit increases in volume, and extends upwards on each side of the chorda. At the same time similar deposits of cartilage appear in the dorsal region, and all of them increase greatly in thickness and join up so as to form a complete ring round the notochord except for a mid-dorsal break (*cp.* fig. 2). This tube is thinnest at the mid-ventral line, and constitutes the soft parachordal cartilages; but there is no break between them ventrally. Whilst the parachordals are increasing in volume the notochord is diminishing in size, and its membranes become gradually reduced. In front, the parachordals gradually merge into the hard cartilage of the auditory capsule, but the latter capsules are always separated in the mid-ventral line by a zone of soft parachordal

cartilage, and there is always a narrow ring of the same cartilage surrounding the embedded notochord. Opposite the posterior boundary of the auditory foramen the parachordal tube is completed dorsally for a very narrow space, so as to complete the chordal roof (*cp.* fig. 2). This roof is much more extensive in *Bdellostoma*, so that in this respect *Myxine* is the more primitive. In *Myxine*, MÜLLER* did not find the dorsal fusion of the parachordals at all; but I am inclined to think that it practically invariably occurs, in spite of the fact that PARKER did not find it also. At the same region independent nests of soft cartilage appear *within* the notochordal sheath, and the notochordal membranes almost entirely disappear. The now cartilaginous notochord is, in places, in contact with the parachordal tube in which it lies, *but there is nowhere any fusion between them.*† The cartilaginous tip of the notochord projects freely beyond the anterior border of the parachordals in the median line, as shown in fig. 2. The parachordals are now supposed to split, and to extend forwards as diverging arms of hard cartilage on each side, forming the inner boundary of the auditory foramen (*au. f.*), and meeting the trabeculæ opposite the anterior border of the auditory capsule. This is certainly the appearance suggested by dissections (fig. 2), but an examination of serial sections seems to me to indicate that the so-called parachordal cartilages terminate with the soft cartilage—*i.e.* they are composed entirely and only of soft cartilage. For it must not be forgotten that the diverging arms are, as far as we can see, nothing more or less than the inner wall of the auditory capsule completing the auditory foramen, which is presumably a perforation in the wall of the capsule and not an enclosure by the capsule with a fused parachordal. AYERS' and JACKSON'S fig. 4 is misleading on this point, as it does not show the distribution of the hard and soft cartilage, which seems to me may have some significance in this connection. The shading of this region in their fig. 7, and also the figure of RETZIUS,‡ support the view suggested above; but it must not be forgotten that the (assumed) complete fusion of the auditory capsule with the trabecula in front admits the possibility of a similar fusion of the capsule and parachordal behind, although there is absolutely no *evidence* for it in either case.

Auditory Capsule (figs. 1 and 2, *au. c.*).—This is fused behind and internally with the parachordal, as above stated. It is an oval-shaped hollow capsule of hard cartilage, sloping upwards and outwards, with its dorsal or inner wall perforated by the large egg-shaped auditory foramen (fig. 2, *au. f.*), which is, however, closed by a tough, fibrous membrane, about half of which consists of the fibrous cranial wall. This membrane is perforated to admit the exit of the auditory nerves. Just opposite the second fenestra of the skull ($f.^2$), the dorso-external border of the auditory foramen is connected by means of an internal column of hard cartilage with the ventro-external wall of the

* See his concluding remark, p. 340.

† Since writing the above, examination of further series of sections, especially of a vertical longitudinal series, indicates that fusion *does* take place ventrally between the cartilage of the notochord and that of the parachordals. In fact, I now question whether the so-called anteriorly projecting tip of the chorda is not after all a part of the parachordals. But these questions are difficult to settle with adult material.

‡ *Das Gehörorgan d. Wirbelthiere*, i., 1881.

capsule. This column passes from above downwards and outwards, and for a few sections divides the auditory chamber into a larger ventro-internal cavity, open widely by the auditory foramen, and a smaller closed dorso-external cavity. As, of course, the connection is only a rod, the division of the auditory chamber into two, seen in a few sections, is apparent but not real. This column has been overlooked by all writers on the myxinoid skeleton except PARKER, who figured it in his sections but failed to understand its real nature. RETZIUS (*op. cit.*) also found it, and states that it corresponds in direction to a continuation of the bridge of soft cartilage described below as connecting the auditory capsule with the hyoid arch. This is quite true, but whether the fact has any significance cannot be determined by adult anatomy. It must, however, be emphasised that the two structures represent two different kinds of cartilage. The dorso-external wall of the auditory capsule behind fuses by means of a very short but wide bridge of *soft* cartilage (forming the posterior boundary of the second fenestra) with the dorsal extremity of the hyoid arch. Where this fusion occurs *the entire thickness of the capsule consists of soft cartilage.* There is, in fact, here a ragged oasis or plug of soft cartilage in the wall of the capsule. In front, the dorso-external surface of the capsule forms the dorsal boundary of the second fenestra of the skull ($f.^2$).* The anterior margin of the capsule fuses with the posterior extremity of the trabecula.

"*Trabecula*" (figs. 1 and 2, *tr.*).—The trabecular bar fuses behind with the auditory capsule, as just described. It then passes almost straight forwards as a stout rod of hard cartilage, its lateral border behind being fused by a very short but wide bridge of *soft* cartilage with the dorsal border of the superior process of the pterygo-quadrate. At this region the trabecula itself is invaded by numerous nests of soft cartilage. The above bridge forms the ventro-anterior boundary of the second fenestra of the skull, and the posterior boundary of the first ($f.^1$). In front, the outer edge of the trabecula forms the entire dorsal or internal boundary of the first fenestra. At its anterior extremity the trabecula becomes gradually converted into soft cartilage, PARKER's figures of the distribution of the hard and soft cartilage at this region being inaccurate according to my dissections and sections. From its inner border in front the trabecula despatches downwards and forwards a rod of soft cartilage (the sustentaculum of NEUMAYER), which fuses with the central expanded portion of the hypophysial plate. This cannot be fully shown in such a view as illustrated in fig. 2, owing to the perspective. The trabecula finally bends outwards to fuse with the posterior extremity of the palatine bar, which here consists entirely of soft cartilage also.

An interesting observation recorded by ALLIS is that in a "12 mm." † *Bdellostoma* the "pharyngeal basket is nowhere connected with the trabecula," the bridges of soft cartilage described above being absent. From the fact that these bridges *are* of soft cartilage (PARKER, however, figures the anterior one as *hard* cartilage in *Bdellostoma*), ALLIS concludes that they must represent later additions. On the other hand, the fact

* All the fenestræ are closed by fibrous membranes or tissue.

† This measurement is obviously erroneous.

that the hard cartilage in the region of the bridges is either wholely or largely invaded by soft cartilage suggests precisely the opposite view that the hard cartilage replaces the soft. A careful examination of the sections of the 6·5 cm. and the 10 cm. Hags undoubtedly reveals the presence of the bridges exactly as in the adult, and as they are also figured so by NEUMAYER, I cannot confirm ALLIS'S observation as far as *Myxine* is concerned. However, in the absence of information as to the size of his embryo, this is not conclusive.

Hypophysial Plate (figs. 1 and 2, *h. p.*).—This occupies the median hypophysial fontanelle at the base of the skull, bounded by the palatine bars and their commissure, the trabeculæ, auditory capsules, and parachordals. Its function is to provide a basal support for the hypophysial canal or naso-palatine duct. It consists of a central plate, fused with processes from the trabeculæ and the nasal capsule, which sends out a rod in front and a wider process behind. The anterior half of the rod is sometimes formed of hard cartilage (as shown in fig. 2), thus differing from the remainder of the plate, which is of soft cartilage. J. MÜLLER states that it is composed of hard cartilage in *Bdellostoma*, but PARKER found no hard cartilage in it either in *Bdellostoma* or in *Myxine*. It commences under the nasal chamber, just behind the palatine commissure, as a circular deposit of cartilage in the stout membrane connecting the palatine bars. Where the hypophysial canal separates from the nasal chamber, and during its association with the hypophysial plate, it forms a tri-radiate tube, T-shaped in transverse section, and situated immediately below the floor of the membranous cranium, ventrally partly fitting for a time into a median groove in the roof of the pharynx. The hypophysial plate is situated at the base of the upright piece of the T. It is at first slightly saucer-shaped, but further, posteriorly, it becomes bent up sharply at the sides of the hypophysial canal in the form of a V. At the external margin of its widest part it fuses first with the backwardly coursing rod of soft cartilage from the posterior transverse bar of the nasal capsule, and immediately afterwards with a forwardly coursing similar rod from the trabecula, as elsewhere described. PARKER does not describe either of these fusions in *Myxine*; but his fig. 3, pl. 10, seems to indicate that he saw something of the trabecular fusion, and this one is described and figured in *Bdellostoma*. Behind this region the hypophysial plate narrows down into the posterior process, which is bent upwards from below on each side of the vertical limb of the hypophysial canal in the shape of a U. This process expands at its posterior extremity, and is fenestrated to a greater extent than is shown in fig. 2.

NEUMAYER'S figures of the preceding region agree with mine except that the parachordals are figured and described as completely fusing dorsally, and a fusion is described between the hypophysial plate and the subnasal bar. The latter statement is dealt with elsewhere; and, as regards the former, I am quite unable to confirm it. The dorsal parachordal fissure is quite characteristic of *Myxine*.

Superior Lateral Cartilage (figs. 1 and 2, *s. l. c.*).—Consists of soft cartilage, and in front fuses with the dorso-posterior border of the hyoid arch (*hy.*), there containing a

few nests of hard cartilage. It soon fuses below with the dorsal extremity of the first branchial arch, and then passes downwards and backwards over the roof and on to the lateral wall of the pharynx under the constrictor muscle to subsequently fuse below with the upper end of the second branchial arch. It is continued beyond the latter arch as a small rod, rising slightly until it again reaches the roof of the pharynx, where it terminates sometimes in a bifid extremity. This, however, may vary in the two sides of the same animal, as shown in fig. 2.

Inferior Lateral Cartilage (figs. 1 and 2, *i. l. c.*).—Also consists of soft cartilage, and widens considerably in front to fuse with the ventro-posterior border of the hyoid arch, the latter at this region being likewise formed of soft cartilage. The inferior lateral courses almost straight backwards on the ventral wall of the pharynx, and between it and the posterior segment of the basal plate, but slightly external to the latter. It passes internal to the first branchial arch without however being in any way connected with it, and by its dorsal border fuses behind with the ventral extremity of the upper division of the second branchial arch. The inferior lateral cartilage may terminate in this way, as shown in fig. 2, or, as perhaps is more generally the case, and as in *Bdellostoma*, it may be extended beyond the second branchial arch as a tapering rod coursing upwards on the *lateral* surface of the pharynx, where it terminates after a shorter course than the superior bar. NEUMAYER figures its absence behind the second branchial arch, but its presence here is shown by P. FÜRBRINGER and PARKER. I have added it in fig. 1 (although it was not present in the specimen from which the drawing was made) in order that both conditions may be represented.

"*Branchial*" *Arch 1* (figs. 1 and 2, *br. a.*^{1}).—This is of soft cartilage in *Myxine* but of hard in *Bdellostoma*, according to J. MÜLLER. It arises dorsally, as above described, from the superior lateral cartilage, and courses in a half-ring round the lateral wall of the pharynx, bending first backwards and then forwards over the inferior lateral cartilage. It finally fuses with the lower division of the second branchial arch, when this is present (fig. 1), and at once becomes gradually merged into the hard cartilage of the middle segment of the basal plate, as elsewhere described.

"*Branchial*" *Arch 2* (figs. 1 and 2, *br. a.*^{2}).—Formed of soft cartilage. There is a somewhat surprising variation in this arch, since the *lower division* is not always present. It was, for example, undoubtedly absent in the specimen on which figs. 1 and 2 were based; but I have found it in others. It was not found by PARKER in *Myxine*, but is figured by P. FÜRBRINGER and NEUMAYER, and it is present in all my series of sections without exception.* I have therefore added it in fig. 1, which may in this respect be compared with fig. 2. This lower division of the arch, after fusing in front with the first branchial, passes backwards at the side of and external to the dorsal boundary of the posterior segment of the basal plate, finally rising slightly above the latter to terminate freely at about the level of the extremity of the superior lateral cartilage. AYERS and JACKSON describe in *Bdellostoma* a fusion of the posterior extremity of the

* In two museum preparations made by FRIC of Prag, it was present in one and not in the other.

lower division with the inferior lateral cartilage, which eliminates the break in the course of the arch. As this fusion was not found by J. MÜLLER or PARKER in *Bdellostoma*, and has never been seen in *Myxine*, it must represent another and an important variation in the structure of the arch. The *upper division* of the second branchial arch is very short; it fuses above and below with the superior and inferior lateral cartilages, as above described, and in one series of sections despatched forwards in front a blunt process similar to that figured and described in *Bdellostoma* by AYERS and JACKSON. As, however, this seems to be not of general occurrence, I have not introduced it into the figures.

"*Hyoid*" *Arch* (figs. 1 and 2, *hy.*).—The connections of this arch above and below with the auditory capsule and superior and inferior lateral cartilages, have been already described. Dorsally in front, it fuses also with the hard cartilage of the posterior extremity of the superior process of the pterygo-quadrate, and similarly below and in front with the inferior process of the pterygo-quadrate. The hyoid arch is short but wide, and is bent round the lateral wall of the pharynx, lying just external to it. It is composed of hard and soft cartilage, distributed as shown in figs. 1 and 2. According to PARKER, it consists almost entirely of soft cartilage in *Myxine* and of hard in *Bdellostoma*; but I have succeeded in confirming the distribution of the two kinds of cartilage shown in my figures in serial sections, and, further, PARKER'S own sections *do not bear out his dissections.* Posteriorly, the hyoid arch sends backwards a broad, blunt process of soft cartilage, which projects into the fourth fenestra of the skull, as in *Bdellostoma.* Anteriorly, a corresponding process, although a very slight one, is despatched forwards into the third fenestra, which represents the much more extensive process in *Bdellostoma* described by PARKER and AYERS and JACKSON, but not found by J. MÜLLER. In one series of sections, the ventral margin of soft cartilage at the region of the junction of the hyoid with the inferior process of the pterygo-quadrate sent downwards and forwards a blind rod of cartilage; but this seems to be a variation of little importance, beyond that it is one of the numerous examples of the sporadic appearance of soft cartilage in the connective tissues of *Myxine* generally. It is, however, also figured by NEUMAYER.

The posterior boundary of the hyoid assists the superior and inferior lateral cartilages, and the upper division of the second branchial arch, in forming the large and somewhat irregular fourth fenestra of the skull ($f.^4$), whilst its anterior border forms the posterior boundary of the third fenestra ($f.^3$).

"*Pterygo-quadrate*" (figs. 1 and 2, *p. q.*).—This is a tri-radiate structure formed mostly of hard cartilage. It sends upwards and forwards a thick bar or *anterior process* which forms the ventral or external boundary of the first fenestra ($f.^1$), and fuses in front with the zone of soft cartilage (absent in *Bdellostoma*, according to PARKER) forming the posterior extremity of the palatine bar. The second one is the *superior process*, which, with the anterior process, forms the *subocular arch* of AYERS and JACKSON. The superior process passes backwards and slightly upwards to complete

the ventral or external boundary of the second fenestra, to constitute the dorsal boundary of the third fenestra, and to fuse behind with the hyoid arch as above described. Both the above processes consist entirely of hard cartilage. The third or *inferior process* forms the ventral boundary of the third fenestra, and fuses behind with the hyoid arch. It exhibits at about the middle of its course a conspicuous zone of soft cartilage absent in *Bdellostoma*, according to PARKER, and is in fact more or less generally invaded by nests of soft cartilage. Its posterior upper inner surface receives the rod of soft cartilage from the external lateral velar bar, as elsewhere described. The pterygo-quadrate, lying nearer the surface, takes no part in the skeletal support of the pharynx, except to a slight extent the inferior process.

"*Palatine*" *Bar* (figs. 1 and 2, *pl.*).—Forms with the above the palato-pterygo-quadrate of AYERS and JACKSON, and commences behind by a wide stout base of soft cartilage fused with the trabecula and anterior process of the pterygo-quadrate, as above described. It then passes forwards and somewhat inwards, lying at the lateral margin of the ventral wall of the cranium and nasal capsule, to expand in front and to fuse, as the *palatine commissure*, with its fellow of the opposite side at the level of the anterior border of the nasal capsule. The commissure is a wide thickish bar of hard cartilage, and somewhat arched, with the convexity dorsal. Into the ventral concavity fits the base of the *median dorsal tooth*. PARKER figures an anterior "ethmoid" tract of soft cartilage in the commissure, and I also find some evidence of this in my sections. The cornual cartilage fuses irregularly with the external angle of the commissure, and where this occurs there is an invasion of the hard cartilage by nests of soft cartilage. AYERS and JACKSON state that in *Bdellostoma* the cornual cartilage is "attached" to the palatine, and I take it this does not mean fusion.

Immediately in front of the palatine commissure and the median dorsal tooth a median pad, consisting of soft pseudo-cartilage, is seen, which passes forwards over the roof of the pharynx for a short distance. It is invaded, especially in front, by several nests of true soft cartilage. Anteriorly, it lies between the diverging palato-ethmoidalis profundus muscles, with which it is very closely connected. Its true relations are shown in vertical longitudinal sections, when it is seen to arise from the anterior border of the base of the median dorsal tooth, pass forwards for a short distance, and then curve backwards round the anterior margin of the palatine commissure to be inserted into the posterior extremity of the subnasal bar. Only the former part consists of pseudo-cartilage. I shall refer to it again in my next part on the muscles. In the meantime I need only point out that it corresponds to the occurrence of soft pseudo-cartilage in the tendons of other muscles, such as those of the "lingual" apparatus, as described elsewhere.

Cornual Cartilage (figs. 1 and 2, *c. c.*):—Fuses behind, as above described, with the palatine commissure, and consists of soft cartilage. It passes forwards and outwards in a curve immediately internal to the M. tentacularis posterior, its anterior free extremity coursing parallel and just external to the lateral labial cartilage. In the sections the

dorsal surface, near the tip, gave off on both sides a short backwardly projecting hook, which then curved upwards and inwards and finally downwards, to be connected by a short stout ligament with the lateral labial cartilage just behind where the latter fuses with the cartilage of the third tentacle. It certainly appeared to express a tendency to, or to be a relic of, a *cartilaginous* connection between these two elements. The cornual cartilage terminates in front opposite the fourth nasal ring.

F. Nasal Tube and Capsule. (Figs. 1, 5, and 6.)

The nasal aperture is a large opening situated terminally on the dorsal surface of the head. It is guarded laterally by two short, pointed tentacles on each side—the nasal "barbels"—and dorsally by a truncated lip. I am not able to follow W. K. Parker in distinguishing "three nasal barbels on one side, and four on the other."* This external opening leads into a long dorsal tube (*n. tb.*), which in a $45\frac{1}{2}$ cm. Hag measured 14 mm. The latter, in its turn, passes first into the olfactory chamber and then into the hypophysial duct or naso-palatine canal. As in *Bdellostoma*, according to J. Müller and Ayers and Jackson, the nasal tube is widest in front, and gradually tapers as it approaches the olfactory capsule (*n. c.*). Parker, however, figures it as being much narrower anteriorly. It is remarkable in the myxinoids on account of its strong superficial resemblance to a trachea, and it is supported at intervals by cartilaginous rings, which are, however, imperfect ventrally. In some sections of a 6·5 cm. Hag, lent me by Dr Beard, the nasal rings so closely approximated in the mid-ventral line as to be almost in contact, whilst in another series of a 25 cm. Hag the posterior rings overlapped. The number and form of these rings are subject to variation (*cp.* figs. 1 and 6). In the specimen and in the series above, there were eleven in both cases. Parker also describes and figures the same number, but, as he failed to find the first one, his total should be twelve. Ayers and Jackson state that there are normally nine in *Bdellostoma*, with occasional variations of eight and ten. J. Müller gives ten for *Bdellostoma*, and Parker distinguishes twelve. The latter author figures all these arches in *Myxine* as independent; but, as shown in figs. 1 and 6, about the first and last three are usually connected up, the last being further fused at intervals with the anterior transverse bar of the olfactory capsule. These connections I have found both in dissections and in serial sections. As, however, stated by Ayers and Jackson for *Bdellostoma*, "the nasal arches are found to vary to a considerable extent in number, form, and size, both relative and absolute."†

The first *nasal ring* (1) is situated at the anterior edge of the dorsal lip of the external aperture. It is connected dorsally with the second ring by a backward median process (fig. 5). These parts, as well as the dorsal portions of the other nasal rings, can be seen in the living fish showing through the skin. In the neighbourhood of the median process there projects, downwards and forwards into the cavity of the tube, a

* *Op. cit.*, p. 386.

† *Op. cit.*, p. 199.

dorso-median fold of its lining. This lies near the external nasal opening, and is supported by a delicate strip of free cartilage (*cp.* fig. 1). It is probably not the structure referred to by PARKER as the nasal valve, which is merely an adventitious fold of the lining mucous membrane of no importance, and occurring capriciously at all parts of the tube. It is, doubtless, a very small tentacle. Latero-ventrally, on each side, the first four rings were all connected up by a longitudinal bar (fig. 1).* In some serial sections prepared by the late Dr POLLARD and lent me by the late Prof. G. B. HOWES, the third arch sent back and the fourth arch despatched forwards processes which overlapped but did not fuse. Further, the fifth and the sixth arches sent forwards prominent projections which did not, however, quite reach the arches in front. As also described by J. MÜLLER and AYERS and JACKSON for *Bdellostoma*, the lateral connecting bar in front sends forwards a strip of cartilage (fig. 1), which the latter authors found in one case to fuse with its fellow of the opposite side in the mid-ventral line. These cartilages therefore express a tendency to form a complete ring round the external nasal opening, but may represent a pair of vestigial tentacles. Arches five to eight are quite independent; but the last three are connected ventro-laterally by a longitudinal bar with each other, and the last with the nasal capsule (fig. 1). Dorsally the tenth arch is Y-shaped, and also connected in the middle line with the eleventh (fig. 5). The last or eleventh arch is fused with the anterior transverse bar (*a. t. b.*) of the olfactory capsule at five places—by one median and two lateral pairs of rods (figs. 1, 5, and 6). As a result, four fenestræ are formed—a small dorsal and a large lateral one on each side. This corresponds exactly with the condition in *Bdellostoma* described by J. MÜLLER,† and also practically with the figures of AYERS and JACKSON.

In a series of sections of a 25 cm. Hag the following variations from the above description may be noted (*cp.* fig. 6). Arches 7 and 8 were fused ventrally, but 9, on the other hand, was independent, although underlapping 10 below. Ring 10 joined not directly with 11 to complete the large anterior fenestra (but it *did* on the other side), but with the process connecting the latter ring with the anterior transverse bar of the nasal capsule (*a. t. b.*). At this region an additional independent cartilage was present on the left side only, and there was also an additional fenestra dorsally separating rings 10 and 11. Anteriorly, this specimen agreed even in the smaller details with the above description, and hence these posterior variations become the more interesting.

The *olfactory capsule* itself (*n. c.*), like the nasal rings, protects only the roof and sides of its cavity. In front the latter is connected with the lumen of the nasal tube, and below with the hypophysial canal. It is shut off posteriorly from the cavity of the brain-case by the double wall of the cranium, as described by J. MÜLLER. The capsule consists essentially of a series of longitudinal rods fused in front and behind respectively with an anterior (*a. t. b.*) and a posterior (*p. t. b.*) transverse bar. There are nine of these

* In one series of sections the fourth ring was not included on one side.

† *Op. cit.*, p. 109.

longitudinal rods, of which the two lateral are larger and more irregular in shape than the rest, and are called by AYERS and JACKSON the *lateral plates* (fig. 6, *l. p.*). The seven dorsal ones are all very narrow and straight, the middle one occupying the mid-dorsal line. The spaces between these rods are of regular shape except the two lateral, and they are perceptibly wider than the rods themselves. There are seven *olfactory laminæ* in *Myxine*, produced by a corresponding number of longitudinal invaginations from the roof of the olfactory chamber. The lateral plates lie at the side of the bases of the most lateral laminæ, whilst the other seven bars are situated immediately above the dorsal bases of the laminæ. Thus the form of the capsule is clearly identified with the conformation of the olfactory organ, and is essentially the same both in *Myxine* and *Bdellostoma*.

Whilst examining a series of transverse sections I discovered a cartilaginous connection between the posterior transverse bar of the olfactory capsule and the hypophysial plate, which I afterwards found by careful dissection of the adult animal (figs. 1 and 2, *h. p.'*). It was not seen by J. MÜLLER, PARKER, or AYERS and JACKSON, but I learned afterwards that it was described for *Myxine* by NEUMAYER,* and it has lately been independently mentioned by ALLIS in *Bdellostoma*.† It consists of a small cylindrical rod passing from the ventral extremity of the posterior transverse bar downwards and backwards to fuse with the hypophysial plate just where the latter fuses with the trabecula. There is thus at this point a complete ring round the nasal organ and the hypophysial tube—formed above by the olfactory capsule, at the sides by the connections now in question, and below by the hypophysial plate.

The nasal rings consist of the white soft cartilage, but the olfactory capsule, with the exception of the anterior transverse bar, is formed of the brown hard cartilage. The anterior bar seems to represent a transition condition, whilst the larger lateral bars of the capsule are the toughest of all.

NEUMAYER's results on the nasal tube and capsule are sufficiently remarkable to call for special notice. The material he used was that collected by O. MAAS for his well-known work on the renal organ of *Myxine*, but he does not state the size of the specimen on which his wax model was based. It was, however, probably MAAS' 8·5 or 9·8 cm. Hag. The nasal tube is described and figured as a continuous cylinder with a few irregular perforations, and no indications of its tracheal nature, except perhaps as regards the first three rings. If this description is confirmed, then we must regard the nasal tube of *Myxine* as primitively a more or less continuous structure which has secondarily become differentiated into rings. Similarly, the anterior transverse bar of the capsule would seem to belong rather to the tube, as indeed is otherwise probable, since two of the nine longitudinal rods fail to reach it. NEUMAYER also figures, but does not describe, a fusion of the anterior transverse bar of the capsule with the palatine bar and of the postero-ventral region of the tube with the subnasal cartilage ("inter-

* *Op. cit.*, p. 3 and fig. 4. Apparently also by POLLARD (p. 396).
† *Anat. Anz.*, xxiii., pp. 269–270.

trabeculare anterius"). With special reference to these unexpected results I worked over very carefully Dr BEARD's sections of a 6·5 Hag—a younger form than any at NEUMAYER's disposal. I must point out that this specimen had been somewhat damaged before coming into Dr BEARD's hands, that it was not, in fact, properly preserved, and that the sections are very irregular. Nevertheless I think I can positively state that, excepting to a certain extent the nasal tube, NEUMAYER's results are either hopelessly inaccurate or that there must be some explanation of them that does not occur to me. It is difficult to believe that a careful and laborious worker as Dr NEUMAYER is well known to be could be so far in error, and we must therefore await a re-investigation of the embryonic skeleton of *Myxine*. As we should expect, and as pointed out by BEARD * and NEUMAYER, the nasal skeleton is relatively very large in the young forms. The reconstruction of the nasal skeleton given by POLLARD † is in exact agreement with my fig. 1, based on dissections.

G. THE TENTACULAR APPARATUS. (Figs. 1 and 2.)

Omitting the problematical tentacles mentioned in connection with the nasal skeleton, there are four tentacles on each side both in *Myxine* and *Bdellostoma*. These are the nasal and oral barbels of W. K. PARKER. Of them only one, the fourth, has an independent skeleton, that of the other three being fused with portions of the internal framework. I therefore describe under the above head the subnasal cartilage or bar and the lateral labial cartilage, but this is done simply as a matter of convenience. The whole of the apparatus consists of soft cartilage, except the central portion of the subnasal bar and the free internal extremity of the cartilage of the fourth tentacle. All the cartilages extend to the tips of the tentacles.

The *cartilage of the first tentacle* (1), morphologically the second, passes downwards and backwards at the side of the nasal opening, crosses externally the base of the second tentacle, and terminates blindly on the surface of the muscles at about the level of the subnasal bar. A short distance before it terminates, it fuses by its posterior surface with the *lateral "labial" cartilage* (*l. l. c.*). The latter passes at first upwards and backwards to give off a projection, the *internal process*, into which the M. nasalis is partly inserted. AYERS and JACKSON state that this process in *Bdellostoma* is attached to the nasal tube by a membranous ligament, but in *Myxine* it is only *indirectly* connected with the nasal tube and skeleton by means of the insertion of the M. nasalis. Also in *Bdellostoma*, according to J. MÜLLER and AYERS and JACKSON, the anterior extremity of the lateral labial is connected by ligament with the tip of the cornual cartilage.‡ Behind the internal process, the lateral labial bends downwards and backwards in a slight curve, and receives ventrally the *cartilage of the third tentacle* (3), which fuses with it. The latter cartilage is a stout rod, thicker at its base than the

* *Anat. Anz.*, viii., 1893, p. 59.
† *Zool. Jahrb.*, Abt. Morph., viii.; Taf. xxv., fig. 11, 1895.
‡ *Cp.* the description of the latter cartilage.

lateral labial itself, and shows in the sections as it approaches the labial some nests of hard cartilage. The stout base passes downwards and forwards, and lies within the contour of the body, forming more than half the length of the cartilage. The tentacle itself in the living animal is either almost perpendicular, or it can be rotated forwards; hence the inclination of the external portion of the cartilage is subject to muscular control, and consequently varies in preserved material. The cartilage of the third tentacle is the longest of all. After receiving it the lateral labial, connected by ligament in *Bdellostoma* with the tip of the cornual cartilage, according to J. MÜLLER, passes at first backwards, downwards, and inwards in a gentle curve, until it almost reaches the median plane, and thus arrives at the level of the pad of soft pseudo-cartilage at the anterior end of the basal plate. Here it makes a sudden downward and external sigmoid twist over the outer surface of the above pad, to fuse with the external bar of the anterior segment of the basal plate, as described elsewhere. Hence the lateral labials and basal plate form a cartilaginous circle round the mouth that is only broken for a short distance in the mid-dorsal line.

The *cartilage of the second tentacle* (2), morphologically the first, is connected with the first by ligament in *Bdellostoma*, according to J. MÜLLER. On entering the body it passes at first straight backwards but soon takes a sharp bend inwards and downwards, underneath the nasal tube, to fuse with its fellow of the opposite side, and in that way to form the median *subnasal cartilage* or bar (*sn. b.*) passing backwards in the middle line underneath the nasal tube (fig. 2). In *Bdellostoma* the conditions are apparently somewhat different, the cartilage passing gradually into a transverse bar placed at right angles to the anterior extremity of the subnasal bar, and which AYERS and JACKSON call the *transverse "labial" cartilage*. J. MÜLLER figures the transverse labial in *Bdellostoma* as suturally distinct from the subnasal bar, and in his description he says it is "strongly connected" with the latter bar. AYERS and JACKSON confirm this, and state that the "transverse labial cartilage is attached to the anterior end of the subnasal cartilage." PARKER also figures it as distinct from the subnasal. NEUMAYER's figure of *Myxine* certainly allows a transverse labial to be delimited, and I find his figure to a certain extent confirmed by the sections of the 6·5 cm. Hag, and also by the sections of a very small *Myxine* kindly presented to me many years ago by Mr J. T. CUNNINGHAM.* According to NEUMAYER's figure, and the figure and description of POLLARD,† there is, as I find also, no break between the skeleton of the tentacles and the subnasal bar, and it is difficult to escape the conclusion that this must likewise apply to *Bdellostoma*, in spite of the concensus of opinion above. The subnasal bar being thus formed by the fusion of the cartilages of the second pair of tentacles, is composed at first of soft cartilage. It soon flattens out so as to become narrow from side to side and deep from above downwards (*cp.* figs. 1 and 2). It lies a short distance below the nasal tube,

* I have no actual record of the size of this specimen, but it would be about 10 cm. I have already referred to it as the 10 cm. Hag.

† *Anat. Anz.*, ix., p. 351; and *Zool. Jahrb.*, Anat. Abt., viii.

which it supports, and is soon gradually converted into typical hard cartilage. Posteriorly it becomes first of all more rounded, and then opposite the ninth nasal ring it consists again of soft cartilage, of which the remainder is formed. It now flattens out from side to side and becomes narrow from above downwards, and its expanded posterior free extremity lies underneath the junction of the nasal tube with the nasal capsule—that is, between the latter and the palatine commissure, with which commissure it is connected by stout ligaments. It may even project slightly behind the commissure, as figured by PARKER.

The *cartilage of the fourth tentacle* (4) is quite independent; but, according to J. MÜLLER in *Bdellostoma*, it is connected by ligament with the basal plate, and according to AYERS and JACKSON with the base of the third tentacle also. It consists of a slightly curved and somewhat vertical rod of soft cartilage, situated *entirely* in the curiously shaped tentacle, and rather tilted towards the middle line. To this part is fused, at somewhere about the middle of its length, a stout rod which passes within the contour of the body outwards, upwards, and backwards on the surface of the muscles, where it terminates. The latter internal rod consists largely of *hard* cartilage. According to all published accounts, the shape of this tentacular cartilage of *Myxine* is different from the corresponding one in *Bdellostoma*, where it forms an irregular plate. I have, however, seen indications of a similar shape in some specimens of *Myxine*.

PARKER went seriously wrong on the tentacular skeleton of *Myxine*. He figures all the tentacular cartilages as *independent*, and altogether missed the lateral labials. It is difficult to understand how so wonderfully skilled a dissector as PARKER could have made these mistakes, *especially as the cartilages are actually shown in his sections* (which, however, he entirely misinterprets), and as they are by no means difficult to dissect. His description of *Bdellostoma* is much happier, although not quite correct, and he is also inaccurate in figuring and describing the subnasal bar of *Myxine* as consisting *entirely* of hard cartilage. NEUMAYER figures and describes a fusion between the posterior end of the subnasal bar and the anterior end of the hypophysial plate, but I find no traces whatever of this either in the 6·5 cm. or in the 10 cm. Hag. Apart from this, his description of the tentacular apparatus, as far as it goes, agrees exactly with mine. P. FÜRBRINGER inaccurately describes the lateral labial of *Myxine* as a connective tissue connection, but his figure and description of the tentacular apparatus (the skeleton not directly concerning him) is clearly inspired by J. MÜLLER's. HOWES' account of *Bdellostoma* is also wrong on practically all points, as pointed out by POLLARD, whose description of *Myxine* (*op cit.*) was the first to exhibit any degree of accuracy.

H. THE BASAL PLATE. (Fig. 10. Also figs. 1 and 2.)

The base of the cranial skeleton is formed by the stout "beam" underneath the gut, called by J. MÜLLER the "tongue bone" and referred to by HOWES as the

"dominant monster of the Hag." It may be regarded as consisting of a linear series of three pieces—the anterior, middle, and posterior segments of the basal plate (*b. p.*$^{1-3}$). Looked at from the side (fig. 1) the posterior segment is horizontal, whilst the two anterior segments constitute a dorsally inclined plane.

The *anterior segment* is the most complex. It consists of three pieces—a median, the *internal bar* of the anterior segment (*i. b. p.'*), and two lateral, the *external bars* of the segment (*e. b. p.'*). The two latter, and to a slight extent the former, bear in front pads of soft pseudo-cartilage (uncoloured and obliquely striated in the figures) which contain nodules of true soft cartilage, especially near the dorsal border. The lateral labial cartilage (*l. l. c.*), composed of soft cartilage, courses downwards in a sigmoid twist over the dorso-external face of the large outer pad to merge gradually and without any break into the hard cartilage of the external bar, which extends a short distance in front of the internal bar. The anterior border of the latter, covered with a layer of smooth soft pseudo-cartilage, forms a concave pulley surface for the tendon of the M. copulo-glossus profundus (= the protractor of the dental plate—AYERS and JACKSON), which tendon is shown cut across in fig. 10 (*c. g. p.*). This tendon passes forwards from its muscle *under* the internal bar, doubles round the pulley border on to its dorsal surface, and then courses backwards *over* the bar, to be inserted into the anterior arch of the dental plate. The dorsal surface of the internal bar also bears a thick pad of soft pseudo-cartilage anteriorly, against which the dental apparatus glides backwards and forwards. Just in front of the posterior fenestra this pad thins down and splits into a paired structure, which assists in forming the angular depression in which the V-shaped dental skeleton works. The internal bar is composed entirely of hard cartilage except where it passes into the middle segment behind, and consists of one piece only; for whilst it thins down somewhat in the middle line, there is no break or even a change in the character of the cartilage. Posteriorly, there is a large elongated fenestra closed by fibrous tissue, and we find a zone of soft cartilage interposed between the internal bar and the hard cartilage of the middle segment. But here, again, it is a gradual transformation, and there is no break or suture between the two segments. On the other hand, the external bar is quite independent of the internal bar and of the middle segment, being separated from the latter by a pad of soft pseudo-cartilage (uncoloured and obliquely striated in figs. 10 and 1). A transverse section through the anterior segment of the basal plate shows that the three pieces form a deep crescentic trough which lodges the dental apparatus. The internal bar is slightly curved, with the concavity dorsal; and the external bar, separated from the internal by a fibrous packing, projects upwards and outwards (*cp.* fig. 1).

The *middle segment* (*b. p.*^{2}) is comprised of *one* piece of hard cartilage, but in the middle line it is very thin and there consists of soft cartilage (fig. 10). There is, however, certainly no break or suture. In transverse section the cartilage is seen to be very thick laterally and to form a shallow cup, with the concavity dorsal. In order to provide a groove for the keel of the dental skeleton, and behind for the retractor tendon,

it is seen that the narrow trench in which these parts play is formed by the posterior continuation of the paired rails described above, which are here much deeper vertically and are raised up more perpendicularly from the basal plate. In this way a deep narrow trough arises—the sides represented by the paired rails and the floor by the basal plate. Posteriorly, as above mentioned, this trough transmits the tendon of the M. longitudinalis linguæ, which is thus kept in the middle line. Here the rails become almost entirely fibrous, and finally pass over into the fibrous roof of the canal formed by the posterior segment of the basal plate, as described and figured for *Bdellostoma* by J. Müller. The rails therefore represent the bifurcated fibrous roof of the posterior segment of the basal plate continued forwards over the dorsal surface of the middle and anterior segments, as is evident from their relations and histology. The appearance of soft pseudo-cartilage in them thus corresponds to the existence of the same tissue in the tendons of some of the muscles. The postero-external angles of the middle segment receive the ventral extremity of the first branchial arch, consisting of soft cartilage, which, however, gradually mixes with the hard cartilage of the basal plate without a break. When the detached lower division of the second branchial arch is present (fig. 1), it fuses with the first arch shortly before the latter reaches the basal plate, as described for *Bdellostoma* by Ayers and Jackson.

The *posterior segment* (*b. p.*[3]) is about half again as long as the other two segments together, and is immovably attached to the middle segment. Even here, at the junction-place of two skeletal tissues of different character, it is not possible to establish a joint. The posterior border of the middle segment is ragged, and bears small outgrowths of *soft* cartilage. The matrix of the posterior segment is very closely connected with this border, also contains nests of soft cartilage, and even in parts seems to be in direct organic connection with the middle segment. The posterior segment is formed of a thick sheet of hard pseudo-cartilage bent up longitudinally at the edges so as to form in transverse section the figure of a U, and roofed over dorsally in front by fibrous tissue, as above described, and behind by the anterior extremity of the M. copulo-copularis. Its cavity transmits the tendon of the M. longitudinalis linguæ. Posteriorly, the segment narrows down in the vertical plane; the sides of the U first of all diverge and then disappear, and in this way leave only the solid keel, which tapers down to a point and vanishes.

In *Bdellostoma*, according to J. Müller, the anterior segment of the basal plate consists of *four* pieces, the internal bar being divided in the middle line, and the two halves connected by ligament. Further, it is separated by a *movable joint* from the middle segment, the latter segment in its turn consisting of *two* pieces meeting at a median suture. This description is confirmed by Ayers and Jackson, except that the inner bars of the anterior segment are stated to be fused in the middle line anteriorly, where, further, they are divided by a *transverse* suture. Again, according to Ayers' and Jackson's figure and apparently their description, the lateral labial cartilages are not fused to the external bars of the anterior segment but are only "attached"

to them.* Finally, there are indications of a division of the middle segment into *four* pieces. PARKER also describes the two anterior segments of *Bdellostoma* as consisting of six pieces "*all connected together by tracts of soft cartilage*," but he also quite erroneously makes the same statement with regard to *Myxine*, although it is contradicted by his own sections, which he misinterprets. PARKER entirely missed the lateral labials in *Myxine*, and failed to observe their correct relation to the anterior segment of the basal plate in *Bdellostoma*. P. FÜRBRINGER also describes six pieces in the first two segments of *Myxine*; but his paper is not directly concerned with the skeleton, and his mind was evidently prejudiced by J. MÜLLER'S work. NEUMAYER'S description and text-figure of *Myxine*, as far as they go, agree exactly with mine. In this connection I felt it important to examine very carefully the condition of the basal plate in Dr BEARD'S sections of a 6·5 cm. Hag, and found that it agreed *absolutely* with the condition described above for the adult—*i.e.* the anterior segment consisted of three pieces and the middle segment of one, nor was there any difference in the connections between these parts and in the distribution of the hard and soft cartilage. We can therefore only conclude either that the basal plate of *Myxine* differs in some important respects from that of *Bdellostoma*, or that the latter has still to be accurately described. I am convinced from my dissections, and from the examination of three series of sections (apart from those of Dr BEARD'S small *Myxine*), that the basal plate of *Myxine* is here correctly described for the first time.

I. SKELETON OF THE DENTAL APPARATUS. (Figs. 7, 8, 9, and 1.)

This portion of the skeleton has been picturesquely described by PARKER as a "curious *apron* with slits in it and short strings projecting from it." The teeth are laid on a somewhat complex skeletal framework, bent up from the middle line at an angle so as to form a V-shaped figure in transverse section. Its natural position—or, rather, one of them—in longitudinal space is seen in fig. 1 (*a. d. p.*, *p. d. p.*). The whole apparatus, however, slides backwards and forwards in the trough formed by the basal plate, as already described, being drawn forwards by a *protractor muscle*, the copulo-glossus profundus (*c. g. p.*), and withdrawn by a *retractor muscle*, the longitudinalis linguæ (*l. l.*), the tendons of which muscles only are shown in the figures. The teeth as a body thus move backwards and forwards, so that the Hag *rasps* its way into the body of its victim; and not only this, but the whole apparatus with the teeth can be actually protruded entirely out of the mouth. This was first observed by GUNNER, who described and figured the everted teeth in 1766. His statements were stoutly controverted by J. MÜLLER, who, on purely anatomical grounds, asserted that GUNNER'S work was "entirely inaccurate." I can, however, with other observers, fully confirm this old writer, for I have repeatedly observed the living Hag protrude its teeth in the way described, and have succeeded in preserving several with the teeth out. By this

* This agrees with J. MÜLLER'S figure and description. Nevertheless I doubt it.

very curious device the rasping action is, of course, made immensely effective, and a few Hags will completely clear out a full-sized cod in an hour and a half. I shall give a full description of the process in my second part on the muscles, but in the meantime one may point the moral that to make physiological deductions from anatomical data is attended with some risk.

The dental skeleton is composed entirely of soft cartilage, except the greater portion of the posterior arch, which consists of hard cartilage. It commences in front as an irregular deposit of soft cartilage in the tendon of the M. copulo-glossus profundus,* and is at first almost flat, but slightly arched (with the concavity dorsal), taking no part in supporting the first tooth of the outer row, which is raised up almost at right angles to it. The cartilage soon widens out, and despatches a process forwards on each side to support the anterior teeth of the outer row (*o. r. t.*). This is the *anterior arch of the dental plate* (*a. d. p.*), and it at once assumes the characteristic shape of an *obtuse* V, the teeth resting on the inner surfaces of the two arms. The tendon of the M. copulo-glossus profundus (*c. g. p.*) widens out very considerably behind, to be inserted into practically the whole of the anterior border of the anterior arch of the dental plate (fig. 7), the posterior wide portion of the tendon consisting of a tissue very similar to soft pseudo-cartilage. The anterior "fine comb of horny spikes" described and figured by PARKER in *Myxine*, but not in *Bdellostoma*, is nothing more than this tendon cut across (*cp.* figs. 7 and 10). The arch never extends laterally beyond the bases of the outer row of teeth, and hence its lateral surface has a curve similar to that of the fused bases of this row of teeth. Both rows of teeth with their papillæ may be said to rather *rest* on the arch, since they are only loosely attached to it. The dental skeleton is, in fact, always more or less completely separated from the teeth, as emphasised by J. MÜLLER, by a series of spaces sometimes containing blood. I have not yet investigated the vascular system of the head, and therefore cannot say whether these spaces are blood sinuses or not. The anterior arch bears two fenestræ, and as I find these both in dissection and in serial sections, they must be of constant occurrence. The larger one is elongated from before backwards so as to almost divide the anterior arch into two, and is covered over by fibrous tissue, while the smaller one transmits the nervus dentalis of J. MÜLLER. The postero-external angle of the arch gives off two rods—the *posterior external* (*a. d. p.'*) and the *posterior internal* (*a. d. p.''*) *processes* of the anterior arch. The former is plate-like and much the larger of the two, crossing over the latter dorsally to it, and courses inwards and upwards in the fold of mucous membrane situated over and almost obscuring the posterior teeth of the inner row (*i. r. t.*), to terminate blindly in this fold immediately dorsal to the last tooth of the inner row. The internal process is a small rod which passes almost straight backwards below the external border of the inner row of teeth, and finally turns inwards to fuse with the posterior arch of the dental plate. The product of this fusion then continues backwards as a stout rod (turning slightly upwards) along the ventro-lateral border of

* This is the "median dorsal bar' of AYERS and JACKSON. It is, of course, ventral.

the pharynx, where it soon terminates. As shown in fig. 8, the external boundary of the fusion rod may be somewhat irregular and perforated.

The posterior arch of the dental plate (*p. d. p.*) is not concerned with the support of the teeth, but serves for the attachment of the tendon of the M. longitudinalis linguæ, which fans out as it approaches the arch so as to be inserted into practically the whole of its posterior border. The actual appearance of the posterior arch is not shown in any of the figures, for it must be remembered that all the figures represent the dental skeleton flattened out. The floor of the mouth sends down a longitudinal gully-like evagination or keel, into which, doubtless, the food drops after being liberated from the teeth in order to be passed backwards into the œsophagus. This evagination is very compressed from side to side, in the empty mouth, and lies entirely below the level of the teeth. In front, its base rests on the anterior arch, which, however, does not support its sides (except for a very abbreviated space posteriorly), since the arch, which is not bent, has to make too wide a deflection in order to take up a position external to the bases of the teeth. Hence the *obtuse* V. The posterior arch, having no connection with the teeth, is not deflected away from the diverticulum, but is sharply bent up on each side of it from the middle line so as to form in transverse section an *acute* V-shaped figure. The dorsal extremities of the arms of the V are continued backwards and outwards as spherical rods just internal to the inner row of teeth, to fuse with the internal process of the anterior arch as above described. In the sections, the posterior arch was cleft by a deep fissure behind, but there were no further indications of a separation of the arch in the middle line into two halves. The arch consists mostly of hard cartilage, but there is a median block of soft cartilage, and more soft cartilage where it fuses with the anterior arch. The tendon of the M. longitudinalis linguæ, as it fans out to be inserted into the arch, exhibits the same soft pseudo-cartilage-like appearance in its median portion as in the tendon of the protractor muscle.

The large space between the anterior and posterior arches is entirely filled in by ligamentous tissue containing the same soft pseudo-cartilage-like tissue. It is, in fact, the direct continuation of the tendon of the M. longitudinalis linguæ, as described by J. Müller; and its presence is obviously necessary, or the pull of the tendon would break the slender *cartilaginous* connections between the two arches.

I am not describing the teeth here, since, as J. Müller first pointed out, they do not belong to the skeleton, but only rest on the skeletal parts of the "tongue." They will be described with the skin in my third part. In the meantime I may point out that according to my dissections, and also a reconstruction of the teeth from serial sections, there are nine apparent teeth in the outer row (*o. r. t.*) and ten * in the inner row (*i. r. t.*). *Cp.* fig. 9. In both rows, however, the first two teeth are fused at the base, and for this and other more important reasons *each pair corresponds morphologically to one tooth only.* Hence the above numbers should be reduced by one in each row. The last tooth in each row is liable in some specimens to be overlooked,

* Parker gives for *Myxine* 7 and 9, and J. Müller 8 and 8–9, which illustrates my point below.

owing to a covering fold of mucous membrane, and this explains some of the so-called variations in the number of teeth referred to by systematists. As pointed out by AYERS and JACKSON, it is quite a simple matter to obtain excellent microtome sections of the teeth, *provided they are embedded in celloidin.* And even with paraffin embedding I have serial sectioned three heads of moderately sized Hags (about 25 cm.) without in any way damaging the teeth or losing a section.

J. MÜLLER's description of the dental skeleton of *Bdellostoma* calls for no comment, except that he does not figure or mention the small perforation transmitting the dental nerve, but refers to it in his later work on the nerves. It is not described by AYERS and JACKSON or by PARKER in *Bdellostoma*, although the latter author figures it in *Myxine*. PARKER also describes and figures in *Myxine* a long median rod of soft cartilage projecting backwards from the posterior arch, but not found in *Bdellostoma*, and in the latter type he describes the anterior arch as formed largely of *hard* cartilage. NEUMAYER figures and describes the two arches in *Myxine* as *unfused* and only joined up by connective tissue. The slender cartilaginous connections of the adult may well be secondary.

J. SKELETON OF THE VELUM OR PHARYNGEAL VALVE.

(Fig. 16. Also figs. 2 and 1.)

The skeleton of the velum, of which J. MÜLLER says nothing similar is known in the animal kingdom, commences, as the *external lateral velar bar* (*e. l. b.*), by a club-shaped ventrally scooped out head (fig. 16) of *hard* cartilage at the posterior end of the third fenestra of the skull. This is connected near its extremity by a short bridge of *soft* cartilage (*e. l. b.'*) arising from the inner or ventral edge of the bar, which fuses near the ventral border of the fenestra with the junction of the inferior process of the pterygo-quadrate with the hyoid arch. This bridge is not present in *Bdellostoma*, according to AYERS and JACKSON, but the head of the external bar is connected by *ligament* only with the pharyngeal wall. J. MÜLLER's description of *Bdellostoma* more nearly approaches the *Myxine* condition, there being precisely the same connection, but formed, however, partly by a cartilaginous articular tubercle and partly by ligament. In *Myxine*, PARKER says that the bar is "joined to the general thickness of cartilage in the hind part of the oval fenestra," but does not state the nature of the junction; whilst in *Bdellostoma* he says that it is "confluent with the hyomandibular." His figures give no assistance on this point.

At first the external bar lies above and external to the pharynx and anterior to the base of the velum, but it soon assumes a position internal to an anteriorly directed blind diverticulum of the gut which surrounds it on all sides except internally. This diverticulum then fuses below the bar with the naso-palatine duct or hypophysial canal, and above the bar it approximates very closely to the same canal but does not fuse with it. As a result the bar is surrounded on all sides, except for a very narrow

breach, by a double mucous enclosure. At about section 700 in the chart (fig. 16) the hard cartilage becomes gradually replaced by soft cartilage, of which the whole of the remainder of the velar skeleton is formed. Subsequently the fused diverticulum and hypophysial canal join up with the wall of the pharynx in such a way that the ventral wall of the canal portion and the inner wall of the diverticular portion first fuse with the dorso-lateral wall of the pharynx, and then the product of this fusion disappears. In this way the connection between the cavities of the three structures is established. It follows, therefore, that the velum is a double structure, each half supported by the external lateral velar bar and each formed as an evagination of the dorso-lateral wall of the pharynx. These two halves are connected by a dorsal median double partition, called by J. MÜLLER the "suspensory ligament" of the velum; and a transverse section of this region would suggest (of course, wrongly) that the velum had been formed by a median dorsal *invagination* of the roof of the pharynx, and that this invagination had then sent out on each side into the cavity of the pharynx a lateral extension.

The external bar now gives off the *internal lateral velar bar* (*i. l. b.*), which courses laterally in the velum internal and ventral to the external bar. PARKER figures and describes the internal bar as *independent* of the external in *Myxine*; and although this condition is sometimes suggested, as on the right side of fig. 2, I have always found it *fused* with the external bar, and as such PARKER figures and describes it in *Bdellostoma*. The two internal bars subsequently become connected by a transverse bridge, the *anterior transverse velar bar* (*a. t. v. b.*), which traverses the now enlarged isthmus connecting the lateral wings of the velum. From this transverse bar three processes arise, as follows: (1) a pair of narrow rods which pass forwards and gradually ascend dorsally in the median partition until they reach the roof of the pharynx, where they form a portion of the *suprapharyngeal skeleton* (*sp. sk.'*, fig. 16) suspending the velum from the dorsal pharyngeal wall. Here each rod gives off externally a long blind process, which passes outwards and backwards over the roof of the pharynx, the main stem being then continued almost straight forwards, but slightly outwards, below the notochord and over the pharynx; (2) a narrow rod which arises from the dorso-posterior surface of the bridge, and passes forwards and upwards in the median partition until it reaches the roof of the pharynx, where it constitutes the remainder of the suprapharyngeal skeleton (*sp. sk.''*). It at once bends sharply backwards on itself, and extends posteriorly as a wide plate in the middle line under the chorda and over the pharynx. Behind the suprapharyngeal skeleton the position of the longitudinal bars is reversed, the external bar having now crossed over above the internal so as to occupy the more median position. In *Bdellostoma*, according to the figures of J. MÜLLER, PARKER, and AYERS and JACKSON, there is no such crossing; but I suspect the drawings of these authors must be diagrammatic in this respect, or the velum must be widely different in *Bdellostoma*. The two internal bars are finally connected up a second time by the *posterior transverse velar bar* (*p. t. v. b.*), which

lies near the ventral surface of the now much folded velum, and which has a very ragged or "mossy" posterior border. Behind this border there may be several detached nodules of cartilage. The posterior portions of the external bars are placed asymmetrically somewhat near the middle line, under the dorsal surface of the velum, and continue so for the remainder of their course. The internal bars are prolonged beyond the posterior transverse bar, and curve first outwards and then upwards and inwards under the surface of the velum, until they reach the dorsal surface, where they terminate over the extremity of the external bars.

The skeleton of the velum is subject to some variation, especially as regards the suprapharyngeal skeleton and the irregular processes from the posterior border of the posterior transverse bar. In one specimen there were three distinct rods arising from the latter bar—a median shorter one and two longer symmetrical ones. In this connection, compare figs. 2 and 16. In *Bdellostoma*, according to J. MÜLLER, PARKER, and AYERS and JACKSON, there is only one process from this bar—a median one bifid behind.

Myxine agrees with AYERS' and JACKSON'S account of *Bdellostoma* except in one striking respect. The suprapharyngeal plate or cartilage of these authors is clearly the expanded posterior plate of the rod *sp. sk.''*, but the anterior rods *sp. sk.'* do not end blindly but are fused to the antero-external angles of the above plate. Without wishing to cast doubt on this description, which may be based on a variation or on a different species, it must be mentioned that the descriptions of the suprapharyngeal skeleton of *Bdellostoma* given by J. MÜLLER and W. K. PARKER agree essentially with mine of *Myxine*—except that PARKER missed the plate *sp. sk.''* in *Myxine*, and describes it in *Bdellostoma* as a thin rod bifid behind.

K. SKELETON OF THE CLUB-SHAPED MUSCLE.

This consists of two bars, placed one above the other, at the posterior extremity of the muscular complex known as the club-shaped muscle. I describe them now as found in a 35 cm. Hag.

Inferior Chondroidal Bar.—This is composed of perfectly typical hard cartilage throughout, with the matrix very strong superficially in older specimens. The posterior portion of its length (8 mm.) comes to the surface of the M. perpendicularis (P. FÜRBRINGER) at the mid-ventral line. This portion lies *behind* the posterior extremity of the M. copulo-copularis (P. F.; M. constrictor musculi mandibuli, AYERS and JACKSON), and gives origin to the fibres of the perpendicularis. In front of this region the bar disappears into the copulo-copularis for about 3 mm., the posterior fibres of the latter muscle being inserted into it. It was a slightly asymmetrical laterally compressed rod pointed behind and blunt in front, 11 mm. long and 1 mm. deep. It has no connection with the fibres of the M. longitudinalis linguæ (P. F.; M. retractor mandibuli, A. and J.).

Superior Chondroidal Bar.—Consists of hard pseudo-cartilage exactly as in the posterior segment of the basal plate, *but may contain here and there small nodules of*

true (*soft*) *cartilage*. The bar comes to the surface of the M. longitudinalis linguæ in the mid-dorsal line, and extends in front between the posteriorly diverging halves of the M. copulo-copularis, with which, however, it is not connected. The whole edge of the bar and the lateral portions of its ventral surface provide an origin for some of the dorsal fibres of the longitudinalis linguæ, whereas the median portion of the ventral surface gives attachment to the M. perpendicularis. Behind the latter muscle the bar sends downwards and backwards in the mid-longitudinal vertical plane a thin tough sheet, the two sides of which give origin to a considerable bulk of the fibres of the longitudinalis linguæ, and which separates the two halves of this muscle in the middle line posteriorly. In front of this they are separated by the perpendicularis. This tough sheet in the specimen dissected consisted of hard pseudo-cartilage like the remainder of the bar, but in a series of sections of a 25 cm. Hag it was composed of connective tissue. It thus seems to chondrify as the animal grows. The superior chondroidal bar is a broad dorso-ventrally compressed plate rounded behind and pointed in front, 11 mm. long and 5 mm. at its widest part.

In *Bdellostoma*, according to AYERS and JACKSON, there is a difference of some importance. *Both* bars consist of the hard pseudo-cartilage, similar to that of the posterior segment of the basal plate, and, to quote AYERS and JACKSON, "are not, therefore, to be regarded as skeletal derivatives of the visceral or branchial arches, but simply as chondroidal modifications (*i.e.* condensations of connective tissue) in the muscular fascia" (*op. cit.*, p. 212). Now we have seen that the inferior chondroidal bar of *Myxine* is composed entirely of typical hard cartilage, and further, that the superior bar may contain nodules of true soft cartilage, as in the so-called branchial arches. We must therefore conclude either that these bars in *Myxine* and *Bdellostoma* are not homologous, which is surely incredible; or that the histology of the skeleton, as we have the best reason for supposing, is but an equivocal morphological guide. There are, however, strong grounds for believing that AYERS' and JACKSON's description of the inferior bar is inaccurate; for J. MÜLLER's description of the two bars in *Bdellostoma* agrees exactly with mine in *Myxine*, and W. K. PARKER, who seems to have missed the superior bar altogether, describes the inferior bar both of *Myxine* and *Bdellostoma* as formed of "hard cartilage" and colours it green in his figures.

L. THE BRANCHIAL SKELETON. (Figs. 11–15.)

This was first described for *Myxine* by BURNE,* having previously been missed by J. MÜLLER (who first found it in *Bdellostoma*, but whose language is ambiguous as regards *Myxine*) and W. K. PARKER. SCHREINER† notes its presence in *Myxine*; and it is, in fact, quite easily seen by any one accustomed to careful dissection. On the other hand, BURNE failed to find the so-called "gill bars" of AYERS and JACKSON in *Bdellostoma*, and I am able to supplement his description of *Myxine* in several important respects.

* *P. Z. S.*, 1892, p. 706. † *Bergens Museum Aarbog*, 1898, No. 1, p. 6.

The branchial skeleton is situated entirely posteriorly in relation to the fused gill ducts, and on the left side to the ductus œsophago-cutaneus also. To take the latter side first (fig. 11), we find in dissections that about half the skeletal apparatus in its fully developed form is connected with the fused gill ducts, and the other half with the ductus œsophago-cutaneus and the œsophagus itself. The latter portion lies on the external lateral wall of the œsophagus and the ductus, and passes generally downwards and backwards. Above, it has two processes—a dorsal posterior one (x^1), which passes under the constrictor muscle on to the roof of the œsophagus; and a ventral anterior one (x^2), which passes straight on to the ventral surface of the œsophagus. Below, it has one process (x^3)—a posterior one which passes backwards towards the caudal wall of the ductus—and a vertical rod (x^4), which connects it with the second portion of the apparatus. The latter commences with a horizontal rod (y^1), which passes forwards in a curve over the branchial cloaca, bears a blunt process in front (y^2), and then suddenly dips down, bends round under the ventral border of the first gill duct, to terminate in a lanceolate plate (y^3), which turns upwards and forwards and lies on the inner surface of the first two efferent gill ducts. The plate in the specimen figured was pierced by two fenestræ.

We may now turn to fig. 13, which represents this apparatus as reconstructed from serial sections, and we note at once that it consists of two perfectly distinct parts. The dorsal one commences ventrally on the anterior external wall of the ductus, and below, it sends inwards a hook-shaped process underneath the ductus (x^4). At about section 3070 (*cp.* the chart) the process x^1 has reached the œsophagus, on the outer wall of which it lies for the rest of its course. The process x^3 passes backwards and downwards over the external surface of the ductus, and terminates a short distance beyond it. The ventral part begins anteriorly by a perforated plate (y^3), which lies immediately *internal*, and is related, to the first four efferent gill ducts (including the fifth posteriorly), instead of the first two, as in fig. 11. The knob y^2 is represented in the sections by a long blind process extending forwards for some distance *external* to the first four efferent gill ducts. The bar y^1 bends round under the ventral wall of the branchial cloaca to fuse with y^3; whilst above, it passes backwards external to the branchial cloaca, exhibiting a longitudinal fenestra, and behind ends blindly before the branchial cloaca fuses with the ductus.

A comparison of these two figures renders it abundantly clear that the branchial skeleton of the left side is a complex of *at least* two parts (see below), the point of junction in fig. 11 being where x^4 meets y^1. In fig. 13, therefore, the greater portion of x^4 has been lost, in this way separating the two sections—one of which clearly belongs to the efferent gill ducts, and the other to the ductus œsophago-cutaneus. This view of the branchial skeleton is borne out by its wide range of variation, and I have dissected two specimens in which *each half of the apparatus was respectively missing.* One of these variations is represented in fig. 12, which obviously represents the ductus portion of the skeleton, and corresponds exactly with the dorsal portion of fig. 13, except that

in the latter the extension x^2 is wanting. BURNE and AYERS and JACKSON also mention that it varies in *Bdellostoma*.

On the right side (fig. 14) the branchial skeleton also varies considerably, but I figure and now describe its most highly differentiated condition as I have found it. The apparatus is at once simpler and more complex on this side—simpler in so far as the portion related to the ductus œsophago-cutaneus is necessarily wanting, as there is no ductus on this side; and more complex in as much as a complete ring is formed round the branchial cloaca. That this ring is a secondary formation is indicated by the fact that in one specimen dissected it was incomplete; but there were present an anterior and a posterior process from the perforated plate which did not quite meet external to the branchial cloaca to form a perfect ring (*cp.* below and fig. 15). The posterior process of fig. 14, which passes upwards separate from and posterior to the last efferent gill duct, represents y^1 of fig. 11, whilst the bar lying external to the branchial cloaca is clearly y^2 of figs. 11 and 13. The fenestrated plate y^3 on the inner surface of the first two efferent gill ducts will naturally correspond with the same structure on the other side, but instead of having two large perforations it had four small ones.

If we now turn to the reconstruction from the serial sections (fig. 15), we observe that the branchial skeleton here consists of two separate pieces—one external and the other internal to the branchial cloaca, as well as two small detached cartilages (z^1, z^2). This condition must not be confused with the imperfect ring just described, where two processes from the perforated plate embraced the branchial cloaca externally. These two processes are doubtless similar to the two in fig. 15 seen underneath y^2, and which seem to represent an attempt to complete the circle—not by extending round on to the external surface of the branchial cloaca, as above described, but by fusing with y^2. It therefore seems as if the circle may be formed in more ways than one. In fig. 15, y^2 and y^3 are related rather to the branchial cloaca than to individual gill ducts,* except that the last efferent gill duct passes between y^1 and the posterior extension of y^3. The two latter processes serve to support the ventrally directed portion of the branchial cloaca just above its external opening. The separate cartilage z^1 curls round under the ventral edge of the branchial cloaca as if to fuse with y^2, but it does not do so. It is evidently a detached portion of the backward blunt process given off from the perforated plate in front. The posterior downward process from the same plate is separated from y^2 by the ventral extension of the branchial cloaca. The detached cartilage z^2 is situated behind and above the external branchial opening, and, perhaps, represents the extremity of the line of chondral deposit forming y^1.

A comparison of figs. 13 and 15 shows at once that the corresponding parts of the two sides are essentially similar, and further, that the efferent gill duct portion of the branchial skeleton may itself be a complex formed of *at least* two pieces; and hence on

* It is, perhaps, rather a refinement of description to associate any part of the branchial skeleton with individual gill ducts, since, apart from the ductus œsophago-cutaneus portion, its function on both sides and in all cases is to strengthen the wall of the branchial cloaca, and it is possible that *all* the gill ducts have contributed to it.

the left side there are perhaps no less than three parts in the apparatus. The only important difference between the two sides in the reconstructed figures is that the connection between y^1 and y^3 is wanting on the right side; but we have seen that the ring condition is subject to variation. The two posterior rod-like extensions of the perforated plate on the left side are represented by only one on the right—but this is quite a minor point; whilst we have already seen that the condition illustrated in fig. 14 may be easily deduced from that shown in fig. 15.

It now becomes a matter of interest to enquire how this skeleton may have been built up. AYERS and JACKSON suggest that on the left side it is a complex of the skeleton of the *last* efferent gill duct, together with that of the ductus œsophago-cutaneus. This may, indeed, be true for *Bdellostoma*; but in *Myxine* it would seem as if *all* the gill ducts had a share in it, and this may explain why the apparatus is the more complex in *Myxine*. In any case, we may question whether the perforated plates of *Myxine* are represented in *Bdellostoma* at all, and, as *Myxine* is obviously the more specialised form as regards its gills, it would follow that this portion of the branchial skeleton is a neomorph of no special significance as regards the gill cartilages discovered in *Bdellostoma* by AYERS and JACKSON. It is, however, conceivable that, by a concentration of the latter cartilages due to the confluence of the gill ducts, the branchial skeleton of *Myxine* may owe its existence. The bearing which these conclusions have on the branchial skeleton of cyclostomes generally is not without interest, for it would seem that every cyclostome must be considered on its merits; and we cannot, for example, say that *Myxine* is intermediate between the lamprey and *Bdellostoma*.

The cartilage of the branchial skeleton is histologically the feeblest of the true cartilages, even if it can be called such, in the whole body, being distinctly weaker than that of the tentacles and somewhat weaker than that of the caudal fin. The cartilage cells are relatively large, and are embedded in a rather delicate network which seems in places to be continuous and in others to consist of capsules around the cells, each of them independent. There is only a slight deposit of cement—provided the above network does not, as I think, represent that substance. In this connection, compare the cartilage of the caudal fin. The branchial cartilage is, in fact, one of the numerous transition connective tissues of *Myxine*, and this is indicated by its staining reactions, since it colours neither a distinct blue nor red, but an indefinite colour suggesting both these dyes. It may, however, be regarded as an extreme variety of the soft cartilage.

M. SKELETON OF THE CAUDAL FIN. (Fig. 17.)

Of the so-called "fins" of *Myxine* the adipose dorsal fin has no skeletal support, or, at the best, but a very few short detached rods extending only a very short distance beyond the contour of the back muscles ("*fin*" 1). This passes without a break into the caudal fin ("*fin*" 2), which possesses an elaborate skeletal framework and which passes round the extremity of the tail and then forwards as far as the cloaca. In living

material the fin rays and the pulsating caudal heart are plainly visible. In front of the cloaca there is an adipose median pre-anal fin, with no skeleton ("*fin*" 3).

It is very surprising that although J. MÜLLER dissected the tail of *Bdellostoma*, he should have failed to recognise its skeletal features. In a brief reference * he describes the median cartilages as fibrous vertical sheets—that is, merely local thickenings of the fibrous septum separating the two halves of the body musculature. SCHNEIDER, in 1879, for the first time briefly describes the caudal skeleton of *Myxine* and *Bdellostoma*; and CLELAND,† unaware of SCHNEIDER'S work, subsequently published two descriptions of *Myxine*. These are, however, very inaccurate. The first detailed account of the tail of a myxinoid was published by G. RETZIUS,‡ whose work I can confirm except in some minor points.

Myxine possesses, as far as external features go, a diphycercal caudal fin. When, however, a dissection of the tail is made it is seen that the notochord (*nt.*) near its extremity takes a slight downward turn, and that its tip is buried in a median vertical sheet of cartilage (*m. d. b.*, *m. v. b.*) which in front splits up dorso-ventrally so as to extend forwards under the notochord, over the neural tube, and also slightly between these two structures. This cartilage is unevenly distributed above and below the notochord, passing further forwards above and being deeper below. To it are fused, above and below, the posterior *fin rays* (*f. r.*), which are, of course, not comparable to the true fin rays of the higher fishes, the whole apparatus being characteristic of the myxinoids. J. MÜLLER and AYERS and JACKSON describe an imperfect segmentation of the fin rays of *Bdellostoma* as in the bony fishes, but I have seen no traces of this in *Myxine*. Further, there is no indication of a segmental arrangement of the fin rays, except anteriorly. The cartilage partly encloses the bulbous extremity of the neural canal (containing the enlarged termination of the spinal cord), since the latter extends further backwards than the notochord. It may be conveniently divided into the following two portions :—

Median Dorsal Bar (*m. d. b.*).—This is narrower than the median ventral bar, but extends further forwards. The posterior portion of it is attached in the middle line by an expanded base to the roof of the neural tube; but the anterior half is lifted up above the neural tube, and is there merely a thin bridge of cartilage connecting up the bases of the fin rays. There were forty eight fin rays connected with this cartilage in the specimen figured. RETZIUS gives about thirty for *Myxine*, and AYERS and JACKSON twenty-five to forty for *Bdellostoma*.

In front of the median dorsal bar there are about forty-five free fin rays not connected with any longitudinal cartilage, but inserted by their bases into the septum between the two halves of the body musculature and connected by fibrous tissue. In *Bdellostoma*, according to AYERS and JACKSON, the anterior dorsal fin rays have such expanded bases as to almost complete the dorsal bar in front. There are apparently

* *Op. cit.*, p. 91. † *Fourth Ann. Rep. Fish. Bd., Scotland*, 1885. Also *Rep. Brit. Assoc.*, 1885.
‡ For the complete paper, see *Biol. Unters.*, vii., 1895, p. 26.

many more fin rays in *Myxine* than in *Bdellostoma*—as, for example, in the specimen figured there were altogether 147 fin rays in the caudal fin as against a maximum of ninety-four in *Bdellostoma*, according to AYERS and JACKSON.

Median Ventral Bar (*m. v. b.*).—Commences as a short thin rod closely attached to the mid-ventral region of the notochord. In front, one or two nests of cartilage cells may occur in the same position (fig. 17). This rod soon widens out, by a concave forward sweep, into a wide plate (with an expanded chordal base) to which the ventral fin rays are fused. In *Bdellostoma*, according to AYERS and JACKSON, the bar arises *far forwards* as a *pair* of long slender rods which fuse behind in the mid-ventral line. This is, therefore, very different from *Myxine*. The bar sends forwards a prominent projection with a rounded extremity which gives origin to the pair of wide, somewhat diffuse, muscle sheets of *small* fibres which pass one on each side of the caudal hearts externally, as described by G. RETZIUS, and which are responsible for the pulsation of these organs. In one specimen there was a perforation in this bar near the anterior end which I have since failed to find either by dissection or in serial sections, and which transmitted an anastomosis between the caudal hearts. As the median ventral bar passes backwards, and the notochord is tapering down, its base becomes more expanded and begins to creep up at the sides of the chorda. At the same time, a small rod of cartilage is deposited dorsally on each side in the angle formed by the base of the fibrous neural tube (*sp. c.*) and the roof of the notochord, one or more nests of cartilage cells being found at short intervals in front of these rods (two are shown in fig. 17). The rods and the median ventral bar then suddenly fuse, so that the chorda is now completely invested with cartilage, except dorsally. Shortly afterwards this compound rises up and fuses with the median dorsal bar, *thus forming a complete cartilaginous neural tube* except that its floor is formed by the roof of the chorda. The latter itself becomes here gradually invaded by cartilage cells and is soon almost entirely merged into the median ventral bar, with the result that the cartilaginous neural tube is finally completed at its only lacuna—the base. However, nests of notochordal tissue occur at intervals embedded in the cartilage behind this region, thus indicating that the chorda extended further back than appears in the adult. Immediately behind the termination of the chorda a large fenestra arises on each side of the tube formed by the fused median dorsal and ventral bars, and the spinal canal is thus exposed laterally. The latter, however, does not entirely fill this fenestra, even at its expanded termination lodging the curious dilated filum terminale,* as shown in the figure. The ventral edge of the bar bears thirty-three fused fin rays, of which only one was bifurcated in the specimen figured, as against the five bifurcating rays attached to the dorsal bar. G. RETZIUS gives about thirty, but does not figure any bifurcating rays although such are mentioned in the text. In *Bdellostoma*, according to AYERS and JACKSON, there are only about twenty fused ventral rays, of which nearly all are bifurcated, and the median ventral bar extends forwards under the notochord as far as the cloaca without fusing with the

* In *Bdellostoma*, according to AYERS and JACKSON, the spinal cord is *not* dilated.

anterior ventral fin rays. According to BASHFORD DEAN, the fin rays arise as *unbranched* structures in *Bdellostoma*.

In front of the ventral bar there were in the specimen figured twenty-one free fin rays, all situated behind the cloaca, which extend a short distance within the contour of the body between the slime sacks, and none of which were bifurcated. AYERS and JACKSON figure fourteen, with all but one bifurcated.

I may mention here that in *Bdellostoma* AYERS and JACKSON describe "a very thin irregular sheet of cartilage in the wall of the cloaca, especially in the anal region," which they believe possibly "serves to expand the anal opening in anal respiration" (p. 217). I have failed to find this cartilage in *Myxine* either by dissection or in serial sections.

The minute structure of the cartilage of the fin rays and dorsal and ventral bars indicates that it is an extreme variety of soft cartilage, but not so primitive as that of the branchial skeleton. Its staining reactions resemble those of the latter cartilage, but there is a tendency in the direction of differentiation. It consists of large cells, each surrounded by a distinct deeply staining zone that I take to represent the cell capsule, embedded in a practically continuous reticulum of what is clearly cement. The intercellular substance or cement is, however, distinctly weaker than in typical soft cartilage, but it is, on the other hand, more continuous or reticular than in the hard cartilage, thus accentuating the difference between typical hard and soft cartilage already referred to.

April 26, 1905.

N. EXPLANATION OF THE PLATES.

REFERENCE LETTERS.

a. d. p. Anterior arch of the dental plate.
a. d. p.′ Posterior external } processes of above.
a. d. p.″ Posterior internal }
a. t. b. Anterior transverse bar of the nasal capsule.
a. t. v. b. Anterior transverse velar bar.
au. c. Auditory capsule.
au. f. Auditory foramen.
b. p.$^{1-3}$ Anterior, middle, and posterior segments of the basal plate.
br. a.^{1} First "branchial" arch.
br. a.^{2} Second "branchial" arch. In fig. 1, consisting of separated upper and lower divisions.
br. ap.′ Left } branchial apertures.
br. ap.″ Right }
c. c. Cornual cartilage.
c. ct. c. Capsule of the cartilage cell.
c. g. p. Tendon of the M. copulo-glossus profundus (P. FÜRBRINGER) cut across (= the tendon of the protractor muscle of the dental plate, AYERS and JACKSON).
ch. c. Chordal cells forming the notochordal "jelly."
ch. ep. Chordal epithelium.
cl. Cloaca.
cr. Membranous cranium.
cs. Centosomes of the cartilage cell.
c. sb. Cement substance of the hard cartilage.
ct. c. Cartilage cell.
d. œs. ct. Ductus œsophago-cutaneus.
d. t. Median dorsal tooth.
e. b. p.′ External bar of the anterior segment of the basal plate.
e. l. b. External lateral velar bar.
e. l. b.′ Short rod of *soft* cartilage connecting above with the posterior extremity of the inferior process of the "pterygo-quadrate" (*cp.* figs. 1 and 2).
el. ext. Elastica externa of the notochordal sheath.
f.$^{1-4}$ The four fenestræ of the skull.
"*fin*"$^{1-3}$ Dorsal, caudal, and the adipose pre-anal "fins."

Mr. F. J. Cole on the Morphology of Myxine.—— Part I, Plate I.

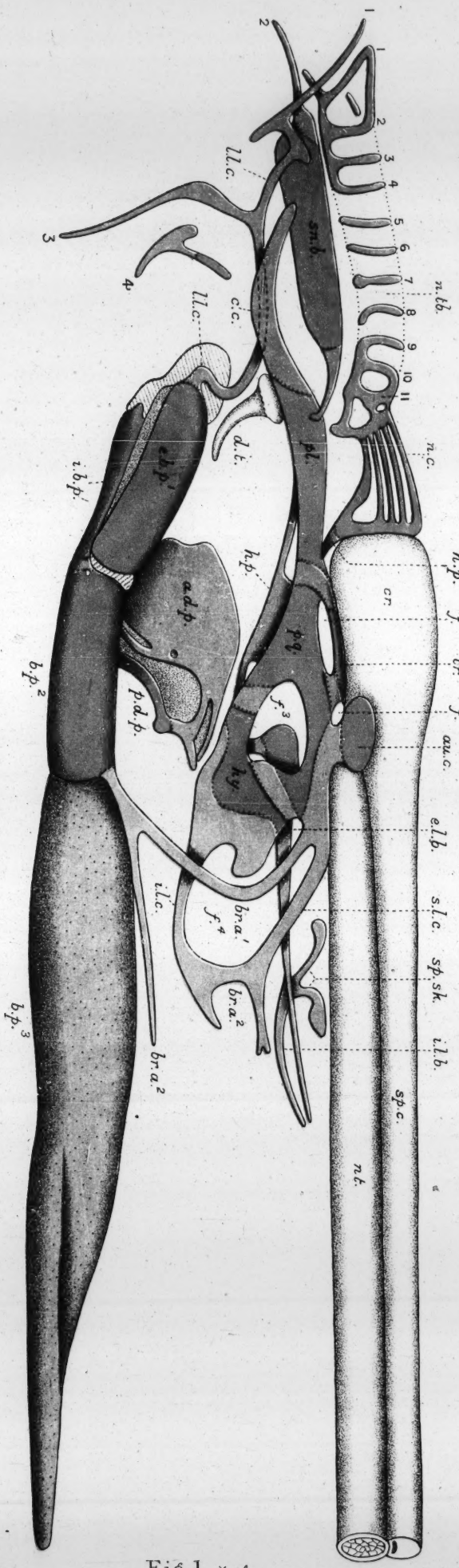

Fig. 1 × 4

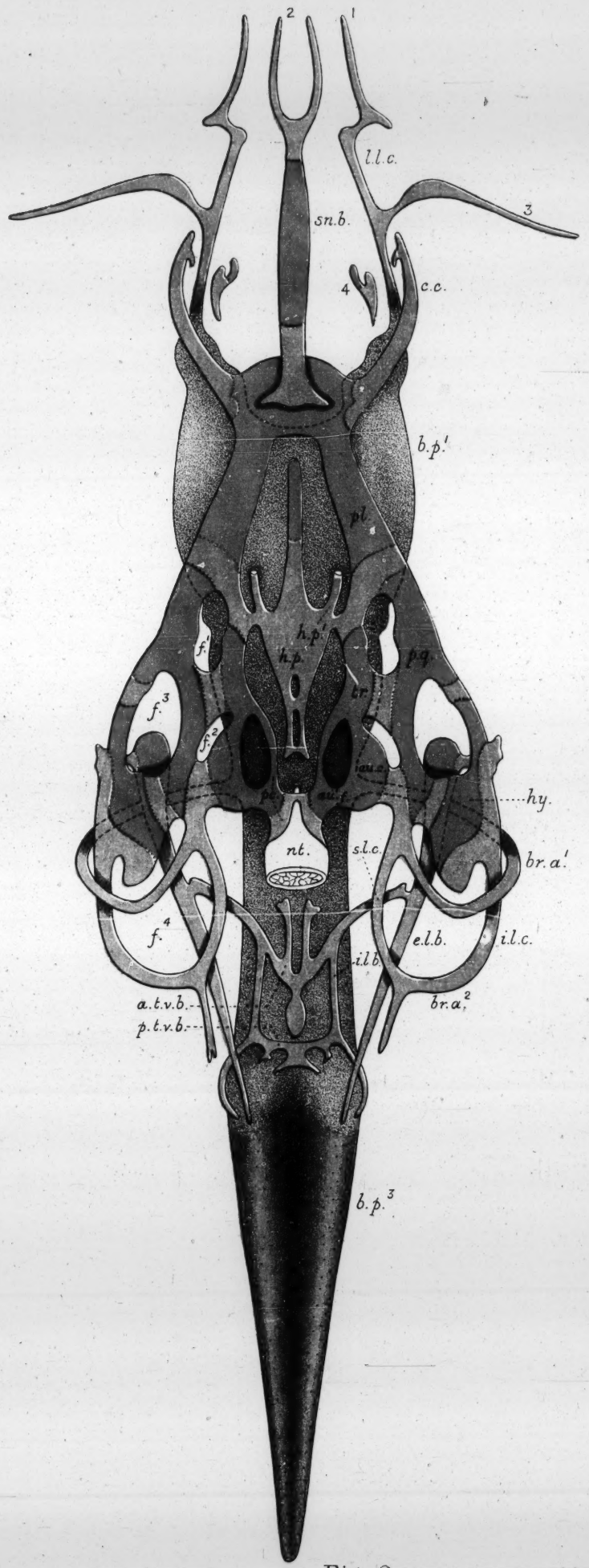

Fig. 2 × 4

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Mr F. J. Cole on the Morphology of Myxine.— Part I, Plate II.

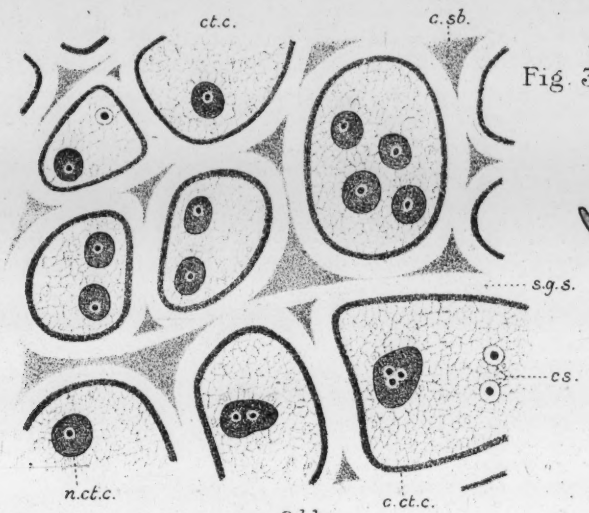


Fig. 3.

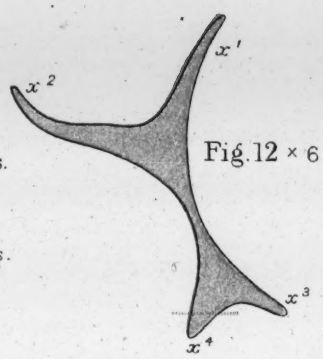


Fig. 12 × 6

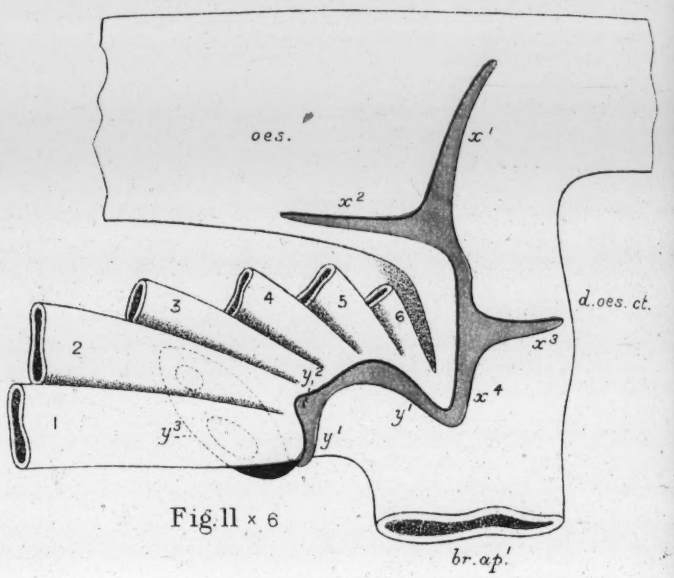


Fig. 11 × 6

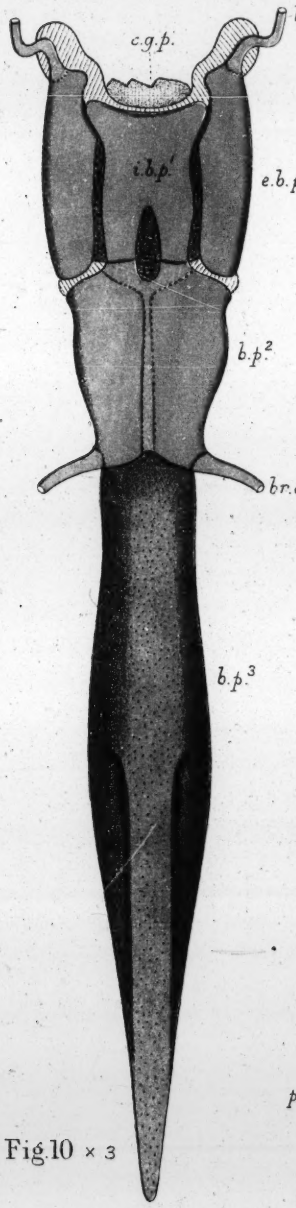


Fig. 10 × 3

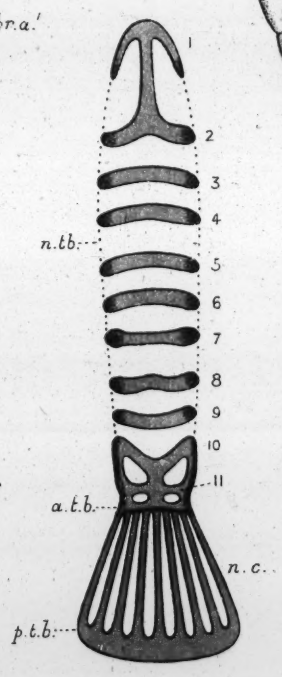


Fig. 5 × 4

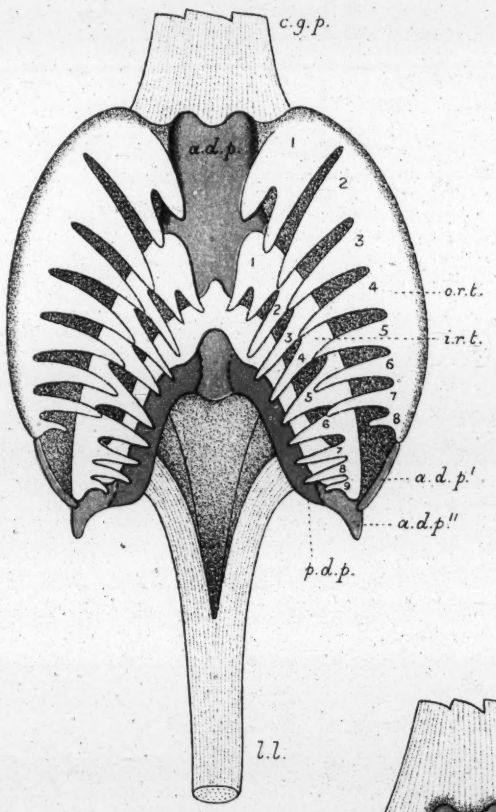


Fig. 9 × 4½

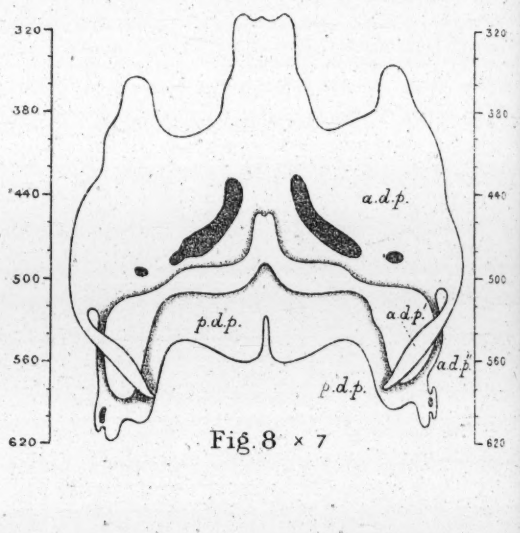


Fig. 8 × 7

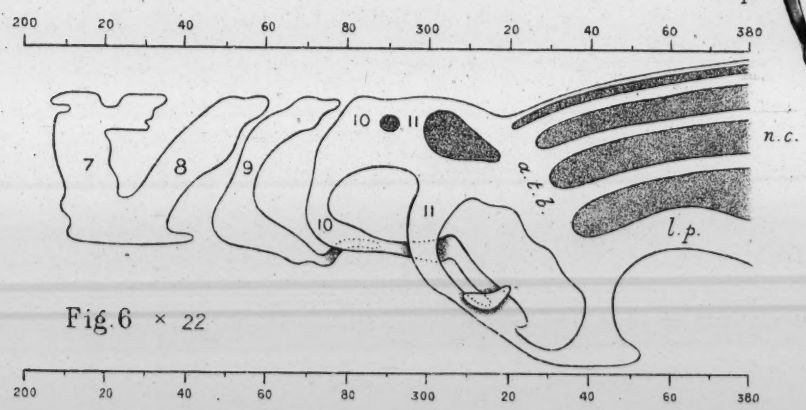


Fig. 6 × 22

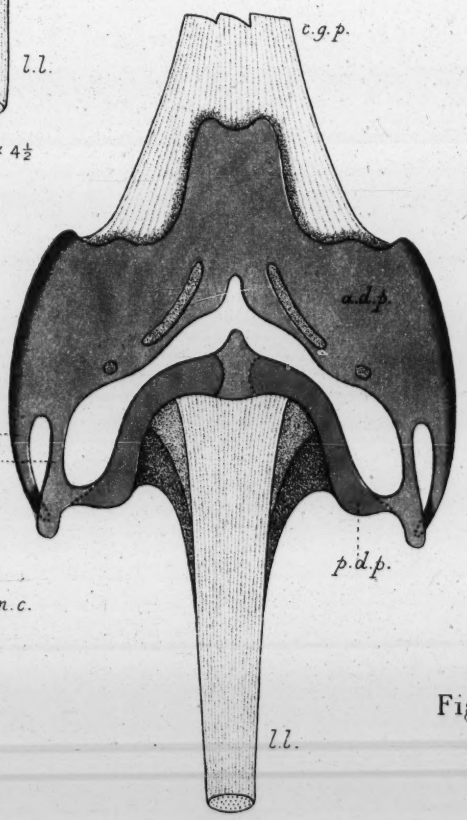


Fig. 7 × 4½

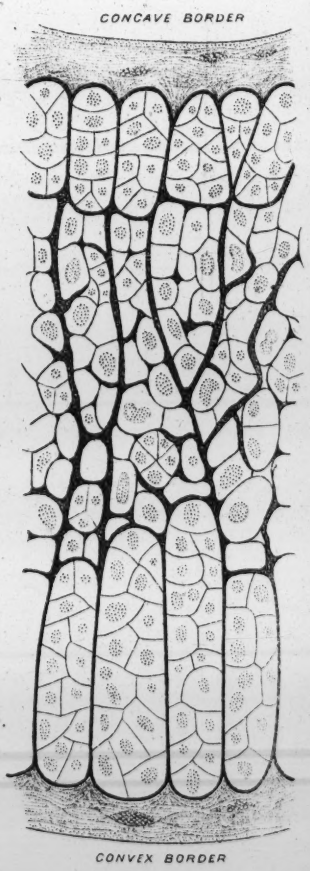


Fig. 4.

f. r. The so-called "fin rays" composed of soft cartilage (although not coloured).
h. p. Hypophysial plate.
h. p.' Rod connecting above with the posterior transverse bar of the nasal capsule (figs. 1 and 2).
hy. "Hyoid" arch.
i. b. p.' Internal bar of the anterior segment of the basal plate.
i. l. b. Internal lateral velar bar.
i. l. c. Inferior lateral cartilage.
i. r. t. Inner row of ventral teeth.
l. l. Tendon of the M. longitudinalis linguæ (P. FÜRBRINGER) cut across (= the tendon of the M. retractor mandibuli, AYERS and JACKSON).
l. l. c. Lateral labial cartilage.
l. p. Lateral plate of the nasal capsule.
m. d. b. Median dorsal bar of the caudal fin skeleton.
m. v. b. Median ventral bar of the caudal fin skeleton.
n. c. Nasal capsule.
n. ct. c. Nucleus of the cartilage cell.
nt. Notochord.
nt. sh.$^{1-3}$ External, middle, and internal layers of the fibrous notochordal sheath.
n. tb. Nasal tube.
œs. Œsophagus.
o. r. t. Outer row of ventral teeth.
pc. "Parachordal" cartilage.
p. d. p. Posterior arch of the dental plate.
pl. "Palatine" bar.
p. q. "Pterygo-quadrate."
p. t. b. Posterior transverse bar of the nasal capsule.
p. t. v. b. Posterior transverse velar bar.
s. g. s. Secondary ground substance of the hard cartilage.
sk. l. Skeletogenous layer of the notochordal sheath containing elastic fibrils externally.
s. l. c. Superior lateral cartilage.
sn. b. Subnasal bar.
sp. c. Membranous neural tube.
sp. cd. Spinal cord. Note the expanded termination.
sp. sk. Suprapharyngeal skeleton (fig. 1).
sp. sk.' Anterior, *sp. sk.''* Median } connecting processes of the suprapharyngeal skeleton. In *Bdellostoma*, according to AYERS and JACKSON, all three fuse with a median dorsal *suprapharyngeal plate* represented in *Myxine* by the expanded extremity of *sp. sk.''*
tr. "Trabecula."
x.$^{1-4}$, *y.*$^{1-3}$, *z.*$^{1-2}$ } The various portions of the branchial skeleton. *Cp.* text.

PLATE I.

Fig. 1. Dissection from the left side of the skull of a 45½ cm. Hag. × 4. Tentacular cartilages and the rings of the nasal tube numbered from before backwards. The colours and shading indicate the different kinds of cartilage, and also the staining reactions of the same with MANN's methyl-blue-eosin: hard cartilage, red; soft cartilage, blue; hard pseudo-cartilage, blue (dotted); soft pseudo-cartilage, uncoloured and obliquely striated.

Fig. 2. Dissection from the dorsal surface of the skull of a 45½ cm. Hag. × 4. Tentacular cartilages numbered from before backwards. The third tentacle, branchial arches, and suprapharyngeal + velar skeleton have been displayed for the sake of clearness. Nasal tube and capsule removed, and dental skeleton not shown. The correct relations of the parts have been maintained (*cp.* in this respect AYERS' and JACKSON's fig. 7). Colours, etc. as in fig. 1.

PLATE II.

Fig. 3. Transverse hand section of the "hard (brown) cartilage" taken from the middle segment of the basal plate of a 35 cm. Hag, and stained with MANN's methyl-blue-eosin. Zeiss apochr. 1·5 mm., compens. oc. 4.

Fig. 4. Portion of transverse hand section through the posterior segment of the basal plate of the same fish, to illustrate the structure of the "hard pseudo-cartilage." Methyl-blue-eosin. The upper border is the dorsal concave and the lower the ventral convex border of the cartilage. Same lens and eye-piece as fig. 3, but much less magnification.

Fig. 5. Dissection from the dorsal surface of the nasal skeleton of a 45½ cm. Hag. × 4. Nasal rings numbered from before backwards. As only those portions of the skeleton visible in a mid-dorsal view are shown, the figure should be compared with fig. 1.

Fig. 6. Reconstruction from serial sections of the skeleton of the posterior portion of the nasal tube, and of the anterior portion of the nasal capsule of a 25 cm. Hag as seen from the left side. The scale in this and subsequent figures refers to the enumeration of the sections—numbered consecutively from the anterior extremity backwards. Nasal rings numbered from before backwards. *Cp.* with figs. 1 and 5. × 22.

Fig. 7. Dissection from the ventral surface of the dental skeleton of a 45½ cm. Hag. × 4½. The parts have been displayed for the sake of clearness—*cp.* fig. 8 for the natural relations. Colours as in fig. 1.

Fig. 8. Reconstruction from serial sections of the dental skeleton (without the teeth) of a 25 cm. Hag as seen from the dorsal surface. The two halves of the dental plate in transverse section form a distinct **V**, but in the figure they are represented flattened out. Otherwise the fig. is not diagrammatic. × 7.

Fig. 9. Dissection from the dorsal surface of the teeth and dental skeleton of a 45½ cm. Hag. × 4½. The parts have not been displayed to such an extent as in fig. 7. Teeth numbered from before backwards. Colours as in fig. 1.

Fig. 10. Dissection from the ventral surface of the basal plate of a 45½ cm. Hag. × 3. Colours, etc. as in fig. 1.

Fig. 11. Dissection from the left side of the branchial + œsophageal duct skeleton of a 35½ cm. Hag. × 6. Efferent gill ducts numbered from before backwards. Cartilage blue.

Fig. 12. The corresponding cartilage, with a similar orientation, of a 35 cm. Hag, to show the absence of the ventro-anterior (efferent gill duct) portion of the preceding figure. × 6.

Plate III.

Fig. 13. Reconstruction from serial sections of a 25 cm. Hag of the branchial + œsophageal duct skeleton of the left side seen from the external surface. The orientation is exactly the same as in figs. 11 and 12. × 22.

Fig. 14. Dissection of the same 35½ cm. Hag of fig. 11 from the right side to show the branchial skeleton. × 6. Efferent gill ducts numbered from before backwards. Cartilage blue.

Fig. 15. Reconstruction from serial sections of a 25 cm. Hag of the branchial skeleton of the right side seen from the external surface. The orientation is exactly the same as in the preceding figure. × 22.

Fig. 16. Reconstruction from serial sections of the skeleton of the pharyngeal velum of a 25 cm. Hag as seen from the ventral surface. The natural relations of the parts have not been disturbed, and this figure should therefore be compared with fig. 2, where the velar skeleton is represented displayed. There was a series of detached nodules of cartilage in the region of the posterior transverse velar bar (*p. t. v. b.*) which have been omitted from the figure. × 15.

Fig. 17. Dissection from the left side of the skeleton of the "caudal fin" of a 31 cm. Hag. × 2. Where the "fin rays" are not fused with a longitudinal cartilage, *i.e.* as they are posteriorly, they are connected up by fibrous tissue (not shown in the figure). Spinal cord black. Slime sacks omitted.

Fig. 18. Thin transverse section of the notochord and its sheath of a 34 cm. Hag taken at about the middle of the body. Two of the chordal cells are shown as they appear in *thick* sections, in which the whole of one wall of a cell with the opposed nucleus may be seen. In two others, the nucleus is shown in section embedded in a thin layer of protoplasm. Zeiss apochr. 1·5 mm., compens. oc. 4.

Mr F J. Cole on the Morphology of Myxine.—— Part I, Plate III.

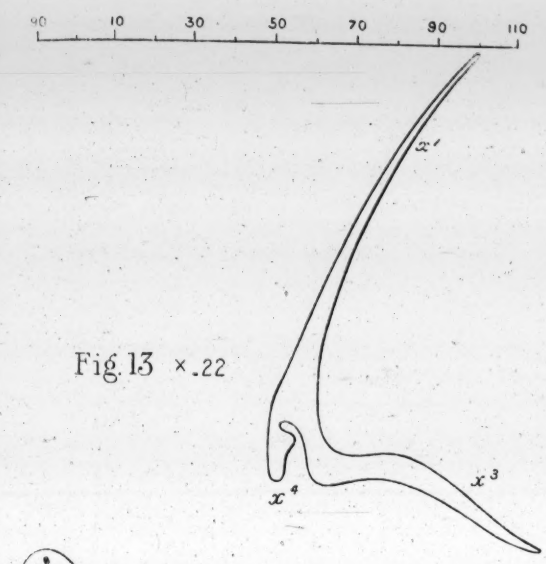
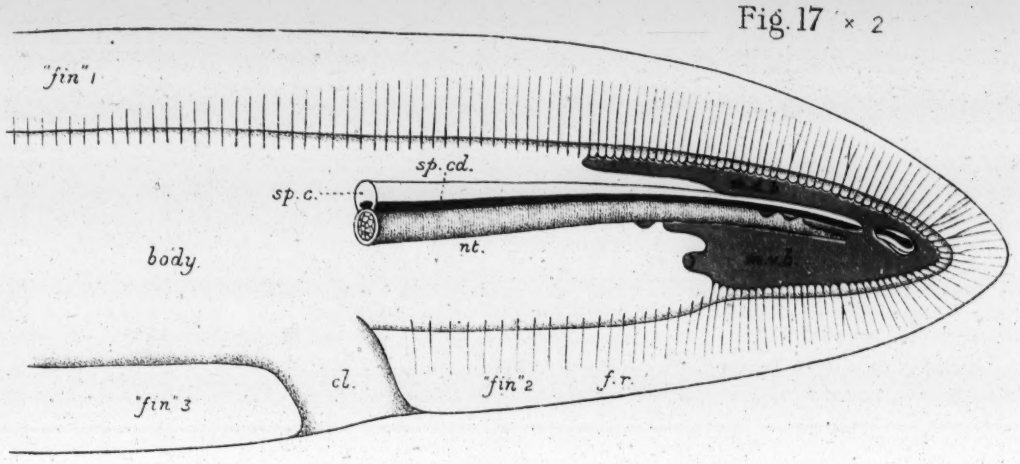
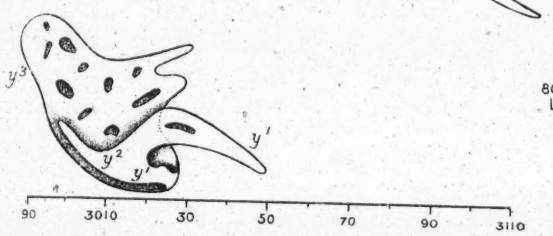


Fig. 13 × 22

Fig. 17 × 2





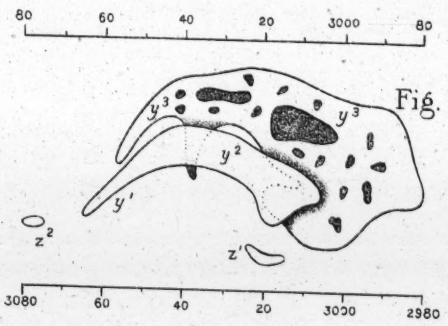


Fig. 15 × 22

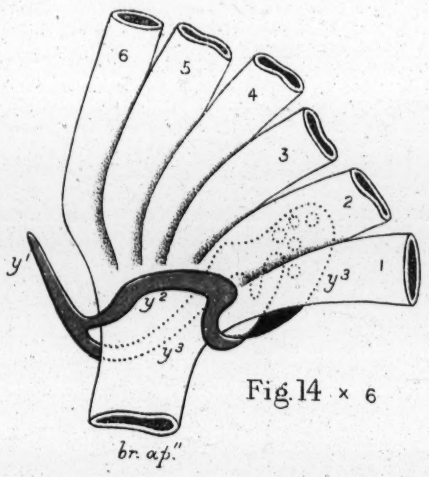


Fig. 14 × 6

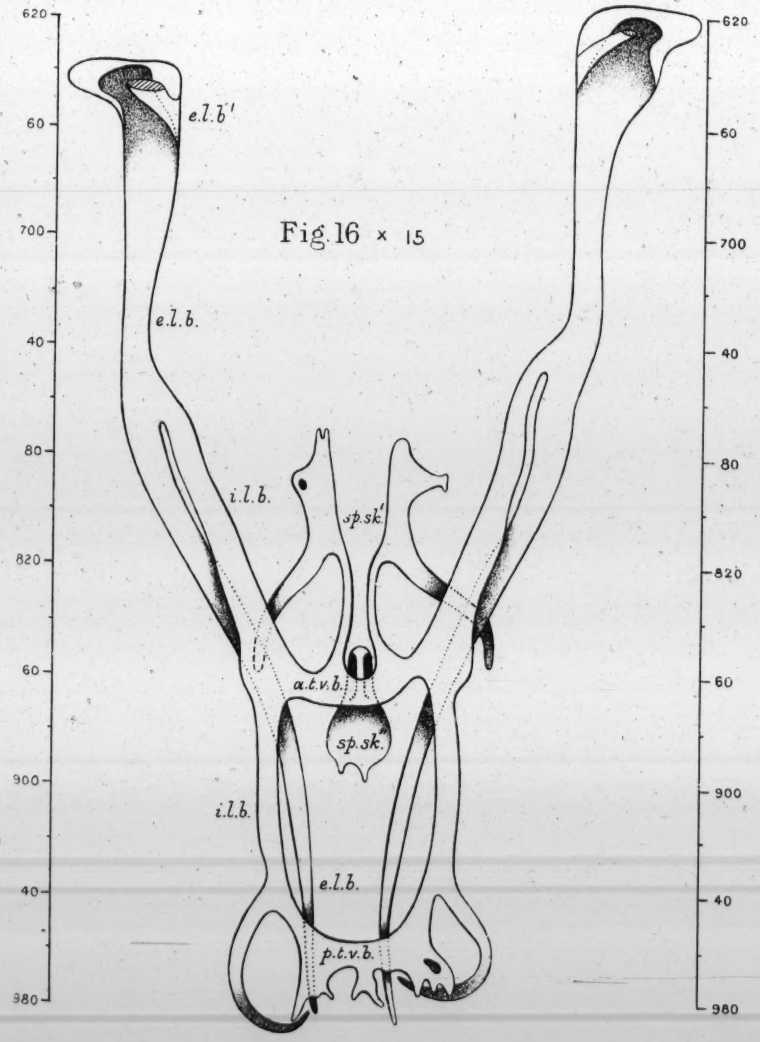


Fig. 16 × 15

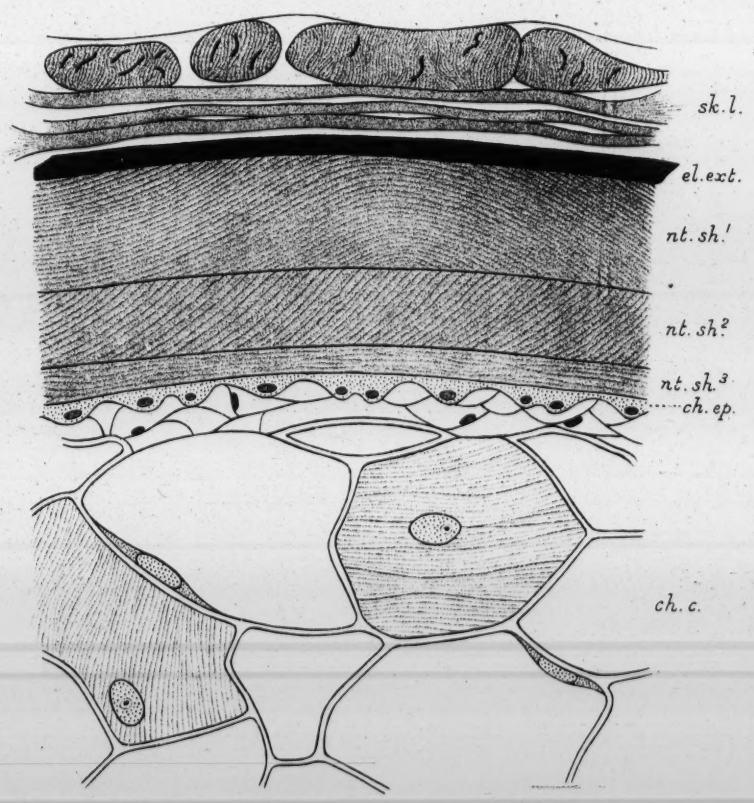


Fig. 18.

F. J. Cole. del.

M'Farlane & Erskine. Lith.rs Edin.r

XXXI.—The Life-History of *Xenopus lævis*, Daud.

By **Edward J. Bles**, B.A., B.Sc., Assistant in Zoology at the University of Glasgow. (With Four Plates.)*

(Read January 18, 1904. MS. received January 11, 1905. Issued separately November 8, 1905.)

Introduction.

The present communication is intended to be the first of a series dealing with observations on the life-history of the Anura Aglossa and their anatomy at different stages of development. *Xenopus lævis*, with its small ova and protracted larval free-swimming stages, must necessarily form a basis for the study of the development of that other remarkable Aglossan, *Pipa americana*. Although the adult Aglossan is an aberrant and specialised Anuran, there are Urodele features in the development of *Xenopus* which make its embryology of great general interest. These primitive features, combined with others peculiar to the genus, impress a character upon the early life-history of this frog which is widely divergent from that of the Phaneroglossa with small ova.

The fullest account of the development of *Xenopus* is contained in a short paper by Beddard, published in 1894. He has cited and reviewed the scanty earlier literature. Nothing has since been contributed to the subject but a note on the breeding habits by myself (1901). Beddard's observations were made on material obtained at the gardens of the Zoological Society of London. Specimens of *Xenopus lævis* from Zanzibar spawned there a few months after their arrival. The earliest stage observed was the larva shortly after hatching; some frogs were reared from the tadpoles. The most important new fact made known in the paper was the presence of a cement organ ("sucker"). Its structure was described. W. K. Parker's observations on the presence of external gills and the absence of so-called internal gills were confirmed. Some details of the internal structure were described. Figures drawn from fresh specimens are given of three tadpole stages, early and late.

Beddard confirms the absence of horny teeth already noted by Parker ('76) and Leslie ('90). But he did not connect this deficiency with the absolutely different method of feeding which must necessarily follow. The food of all our European tadpoles is obtained by the scraping action of the lips with their rows of horny teeth, sometimes, but rarely, aided by the biting horny jaws. The teeth act exactly like the radula of a gasteropod and are used to rasp away animal or vegetable matter from any substratum. Beddard found numbers of Cyprids and nothing else in the alimentary canal of the *Xenopus* tadpoles, and concluded

* Grateful acknowledgment is due to the Carnegie Trust for generously defraying the cost of reproducing the plates illustrating this paper.

that they were purely carnivorous and adopted this diet from choice, and states that "there was plenty of water-weed on which they could have fed." These statements it is impossible to bring into harmony with the observations recorded in this paper, and it is difficult to conceive, taking the facts as known to BEDDARD, how a tadpole without any buccal hard parts could feed on water-weed. The other observations which BEDDARD records I can confirm, with the exception of only one or two minor points regarding the cement organ.

It may be convenient to give here a brief summary of the main observations described and conclusions reached in this paper.

(1) The conditions are enumerated and discussed under which *Xenopus* and other Amphibia can be induced to breed freely in captivity (pp. 795 and 796).

(2) A detailed account is given of the breeding habits of *Xenopus* (pp. 797–798).

(3) The remarkable method of oviposition is described in some detail (p. 798).

(4) It is concluded that fertilisation is external in *Xenopus* (p. 799).

(5) The egg-envelopes are described and the occurrence of a rudimentary egg-shell noted (p. 800).

(6) The segmentation of the ovum and the development of the embryo within the egg are described and figured for the first time (pp. 801–806).

(7) The late embryo is shown markedly to resemble corresponding stages in Urodele development.

(8) The posterior ends of the medullary folds are found not to enclose the blastopore nor meet behind the anus (p. 803).

(9) The early development of the face is described and frontal views figured (pp. 806 and 811).

(10) The "frontal gland" and its secretion are shown to be functionally concerned in the hatching process (pp. 807–809), and the development of the gland is described (pp. 804, 805, and 806).

(11) It is shown not to be connected with the formation of the neuropore and, as it is not a sense-placode, lends no such support as v. KUPFFER claimed it did to his theory of monorhiny and amphirhiny (p. 809).

(12) The process of hatching in an Anuran is described for the first time (p. 807).

(13) An account is given of the habits of the tadpole after hatching and before it begins feeding (p. 810).

(14) The development of the pectoral lymph-hearts at this early stage is noted for the first time in an Amphibian (p. 812).

(15) The *Xenopus* tadpole is shown to breathe by its lungs as soon as it begins to feed (p. 812).

(16) The development of the cement organ is traced from the first appearance to its disappearance (pp. 803, 804, 805, 806, 811, and 813).

(17) A brief account is given of the feeding habits. The manner of taking food is seen to be remarkably similar to that of *Ammocœtes* (pp. 813–814).

(18) A few cases of branching tentacles in old tadpoles are figured, and it is argued from the frequent symmetry of the branching that there is a congenital tendency to branch and that the branching is not due to regeneration after injury. The branching, if the above view is correct, is evidence in support of the theory that the tentacles are external gills of the mandibular arch (pp. 814–816).

(19) A very curious difference between the behaviour of the dark chromatophores of the head and abdomen and those in the distal part of the fin-fold is described. The latter expand at night, while the former contract (p. 816).

(20) The arm is developed in a sac shut off from the gill-chamber. When it is protruded it is found that the action of the branchial current is not interfered with as in terrestrial Anura and feeding by the branchial current goes on as before (p. 817).

(21) The external features of the process of metamorphosis are described (p. 817).

(22) The young frog is found to feed on small Crustacea, chiefly *Daphnia*, like a young Urodele.

(23) One specimen, a male, was seen to become sexually mature when two years old.

In this paper very little is said about the internal anatomy of embryos and tadpoles. It is hoped that this omission will be made good later.

Methods of Preservation, Examination, etc.

It is not proposed to give full particulars here of all the methods used in this investigation. There are, however, one or two new devices which may be useful to others and are therefore worth recording.

The early stages of *Xenopus* (segmentation, gastrulation) are best preserved in a 4 per cent. formaldehyde solution, after stripping all the jelly from the vitelline membrane. The latter is so close-fitting that it cannot be removed from the living egg in these early stages. When the embryo elongates, the vitelline membrane swells up and can easily be removed. From this stage onwards the best preservative for general purposes is the one formulated below. Of course, for some special stains, special preserving fluids are indicated, such as corrosive sublimate for Heidenhain's iron-hæmatoxylin, and so on.

The following mixture was made as the result of experiments to discover a killing and fixing fluid with the advantages and without the faults of Perenyi's fluid. As in Perenyi's fluid, the basis is strong alcohol, but glacial acetic acid replaces the nitric acid, and formalin the chromic acid. The fluid has greater penetrating, fixing, and hardening power than Perenyi's, and it has the same great advantage of not making the yolk hard and brittle. The nuclear structures are far better preserved than they are by Perenyi's fluid, mitotic figures are often perfectly fixed, and the embryos and tadpoles of all ages are killed almost instantaneously if they are transferred to the preservative with a minimum quantity of water.

Mix 90 c.c. of 70 per cent. alcohol with 3 c.c. of glacial acetic acid. Any

quantity of this mixture can be made as a stock solution, as it keeps indefinitely. Just before use add 7 c.c. of formalin (40 per cent. formaldehyde solution) to each 93 c.c. of the above stock solution. This killing and preserving fluid contains :—

90	vols.	70 per cent. alcohol,
3	,,	glacial acetic acid, and
7	,,	formalin, in every
100	vols.	

The fluid cannot be used with confidence when more than a fortnight old for killing, but embryos and larvæ of Anura, if killed in a large quantity of the fresh fluid, may be left in it indefinitely for preservation. The same reason can be given for both these statements, viz. that the fluid after a fortnight has begun to decompose, which impairs its killing and hardening powers but not its preserving property. The formalin and acetic acid both disappear from the fluid sooner or later, as can easily be proved by the disappearance of their characteristic odours; these are replaced by an aromatic odour mixed with that of the spirit, and this mixture is as good a preservative as pure 70 per cent. alcohol. This property of purifying itself, as it were, makes the fluid particularly useful for recommending to collectors at a distance. Specimens can be killed in it and then either sent off in the same fluid or forwarded in a change of the fluid after twenty-four hours, according to the bulk of the specimen and the relative size of the bottle or jar. It will then travel any distance without further preparation.

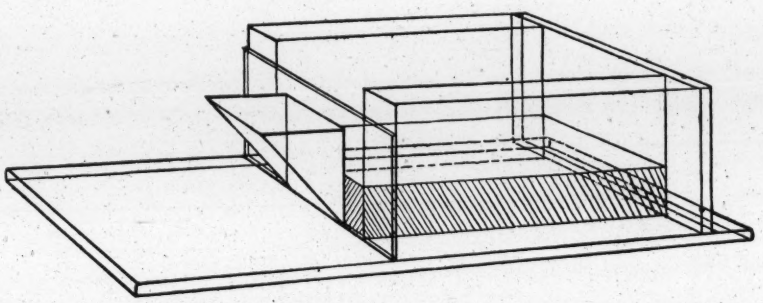
From the reports of friends and colleagues who have used this mixture and from my own experiments, I gather that it is useful for the most diversified objects, from the eggs and yolky larvæ of Echinoderms to the larvæ and adults of *Anopheles*, the newly hatched fry of *Salmo fario*, and a full-grown *Ammocœte*. It is, judging from these examples, worth a trial on almost any object, especially yolky embryos. Of course it decalcifies and is useful for preserving and decalcifying small Craniates or their heads when required for sections. The specimens should be transferred from this fluid to 50 per cent. alcohol, washed and passed into 70 per cent., and can then be treated as required.

A Simple Prism Reflector.—All students of Anuran tadpoles have sooner or later felt the want of a convenient method of examining these objects from all points of view without the risk of damaging the specimen. A simple and cheap means of carrying this out is to build a trough as shown in Text fig. 1. The base is an ordinary 3 inches by 1 inch or 3 inches by $1\frac{1}{2}$ inch glass slide. The sides of the trough are of plate glass; the right-hand end is a piece of ordinary thin glass; the end over the middle of the slide is a piece of No. 2 or No. 3 cover-glass.

The cementing can be done in a few minutes if marine glue is used; the objects can then only be examined in water or formalin, but if the trough is to contain spirit the cement used must be carefully applied and allowed to thoroughly harden before use. Bichromated gelatine, Lovett's cement, or some such spirit-proof cement can be used.

When an object is to be examined under the microscope, a thick piece of glass or several thin pieces are laid in the bottom of the trough, the appropriate fluid poured in, and the object then placed as close as possible to the cover-glass end of the trough, with the surface of the object so placed that by looking through the end of the trough horizontally the required view of the object can be obtained. But the same view can be seen by fixing a right-angled prism against the cover-glass and examining the reflection of the object thrown up vertically from the internal surface of the hypothenuse side, and the whole arrangement can be put on the stage of a vertical binocular microscope with the objective over the horizontal face of the prism and the reflected image enlarged by the use of low powers. With the arrangement shown in Text fig. 1 it is possible to use a Zeiss A objective. The prism is fixed to the cover-glass with a drop of cedarwood oil or castor oil or glycerine. The object is easily moved about into any required position while the eyes are at the microscope, as the lower end of the tube does not come in the way of the right hand.

The side views of eggs and the frontal views of larvæ figured on the plates were drawn with the help of this little appliance.



TEXT FIG. 1.—Glass Trough with prism for internal reflection. (Natural size.)

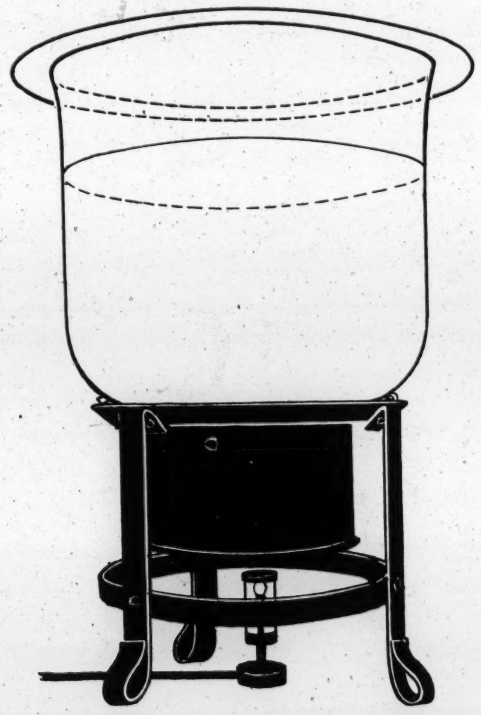
Before concluding these introductory remarks, I wish to acknowledge very gratefully the help of the artists who have so patiently and carefully carried out my wishes in making the illustrations for this paper. A member of the zoology class of 1903, Mr HORATIO MATTHEWS, kindly provided the drawings of figs. 12, 13, and 14 from the living embryos. The rest of the drawings (excepting four) are from the skilful brush of Mr A. K. MAXWELL, whose work I have controlled and confirmed throughout. Figs. 16, 17, 18 and 19 were sketched by myself from the larva as it hatched, and these sketches have been carefully elaborated, with the help of the identical specimen (killed five minutes after hatching), by Mr E. WILSON of Cambridge.

OBSERVATIONS ON BREEDING AND DEVELOPMENT.

Breeding Habits.—My embryological material has all been obtained from specimens of *Xenopus* which I have now kept in captivity for seven years, since December 1896. Spawn was first obtained in February 1899, when the frogs were in the Tropical Lily Tank of the Cambridge University Botanic Garden, and a note on the

observations then made has already been published (Bles, '01). Since then the same female has spawned during the spring and summer of 1901, 1902, and 1903, under conditions easily established anywhere. The methods adopted are possibly more or less applicable to the breeding of other Amphibia in confinement, and I will therefore enter into details.

In the first place, the most necessary condition of success in this and similar cases is that the frogs should be allowed to hibernate. But, in order to accomplish this successfully, a frog must be in the best health and condition when the winter sets in, and must have passed the summer in the best circumstances as regards heat, light, and food supply.



Text Fig. 2.—Tropical Aquarium described in text. The diameter of the bell-jar is 20 inches; other parts are in proportion.

Xenopus is practically a purely aquatic animal, probably more so than any caducibranchiate Urodele, and should be kept in an aquarium at a tropical heat during the summer in a place which is reached by the early morning or the evening sunshine. The direct rays from the sun will thus not strike the aquarium for more than an hour or so. The form of tropical aquarium I have found perfectly successful is one devised by my friend the late Mr J. S. Budgett, who kindly gave me permission to describe it here.

Text fig. 2 shows a bell-jar 20 inches in diameter standing on an iron tripod. The circular ring at the top of the tripod is slightly dished inwards to adapt it to the bottom of the bell-jar. Upon the upper surface of this ring rests the flange of a galvanised iron tank containing water, and heated below by a Zeiss micro-burner (Bunsen). This tank acts as a water-bath, and is kept 10°-15° C. hotter than the aquarium, according to the quantity of water in the latter. I find that this particular

burner is very convenient, as the temperature of the aquarium remains constant sometimes for weeks together, and a variation of a degree can be put right by slightly lowering or increasing the flame. Between the bottom of the aquarium and the flange of the water-bath I have a coil of three turns of asbestos cord, which keeps the glass away from the hot metal, and prevents evaporation from the water-bath.

In such aquaria my *Xenopus* have now lived for years, and in them they have spawned, and the tadpoles been reared, in one case to maturity.

The bottom of the aquarium is covered with earth and stones, and *Vallisneria* thrives in it. During the summer the *Xenopus* are kept at about 25° C., and the temperature may rise occasionally to 28°–30° C. They are fed daily with small earthworms or thin strips of raw calves' liver, and are fed until they refuse to eat more, which they do in a comical manner, by pushing aside the food with the palm of the hand when it is held near them. The water in the aquarium is never changed.

In December the temperature is allowed to sink to 15°–16° C. during the day, and it may sink to 5°–8° during the night. The frogs then become lethargic and torpid, take no food for days, eat very little at a time and move about rarely, spending very little time, at any rate during the day, at the surface of the water. I have on a few occasions approached the aquarium very quietly and found one or the other of the frogs lying flat on the stones with what appeared to be a translucent film dimming the brightness of its eye.

A sudden noise, however slight, makes the frog start up, and the film moves slowly outwards and forwards, uncovering the eye, and is recognised as the lower eyelid. The frog then moves away. Judging from analogy, one might conclude that the frog had been aroused out of its sleep. Whether this conclusion is justified or not, the facts seem worth recording, as the evidence of the occurrence of sleep in the lower vertebrates is somewhat slender.

When the temperature of the aquarium rises in the spring and the days become brighter, a change in the behaviour of the frogs becomes evident. The female and males spend a great part of the day at the surface of the water with eyes and nostrils above the surface. The males become exceedingly restless, swimming about with an air of wishing to escape from the aquarium. Both sexes are now very shy, and difficult to feed.

There may or may not already be attempts at pairing, but by taking the following measures pairing will take place immediately (with or without spawning), or at least the male, after being silent the whole winter, will commence to croak at once. First, the temperature of the aquarium is raised to 22° C. and, secondly, when it has become constant, a certain amount of the water, say 2 gallons, is drawn off morning and evening, allowed to cool for twelve hours, and then run in slowly in the following manner, in order to simulate the fall of rain. The cooling vessel is raised above the level of the aquarium, and a syphon is used to run off the water. The lower end of the syphon is drawn out to a fine point, and turned up in such a way that the water rises up like from a fountain, and falls as spray into the aquarium. The third condition

to be established is to bring the water into such a state that the larvæ will find their food when it is required. This condition will be explained later (pp. 813–14).

By carrying out such measures I obtained from one female between April and July 1903 more than fifteen thousand eggs. Of these, twelve thousand were taken out of the aquarium and counted, and the remainder were estimated at three to five thousand.

Some general significance may, I believe, be attached to the results in breeding this frog, the more so as they are in accordance with other similar results obtained by SEMPER (1878) with Axolotls and by myself with Axolotls, *Triton waltlii*, and *Discoglossus*.* SEMPER showed that by feeding them copiously, and by keeping Axolotls crowded together in small vessels, he could obtain spawn from the same individual three or even four times a year after a sudden transference to an aquarium stocked with growing plants, with stones on the bottom and supplied with running water. I have repeated his experiments and can fully confirm his statements. With *Discoglossus pictus* I have had a similar experience. Specimens kept in a small vivarium for four and six years have been given a superabundance of food during the summer, allowed to hibernate, and, when they showed signs of readiness to breed, a little tank in a corner of the vivarium has been filled with suitable pond water, and invariably within forty-eight hours the frogs have spawned. On two occasions the males have taken to the water in the spring and assumed their nuptial characters, but for several weeks the water which had stood in the tank during the winter has been allowed to stand. No pairing took place, but as soon as the water was changed spawn was deposited in twenty-four hours. Two female *Discoglossus* have each spawned twice every summer for the last three years, just as they do when free. Two pairs of *Triton waltlii*, which I have reared from larvæ, have spawned when two years old. In their case the same treatment was carried out. They were well fed in the summer, kept cold in the winter, and fresh water added to the aquarium when they showed readiness to pair.

These various experiences appear to indicate that the difficulty met with in breeding Amphibia kept in confinement is not due to any toxic influence on the gonads due to the results of close confinement. Darwin was inclined to believe that the functions of the generative organs were sometimes impaired by captivity, but unless and until concrete evidence is given to show what specific influence is at work, it would very often be simpler to assume that the external conditions are unfavourable for breeding, or deficient.

In the case of *Xenopus* all the other conditions may be present, but if there is no daily change of the water there is no oviposition, and although the male may embrace the female, the behaviour of the latter clearly shows that she is not ready to spawn.

If the view is correct that breeding is brought about in animals, especially in those with a fixed breeding season, as the response to a certain set of definite external stimuli on the sexually mature of the species, it may help to explain

* And also by P. KAMMERER ('04) with *Salamandra maculosa* and *S. atra*.

why some animals appear in great numbers in one year and are much less numerous in another. Entomologists are familiar with this phenomenon, and it may be worth considering whether such fluctuations in numbers are not due to causes of the nature indicated.

It is obvious that changes in the environment affecting the breeding habits might lead to rapid divergence through the action of Natural Selection, and the diversity in the breeding habits of allied tailless Batrachians has perhaps been established through the agency of such induced sterility.

The male *Xenopus* begins to assume nuptial characters a couple of days after the temperature is raised to 22° C. The dorsal surface of the hand darkens and the area covered with nuptial asperities extends along the arm towards the axilla; the whole patch blackens from the hand inwards in the course of about two days. The shape of the patch has already been figured (Bles, '01).

The abdomen of the female becomes very much distended during the winter by the enormously enlarged ovaries, so much that the lungs are displaced upwards and raise the dorsal body-wall on either side of the vertebral column into two great projecting longitudinal humps. The three flaps of skin surrounding the cloacal aperture are flaccid until the spring, when they become swollen and turgid and more highly vascularised. I was unable to detect any change in the epidermis of the breeding female until last year (1903), when the back of the hands became darker at the same time as the nuptial asperities appeared in the male. Special attention was paid to this point in the seasons before, and it is certain that nothing of the kind occurred then, so that it appears that a secondary sexual character of the male is making its appearance with age in the female (see Boulenger, '97, p. 72, for similar cases).

During the first week of the newly established spring conditions the males become vocal. They have been silent throughout the winter, and their first attempts are intermittent and low in tone. Their voice strengthens from day to day, and at night-fall, especially if fresh water has been added, becomes a loud and continuous metallic rattle, kept up for hours with hardly a break. The noise made by a single frog is loud enough to be heard at a distance of 100 yards or more in the open.* It resembles the voice of *Hyla arborea* more than that of any European frog, but has two alternating notes extremely like those made in winding up an old grandfather's clock with a crank handle. By rubbing the corrugated handle of a pair of large forceps backwards and forwards against the rounded edge of an empty tin tobacco box, I have imitated the sound so exactly that the frogs have responded. The croak is produced under water, and although air is no doubt passed to and fro between the lungs and the buccal cavity, there is no movement of the pectoral or gular region visible externally.

Normally, pairing only occurs at night. The male croaks loudly and incessantly

* I had not heard its full strength when I made the statement in my former account (Bles, '01, p. 211).

during the twilight until he seizes the female. There is nothing resembling the courtship of Urodeles. The male makes a sudden dash and clasps his mate round the lumbar region; his arms are too short to meet on the ventral surface. In the amplexus being inguinal, *Xenopus* resembles the Discoglossidæ and Pelobatidæ (Boulenger, '97, p. 69). The note of the male changes at the commencement of pairing. In a low tone he utters "cô, cô, cô, cô, ... " and at each syllable strikes the under side of his head against the back of the female. Between each stroke the floor of the mouth of the male is seen to bulge considerably so as to carry his head away from the back of the female. When spawning begins he is silent, but every now and again while the amplexus lasts he croaks loudly during the short intervals when oviposition stops.

The account given by Leslie ('90) of the breeding of *Xenopus* speaks only of spawning taking place in August, *i.e.* the South African spring.

According to my observations, I should conclude that, like Discoglossids, the same female may spawn in the wild state several times during the spring and summer, and that the males are ready to pair at any time in those seasons. My female *Xenopus* had spawned for the first time in the year in February (Cambridge), April (Cambridge), and May (Glasgow), in successive years, for a second time in June, and a third time at the end of August in the same summer (Glasgow, 1902); in the following year three batches of eggs were laid at corresponding times. Thus the animals became acclimatised to a difference of six months in the seasonal changes.

Each batch of mature eggs was usually deposited in the course of five days, one night in which pairing did not take place intervening between the nights when eggs were laid. On one occasion spawning took place on five nights; between the third and fourth night of spawning there was an interval of three nights; the others were on alternate nights as usual.

It has so often happened that male and female have been seen to cast their skin the morning after pairing, that it is probable that ecdysis is usual at this time. The skin is loosened all over the body, the legs are then freed, and the skin which is attached to the snout is drawn forward, crammed into the mouth, and eaten in exactly the same way as it is by many terrestrial frogs.

Oviposition.—The amplexus is continued from the evening until the next morning, and may last until 9 A.M. Spawning does not commence immediately, but may begin an hour after pairing. From this time onwards eggs are laid at frequent intervals all through the night. As a rule the eggs are laid singly, and the pair swim about or come to the surface to breathe between each act of spawning. But occasionally three or four eggs are laid in quick succession in the same spot, and somewhat rarely eight to ten eggs will be emitted in a group.

The egg is held between the three protruding flap-like lips of the cloacal spout, while the pair swim about restlessly for half a minute to two minutes; the female then grasps a long leaf or a twig of water-plant between her outstretched feet, and the pair

come to rest in such a position that the cloacal spout of the female becomes applied to the anterior end of a shallow median groove on the ventral surface of the male, which runs back to the cloacal opening of the male for about three-quarters of an inch. This groove is formed by two skin folds over the ventral edge of the pelvic symphysis. The egg is passed out, travels rapidly along this groove, over the cloacal aperture of the male and directly backwards about 4 inches to the weed held by the outstretched legs of the female, where it adheres. The egg has to travel about 5 inches from the female to the weed, and is carried this distance in a straight course. This is partly due to the fillip it receives from the tumid lips of the cloaca as it passes out, and partly to a backwardly directed current in the water, created by gentle swimming movements of the feet of the male. The pair immediately swim away, another egg appears in the cloacal spout of the female, and the process is repeated. As each egg or group of eggs is laid, a spasmodic quiver can be seen momentarily passing over the body of the male, and at this time, I have reason to believe, a very small number of spermatozoa are emitted.

Fertilisation.—The curious method of oviposition resembles in the action of the female the spawning of Urodeles and is so unlike that observed in the Phaneroglossa, that the question of fertilisation is raised. Quite a number of considerations point to the conclusion that each egg is fertilised as it is laid and after it has passed into the water, but all attempts to secure spermatozoa in the water as the eggs were laid proved unsuccessful, as were also attempts made to observe fertilisation in the living egg.

One set of observations repeated at different times proves, I believe, that fertilisation does occur after or during deposition of the egg, as in other Anura. The eggs when attached always have the dark pole below, and within half an hour rotate within the egg-membranes, so that the dark pole is above and the light pole below. In *Xenopus* eggs which are unfertilised this rotation does not take place at all as a rule, or may be incomplete or take an hour or more to complete. This agrees exactly with the rotation described in eggs of other Anura with external fertilisation (R. Hertwig, '03, p. 534).

Further, there are at each spawning a number of eggs (100–200) which do not become attached, presumably by accident, but fall to the bottom of the aquarium. It is exceedingly rare to find a fertilised egg among these. This seems to point to the conclusion that a very limited number of spermatozoa are emitted, otherwise it is difficult to understand why these eggs should not have spermatozoa carried to them in a small aquarium with water kept in constant motion by the active pair of frogs. Every now and then during spawning an egg is passed which does not pass along the ventral groove of the male in the normal manner, and these drop to the bottom. This would account for these eggs not being fertilised. It is hardly probable that they are all immature eggs; that would not account for them not having been attached, as they are in other respects quite normal. One of these ova has been figured in fig. 1, Plate I.

The Egg-envelopes.—The diameter of the whole egg when laid varies between 2·75 and 3·0 millimetres. It is surrounded by a layer of transparent jelly-like substance,

the outer coat being extremely adhesive. One result of this is that the eggs stick to the first foreign body they touch in the water. Another consequence is that the eggs become coated with a thin layer of mud when laid in turbid water; the suspended particles stick to the surface. The appearance of an egg laid in fairly clear water is shown in fig. 11, Plate II. Within a few hours the outer layer changes in consistency; it becomes hard and horny, and floating particles no longer stick to it. The horny coat is exceedingly tough, and might be regarded as a rudimentary egg-shell. In this capacity its function would only be transitory, as on the second day after spawning this outer envelope splits and its contents ooze out. The substance of the outer coat is so tough and unyielding that the contents are tightly pinched as they issue; the jelly and the soft embryo itself are both constricted between the lips of the chink. The egg thus freed is composed of a thick outer layer of jelly, the vitelline membrane and the elongated embryo lying in the fluid within the vitelline membrane (see fig. 12, Plate II). The whole remains adherent to the original place of attachment; the outer coat shrinks and forms a shrivelled ring round the place of attachment of the egg to the substratum. The escape of the egg-contents only occupies a few minutes.

A very similar shell-like structure, which seems to be undescribed, is found in the egg of *Hyla arborea var. meridionalis*. Here it is not superficial, but enclosed in a thin, soft, adhesive layer which holds together a number of eggs in a clump. Inside this layer is a tough, thin, whitish translucent membrane; then follows a layer of jelly and then the vitelline membrane. This "egg-shell" is also burst at an early stage. On the third or fourth day after the spawn is laid, the spherical shell is found in two hemispherical pieces lying embedded next to each egg in the jelly of the clump. In this case the split extends meridionally completely round the "shell." The splitting of this membrane in the case of both *Xenopus* and *Hyla* is most likely due to the absorption of water by the inside jelly and its consequently swelling until the internal pressure bursts the non-extensible membrane. This membrane is comparable with the true membranous egg-shell discovered by GUPPY in the large eggs of *Rana opisthodon*, which are laid in the crevices of rock and enclose the young frog until perfectly formed.

The Ovum.—The history of the ovum is here taken up at the period of oviposition. It then measures 1·5 millimetres in diameter, so that it ranks among the smaller Anuran eggs.

While rotating within the egg-membranes after fertilisation, it can easily be seen that the pigmented and unpigmented areas of the surface are approximately equal (see fig. 1, Plate I). The pigmented area usually covers rather less than a complete hemisphere. The dark half is of a rich brown colour, while the yolk is of a very pale greenish-blue colour.

Fig. 1 represents an unfertilised egg about twenty-four hours after oviposition. It is easily recognised by the patch of unpigmented protoplasm which has risen with the egg-nucleus inside it to the surface, and has displaced the superficial layer of dark pigment from the upper pole. This appearance is very characteristic of unfertilised

eggs on the second day after spawning. In other respects the egg shows the normal appearance before segmentation commences.

Segmentation.—The details of segmentation do not appear to differ much from those seen in small eggs of other Anura, and are, therefore, not described in detail. I have had figures of a few stages made as accurately as possible to show the general course of the processes.

The first furrow is, as usual in Anura, meridional and divides the eggs into two equal blastomeres. It is completed one hour to one and a half hours after fertilisation. The next vertical furrow appears within two and a half hours after fertilisation, and the third (horizontal) one within the next hour. The third furrow does not form exactly parallel to the equator, but is bent in the manner shown in fig. 2, Plate I. The egg is viewed here from the right side, according to the statement made by O. SCHULTZE and KOPSCH to the effect that the unpigmented portion of the egg reaches much nearer the upper pole on that side of the egg which is to become the posterior end of the embryo. In this egg there is at the upper pole a marked departure from the radial symmetry which, as shown in fig. 2A, is still present at the lower pole. The egg has become bilaterally symmetrical with an elongated and a more rounded cell on each side at the upper pole. The pigment is not altogether confined to the cells of the upper hemisphere; there is a patch posterior to them (fig. 2, Plate I). The egg represented in figs. 3 and 3A, Plate I, is at a stage reached about four hours after fertilisation. The segmentation has become irregular, especially of the yolk cells. There is still a marked difference in the bulk of the cells of the upper and lower hemispheres, but the latter are now rapidly dividing. Furrows start from the edge of the pigmented areas and extend downwards over the yolk cells until they meet an existing furrow near to, but not accurately at, the lower pole of the egg. At the stage shown in fig. 4, the cells of the upper and lower hemispheres are almost equal in size and are again arranged fairly regularly. This return to a regular arrangement must be a result of the mobility of the superficial cells, together with the need for a geometrical arrangement in order to accommodate a given number of cells of a certain size in a limited space. That the cells of both the upper and lower poles are movable to a certain extent can easily be observed in the living egg. Newly divided cells may push apart two cells which were in contact with each other until the edges which were touching are separated by the whole diameter of the intruding cells. If there are intercellular strands of protoplasm at this stage they must certainly become broken in the shifting about of the cells.

At the stage shown in fig. 5, the cells have become smaller and have lost both the regular arrangement and the roughly hexagonal outlines seen in the earlier stage. The yolk cells near the equator are dividing more rapidly than those at the yolk pole and are appreciably smaller. The living segmenting egg at this stage has a feature which is shown here in the figure (fig. 5) of a preserved specimen. A number of cells at the margin of the spreading epiblast are of much paler brown

than the neighbouring epiblast cells. This difference in tint is found, by watching the cells in a living egg, to be the sign of an approaching division of the pale cell. The pigment reappears apparently undiminished in quantity after division. The cause of such cells turning pale at this particular time I hope to discuss later, as there appears to be some parallel between this process and that of contraction in the chromatophores.

Gastrulation.—A stage of the development of the blastopore is shown in figs. 6 and 6A. In the side view (fig. 6), the egg is shown in the vitelline membrane and lying with its orientation as in life, the lowest point on the egg sphere being 30° behind the middle of the dorsal lip of the blastopore. This view shows very clearly features not easily shown in the ventral view, namely, the extent of the depression of the posterior lip of the blastopore and the sharp, slightly puckered edge of the dorsal lip, which is not depressed, but remains up to the edge coincident with the surface of the sphere of the egg. At this stage indications are already noticed of the arrangement of ectoderm cells in rows, forming alternating light and dark streaks lying in the position of great circles roughly coaxial with the great circle passing through the longitudinal axis of the embryonic area. These streaks are indicated in fig. 6, passing from the posterior edge of the upper pigmented area towards the blastopore. In fig. 6A, the lower ends of these streaks are seen near the right-hand angle of the blastopore. In this figure (6A) the blastopore has reached the greatest lateral extension it attains as a crescentic opening.

In fig. 7 the egg is seen from behind, and a stage is represented where the blastopore has just become circular. The yolk plug does not protude; at no time does it become prominent in *Xenopus*. The pigmented cell-area now extends back to the edge of the upper lip of the blastopore. A peculiar feature of this stage is the constant occurrence of alternating bands of darker and lighter cells arranged as described above in the previous stage figured (fig. 6). The observation that epiblast cells undergoing mitotic changes become pale at certain stages of the process leads to the interesting conclusion that the lighter patches are areas in which the epiblast cells are proliferating simultaneously.

The Embryo.—In figs. 8 and 8A, representing views of the same egg seen from behind and from before, an early stage of the development of the medullary folds is shown. The view from behind (fig. 8) shows that the medullary folds are externally much less obvious at the hinder end than they are in the head region seen in fig. 8A. On the dorsal contour of the latter figure, the gentle elevations are due to the swollen medullary folds, and the slight notch between them indicates the neural groove (*Rückenrinne*). The neural groove extends far forwards, as far as the anterior wall of the brain, and can be seen between the paired dark patches on the exposed floor of the thalamencephalon. These areas are occupied by the pigmented cells of the optic vesicles, and closely resemble those described by EYCLESHEIMER in *Rana palustris* ('93). Returning to the medullary folds, it will be seen from fig. 8A that at about the middle of their length the medullary folds have converged from the anterior end to lie side by side. It is thus possible to identify

the elevations on the upper contour of fig. 8, Plate I, near the middle line as the medullary folds, otherwise it would be difficult to make out their position in the posterior view of this stage. They flatten out and disappear towards the blastopore. On the outer side of each medullary fold is a band of pale ectoderm cells which can be seen at the points *a* and *b* to be slight ridges. These bands of proliferating ectoderm cells meet at a point just anterior to the future anal perforation, marked at this stage by a small pigmented area at the hinder end of the closed and elongated blastopore. As these bands lie external to the medullary folds, it is certain that at this stage the medullary folds do not enclose the blastopore or meet behind the anal opening. The archenteron is now completely closed to the exterior, as for a short time the blastopore is quite closed ; there is no anus. The anal opening is formed in a few hours after this stage has been passed through. The neurenteric canal and postanal gut persist for a long time ; they are present at the stage shown in fig. 14. The dark area (fig. 8A) anterior to the medullary folds, and extending a short distance back along the sides of the fore-brain region, is a region covered with modified, deeply pigmented ectoderm cells, the greater number of which are destined to become secretory epithelium cells. The median ventral portion of this area is the precocious rudiment of the cement organ ("Sucker") ; the lateral dorsal portions contribute to the formation of the frontal gland ("Stirnknospe," v. KUPPFER, "Stirnstreifen," HINSBERG). This area corresponds in its position and relations to other parts exactly to O. SCHULTZE'S "Sinnes-platte," MORGAN'S "Sense-plate." It is not to be regarded as a lateral extension of the medullary plate (MORGAN, '97, p. 57). It is a part of the general ectodermal covering without any obvious connection with the central nervous system and formed by differentiation in the cells of the superficial epidermic layer of the ectoderm only. The deeper nervous layer of the ectoderm is not modified and there would seem to be no justification for the term "Sinnes-platte."

The portions of the ectoderm from which the epithelium of the nasal pits is derived are enclosed by or are possibly included in this dark area ; it is not possible to identify them at this stage.

The medullary folds have arched over and their edges have met along their whole length in the next stage figured (fig. 9). The ectoderm is raised as a gentle ridge over the closed neural tube, and there is a line of deeply pigmented ectoderm cells along the median external line of junction of the folds, and at the bottom of the shallow groove formed by the rounded edges of the folds. The anterior end of this pigmented groove marks the position of the neuropore. A short distance posterior to the neuropore, the pigmented groove is intersected by a crescentic band of pigmented ectoderm, the early rudiment of the frontal gland. Below the position of the neuropore is a large pigmented patch of ectoderm, the rudiment of the cement organ. It shows some indication of a paired nature in the presence of a more deeply pigmented patch at each end of the transversely elongated area. This figure (9), compared with fig. 8A, shows that the greater part of the pigmented area in front of the medullary folds at the earlier stage becomes

cement organ at the later stage. And the postero-lateral horns of this crescentic area (fig. 8A) are carried inwards towards the median dorsal line as the roof of the fore-brain closes in and give rise to the transverse band of cells I have called the *frontal gland*. This will be dealt with in detail later. We shall see that the area shown in fig. 9 enclosed by the line of the frontal gland and the upper edge of the cement organ is the part from which the anterior organs of the face develop, viz.—the stomodœum and the nasal pits. At this stage the egg has just begun to elongate in the direction of the future axis of the body.

In fig. 10 an early stage in the elongation of the larva is represented, and it will be at once observed that the dorsal concavity which is so marked a feature of corresponding stages in all such forms as *Rana*, *Bufo*, *Hyla*, *Bombinator*, and *Discoglossus* is not only absent here, but there is a distinct convexity of the dorsal contour in side view—the two ends of the embryo are slightly bent ventralwards, so that there is a shallow ventral concavity. It is this feature in its general appearance which in early embryonic stages of *Xenopus* recalls rather a Urodelan embryo than one of the familiar Anuran embryos. The difference is due mainly to the fact that while the embryo of the typical Phaneroglossa owe their increase in length from the beginning to the elongation of the tail and sometimes of the head also, the abdominal region remaining short, the embryo of *Xenopus* for sometime grows in length in the trunk region, the tail remaining short and stunted. Connected with the checked growth of the trunk in the forms mentioned above is the persistence of the anus in its position high up on the posterior surface of the egg, which involves the outgrowth of the tail, so as to make it sprout not directly backwards but obliquely upwards at an obtuse angle with the long axis of the body. In *Xenopus*, on the other hand, the growth takes place in such a way that the anus is swung round from its equatorial position on the spherical egg to a ventral position in the elongating embryo, as indicated in fig. 10. As indicated in figs. 10 and 12–14, the tail remains a mere stump, while the trunk elongates considerably. A form of embryo results, bearing considerable resemblance in the proportions of its main regions to Urodele larvæ and also to young Dipnoan larvæ, where the anus is placed close to the posterior end of the body at the root of a very short tail. This short-tailed, long-bodied phase of the development of *Xenopus* may, with some confidence, be looked upon as primitive.

Returning to stage 10, it will be seen that the region of the spinal cord is pinched up, as it were, from the ventral part of the trunk, and the fore-brain swells out the tip of the head. Behind the fore-brain is an elevation of the whole branchial region. There is an accumulation of yolk at the posterior end producing a swelling out of that part. The fin-fold has not made its appearance. The cement organ is beginning to assume its characters, showing a compact group of cells with densely pigmented inner ends and outer ends filled with a clear mass of secretion. At the extreme anterior end is placed the frontal gland, seen better in the frontal view (fig. 10A). This shows how the frontal gland has become continuous with the pigmented band of cells along the sagittal line.

The position of the neuropore is indicated by a few pigmented cells forming a short longitudinal streak exactly at the extreme anterior end of the animal. Between the neuropore and the cement organ is the region of the stomodœum.

The next change in the external appearance of the embryo is due to the outgrowth of the median fin-fold of the tail. An early condition is shown in fig. 12. The trunk is slightly longer than before (stage of fig. 10), and the back of the embryo has become straighter.

The growth in length of the embryo has stretched out the vitelline membrane into an ellipsoidal form. In the particular embryo drawn the cement organ happened to be larger than it is usual to find it at this stage. The line of the frontal gland is well defined; it extends ventrally as far as the edge of the cement organ, and between it and that edge is enclosed a patch of deeply pigmented ectoderm. The portion of the median fin-fold to develop first is that bordering the tail region; in fact, the extent of the tail region is fairly well defined by the limits of the fin-fold at this stage. The fold is deepest in the post-anal ventral part, extends round the posterior end of the embryo, and fades away just before reaching the dorsal surface.

When the embryo has reached a length of 3·8 millimetres (fig. 13) the fin-fold has grown up along the dorsal surface and extends forwards as far as the part over the hind-brain, practically marking out the whole length of the organ. The fin is still deepest in the post-anal ventral portion. It is necessary at this stage to distinguish between a ventral post-anal abdominal part of the tail into which the posterior end of the yolk mass is drawn out, and a dorsal part which, it will be seen, grows out more vigorously and gives rise to the segmented, muscular part of the tail. In the head there is remarkably little indication externally of the developing eye, brain, and visceral arches. The position of the eye can just be discerned by the dark patch drawn in the figure; the mandibular arch is slightly raised above the general surface, and the posterior group of visceral arches is just discernible as another broad and gentle elevation on each side of the head. The line of the frontal gland is very distinct and obviously continuous with the cement organ at its lower ends. The cells of the cement organ are filled with clear secretion at their outer ends, and thus produce an appearance of a low translucent ridge running across the dark cement organ from side to side. No indications of nasal pits or of the stomodœum are to be seen. The myotomes are just beginning to be visible externally in the hinder trunk region.

Further growth is shown in the embryo of fig. 14, drawn after removal from the egg. The tail has grown out more especially in its dorsal muscular part, and the fin-fold of the tail has now become widest along this muscular part. The post-anal abdomen has also been slightly drawn out in length by the growth of the tail. The dorsal fin has become higher, and now reaches forwards on the head as far as the hind level of the fore-brain. Numerous myotomes can now be seen in the living embryo. The eye is plainly seen and also indications of the lens thickening. A bulging behind the branchial region indicates the position of the heart; more dorsally the ear vesicle

may be seen faintly through the skin, and a swelling shows the position of the pronephros.

All the preceding stages are still enclosed within the egg. We now pass on to the larvæ at the time of hatching. Fig. 16, Plate III, shows the larva at the normal stage of hatching with the vitelline membrane now enormously swollen out. The more or less thin layer of jelly which surrounds the membrane has not been drawn. Compared with the last stage illustrated (fig. 14), the muscular portion of the tail has grown, while the abdominal portion of it has become insignificant. The length of the tail (fig. 15A) is still only about half the length of head and trunk together; the fin-fold has not yet become a powerful swimming organ. The head is now for a time divided from the trunk by a constriction, a neck, which is the more marked since the trunk is swollen just behind the constriction by the bulging of the skin over the pronephros (see fig. 16, Plate III). On the somewhat flattened anterior surface of the head the following can be made out (fig. 15, Plate III). On the ventral surface of the head is a conspicuous cement organ projecting downwards and forwards, exceedingly deeply pigmented at its base, so as to appear almost black. Running round the anterior edge of the protuberance is a clear-looking ridge with its outer end curved backwards; this is composed of the outer ends of the tall columnar cells of the cement organ filled with the cell-secretion. The oval patch behind the crescentic ridge is formed by the inner ends of the pigmented gland cells shining through the epidermis. The outline of the cement organ is thus crescentic and not circular, as BEDDARD described it in his *Xenopus* tadpoles. Running up from the anterior border of the cement organ, and passing obliquely outwards, is the pair of bands of pigmented cells connecting the frontal gland with the cement organ. The cells of all three structures are found to be essentially similar when examined in sections, so that there is a continuous line of mucus-secreting cells enclosing an area shaped roughly like an inverted trefoil on the anterior surface of the head. The base of the trefoil above contains the paired rudiments of the nasal pits, and the apex contains the stomodœum. The lateral bands of mucus cells are narrow and meet the ends of the broad transverse band which forms the frontal gland just internal to the level of the outer edge of the nasal pits, so that the nasal pits are bordered dorsally by the frontal gland. By comparing fig. 15 with fig. 10A (Plate I), it will be seen that the frontal gland is dorsal to the neuropore, and that the neuropore, if it persisted or if traces of it remained until this stage, would lie between the centres of the nasal pits. Another point worth notice is brought out by a comparison of the frontal view of these two stages, and that is the very close proximity of the neuropore to the small area from which the stomodœum will be formed later in development. Taking into account the thickness of the ventral wall of the fore-brain, it will be seen how little space is available between the cement organ and the brain for the potential mouth at the earlier stage (fig. 10A).

Hatching.—The larva of *Xenopus* hatches forty-eight hours after the egg is laid, when the eggs are kept undisturbed in a constant temperature of 22° C. I found, for instance, that a batch of some hundreds of eggs, laid on the night 24th–25th August 1903, had all hatched on the morning of 27th August, with the exception of twenty or thirty. As the larvæ might be expected to emerge from these during the course of the day, they were placed for observation in a little tank kept at a temperature of 22° and were watched from the side through a horizontal binocular microscope (Zeiss' Braus-Drüner model) under a low power. The view of the egg obtained in this way is represented in fig. 16, Plate III. The jelly on the surface is not shown; a very thin layer of it covers the vitelline membrane, the thinness being partly due to the way in which the vitelline membrane has swollen up, the jelly having now to cover a much larger surface. At an early stage the membrane loses its spherical form and becomes ellipsoidal (see fig. 12); it is elastic and becomes swollen up by the pressure of its fluid contents. It is easy to test this by tearing a hole in the membrane; a jet of the fluid is forced out through the hole and the membrane collapses and shrinks. The egg is, then, in a condition of turgidity, and this forms a factor in the hatching process. The larva lies on the lowest part of the vitelline membrane, the side of its body in contact with it and roughly parallel to the long axis of the egg, which is always horizontal. In this position the larva is attached by the secretion of its cement organ to the vitelline membrane about two hours before the hatching occurs. Before attaching itself it seems to have sunk down to a position of equilibrium with its centre of gravity over the lowest point in the curve of the egg membrane. This is indicated by the position of the tip of the tail, which is always at a higher level than the head. The surface of the larva is richly ciliated and the fluid in the egg membrane is kept in rapid rotation, the current over the body of the larva passing from head to tail. The larva lies perfectly still, except that every ten to fifteen minutes it turns over from one side to the other. It will be seen from fig. 16 that the head of the larva only touches the vitelline membrane where the surface of the eye rests upon it; the cement organ is not itself in contact, as a short string of secretion passes from the gland to the membrane. The first sign that hatching is about to take place is a slight bulging outwards of the vitelline membrane opposite the head of the larva. In the course of the next five minutes the membrane under the anterior part of the head softens and the head sinks into the soft place, the membrane partly moulding itself to the contours of the head and partly bulging beyond the head, as drawn in fig. 17A, Plate III. It will be observed that in this position the extreme anterior end of the head where the frontal gland is situated is not actually touching the vitelline membrane, although very close to it. In another five minutes the membrane has moulded itself by a further softening to the anterior contour of the head (see fig. 17B), and now the frontal gland touches the vitelline membrane. When this stage is reached the hatching is rapidly completed. During the next three minutes the pouch of vitelline membrane in which the head lies distends more and more, until an imaginary line drawn in the original smooth curve of the membrane would pass through the middle of the cement

organ (fig. 17c). In this position the tadpole remains for not more than thirty seconds; the stretched part of the membrane then bursts, and the larva is shot out head foremost, as shown in fig. 18. In the same instant the vitelline membrane shrivels up like a burst indiarubber toy balloon. The larva may remain motionless with its tail between the torn edges of the egg membranes for one minute; it then wriggles and frees the tip of its tail, the tail swings round through an arc of 180° and the larva is then seen hanging, as in fig. 19, by a thread of mucus to the still further shrunken egg membranes. This thread of mucus is the short thread which attached the larva to the inside of the vitelline membrane and now drawn out to the length of the larva. It is apparently pulled out when the tadpole bursts out of the membranes, but it is not shown in fig. 18, as it is hidden by the body.

The following suggestions are offered as to the method of this hatching process. The larva attaches itself and fixes its position in relation to the vitelline membrane. Each time the animal turns over it must necessarily straighten itself, and in so doing the frontal gland touches the vitelline membrane, and smears it with a little of the secretion of the frontal gland. The secretion is of a different nature to the secretion of the cement organ; it acts upon the vitelline membrane and softens it. The softened patch is distended by the pressure of the fluid in the turgid egg, and the head of the larva sinks into the pouch which is formed. When the frontal gland comes into contact with the vitelline membrane in the last stages of hatching, the softening process is hastened, and when the larva's head is pressed against the pouched-out membrane, the fluid pressure in the egg must act, not on the membrane, but on the body of the larva. The fact that the softening process only goes on opposite the anterior surface of the head is shown by the vitelline membrane retaining its normal curvature at all other points, even so close to the head as opposite the neck (see figs. 17A and B).

To test the hypothesis that the secretion of the frontal gland acts on the egg membrane, the following experiment was made. Four eggs out of six left unhatched at 12 P.M. on 27th August were hung up, so that the long axis xy (fig. 16) was vertical instead of horizontal, and the head of the larva uppermost. By grasping with fine forceps a little of the jelly at the pole x, and drawing it out of the water against the glass of the tank, the eggs could be fixed in this position. Of the four eggs slung up in this way, none were hatched at 6.30 P.M.; the two eggs left in the normal position hatched out the larvæ between 1 P.M. and 2 P.M. At 6.30 P.M. the four eggs were returned to the normal position, and within half an hour all were hatched. The larvæ were thus prevented from hatching for about five hours.

The reason appears to be this. When the egg is carefully revolved into the new position, the larva remains attached, but slides down towards the pole y; the head is consequently carried away from the egg membrane near the pole x, and when the larva moves, it is easy to see that the head apex, where the frontal

gland lies, swings out away from the membrane and never touches it. There was not the slightest bulging of the membrane opposite the head in the reversed eggs.*

Regarding the action of the secretion of the frontal gland, there seems to be reason to believe that it is digestive, and probably is due to a peptic ferment. Miss R. ALCOCK ('91 and '99) discovered that the external epithelium of the skin in the *Ammocœtes* of *Petromyzon planeri* and in *P. fluviatilis* produces a peptic ferment capable of digesting fibrin in a 0·2 per cent HCl solution. In the frontal gland, then, a similar secretion is probably localised in an appropriately-placed patch of epithelium, and the acid medium requisite for the action of the peptic ferment is, no doubt, supplied by the excretion from the pronephros. But it is not necessary to dwell on this point, as the specific action of the frontal gland can no doubt be tested by experiment.

It is obvious that the frontal gland is actively secreting at hatching time from the coating of secretion which hardens on the surface of the gland in larvæ preserved at this stage. The light band seen (fig. 15) running along the middle of the gland is produced by coagulated secretion. After hatching, the frontal gland begins to atrophy, and has disappeared three or four days later, before the tadpole begins to feed.

Considerable morphological importance has been attached to the frontal gland by VON KUPFFER ('93, p. 78, '94 and '03, pp. 188 and 190). He regarded it as the "unpaarige Riechplakode" of the frog and uses it as evidence of a monorhine stage in the development of an amphirhine form of Vertebrate, believing that it arises as a sensory thickening of the ectoderm at the spot (the neuropore) where the brain retains its connection with the ectoderm longest; the connection he considered to be the primitive unpaired olfactory nerve. It would not be necessary to refer to this view here if VON KUPFFER had not repeated his interpretation of the "Stirnknospe" in his chapter on "Die Morphogenie des Centralnerven-systems" in O. HERTWIG's "*Handbuch der Entwicklungslehre der Wirbeltiere.*" In the same work KARL PETER ('02), in the chapter on "Die Entwicklung des Geruchsorgans in der Reihe der Wirbeltiere," gives weighty reasons for rejecting VON KUPFFER's fascinating theory of Mono- and Amphirhiny (PETER, '02, pp. 12–13, p. 26). PETER, in this chapter, refers to his own paper on VON KUPFFER's theory ('01, p. 654), where he includes observations on *Bufo "cinerea"* (syn. *B. vulgaris*), showing that the "Stirnknospe" has in the toad no connection with the neuropore and that it must be placed in a different category to the sense-organs of the Anura, since it develops from the external layer of the ectoderm, while the sense-organs are all derived from the inner nervous layer (see also PETER, '01*a*). CORNING ('99) and HINSBERG ('01) have described the frontal gland in *Rana temporaria* and *R. esculenta*, and it can be gathered from their descriptions and PETER's of the

* It has been possible to show by experiment on *Hyla* larvæ that the surface of the frontal gland only, and no other part of the ectoderm, has the power when touching it to soften the vitelline membrane.—9th April 1904.

toad that the gland is at the height of its development in each of the three species when the larva reaches the size at which hatching occurs, and that it rapidly atrophies after hatching. Its structure in these forms and in *Xenopus* is essentially similar.

The results arrived at on this subject may be summed up by stating that the main function of the frontal gland in the Anura is to soften by means of its secretion the tough and turgid egg membrane in order to allow the larva to escape at an early stage of development before any external hard parts have been formed which might be used for breaking out. The frontal gland is a transitory structure, like the egg-tooth of lizards, and like it again only actively functional for a few minutes in the life of each individual. Its interest is chiefly physiological, but it may serve as a warning to morphologists of the danger run in assuming that an inconspicuous organ, the function of which is not known, is vestigial.

The Larvæ after Hatching.—The first few days after hatching are spent by the tadpole attached to weeds, etc. Its abdomen contains a considerable amount of yolk which must be absorbed and the alimentary canal opened up before it can begin to swim about and feed. This does not come about for a period varying between three days and a week in an aquarium kept at 22° C. The cement organ is functional during the whole of this time. The tadpoles at this stage never lie on the bottom of the aquarium, where, owing to their pale colour, they would stand out conspicuously against the sediment. Those found on the bottom are prematurely hatched younger larvæ. The secretion of the cement gland is extremely tenacious. If a tadpole is induced to move, the mucus thread breaks away from the surface of the sucker and the animal swims with a curiously stiff flickering wriggle, remarkably like that of a young *Amphioxus*. For the first day or two after hatching a free-swimming larva always tends to assume the vertical position with its head up, and consequently the swimming movements carry it towards the surface. Another physiological character tends in the same direction. Although not clearly discernible at any given time in any particular individual, the larvæ respond to light stimuli in such a way that it could be put down to feeble positive heliotropism. If an aquarium containing some hundreds is disturbed and the larvæ scattered, the great majority of them will, in twenty-four hours, be near the surface or on the light side of the vessel; a few dozen, however, will be distributed at random. The mucus from the cement organ is both very adhesive and very tough. Whatever the cement organ touches it immediately becomes fixed to and the tadpole comes to rest. This applies to so unsubstantial an object as the surface film. A 20-inch bell-jar, four or five days after spawning has taken place in it, will have one hundred to two hundred larvæ hanging from the surface film in the position of the larva in fig. 19, Plate III. The thread of mucus is of varying length, and may either be as long as or rather longer than the larva. The surface film is, of course, drawn down into a slight dimple by the weight of the animal. I have never seen any movement in an attached larva, except when in the act of swimming away; it usually hangs down perfectly

quiescent. A strong continuous current sufficient to sway them 45° from the normal will not induce more than one in twenty or thereabouts to detach itself.

The most obvious changes during the first days of this phase are the darkening of the pigmentation both of the eye and of the nasal pit, which both become more conspicuous, the fin becoming wider, but only very gradually (cf. fig. 22A), and three short unbranched external gills appearing on the three branchial arches. The gill slits are closed. As the yolk is absorbed the larva becomes more and more transparent, and at the same time the ciliation disappears *pari passu* with the yolk, except the ciliation in the nasal pits, which persists. As it becomes more transparent the larva becomes more and more restless, and about four to six days after hatching spends more time swimming about than hanging on by its cement organ. At this time the mouth opens and the branchial current of water is set up with its rhythmic action. By this time the operculum has grown back from the hyoidean arch and fused in the mid-ventral region with the body-wall under the pericardium to form the gill-chamber. The upper, lateral ends of the folds remain free and form the ventral or outer lip of the spiracle on each side of the neck. The spiracle at all stages opens upwards and backwards and is not produced into a spout; the inner wall of the spiracle is the body-wall of the animal.

The mouth opens and the branchial respiratory current begins some hours, probably about twelve hours, before the animal takes in food. It is easy to see that the gut still contains a mass of yolk, and in sections the œsophagus is found to be solid. Exactly the same condition of things has been described in the common frog (MARSHALL and BLES, '90, pp. 223–4), with the difference that in *Rana* the branchial chamber is not completely formed and that feeding begins after a much longer interval.

Drawings of the animal at this stage are given in figs. 22 and 22A. In the front view of the head an attempt has been made to show the commencing transparency of the tissues. It brings out clearly the fact that the chromatophores of the skin are confined to the dorsal surface; the ventral is free from them except in one place to be referred to later. Covering the whole underside of the head there is a large continuous lymph space, well marked off from the other lymph spaces, which I propose to call the *submental lymph sac*. Through its translucent walls can be seen the ventral wall of the buccal cavity. The cement organ is at the extreme tip of the head (see fig. 22A), and immediately above it is the mouth or rather the lower jaw, still showing at the symphysis the junction of the mandibular arches. At this stage the lower lip protrudes in front of the upper; this condition becomes still more pronounced in older tadpoles, where the mouth opening comes to lie on the upper surface. The ventral chromatophores on the skin of the mandible then face upwards and become practically dorsal. This disposition of the mouth vanishes at the metamorphosis and is apparently adapted to the peculiar feeding habits of the tadpole. The oral tentacles have not yet made their appearance. Above the mouth are the large shallow depressions of the nasal pits, with their well-marked

raised rim incomplete behind. The median swelling behind the olfactory pits is caused by the brain. The eyes stand out prominently from the side of the head.

In the side view there are very obvious alterations to be observed in the proportions of the parts of the animal. At hatching the tail was only one-third of the total length; it is now two-thirds, while the abdomen, which was very elongated, now appears very short in proportion. The fin-fold has grown considerably, and has grown out from below the abdomen, drawing out the cloaca, so that the cloacal opening comes to lie at the edge of the fold. The pronephros tubules can be seen quite distinctly through the transparent skin.

The Lymph Hearts.—Just behind the pronephros on each side is the newly developed lymph heart. It lies in a small lymph space immediately below the skin on a level with the inner cœlomic outline of the pronephros. It can be best located in the living animal by the movements of the nearest chromatophore of the skin, which, under the microscope, are more conspicuous than the pulsations of the lymph heart itself. Delicate trabeculæ run across the enveloping lymph space from the heart to the integument, and these pull down the skin at each contraction of the heart. At this stage the pulsations are very irregular; they sometimes cease for one or two minutes, and seldom continue uninterruptedly for even twenty beats, hence it is difficult to time them. When most regular they average forty beats a minute.

This early appearance of the pectoral lymph hearts is after all not very remarkable when the extent and physiological importance of the lymph spaces in the tadpole is considered. However, the find was an unexpected one, as I had already paid a little attention to the subject and found that the pelvic lymph hearts do not appear in *R. temporaria* and *Bufo calamita* before the metamorphosis.

The Tadpole.—It is not difficult to notice the commencement of feeding in a batch of young tadpoles. Those which have not begun, only swim about fitfully and then hang by the cement organ, the breathing movements continuing while they hang. Those which have begun to feed are suspended in mid-water, making little or no progress, and are steadily gulping away; the fæces appear in the cloaca within twenty minutes or half an hour afterwards; thus the time at which the alimentary canal is open to the passage of food can be easily and definitely fixed. This is important, because it makes the following interesting fact easy to determine by watching at the correct time, namely, that within two hours after beginning to feed the tadpoles rise to the surface for air and begin to use their lungs as breathing organs. BEDDARD observed that *Xenopus* does not develop "internal" gills ('94, p. 106) and concluded that respiration was carried on through the blood-vessels of the "filters" placed on the internal side of the branchial arches. His observations are correct, but as he paid no attention to the use of the lungs, he was led to a conclusion which turns out to be of subsidiary importance. It is possible that a certain small amount of oxygenation of the blood does go on in the processes of the filtering apparatus. And until feeding commences respiration is carried on by the external gills. But as the tadpoles are constantly rising to the surface for air

and do so more frequently the warmer the water is, it follows that the lungs are not only hydrostatic in function but also respiratory. Reducing the quantity of water has the same effect as raising the temperature. It is difficult to follow an individual tadpole under normal conditions in a large aquarium, but isolated ones in small aquaria will rise every five to ten minutes.

The tadpole shown in fig. 23 should be glanced at as an intermediate stage before proceeding to the typical and final form. This specimen is in the condition reached after about two days spent in feeding. During the first few hours feeding may be interrupted now and then for a few minutes while the animal suspends itself by its cement organ, then this organ begins to atrophy and by this stage has completely disappeared. The snout has not yet assumed the characteristic shape, which is seen one or two days later. Here it is rounded; but it is important to note that there is no trace in this transitional stage of lips like those bearing the horny teeth in ordinary tadpoles. W. K. Parker and Beddard have already drawn attention to the total absence of horny teeth. The pronephros and the lymph heart are very clearly seen at this stage; the transparency of the tissues is still increasing. The contents of the cœlom are, however, beginning to disappear from view, as the chromatophores are rapidly increasing in the abdominal wall. The hind limb is just forming as a rudimentary bud. The tail has lengthened and its shape is quite typical.

A day or two later the appearance of the tadpole has undergone a very obvious change. The shape of the snout becomes like that of the advanced tadpole shown in fig. 24, Plate IV. It may be described as wedge-shaped, with the lower lip forming the slightly curved edge of the wedge. The tentacles sprout exactly at the angles of the mouth and soon become long slender processes. This stage is one which persists for about two and a half months without any marked changes, apart from the great increase in size and the growth of the hind limb. It is, therefore, the typical larva of *Xenopus*, but for various reasons I must omit its full description here and reserve it for a future communication. The chief reason for so doing is inherent in the tadpole itself. It is so transparent in the head region that almost all the complicated structure of the vertebrate head can be studied in the living animal, and it would be necessary to give accurate figures to make all this clear in a description. The difficulty lies in my inability to produce such figures, and, as fig. 24 shows, the dead specimen allows very little internal structure to be seen. I will confine myself to giving a brief account of the extraordinary feeding habits. The tadpoles I have reared (one male was brought to maturity) were fed exclusively until the metamorphosis on pure culture of the green Flagellate *Chlamydomonas*. They thrive best in water which is thick with the Flagellates. In this they float almost vertically in mid-water, rapidly undulating the posterior third of the tail and at the rate of forty to fifty a minute take in gulps of the water. The water passes out through the spiracle; the *Chlamydomonas* are retained by the filters in the buccal cavity and drawn into a ciliated groove on either side of the pharynx. In this groove the green mass

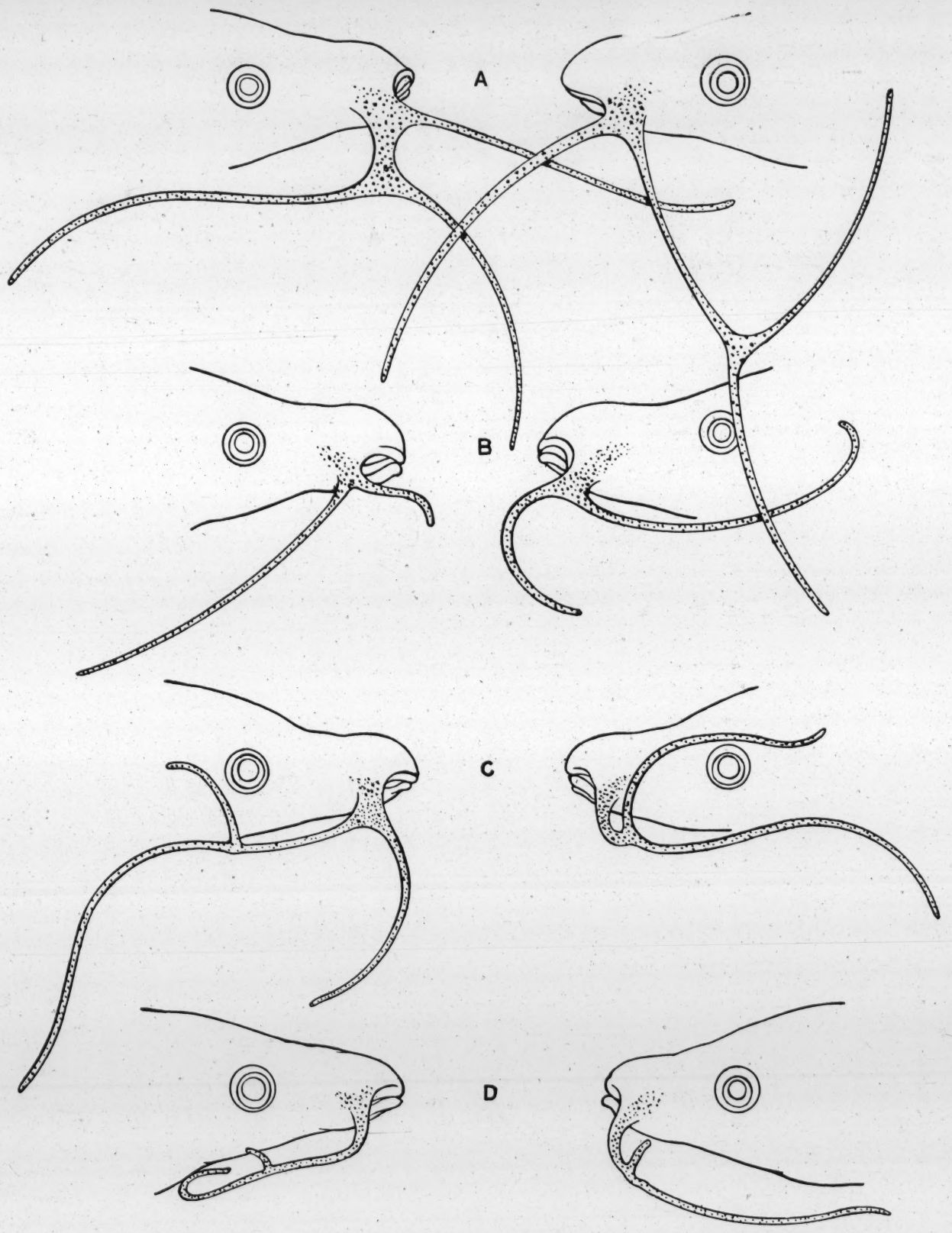
can be seen passing back in a kind of helicoid vortex towards the œsophagus, where the two green currents converge and disappear. Every day for ten weeks fresh culture must be added to the aquarium water; even so few as ten large tadpoles will clear twenty gallons of water in a single day of one added gallon of thick culture. When the water is clear they swim restlessly about like fish, as though searching for food, taking a gulp every now and then, as if to test the water, and then swimming on. As soon as fresh culture is poured in they immediately suspend themselves in mid-water and commence gulping regularly. That the current of water through the gill-slits is not kept up except when feeding confirms the statement made above that it is not a respiratory current. This method of feeding is so remarkable that it is desirable to find out whether it is normal in the natural habitat. It is interesting that the only Craniate known to feed in a similar way is *Ammocœtes*; according to A. SCHNEIDER its chief food is *Euglena*. In both cases the food is filtered out and then collected in a ciliary current. *Xenopus* tadpoles kept in *Euglena* culture were starved to death, however. Mr BEDDARD's tadpoles fed on Cyprids and nothing else; mine invariably rejected any small Crustacean which entered their mouths and starved amid an abundance of Ostracods and Cladocera. The movements of the tadpoles, their way of taking in water, the ciliated bands, the dorsal position of the mouth and the shape of the lips, all point to micro-plankton being a staple item of their diet, quite apart from the fact that they thrived on it. I would like to suggest that the swarms of Cyprids in the Zoological Gardens Tank were feeding on micro-organisms which also formed the staple food of the *Xenopus* tadpoles, the Cyprids being swallowed incidentally.

The Mature Tadpole.—Fig. 24 is introduced to supply a more detailed figure than that published by W. K. PARKER and copied in so many text-books. At the same time it should be observed that this tadpole in one important respect, which is probably diagnostic, differs from PARKER's. It will be seen that the long tentacles in fig. 24 are given off from the angles of the mouth; in fact a groove from the junction of upper and lower lip is often continued up the base of the tentacle. Now, in all PARKER's figures the tentacles are given off above the mouth, from behind the upper lip. The species to which PARKER's specimens belong is, most probably, *Xenopus calcaratus* from Lagos.

The attitude of the tadpole in fig. 24 is that taken when swimming rapidly in a vertical position to the surface of the water for air. The hind leg is stretched back, as in a swimming Urodele, and the resistance of the water as it shoots up seems to sway back the slender tentacles, usually directed straight forwards, into the position figured.

There are three points in connection with this stage still to be mentioned, regarding the tentacles, the coloration, and the fore limb.

BEDDARD mentions that he "more than once observed the tentacle of one side to be bifid." This I found to be quite frequent among a limited number which reached this stage (length of 60 mm.); out of eight specimens, six had both tentacles branched. Four of these, picked at random, are reproduced here. It will be seen that in A, B, and



TEXT FIG. 3.—A, B, C, and D. Tentacles of the right and left sides of four Tadpoles (*Xenopus lævis*) about 60 mm. long.

D the right and left tentacles are symmetrical as far as the general arrangement of the branches is concerned, but the position and sizes of the branches differ on the two sides, so that the symmetry is very imperfect. In C the right tentacle has a secondary branch on its backwardly directed fork which is not represented on the left side. Branched tentacles have only been found in the late tadpole stages over 50 mm. long.

As BOULENGER pointed out (footnote to LESLIE, '90), the tentacle of *Xenopus* may be homologised with the balancers of Urodele larvæ. These must be the representatives on the mandibular arch of the external gills on the branchial arches. If these homologies are correct, the tentacle in *Xenopus* is an external gill, and this conclusion is supported by the fact that from its very earliest appearance it has a capillary loop doubling into it, supplied from the dorsal end of the first branchial aortic arch. The branches figured above are, on this hypothesis, the result of a tendency to branch in a persisting gill similar to that found in all external gills which persist for some time during the life or, as in *Proteus* and *Siren*, for the whole of the life of the individual. In the last two animals the tendency to form branches and secondary branches is especially well marked. If the branching is put down to regeneration after injury, how is the bilateral symmetry to be accounted for?

Coloration.—A most remarkable feature in the behaviour of the chromatophores is found in tadpoles of 15–18 mm. and onwards. As is not uncommon among tadpoles, the dark chromatophores on the head and trunk contract at night into spherical masses, but what is most unusual, if not at present unparalleled in any vertebrate, is the fact that other chromatophores, apparently of the same nature, namely, those in the distal half of the tail, expand at dusk as the others are contracting.

The end of the tail, which in the day-time is so transparent that the presence of chromatophores would never be suspected, becomes, to the naked eye, jet black after nightfall. The expansion takes place in the chromatophores in the fin-fold, but not in those on the myotomes of the same (distal) part of the tail; the latter contract at night in harmony with those in the trunk region and anterior myotomes of the tail. The general effect is well shown in fig. 24, although the contrast is much stronger in the living animal, where the pale regions become of glassy transparency. The explanation is, I believe, due to a need for protective colouring in the transparent part of the tail tip. It is kept undulating constantly; in the daylight it needs no pigmentation—it is protected by its transparency; but the refractive stellate cells of the mesenchyme in the fin-fold would at night be liable to catch and reflect any stray light rays, and the expanded chromatophores effectively prevent this. Moreover, they are absent in the part of the fin-fold which does not move actively. But when the physiology of the case is considered it seems to make the solution of the problem of control over the pigment cells more difficult than ever. Taking tadpoles from daylight into a darkened room has the same effect as the changes from day to night.

In figs. 20 and 21, Plate III, the appearance of the skin is shown under the microscope of the part of the tail marked with a cross in fig. 24, in a specimen killed in the daytime

(fig. 20) and one killed during the night (fig. 21). The pale brown network in fig. 20 suggests, by the sharp double outlines occurring in many places, the presence of intercellular passages into which the chromatophores expand radially and leave pigment granules behind adhering to the walls of the passages.

The Fore-limb.—In fig. 24 the fore-limb is seen lying under a transparent patch of integument, and in fig. 25 this region is shown magnified in a slightly older larva. Here the arm has burst through the thin wall of the sac, the edges of which are still present, and it can be seen that the wall of the branchial chamber immediately in front of the arm-sac is intact, nor is the spiracle affected in any way. This is probably what PARKER was referring to when he stated, "The fore-limbs are not hidden beneath the opercular fold" (PARKER, '76, p. 626). The explanation of the difference between *Xenopus* and the more familiar tadpoles is that in the latter the fore-limb develops in a diverticulum of the gill-chamber which remains in communication with it, so that the developing arm protrudes into the branchial space; in *Xenopus* a similar diverticulum is formed, which becomes completely shut off from the gill-chamber, and the arm cannot encroach on that space. The arm emerges in the common frog by breaking through the wall of the branchial chamber on the right side and by passing through the spiracle on the left side, blocking up this passage completely. The appearance of the arms therefore in typical tadpoles sharply marks the abrupt cessation of branchial respiration. In *Xenopus* the arm appears by the rupture of what may be called the "brachial sac." This event in no way interferes with the habits of the tadpole. It remains floating in mid-water in the same position as before, taking in water at the mouth and passing it out by the spiracles, these being, as shown in fig. 25, Plate IV, quite unaffected by the protrusion of the arms. In fact the branchial current is used here not for respiration, but for nutrition, and is not interfered with during the metamorphosis. The main part of the change into the adult condition is very gradual, and feeding can be continued almost without a break while it is going on.

Metamorphosis.—The metamorphosis is completed ten to twelve weeks after fertilisation, in a constant temperature of 20° to 22° C. At this temperature the whole change from the mature tadpole into the tailless frog is passed through in about 15–20 days, but I have no doubt that in its native African pools the temperature in the late spring will be much higher and the metamorphosis correspondingly much more rapid. The commencement of the metamorphosis is well marked by the protrusion of the arms, the beginning of co-ordinated swimming movements of the legs, and the first appearance of blood-vessels across the width of the fin-fold of the tail. It is a remarkable fact that until this time the whole of the fin-fold is completely non-vascular. The usual sub-vertebral vessels are of course present and supply the tail-myotomes, but they give off, until this period of development, no vessels into the fin. The space between the two bounding layers of integument is filled with a loose trabecular tissue composed of stellate mesenchyme cells, the interstices of which seem to be filled either with fluid or

with a gelatinous matrix. It was not possible to make out any circulation of lymph or movement of lymph-corpuscles in the tail fin. The first capillaries appear near the tip of the tail, and they spread towards the proximal parts, becoming more and more numerous as the time for the resorption of the tail approaches; thus obviously raising the suggestion that the process of resorption is carried on by methods connected with this vascularisation. The tail is not used for the respiratory function.

The limbs now grow rapidly. The arm rotates from the shoulder-joint through 90°, and about forty-eight hours after protrusion has left the position shown in fig. 25, Plate IV. (where the arm hangs down in a plane at right angles to the long axis of the animal) for the adult position: namely, directed forwards towards the mouth and lying in a horizontal plane parallel to the long axis. At first the arms are ridiculously out of proportion to the size of the head; they have lagged behind in development very considerably. The deficiency is made up in three weeks, and by the end of the metamorphosis the length of the arms is such as to allow of the fingers just meeting in front of the head. The legs are developed in the primitive fin position, and, apart from the bending of the limb, are kept in this position throughout life. Soon after the stage of fig. 23, Plate IV, where the leg is still in the primitive position, the thigh is swung round at the hip-joint until it stands out at right angles to the body; the knee-joint is bent so that the tibia remains parallel to its original position, and the ankle is bent so as to turn the foot out through 90°, the sole facing backwards instead of inwards. These changes go on during the three days after the arms emerge; at the same time the black claws appear on the three preaxial digits, some weeks before they come into use. The legs now assist the tail quite efficiently in swimming, and grow rapidly, especially the feet.

The colour of the skin changes gradually in character. Fig. 24 still shows the tadpole coloration; two to three weeks later the adult coloration had been assumed, and then the whole creature becomes a strange mixture of larva and adult. The whole habit is larval; the creature still swims in mid-water in an upright position; there is a long tail, mouth and tentacles are unchanged, spiracles are present, but the body itself, in shape and colour, and also the appendages, are adult in character. In two days after this condition is reached (about fourteen days after the arms emerge) the tentacles have almost disappeared, and then the mouth very quickly transforms from the larval to the adult state. The whole process is finished in four to six hours. When the tentacles are very much shrunken, the angle of the mouth, where they were attached until now, seems to grow back under them; the gape of the mouth is consequently widened, and at the same time the stumps of the tentacles become dorsal to the mouth. A minute basal part of the tentacle persists throughout life in *Xenopus lævis*. As the mouth metamorphoses the spiracles close up.

Immediately after the mouth has transformed, the tailed frog ceases to keep

constantly to its larval free-swimming habits, and spends more and more time lying on the bottom. At this time the tail has begun to atrophy, the blood-vessels in the fin spread, the notochord at the tip becomes wavy, and the pigmentation darkens. At the end of a week the greater part of the tail is absorbed; about one-third of it is left, very deeply pigmented, and the young frog has thus reached the stage at which the typical Phaneroglossan lands and becomes terrestrial. There is not the slightest tendency to land in the case of *Xenopus*. It swims about actively in search of food, and for some weeks lives on small, free-swimming Crustacea. Seven specimens reared to this stage consumed enormous quantities of *Daphnia pulex;* a great swarm of these vanished every twenty-four hours, and the frogs throve.

Their hands are at once used in the grown-up manner to cram the food into their mouths; the arms are not used for progression at all, except to push aside water-weeds—hence one of their functions as limbs has almost disappeared. The size of the arm is altogether out of proportion to the size of the leg, which is an extremely powerful swimming organ. The limbs of *Xenopus* as a frog are paralleled by the limbs of *Macropus* as a marsupial.

When W. K. Parker ('76) described the skull in larval *Xenopus*, he laid stress upon what he considered Chimæroid features in the chondrocranium, and was naturally led to attach morphological importance to the lash-like tail end of the *Xenopus* tadpole. Now, although this close resemblance does not exist, there is a certain degree of resemblance which suggests similarity of function. The end of the tail of the *Xenopis* tadpole has a very narrow dorsal and ventral fin-fold (see figs. 23 and 24, Plate IV), and it is easy to see in the living animal that the constant undulatory movement of this narrow membrane has very little propelling power. The suggestion is, then, that the *Xenopus* tadpole, *Chimæra*, and such fishes with a narrow lash-like tail end as the Mormyridæ, use that part for suspending themselves either in mid-water or, in the case of bottom-feeders or mud-feeders, just over the bottom, by means of a rapid undulatory movement.

Sexual maturity appears to be reached at an early age. One male was kept until two years old, when it began to pair.

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EXPLANATION OF PLATES.

Plate I.

Xenopus lævis, Daud. Segmentation stages, blastopore formation, medullary plate, and early embryo. All figures on this plate numbered alike are drawn from the same egg, and figs. 1–9 are magnified × 28·5.

Fig. 1. Unfertilised egg.

Fig. 2. Egg with eight blastomeres, seen from the right side.

Fig. 2a. Seen from below.

Fig. 3. Egg with about thirty-two blastomeres, seen from anterior side.

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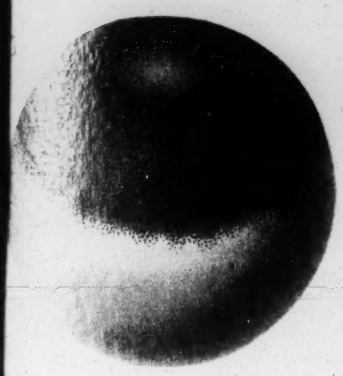

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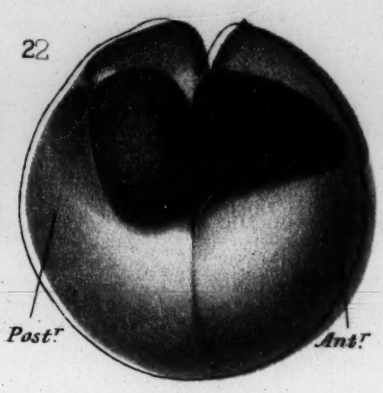

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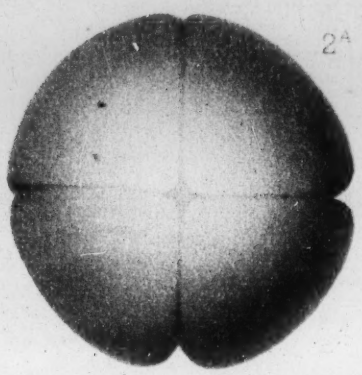

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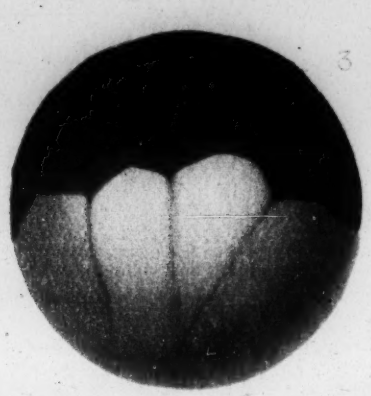

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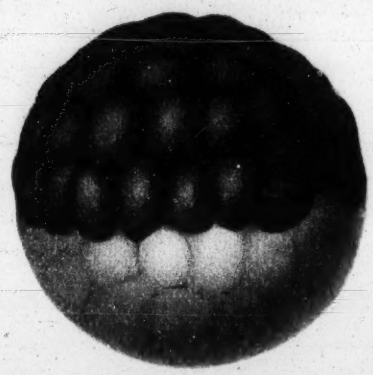

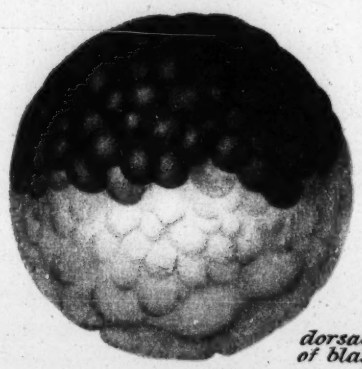

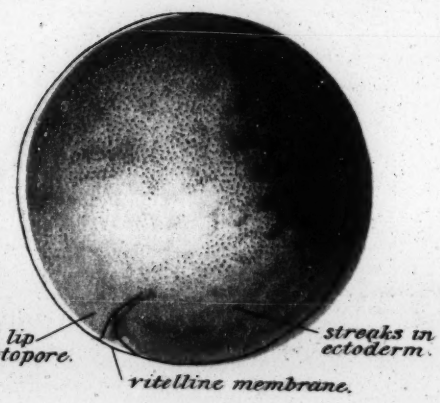

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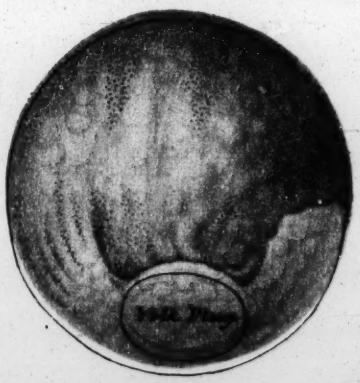

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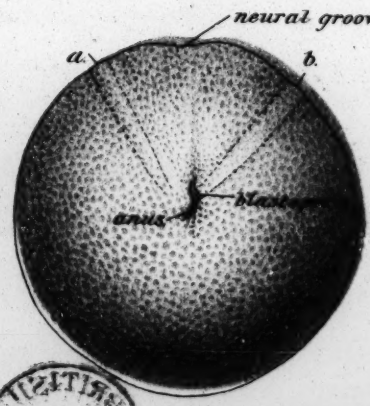

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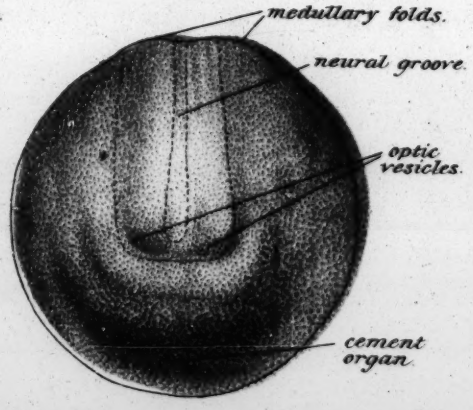

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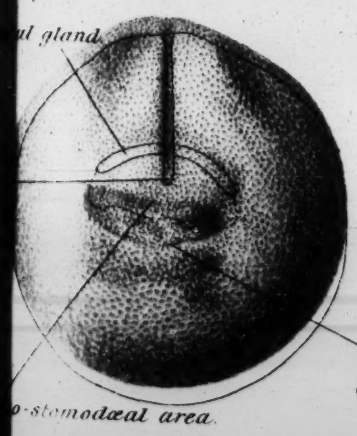

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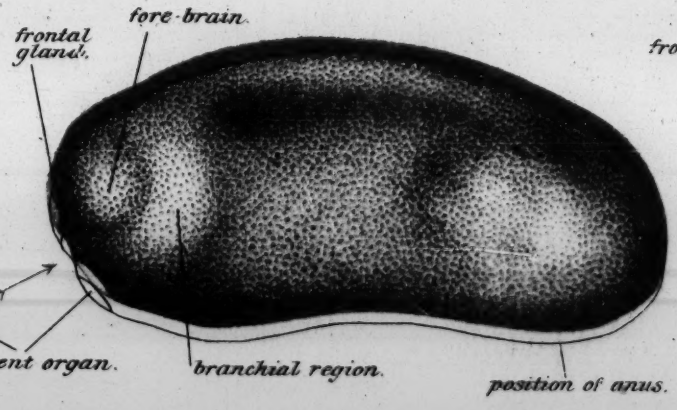

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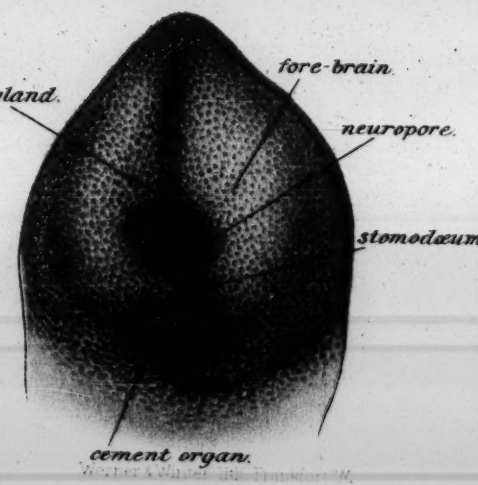

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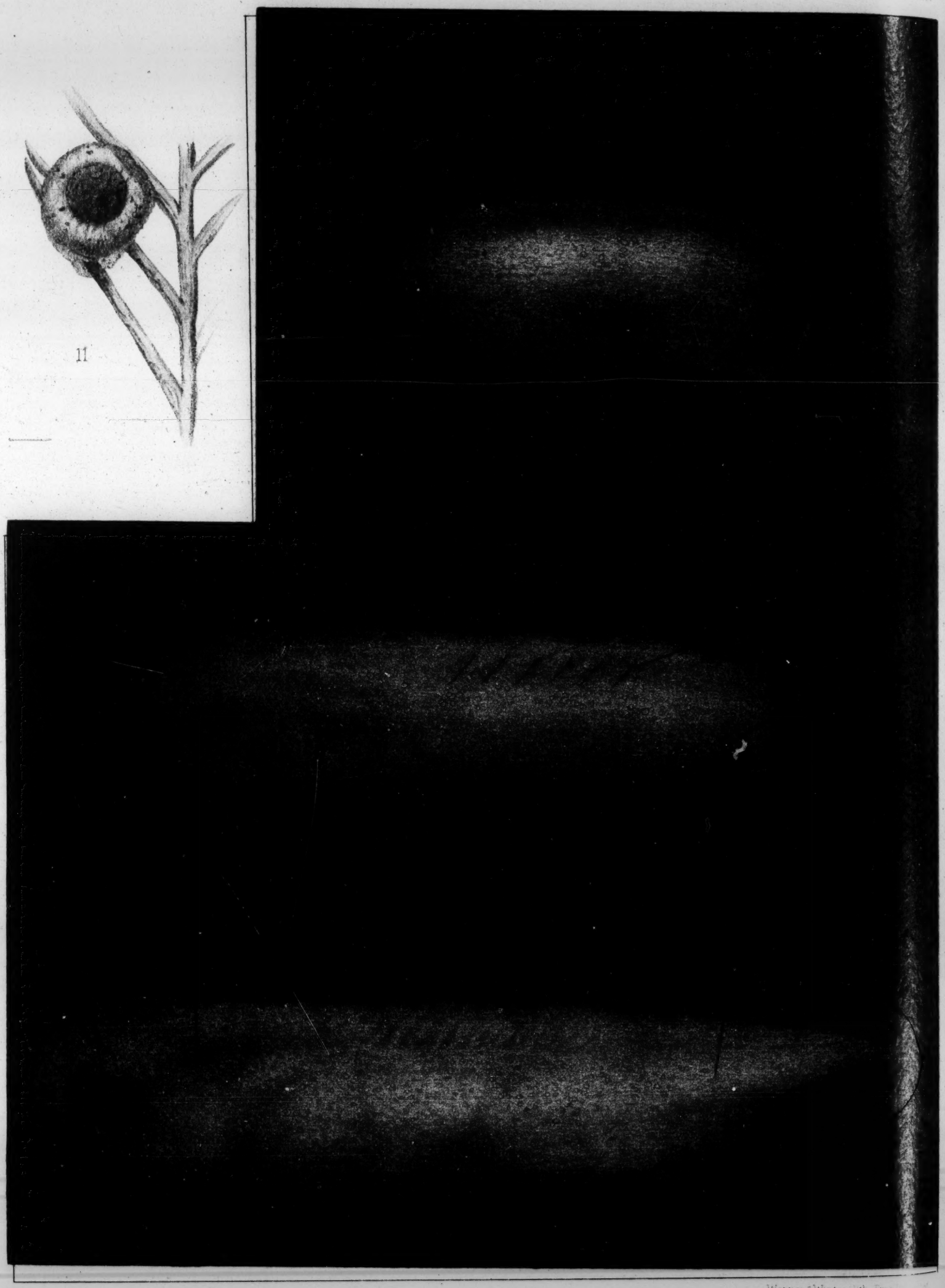

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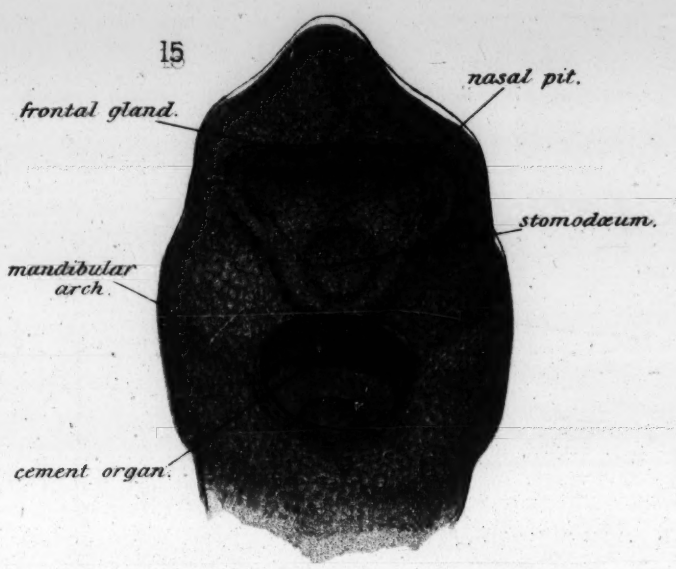

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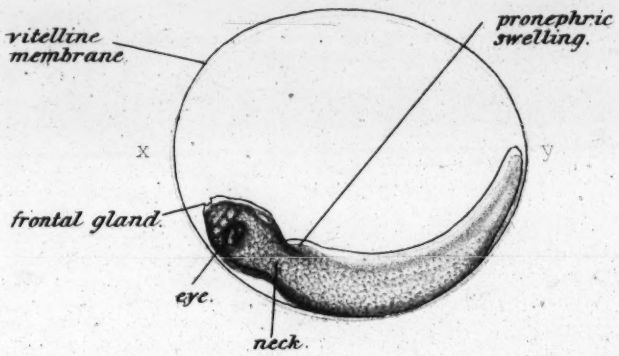

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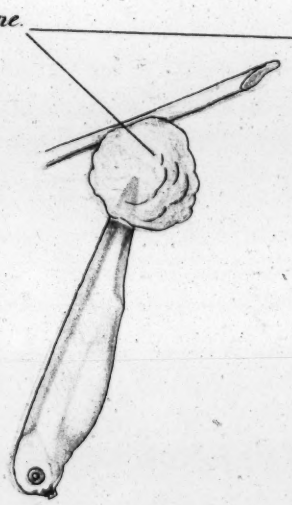

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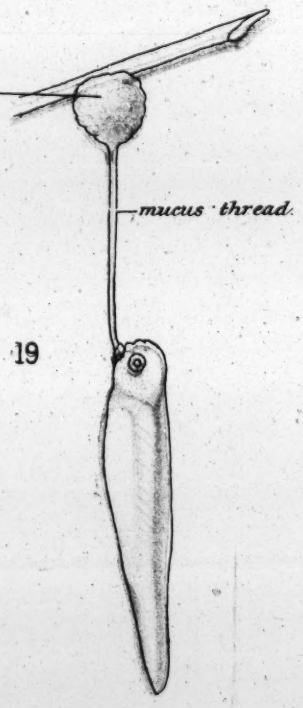

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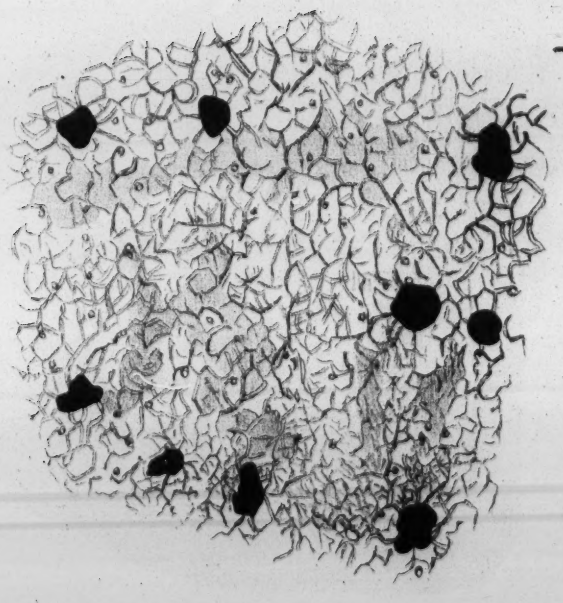

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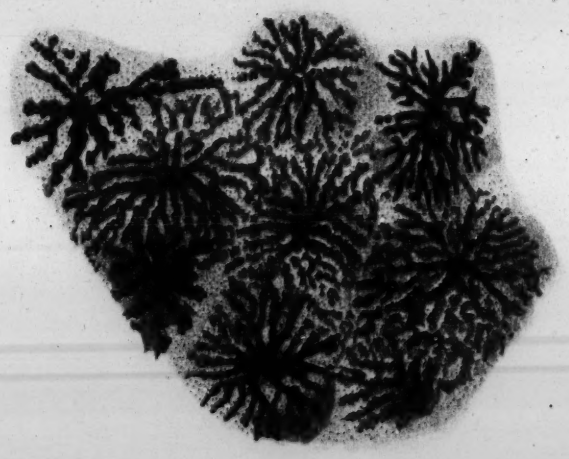

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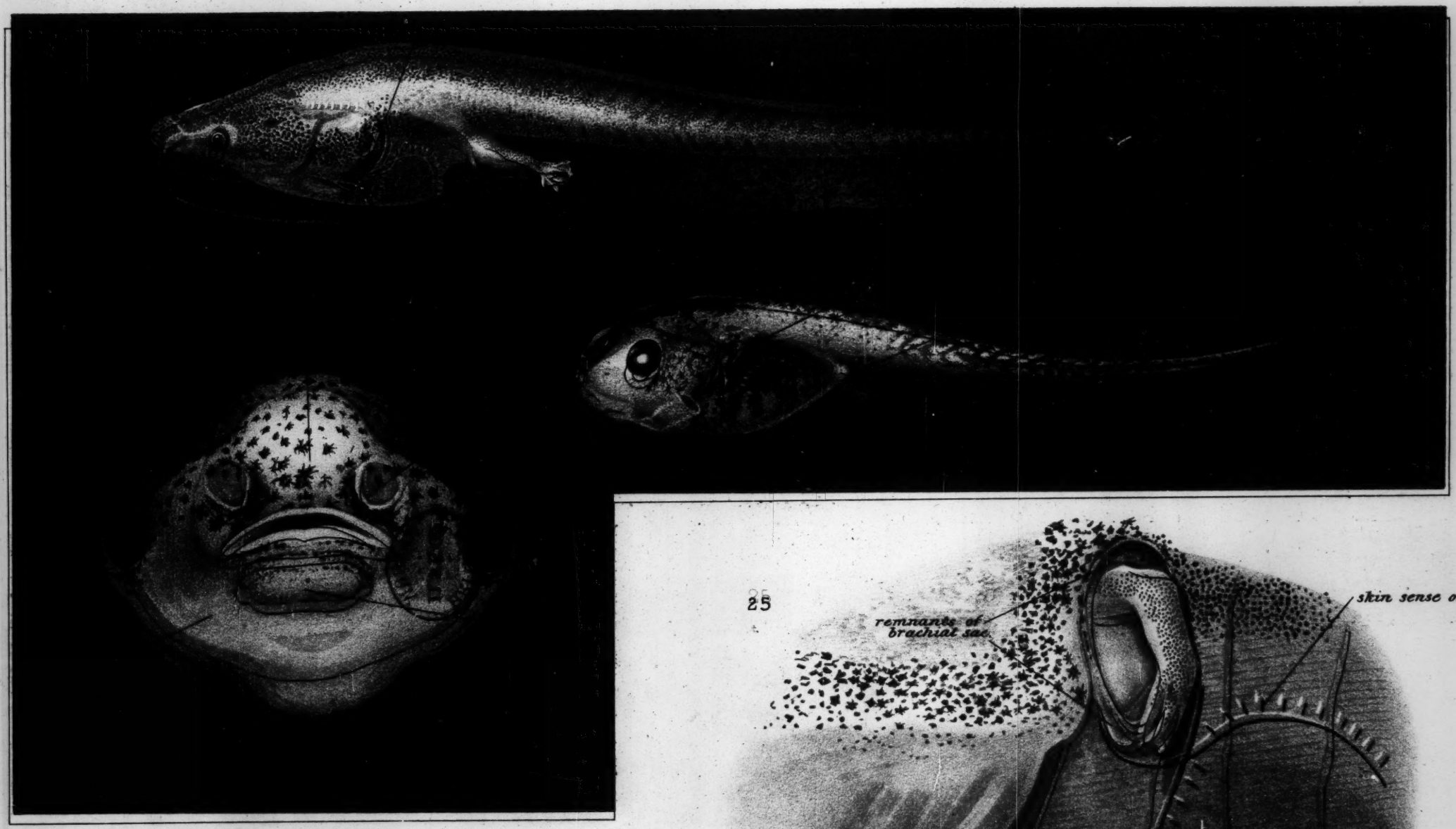
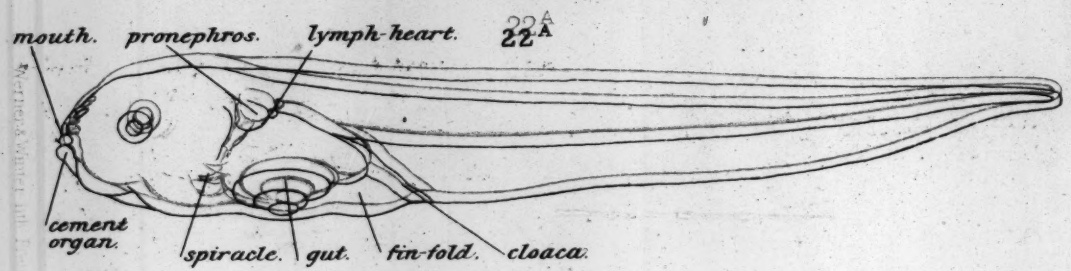



mouth.
pronephros.
lymph-heart.
22A
cement organ.
spiracle.
gut.
fin-fold.
cloaca.

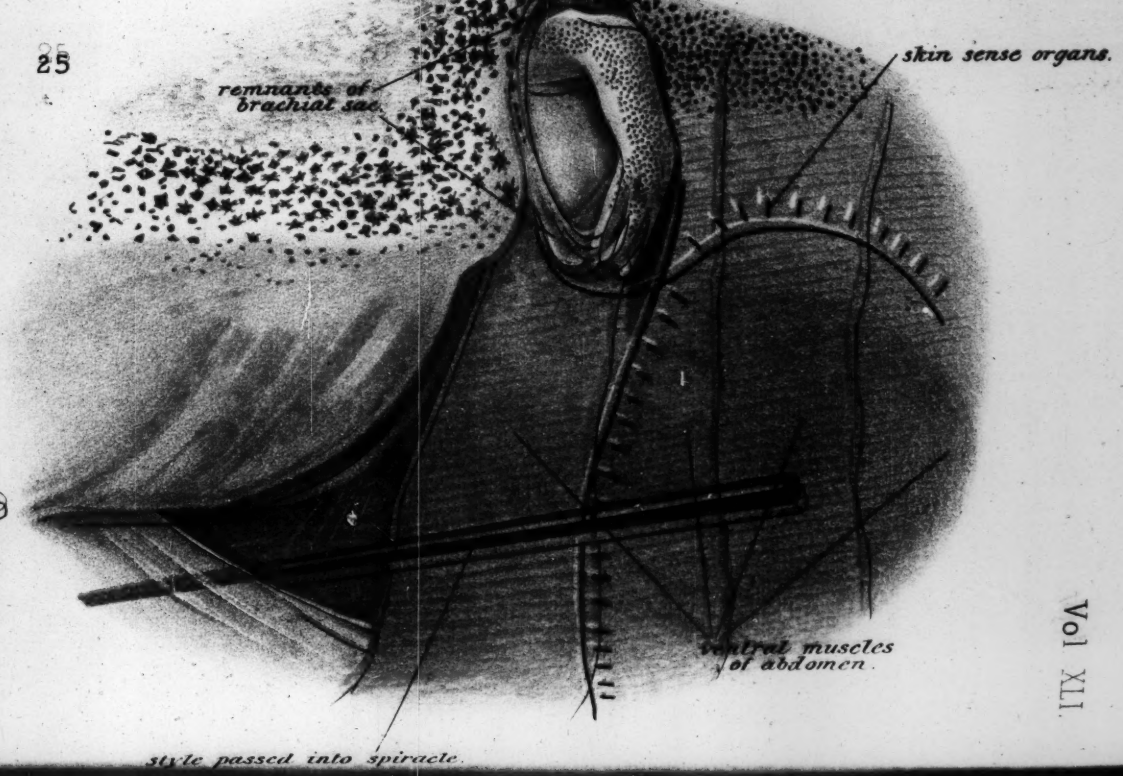

25
skin sense organs.
remnants of brachial sac.
ventral muscles of abdomen.
style passed into spiracle.

Fig. 3A. Seen from below.

Fig. 4. Early blastula stage.

Fig. 5. Late blastula stage.

Fig. 6. Egg with early stage in the formation of the blastopore. The egg is in its natural position, and is seen from the right side.

Fig. 6A. The same stage seen from below.

Fig. 7. Stage showing circular blastopore with large yolk plug.

Fig. 8. Stage with open medullary groove, seen from behind.

Fig. 8A. The same stage seen from before.

Fig. 9. Stage after closure of medullary groove.

Fig. 10. Embryo, 3 mm. long, taken out of egg. (× 25.)

Fig. 10A. Anterior end of the same embryo viewed in the direction of the arrow in fig. 10. (× 48.)

PLATE II.

Fig. 11. Egg deposited on a leaf of *Myriophyllum proserpinnacoides*. (× 6.)

Fig. 12. Embryo, 3·2 mm. long, in vitelline membrane; the jelly surrounding this has been stripped off. Drawn from life. (× 25.)

Fig. 13. Embryo, 3·8 mm. long, taken from egg. Drawn from life. (× 25.)

Fig. 14. Embryo, 5 mm. long, taken from egg. Drawn from life. (× 25.)

PLATE III.

Fig. 15. Head of larva just hatched, seen from before, in the direction of the arrow in fig. 15A. (× 60.)

Fig. 15A. Outline of larva, 5 mm. long, just hatched, seen from side. (× 15.)

Fig. 16. Larva, 4·5 mm. long, lying in the egg attached to the vitelline membrane by the cement organ before hatching commences; the larva is seen from the ventral side. The layer of jelly outside the vitelline membrane is omitted. (× 11.)

Fig. 17. A, B and C. Head of larva during process of hatching, to show the yielding vitelline membrane. A., 8′ 15″ before hatching. B., 3′ 15″ before hatching. C., 15″ before hatching. (× 12.)

Fig. 18. The position of the same larva when it emerged from the egg. The vitelline membrane has shrunk and caught the tip of the tail. (× 10.)

Fig. 19. The same larva as in figs. 17 and 18, as seen a few seconds later, hanging by a thread of mucus to the further shrunken vitelline membrane. (× 10.)

Fig. 20. Portion of the skin of the tail-fin of a tadpole, 62 mm. in length, killed during the daytime, from the middle of the length of the tail. (× 170.)

Fig. 21. A corresponding portion of the tail-fin of the tadpole drawn in fig. 24, which was killed at night. (× 170.)

PLATE IV.

Fig. 22. Anterior view of the head of a larva 10 mm. long. The stage reached is a few hours earlier than that at which the animal begins to feed. (× 38.)

Fig. 22A. Outline sketch of the same larva as fig. 22 in side view. (× 12.) (From a formalin specimen.)

Fig. 23. Side view of a tadpole 12·4 mm. long, which has fed for two days. (× 11.) (From a formalin specimen.)

Fig. 24. Side view of a tadpole, 60 mm. long, with the arm not yet extruded. Drawn in the position in which the rapid ascent to the surface is made to obtain air. (× 3.) (From a specimen killed at night and preserved in alcohol-formalin-acetic acid mixture.)

Fig. 25. The left pectoral region of a tadpole, 62 mm. long, slightly older than that shown in fig. 24, to show the extruded arm and the spiracle into which a style has been passed. (× 13.) (Preservation as described above, fig. 24; killed at nightfall.)

XXXII.—Calculation of the Periods and Nodes of Lochs Earn and Treig, from the Bathymetric Data of the Scottish Lake Survey. By Professor Chrystal and Ernest Maclagan-Wedderburn, M.A. (With Two Maps.)

(Read July 17, 1905. Issued separately November 7, 1905.)

§ 1. In a paper* published in the present volume of the *Transactions* of the Society (p. 659, 3rd July 1905), one of the authors of this communication has given a Hydrodynamical Theory of Seiches, and deduced formulæ with a view to particular applications. The object of the present paper is to give some account of a series of calculations on which we have been occupied for some time, with a view to work out, *à priori*, from the Bathymetric Data of the Lake Survey, the Seiche Constants for the three seiches of lowest nodality in Lochs Earn and Treig.

§ 2. As this is the first time that any attempt has been made to solve completely a problem of the kind, we naturally selected lakes having as simple a configuration as possible. A general idea of the shape of the two lakes, and also of the nature of the data from which we have worked, will be obtained from the 3-in. maps reproduced in figures (3) and (4).

The numbers on the maps indicate the depths in feet. They are mostly arranged in lines (numbered from the deeper to the shallower end, and spoken of as "sounding lines") nearly perpendicular to the longitudinal direction of the lake.

§ 3. The first step is to construct the *Normal Curve* of the Lake (H.T.S. § 20). The line A′ A, drawn as nearly as possible through the deepest soundings, is straightened and taken as the x-axis of the theory. For convenience in identifying sounding lines and intermediate sections, x, measured in arbitrarily chosen units, fixes the distance of any section from the deeper end A′ of the lake. The area, $A(x)$, of the vertical section through each sounding line was calculated or measured by the planimeter. This multiplied by $b(x)$, the breadth of the section at the surface, the units in both cases being feet, gives $\sigma = A(x)\,b(x)$, the ordinate of the normal curve corresponding to each section. The section for which σ is greatest (No. 10 in Earn, No. 5 in Treig) was chosen for the origin, O, of the variable, v. Thus in Earn v is the area between the sounding line No. 10 and the surface line of the section corresponding to x, the sign of v being positive or negative according as the corresponding section is towards the shallower or the deeper end of the lake. The values of v for A and A′ are denoted by α and α'; these and the values corresponding to the various sounding lines were found from the map by means of the planimeter.

The next step was to fix upon two parabolas having a common vertex immediately under O, each passing through one of the ends of the σ-v-curve, so as to get as

* Denoted hereafter by the letters H.T.S.

simple a mathematical approximation as possible. There is here, of course, an element of arbitrariness; but we may expect from the general conclusions of the Hydrodynamic Theory that, within reasonable limits, the results will not be greatly affected by the particular form of approximation selected.

The two parts of the normal curve are now represented by

$$\sigma = h(1 - v^2/a'^2), \quad \sigma = h(1 - v^2/a^2) \quad . \quad . \quad . \quad . \quad . \quad (1),$$

say

$$\sigma = ha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where $a = 1 - v^2/a'^2$ or $1 - v^2/a^2$ according as the section in question is towards the deeper or the shallower end. The depth of the common vertex of the two parabolas is still at our disposal, and its choice is another arbitrary element in our calculations. To avoid any possibility of bias, we elected to determine h so that the sum of the squares of the differences between the observed and calculated values of σ at the sounding lines should be a minimum. That is, σ denoting the observed value, given, for example, in the column of the table on p. 826, we choose h so that $\Sigma(ha - \sigma)^2$ is a minimum. We thus get

$$h = \Sigma\sigma a / \Sigma a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

The biparabolic curve which is to replace the normal curve of the lake is now known. The divergence of the actual curve from the theoretical one will be seen from figures (1) and (2), where the dots indicate the points on the actual normal curve at the various sounding lines. Theoretically it would be necessary to correct for these deviations by RAYLEIGH'S method (H.T.S. § 21); but in practice that has not, as yet, been found necessary.

§ 4. The rest of the work in the case of each lake consists in solving by approximation the equations (44) and (45) for the Periods; and equations furnished by (46) and (47), or by (65), for the Nodes.

The tables on pp. 826, 839 give the principal elements in the calculation of h for the two lakes. To secure accuracy, the logarithms of the various quantities were tabulated in the unabridged scheme; so that any gross error in the work might be readily detected by watching the progression of the differences.

For convenience of reference, we have also inserted in the tables the co-ordinates of the various nodes, which are calculated later on in this paper.

Fig. 1. NORMAL CURVE FOR EARN.

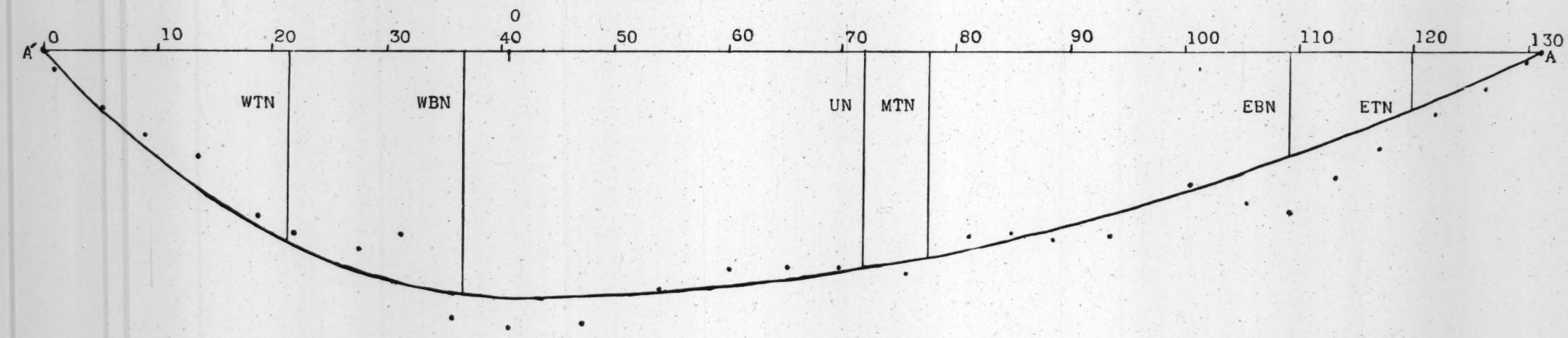
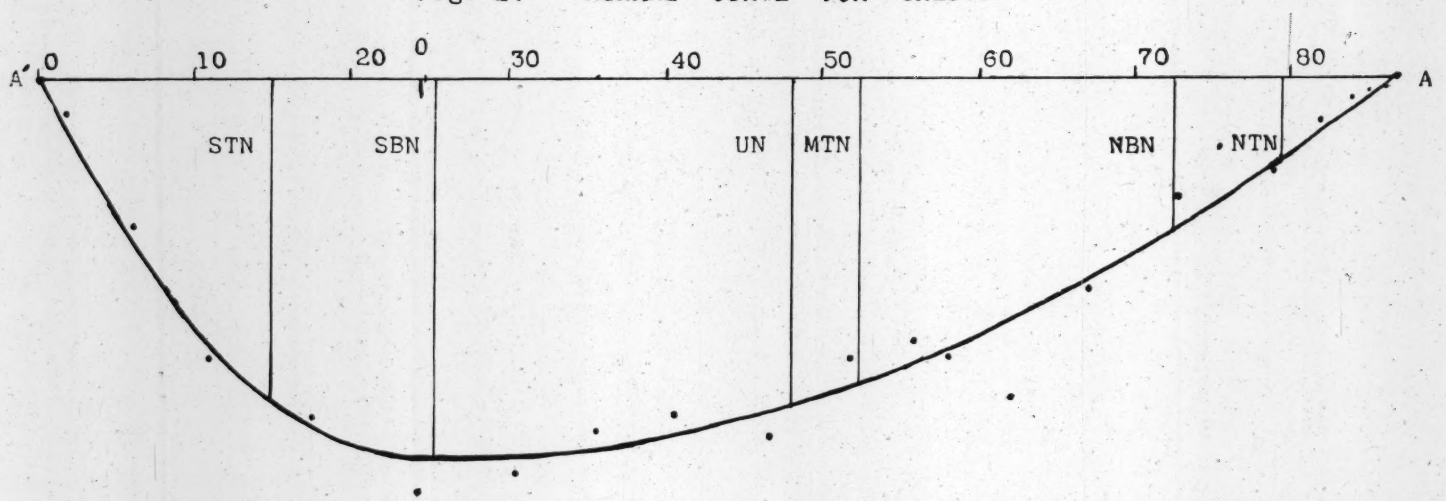


Fig. 2. NORMAL CURVE FOR TREIG.



Data for Loch Earn.

No. of Sounding Line.	x Unit 692·9 Feet.	v Unit 4801×10^2 Sq. Feet.	σ Unit 10^5 Cub. Feet.	a	a^2	σa Unit 10^5 Cub. Feet.	
0	0·0	65·5	0	·0000	·0000	0	
1	0·5	63·9	1467	·0483	·0023	71	$a = 6986 \times 10^4$
2	2·6	57·5	4939	·2293	·0526	1132	$a' = 3145 \times 10^4$
3	4·3	51·2	7422	·3890	·1513	2887	$\rho = a/a' = 2·2214$
4	6·2	43·6	9404	·5569	·3101	5237	$\rho^2 = 4·9345$
5	8·2	35·0	14383	·7145	·5105	10275	$\log \rho = ·34662$
W. Trinode	[9·2]	[30·9]					$h = 2209 \times 10^6$
6	9·4	29·9	16072	·7916	·6266	12722	
7	11·0	21·0	17578	·8972	·8050	15773	
8	12·4	15·3	16119	·9454	·8938	15240	
9	13·7	7·9	23918	·9854	·9710	23571	
W. Binode	[14·0]	[6·0]					
10	15·4	0·0	24853	1·0000	1·0000	24850	
11	16·9	10·6	24494	·9947	·9894	24360	
12	19·1	21·6	21468	·9780	·9565	20998	
13	21·1	31·2	19381	·9540	·9101	18488	
14	22·8	39·5	19220	·9263	·8580	17803	
15	24·4	46·9	19353	·8961	·8030	17340	
Uninode	[25·2]	[50·7]					
16	26·5	56·4	19727	·8497	·7220	16765	
Mid. Trinode	[27·1]	[59·8]					
17	28·2	65·4	16506	·7980	·6368	13175	
18	29·9	71·0	16211	·7619	·5805	12351	
19	31·3	77·1	16702	·7192	·5172	12011	
20	33·1	85·4	16408	·6555	·4297	10757	
21	35·3	96·4	11899	·5610	·3147	6676	
22	37·1	104·7	13415	·4822	·2325	6471	
23	39·0	110·8	14175	·4201	·1765	5957	
E. Binode	[39·1]	[111·0]					
24	40·6	117·2	11110	·3512	·1233	3902	
25	42·0	123·7	8574	·2772	·0768	2377	
E. Trinode	[43·1]	[127·7]					
26	44·0	131·0	5468	·1894	·0359	1036	
27	45·8	138·2	3261	·0978	·0096	319	
28	47·5	142·5	758	·0408	·0017	31	
29	49·4	145·1	21	·0055	·0000	0	
30	50·4	145·5	0	·0000	·0000	0	
$h = \Sigma\sigma a/\Sigma a^2 = 2209 \times 10^6$.					13·6974	302575×10^5	

Periods of Loch Earn.

§ 5. If we calculate T_1, first on the supposition that Earn is a symmetric parabolic lake, secondly on the supposition that it is a semiparabolic lake, we get the following values :—

$$T_1 = \pi(a+a')/\sqrt{(2gh)} = 14·06' \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

$$T_1' = \pi(a+a')/\sqrt{(6gh)} = 16·24' \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

We may take the mean of these (H.T.S. § 35) as a first approximation, viz., $T_1 = 15{\cdot}15'$. A few rough trials shew that T_1 is very near to $14{\cdot}5'$. Taking this as a basis, and using the table of H.T.S. § 38, we get the following table :—

	T_1	T_2	T_3
Parabolic . . .	14·5	8·33	5·91
Semiparabolic . . .	14·5	7·94	5·48
Means	14·5	8·13	5·69

We therefore take $T_1 = 14{\cdot}5$, $T_2 = 8{\cdot}1$, and $T_3 = 5{\cdot}7$ as first approximations in our more rigorous calculations.

§ 6. The most logical procedure would be to calculate the value of c (or c') from the equation

$$\chi(c) \equiv K(c, 1) + \rho K(c', 1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6);$$

and deduce the value of T from the equation

$$T = 2\pi a/\sqrt{(cgh)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Seeing, however, that good first approximations to T are known, it is convenient to assume values of T; calculate the corresponding values of c and c' by means of the equations

$$c = 4\pi a^2/ghT^2, \quad c' = 4\pi a'^2/ghT^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad (8);$$

then calculate the corresponding values of $\chi(c)$; and finally find the value of T corresponding to the root c_1 by interpolation.

The following tables give the principal elements of the calculation. The letters used have the same meanings as in H.T.S. §§ 25 and 39.

§ 7. UNINODAL PERIOD OF EARN.

T	c	a	$\frac{1}{4}(5+a)$	$\frac{1}{4}(5-a)$	$\frac{1}{4}(3+a)$	$\frac{1}{4}(3-a)$	$K(c, 1)$
14·4	3·6284	3·9383	2·2346	·2654	1·7346	−·2346	−1·62507
14·5	3·5785	3·9133	2·2283	·2717	1·7283	−·2283	−1·54970
14·6	3·5297	3·8883	2·2221	·2779	1·7221	−·2221	−1·47850

T	c'	a'	$\frac{1}{4}(5+a')$	$\frac{1}{4}(5-a')$	$\frac{1}{4}(3+a')$	$\frac{1}{4}(3-a')$	$K(c', 1)$
14·4	·7353	1·9853	1.7463	·7537	1·2463	·2537	·69215
14·5	·7252	1·9750	1·7438	·7563	1·2438	·2563	·69685
14·6	·7153	1·9650	1·7412	·7587	1·2412	·2587	·70105

The value of $K(c, 1)$ is readily calculated by means of an ordinary table of logarithms together with LEGENDRE's table for $\log \Gamma(x)$.*

Thus $$K(3{\cdot}6284, 1) = 2\Gamma(2{\cdot}2346)\Gamma({\cdot}2654)/\Gamma(1{\cdot}7346)\Gamma(-{\cdot}2346),$$
$$= -\frac{2 \times 1{\cdot}7654 \times {\cdot}2346 \times {\cdot}7654\Gamma(1{\cdot}2346)\Gamma(1{\cdot}2654)}{{\cdot}2654\Gamma(1{\cdot}7346)\Gamma(1{\cdot}7654)}.$$

Hence $\log\{-K(3{\cdot}6284, 1\} = {\cdot}21087$; and $K(3{\cdot}6284, 1) = -1{\cdot}62507$.

The calculation of $K({\cdot}7353, 1)$ is even simpler.

If $\chi(c) = K(c, 1) + \rho K(c', 1)$, where $\rho = 2{\cdot}2214$, we have now the following table :—

T	$\chi(c)$	Diff.
14·4	−·08755	·08571
14·5	−·00174	·08053
14·6	+·07879	

Hence, interpolating by means of the first difference, we get finally

$$T_1 = 14{\cdot}50'.$$

Also corresponding to the value of T_1, we have

$$c_1 = 3{\cdot}5775; \quad c'_1 = {\cdot}7250.$$

§ 8. BINODAL PERIOD OF EARN.

T	c	α	$\frac{1}{4}(5+\alpha)$	$\frac{1}{4}(5-\alpha)$	$\frac{1}{4}(3+\alpha)$	$\frac{1}{4}(3-\alpha)$	$K(c, 1)$
8·10	11·4673	6·8461	2·9615	−·4615	2·4615	−·9615	+·40305
8·14	11·3541	6·8130	2·9533	−·4533	2·4533	−·9533	+·48754
8·20	11·1893	6·7644	2·9411	−·4411	2·4411	−·9411	+·61286

T	c'	α'	$\frac{1}{4}(5+\alpha')$	$\frac{1}{4}(5-\alpha')$	$\frac{1}{4}(3+\alpha')$	$\frac{1}{4}(3-\alpha')$	$K(c', 1)$
8·10	2·3239	3·2087	2·0522	·4478	1 5522	−·0522	−·23022
8·14	2·3011	3·1944	2·0486	·4514	1·5486	−·0486	−·21271
8·20	2·2676	3·1734	2·0434	·4567	1·5434	−·0434	−·18800

T	$\chi(c)$	Diff. for ·01
8·10	−·11105	·03152
8·14	+·01502	·03014
8·20	+·19525	

$$T_2 = 8{\cdot}135' = 8{\cdot}14' \text{ say.}$$
$$c_2 = 11{\cdot}3598; \quad c'_2 = 2{\cdot}3022.$$

* The abridgment in WILLIAMSON's *Integral Calculus* is convenient. I have noted two errors in it. Log $\Gamma(1{\cdot}573)$ should be 9·949761; and log $\Gamma(1{\cdot}669)$, 9·955750.

§ 9. Trinodal Period of Earn.

T	c	a	$\frac{1}{4}(5+a)$	$\frac{1}{4}(5-a)$	$\frac{1}{4}(3+a)$	$\frac{1}{4}(3-a)$	K(c, 1)
5·700	23·1580	9·6737	3·6684	−1·1684	3·1684	−1·6684	+8·1439
5·725	22·9559	9·6345	3·6586	−1·1586	3·1586	−1·6586	+8·7133
5·750	22·7565	9·5930	3·6483	−1·1483	3·1483	−1·6483	+9·3860

T	c'	a'	$\frac{1}{4}(5+a')$	$\frac{1}{4}(5-a')$	$\frac{1}{4}(3+a')$	$\frac{1}{4}(3-a')$	K(c', 1)
5·700	4·6930	4·4659	2·3665	·1335	1·8665	− ·3665	−4·7315
5·725	4·6521	4·4282	2·3570	·1430	1·8570	− ·3570	−4·2585
5·750	4·6117	4·4099	2·3525	·1475	1·8525	− ·3525	−4·0900

T	(c)	Diff.
5·700	−2·1866	1·4400
5·725	− ·7466	1·0473
5·750	+ ·3007	

$T_3 = 5{\cdot}743' = 5{\cdot}74'$ say.

$c_3 = 22{\cdot}8123$; $c'_3 = 4{\cdot}6230$.

§ 10. Periods by Du Boys' Rule.

The following table gives the details of the calculation of the Uninodal period by the Rule of Du Boys. The distances from the western end of the lake are denoted by x, the unit being $2 \times 1760 = 3520$ feet; while the depth h is entered in feet. The integration $T_1 = 3520/\sqrt{(32{\cdot}2)}\int_0^{19{\cdot}84} dx/\sqrt{h}$ proceeds along the line of greatest depth as indicated by the rule.

[Table.]

x	dx	h	$\sqrt{h}$	Mean $\sqrt{h}$	1/mean $\sqrt{h}$	$dx/\sqrt{h}$
0·00						
0·20	·20	57	7·55	3·77	·2653	·0531
1·00	·80	125	11·18	9·37	·1067	·0854
2·00	1·00	177	13·30	12·26	·0816	·0816
3·00	1·00	231	15·20	14·25	·0702	·0702
4·00	1·00	231	15·20	15·20	·0658	·0658
5·00	1·00	255	15·97	15·58	·0642	·0642
6·00	1·00	273	16·52	16·24	·0616	·0616
7·00	1·00	279	16·70	16·61	·0602	·0602
8·00	1·00	279	16·70	16·70	·0599	·0599
9·00	1·00	273	16·52	16·61	·0602	·0602
10·00	1·00	258	16·06	16·29	·0614	·0614
11·00	1·00	246	15·68	15·87	·0630	·0630
12·00	1·00	246	15·68	15·68	·0638	·0638
13·00	1·00	243	15·59	15·64	·0639	·0639
14·00	1·00	222	14·90	15·25	·0656	·0656
15·00	1·00	207	14·39	14·64	·0683	·0683
16·00	1·00	172	13·11	13·75	·0727	·0727
16·54	·54	163	12·77	12·94	·0773	·0417
17·32	·78	127	11·27	12·02	·0832	·0649
18·04	·72	135	11·62	11·45	·0873	·0629
18·70	·66	45	6·70	9·16	·1092	·0721
19·45	·75	13	3·61	5·16	·1938	·1454
19·84	·39	0	0·00	1·81	·5525	·2155
						1·7234

Hence $_d\mathrm{T}_1 = 3520 \times 1{\cdot}7234/\sqrt{32{\cdot}2} = 1069'' = 17{\cdot}82'$.
Therefore $_d\mathrm{T}_2 = 8{\cdot}91'$,
$_d\mathrm{T}_3 = 5{\cdot}94'$.

The periods given by Du Boys' rule are therefore all greater than those deduced from the Hydrodynamical Theory; but the divergence gets less as the nodality rises, the deviations for $_d\mathrm{T}_1$, $_d\mathrm{T}_2$, $_d\mathrm{T}_3$ being 23 per cent., 9·4 per cent., and 3·5 per cent. respectively.

Nodes of Loch Earn.

§ 11. It appears from H.T.S. §§ 39 and 42 that we may calculate the values of w corresponding to the nodes by means of the equations:—

$$\phi(w) = \mathrm{S}'(c, w)\,\mathrm{K}(c, 1) - \mathrm{C}'(c, w) = 0 \quad . \quad . \quad . \quad . \quad . \quad (9),$$
$$\phi(w) = \mathrm{S}'(c', w)\,\mathrm{K}(c', 1) - \mathrm{C}'(c', w) = 0 \quad . \quad . \quad . \quad . \quad . \quad (10):$$
or $$\mathrm{L}'(c, z) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$
$$\mathrm{L}'(c', z) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12);$$

where w denotes the *scalar* value of v/a or v/a' in all cases; $z = (1-w)/2$; and we use (9) or (10) and (11) or (12) according as we are looking for nodes on the shallower or on the deeper side of the deepest normal point of the lake.

To facilitate calculation we introduce the following notation :—

$$\left.\begin{aligned} &C'(c, w) = -T_1 - T_3 - T_5 - \ldots - T_{2n+1} - R_{2n+1}; \\ \text{where}\quad &T_{2n+1} = A_{2n+1} w^{2n+1}/(2n+1), \\ &A_{2n+1} = c(1 - c/1.2) \ldots\ldots (1 - c/(2n-1)2n), \\ &A_1 = c. \end{aligned}\right\} \quad \ldots \quad (13),$$

$$\left.\begin{aligned} &S'(c, w) = 1 - T_2 - T_4 - \ldots\ldots - T_{2n} - R_{2n}; \\ \text{where}\quad &T_{2n} = A_{2n} w^{2n}/2n, \\ &A_{2n} = c(1 - c/2.3) \ldots\ldots (1 - c/(2n-2)(2n-1)), \\ &A_2 = c \end{aligned}\right\} \quad \ldots \quad (14).$$

If we take n so great that $c/(2n-1)2n$ or $c/(2n-2)(2n-1) < 1$, then

$$|R_{2n+1}| < \frac{|A_{2n+3}|}{2n+3} w^{2n+3}(1 + w^2 + w^4 + \ldots \text{ ad } \infty);$$

$$< \frac{A_{2n+3}}{2n+3} \frac{w^{2n+3}}{1-w^2}.$$

That is,

$$|R_{2n+1}| < |T_{2n+3}|/(1 - w^2) \quad \ldots \quad (15).$$

In like manner,

$$|R_{2n}| < |T_{2n+2}|/(1 - w^2) \quad \ldots \quad (16).$$

Similarly, if we write

$$\left.\begin{aligned} &L'(c, z) = 1 - T_1 - T_2 - \ldots\ldots - T_n + R_n \\ \text{where}\quad &T_n = B_n z^n/n, \\ &B_n = c(1 - 1/1.2) \ldots\ldots (1 - 1/(n-1)n), \\ &B_1 = c, \end{aligned}\right\} \quad \ldots \quad (17),$$

then we can show that

$$|R_n| < T_n/(1 - z) \quad \ldots \quad (18).$$

The formulæ (15), (16), and (18) enable us to estimate the accuracy of the approximation obtained by taking any given number of terms of the respective series. It is obvious that the formulæ (9), (10) are most convenient for nodes near the deepest part of the lake ; and (11), (12) most convenient for nodes near the ends. In most of the calculations tabulated below the equation $L'(c, z) = 0$ is used ; but in many cases we verified our results by working with the other formulæ as well. Other things being equal, the formulæ (11), (12) have an advantage, in respect that tabulation of the steps for the calculation of B is one continuous operation, and there is less chance of error by inadvertence in the entries.

§ 12. As for the periods, so also for the nodes we get first approximations by taking the mean between the extreme cases of a complete symmetric parabolic lake and a semiparabolic lake. As might be expected, these first approximations are not so close for the nodes as they are for the periods. Fortunately the series are very manageable.

If we denote the vertical seiche displacements at the Western end, deepest

normal point, and Eastern end of the lake by $\zeta_{A'}$, ζ_0, ζ_A respectively, we see from H.T.S. § 39 that

$$\zeta_{A'} = -\frac{A}{a\mathrm{S}(c', 1)} \sin nt, \quad \zeta_0 = \frac{\mathrm{AC}(c, 1)}{a\mathrm{S}(c, 1)} \sin nt, \quad \zeta_A = \frac{A}{a\mathrm{S}(c, 1)} \sin nt \quad . \quad . \quad . \quad (19).$$

Hence, if we bear in mind the values of c and c' corresponding to the first three pure seiches, we get the following table of relative signs :—

	$\zeta_{A'}$	ζ_0	ζ_A
Uninodal . . .	−	−	+
Binodal	−	+	−
Trinodal . . .	−	+	+

We infer that the uninode is east of O ; that the binodes are on opposite sides of O ; and that there is one trinode west of O, and two east of O.

§ 13. Uninode of Earn. (East of Deepest Normal Point.)

We have taken $T_1 = 14{\cdot}50'$, $c = 3{\cdot}5785$.

Coefficients of $L'(3{\cdot}5785, z)$.

n	$1 \backsim \frac{c}{n(n-1)}$	$\log\left(1 \backsim \frac{c}{n(n-1)}\right)$	$\log B_n$	$\log \frac{B_n}{n}$
1	$\log c$ =	·55370		
2	·7893	$\bar{1}$·89724	·55370	·55370
3	·4036	$\bar{1}$·60595	·45094	·14991
4	·7018	$\bar{1}$·84621	·05689	$\bar{1}$·57977
5	·8211	$\bar{1}$·91440	$\bar{1}$·90310	$\bar{1}$·30104
6	·8807	$\bar{1}$·94483	$\bar{1}$·81750	$\bar{1}$·11853
7	·9148	$\bar{1}$·96133	$\bar{1}$·76233	$\bar{2}$·98418
8	·9361	$\bar{1}$·97132	$\bar{1}$·72366	$\bar{2}$·87856
9	·9503	$\bar{1}$·97786	$\bar{1}$·69498	$\bar{2}$·79189
10	·9603	$\bar{1}$·98241	$\bar{1}$·67284	$\bar{2}$·71860
11	·9675	$\bar{1}$·98565	$\bar{1}$·65525	$\bar{2}$·65525

Value of $L'(3{\cdot}5785\,,\,{\cdot}32)$.

n	$\log \frac{B_n}{n}$	$\log z^n$	$\log T_n$	T_n +	T_n −
0				1·00000	
1	·55370	$\bar{1}$·50515	·05885		1·14513
2	·14991	$\bar{1}$·01030	$\bar{1}$·16021	·14461	
3	$\bar{1}$·57977	$\bar{2}$·51545	$\bar{2}$·09522	·01245	
4	$\bar{1}$·30104	$\bar{2}$·02060	$\bar{3}$·32164	·00210	
5	$\bar{1}$·11853	$\bar{3}$·52575	$\bar{4}$·64428	·00044	
6	$\bar{2}$·98418	$\bar{3}$·03090	$\bar{4}$·01508	·00010	
7	$\bar{2}$·87856	$\bar{4}$·53605	$\bar{5}$·41461	·00002(6)	
8	$\bar{2}$·79189	$\bar{4}$·04120	$\bar{6}$·83309	·00000(68)	
9	$\bar{2}$·71860	$\bar{5}$·54635	$\bar{6}$·26495	·00000(18)	
10	$\bar{2}$·65525	$\bar{5}$·05150	$\bar{7}$·70675	·00000(05)	
Series error < ·000008				1·15974	1·14513

Hence $L'(3{\cdot}5785\,,\,{\cdot}32) = +{\cdot}01461$.

Calculating in like manner for $z = {\cdot}325$ and $z = {\cdot}33$, we get the following values :—

z	$L'(c\,,\,z)$	Diff.
·320	+·01461	·01254
·325	+·00207	·01248
·330	−·01041	

Interpolating, we find ${}_1z_1 = {\cdot}3258$; and therefore ${}_1w_1 = {\cdot}3484$. It follows that

$${}_1v_1 = {\cdot}3484 \times 145{\cdot}5 \times 480{,}100 \text{ sq. ft.}$$
$$= 50{\cdot}7 \times 480{,}100 \text{ sq. ft.}$$
and $${}_1x_1 = 25{\cdot}24 \times 692{\cdot}9 \text{ ft.}$$

§ 14. Eastern Binode of Earn.

$T = 8{\cdot}14'$, $C = 11{\cdot}3541$.

Coefficients of $L'(11{\cdot}3541\,,\,z)$.

n	$1 \backsim \frac{c}{n(n-1)}$	$\log\left(1 \backsim \frac{c}{n(n-1)}\right)$	$\log B_n$	$\log \frac{B_n}{n}$
1	$\log c =$	1·05515	1·05515	1·05515
2	4·6771	·66998	1·72513	1·42410
3	·8924	$\bar{1}$·95056	1·67569	1·19857
4	·0538	$\bar{2}$·73078	·40647	$\bar{1}$·80441
5	·4323	$\bar{1}$·63579	·04226	$\bar{1}$·34329
6	·6215	$\bar{1}$·79344	$\bar{1}$·83570	$\bar{1}$·05755
7	·7297	$\bar{1}$·86314	$\bar{1}$·69884	$\bar{2}$·85374

Value of L′(11·3541 , ·117).

n	$\log \frac{B_n}{n}$	$\log z^n$	$\log T_n$	T_n +	T_n −
0				1·00000	
1	1·05515	$\bar{1}$·06819	·12334		1·32843
2	1·42410	$\bar{2}$·13637	$\bar{1}$·56047	·36347	
3	1·19857	$\bar{3}$·20456	$\bar{2}$·40313		·02530
4	$\bar{1}$·80441	$\bar{4}$·27274	$\bar{4}$·07715		·00019(4)
5	$\bar{1}$·34329	$\bar{5}$·34093	$\bar{6}$·68422		·00000(5)
6	$\bar{1}$·05755	$\bar{6}$·40912	$\bar{7}$·46667		
Series error < ·000006				1·36347	1·35393

Hence L′(11·3541 , ·117) + = ·00383.

Calculating in like manner, we get the rest of the following table :—

z	L′(c , z)	Diff. for ·001
·117	+ ·00951	·00568
·118	+ ·00383	·00570
·120	− ·00757	

Hence $_2z_2 = ·1187.$
$_2w_2 = ·7627.$
$_2v_2 = 111·0 \times 480{,}100$ sq. ft.
$_2x_2 = 39·05 \times 692·9$ ft.

§ 15. Western Binode of Earn. (West of Deepest Normal Point.)

$T = 8·14'$, $c' = 2·3011.$

Coefficients of S′(2·3011 , w).

$2n$	$1 \frown \frac{c'}{(2n-2)(2n-1)}$	$\log\left(1 \frown \frac{c'}{(2n-2)(2n-1)}\right)$	$\log A_{2n}$	$\log \frac{A_{2n}}{2n}$
2	$\log c$ =	·36194	·36194	·06091
4	·3835	$\bar{1}$·78993	·15187	$\bar{1}$·54981
6	·1151	$\bar{1}$·94689	·09876	$\bar{1}$·32061

Value of $S'(2{\cdot}3011\,,\,{\cdot}09)$.

$2n$	$\log \frac{A_{2n}}{2n}$	$\log w^{2n}$	$\log T_{2n}$	T_{2n} +	T_{2n} −
0				1·00000	
2	·06091	$\bar{3}$·90849	$\bar{3}$·96940		·00932
4	$\bar{1}$·54981	$\bar{5}$·81697	$\bar{5}$·36678		·00002
6	$\bar{1}$·32061	$\bar{7}$·72546	$\bar{7}$·04607		·00000(01)
Series error $<$ ·000001				1·00000	·00934

$S'(2{\cdot}3011\,,\,{\cdot}09) = +{\cdot}99066.$

Coefficients of $C'(2{\cdot}3011\,,\,w)$.

$2n+1$	$1 \sim \frac{c'}{(2n-1)2n}$	$\log\left(1 \sim \frac{c'}{(2n-1)2n}\right)$	$\log A_{2n+1}$	$\log \frac{A_{2n+1}}{2n+1}$
1	$\log c$	= ·36194	·36194	·36194
3	·1506	$\bar{1}$·17782	$\bar{1}$·53976	$\bar{1}$·06264
5	·8082	$\bar{1}$·90752	$\bar{1}$·44728	$\bar{2}$·74831

Value of $C'(2{\cdot}3011,\ {\cdot}09)$.

$2n+1$	$\log \frac{A_{2n+1}}{2n+1}$	$\log w^{2n+1}$	$\log T_{2n+1}$	T_{2n+1} +	T_{2n+1} −
1	·36194	$\bar{2}$·95424	$\bar{1}$·31618		·20710
3	$\bar{1}$·06264	$\bar{4}$·86273	$\bar{5}$·92537	·00008	
5	$\bar{2}$·74831	$\bar{6}$·77121	$\bar{7}$·51952	·00000(03)	
Series error $<$ ·000001				·00008	·20710

$C'(2{\cdot}3011\,,\,{\cdot}09) = -{\cdot}20702$

$\phi({\cdot}09) = -{\cdot}99066 \times {\cdot}21271 + {\cdot}20702 = -{\cdot}00370.$

In this way we get

w	$\phi(w)$	Diff. for ·001
·09000	− ·00370	·00235
·09192	+ ·00080	·00235
·10000	+ ·01970	

${}_2w_1 = {\cdot}0916.$

${}_2v_1 = 6{\cdot}00 \times 480{,}100$ sq. ft.

${}_2x_1 = 13{\cdot}98 \times 692{\cdot}9$ ft.

§ 16. Middle Trinode of Earn. (East of Deepest Normal Point.)

$T_3 = 5{\cdot}743'$, $c = 22{\cdot}8123$.

Coefficients of $L'(22{\cdot}8123, z)$.

n	$1 \backsim \frac{c}{n(n-1)}$	$\log\left(1 \backsim \frac{c}{n(n-1)}\right)$	$\log B_n$	$\log \frac{B_n}{n}$
1	$\log c$ =	1·35817	1·35817	1·35817
2	10·4062	1·01729	2·37546	2·07443
3	2·8021	·44748	2·82294	2·34582
4	·9010	$\bar{1}$·95472	2·77766	2·17560
5	·1406	$\bar{1}$·14799	1·92565	1·22668
6	·2396	$\bar{1}$·37949	1·30514	·52699
7	·4569	$\bar{1}$·65982	·96496	·11986
8	·5926	$\bar{1}$·77276	·73772	$\bar{1}$·83463
9	·6832	$\bar{1}$·83455	·57227	$\bar{1}$·61803
10	·7465	$\bar{1}$·87303	·44530	$\bar{1}$·44530
11	·7926	$\bar{1}$·89905	·34435	$\bar{1}$·30296
12	·8272	$\bar{1}$·91761	·26196	$\bar{1}$·18278
13	·8538	$\bar{1}$·93136	·19332	$\bar{1}$·07938

Value of $L'(22{\cdot}8123, {\cdot}25)$.

n	$\log \frac{B_n}{n}$	$\log z^n$	$\log T_n$	T_n +	T_n −
0				1·00000	5·70310
1	1·35817	$\bar{1}$·39794	·75611		
2	2·07443	$\bar{2}$·79588	·87031	7·41840	
3	2·34582	$\bar{2}$·19382	·53964		3·46450
4	2·17560	$\bar{3}$·59176	$\bar{1}$·76736	·58528	
5	1·22668	$\bar{4}$·98970	$\bar{2}$·21638		·01646
6	·52699	$\bar{4}$·38764	$\bar{4}$·91463		·00082
7	·11986	$\bar{5}$·78558	$\bar{5}$·90544		·00008
8	$\bar{1}$·83463	$\bar{5}$·18352	$\bar{5}$·01815		·00001
9	$\bar{1}$·61803	$\bar{6}$·58146	$\bar{6}$·19949		·00000(16)
10	$\bar{1}$·44530	$\bar{7}$·97940	$\bar{7}$·42470		·00000(026)
Series error < ·0000004				9·00368	9·18497

Therefore $L'(22{\cdot}8123, {\cdot}25) = -{\cdot}18129$.

Continuing the calculation in like manner, we get—

z	L′(c, z)	Diff. for ·01
·25	− ·18129	·04077
·29	− ·01822	·04022
·30	+ ·02200	

Hence $_3z_2 = ·2945.$
$_3w_2 = ·4110.$
$_3v_2 = 59·81 \times 480{,}100$ sq. ft.
$_3x_2 = 27·14 \times 692·9$ ft.

§ 17. Eastern Trinode of Earn.

T = 5·743′, c = 22·8123.

The Coefficients of L′(c, z) are the same as for the Middle Trinode.

Value of L′(22·8123, ·06).

n	$\log \frac{B_n}{n}$	$\log z^n$	$\log T_n$	T_n +	T_n −
0				1·00000	
1	1·35817	$\bar{2}$·77815	·13632		1·36874
2	2·07443	$\bar{3}$·55630	$\bar{1}$·63073	·42730	
3	2·34582	$\bar{4}$·33445	$\bar{2}$·68027		·04789
4	2·17560	$\bar{5}$·11261	$\bar{3}$·28821	·00194	
5	1·22668	$\bar{7}$·89076	$\bar{5}$·11744		·00001(3)
6	·52699	$\bar{8}$·66891	$\bar{7}$·19590		·00000(016)
7	·11986	$\bar{9}$·44706	$\bar{9}$·56692		
Series error < ·0000002				1·42924	1·41664

L′(22·8123, ·06) = + ·01260.

We find the following values in like manner :—

z	L′(c, z)	Diff.
·060	+ ·01260	·01075
·061	+ ·00185	·01062
·062	− ·00877	

Therefore $_3z_3 = ·06117.$
$_3w_3 = ·8777.$
$_3v_3 = 127·7 \times 480{,}100$ sq. ft.
$_3x_3 = 43·1 \times 692·9$ ft.

§ 18. Western Trinode of Earn. (West of Deepest Normal Point.)

$T_3 = 5{\cdot}743'$, $c' = 4{\cdot}6230$.

Coefficients of $L'(4{\cdot}6230, z)$.

n	$1 \frown \frac{c'}{n(n-1)}$	$\log\left(1 \frown \frac{c'}{n(n-1)}\right)$	$\log B_n$	$\log \frac{B_n}{n}$
1	$\log c'$ =	·66492	·66492	·66492
2	1·3115	·11777	·78269	·48166
3	·2295	$\bar{1}$·36078	·14347	$\bar{1}$·66635
4	·6147	$\bar{1}$·78866	$\bar{1}$·93213	$\bar{1}$·33007
5	·7688	$\bar{1}$·88581	$\bar{1}$·81794	$\bar{1}$·11897
6	·8459	$\bar{1}$·92732	$\bar{1}$·74526	$\bar{2}$·96711
7	·8899	$\bar{1}$·94934	$\bar{1}$·69460	$\bar{2}$·84950
8	·9174	$\bar{1}$·96256	$\bar{1}$·65716	$\bar{2}$·75407

Value of $L'(4{\cdot}6230, {\cdot}25)$.

n	$\log \frac{B_n}{n}$	$\log z^n$	$\log T_n$	T_n +	T_n −
0				1·00000	
1	·66492	$\bar{1}$·39794	·06286		1·15574
2	·48166	$\bar{2}$·79588	$\bar{1}$·27754	·18947	
3	$\bar{1}$·66635	$\bar{2}$·19382	$\bar{3}$·86017	·00725	
4	$\bar{1}$·33007	$\bar{3}$·59176	$\bar{4}$·92183	·00084	
5	$\bar{1}$·11897	$\bar{4}$·98970	$\bar{4}$·10867	·00012(8)	
6	$\bar{2}$·96711	$\bar{4}$·38764	$\bar{5}$·35475	·00002(3)	
7	$\bar{2}$·84950	$\bar{5}$·78558	$\bar{6}$·63508	·00000(4)	
8	$\bar{2}$·75407	$\bar{5}$·18352	$\bar{7}$·93759	·00000(09)	
Series error < ·000002				1·19771	1·15574

Hence $L'(4{\cdot}6230, {\cdot}25) = +{\cdot}04197$.

Calculating in the same way for $z = {\cdot}275$ and $z = {\cdot}300$, we get the following table :—

z	$L'(c', z)$	Diff.
·250	+ ·04197	·07289
·275	− ·03092	·06847
·300	− ·09939	

Therefore
$_3z_1 = {\cdot}2644$.
$_3w_1 = {\cdot}4712$.
$_3v_1 = 30{\cdot}87 \times 480{,}100$ sq. ft.
$_3x_1 = 9{\cdot}16 \times 692{\cdot}9$ ft.

§ 19. DATA FOR LOCH TREIG.

No. of Sounding Line.	x Unit 346·5 Feet.	v Unit 7744×10^2 Sq. Feet.	σ Unit 10^6 Cub. Feet.	a	a^2	σa Unit 10^6 Cub. Feet.	
0	0·0	24·47	0	·0000	·0000	0	
1	2·2	22·73	239	·1370	·0188	33	
2	6·0	18·34	972	·4382	·1920	426	$a = 4814 \times 10^4$
3	9·8	13·26	1804	·7064	·4990	1274	$a' = 1895 \times 10^4$
S. Trinode	[12·2]	[9·37]					$\rho = a/a' = 2·5403$
4	13·8	6·79	2227	·9230	·8519	2056	$\rho^2 = 6·4529$
5	18·1	0·00	2704	1·0000	1·0000	2704	$\log \rho = ·40488$
S. Binode	[18·7]	[0·89]					$h = 2507 \times 10^6$
6	22·3	6·11	2568	·9903	·9807	2543	
7	26·2	11·47	2307	·9660	·9332	2229	
8	29·5	15·96	2201	·9340	·8724	2056	
9	34·6	22·36	2357	·8706	·7579	2052	
Uninode	[35·6]	[23·61]					
10	38·7	27·64	1831	·8023	·6437	1469	
Mid. Trinode	[39·2]	[28·03]					
11	42·0	31·24	1772	·7474	·5586	1325	
12	43·9	33·75	1816	·7051	·4972	1281	
13	46·9	37·84	2067	·6293	·3960	1301	
14	51·1	42·72	1394	·5266	·2773	734	
15	56·4	48·20	773	·3989	·1591	308	
N. Binode	[56·7]	[48·40]					
16	60·0	50·98	440	·3274	·1072	144	
17	64·3	54·40	626	·2340	·0548	147	
N. Trinode	[65·2]	[55·12]					
18	68·3	57·57	297	·1420	·0202	42	
19	71·2	59·43	138	·0858	·0074	12	
20	73·2	60·28	107	·0594	·0035	6	
21	75·8	61·40	43	·0240	·0006	1	
22	77·6	61·83	24	·0105	·0001	0	
23	78·0	62·16	0	·0000	·0000	0	
$h = \Sigma\sigma a/\Sigma a^2 = 25072 \times 10^5$					8·8316	22143	

PERIODS OF LOCH TREIG.

§ 20. A few rough trials shew very readily that the uninodal period of the lake is nearly 9′. Using the method already explained for Earn, we derive the following table of first approximations :—

	T_1	T_2	T_3
Parabolic	9·00	5·19	3·67
Semiparabolic . . .	9·00	4·93	3·40
Means	9·00	5·06	3·54

Taking $T_1 = 9{\cdot}00$, $T_2 = 5{\cdot}06$, $T_3 = 3{\cdot}54$ as a basis, we proceed exactly as in the case of Earn, and the results are given below.

§ 21. After the calculations had been made, it was found that the shrinkage error of the 6-inch ordinance map of the lake, on which the soundings were plotted, was not negligible. It amounted, in fact, to about ·5 per cent. As the areas v were measured with a planimeter, and the breadths $b(x)$ with an absolute scale, the linear error $\delta\lambda/\lambda$ enters twice into v, once into $b(x)$, and once into $A(x)$.* The values of a in the table on p. 839 are unaffected, and $\rho = a/a'$ is also unaffected by this error. But $\delta a/a = 2\delta\lambda\ \lambda$, $\delta h = 2\delta\lambda/\lambda$. Since the values of c and c' depend merely on ρ, we have (H.T.S.(45))

$$\partial T/T = \partial a/a - \tfrac{1}{2}\partial h/h = \partial\lambda/\lambda. \qquad (20).$$

Hence, to correct for the map error, we have to multiply each of the periods obtained from the data of the table on p. 839 by the number 1·005.

§ 22. UNINODAL PERIOD OF TREIG.

T	c	a	$\frac{1}{4}(5+a)$	$\frac{1}{4}(5-a)$	$\frac{1}{4}(3+a)$	$\frac{1}{4}(3-a)$	K(c,1)
9·05	3·8433	4·0464	2·2616	·2384	1·7616	− ·2616	− 1·98857
9·10	3·8012	4·0255	2·2564	·2436	1·7564	− ·2564	− 1·91295
9·15	3·7598	4·0049	2·2512	·2488	1·7512	− ·2512	− 1·84013

T	c'	a'	$\frac{1}{4}(5+a')$	$\frac{1}{4}(5-a')$	$\frac{1}{4}(3+a')$	$\frac{1}{4}(3-a')$	K(c′, 1)
9·05	·5956	1·8391	1·7098	·7902	1·2098	·2902	·75462
9·10	·5891	1·8320	1·7080	·7920	1·2080	·2920	·75753
9·15	·5827	1·8251	1·7063	·7937	1·2063	·2937	·76024

T	$\chi(c)$	Diff.
9·05	− ·06162	·07299
9·10	+ ·01137	·07962
9·15	+ ·09109	

$T_1 = 9{\cdot}093' = 9{\cdot}09'$, say.

$c_1 = 3{\cdot}8073$; $\quad c'_1 = {\cdot}5900$.

Corrected for map error, $T_1 = 9{\cdot}14'$.

* The depths were, of course, *read* from the map.

§ 23. Binodal Period of Treig.

T	c	a	$\frac{1}{4}(5+a)$	$\frac{1}{4}(5-a)$	$\frac{1}{4}(3+a)$	$\frac{1}{4}(3-a)$	K(c, 1)
5·06	12·2943	7·0836	3·0209	− ·5209	2·5209	− 1·0209	− ·22608
5·07	12·2456	7·0698	3·0175	− ·5175	2·5175	− 1·0175	− ·18883
5·08	12·1980	7·0564	3·0141	− ·5141	2·5141	− 1·0141	− ·15178

T	c'	a'	$\frac{1}{4}(5+a')$	$\frac{1}{4}(5-a')$	$\frac{1}{4}(3+a')$	$\frac{1}{4}(3-a')$	K(c', 1)
5·06	1·9052	2·9361	1·9840	·5160	1·4840	·0160	− ·06223
5·07	1·8977	2·9310	1·9828	·5173	1·4828	·0173	− ·06714
5·08	1·8903	2·9260	1·9815	·5185	1·4815	·0185	− ·07165

T	$\chi(c)$	Diff.
5·06	− ·06799	·04972
5·07	− ·01827	·04850
5·08	+ ·03023	

$T_2 = 5·0738' = 5·074'$, say.

$c_2 = 12·2275$; $c'_2 = 1·8950$.

Corrected for map error, $T_2 = 5·099$.

§ 24. Trinodal Period of Treig.

T	c	a	$\frac{1}{4}(5+a)$	$\frac{1}{4}(5-a)$	$\frac{1}{4}(3+a)$	$\frac{1}{4}(3-a)$	K(c, 1)
3·550	24·977	10·0453	3·7613	− 1·2613	3·2613	− 1·7613	4·60935
3·565	24·767	10·0035	3·7509	− 1·2509	3·2509	− 1·7509	4·89966
3·575	24·629	9·9758	3·7440	− 1·2440	3·2440	− 1·7440	5·10266

T	c'	a'	$\frac{1}{4}(5+a')$	$\frac{1}{4}(5-a')$	$\frac{1}{4}(3+a')$	$\frac{1}{4}(3-a')$	K(c', 1)
3·550	3·8707	4·0599	2·2650	·2350	1·7650	− ·2650	− 2·03981
3·565	3·8382	4·0438	2·2609	·2391	1·7609	− ·2609	− 1·97823
3·575	3·8168	4·0333	2·2583	·2417	1·7583	− ·2583	− 1·94017

T	$\chi(c)$	Diff. for ·01.
3·550	− ·57233	·30446
3·565	− ·12564	·29974
3·575	+ ·17410	

$T_3 = 3{\cdot}5692' = 3{\cdot}569'$, say.

$c_3 = 24{\cdot}711$; $\quad c'_3 = 3{\cdot}8294$.

Corrected for map error, $T_3 = 3{\cdot}587$.

§ 25. Periods by Du Boys' Rule.

The distances from the southern end of the lake are denoted by x, the unit being $2 \times 1760/2{\cdot}54$ feet. The depth, h, is entered in feet.

x	h	$\sqrt{h}$	Mean $\sqrt{h}$	$1/\text{mean}\sqrt{h} = \frac{dx}{\sqrt{h}}$	x	h	$\sqrt{h}$	Mean $\sqrt{h}$	$1/\text{mean}\sqrt{h} = \frac{dx}{\sqrt{h}}$
0	$dx = 1$ throughout				20	438	20·93	20·89	·0479
1	8	9·22	4·61	·2169	21	433	20·81	20·87	·0479
2	165	12·85	11·03	·0907	22	408	20·20	20·50	·0488
3	240	15·49	14·17	·0706	23	406	20·15	20·17	·0496
4	258	16·06	15·77	·0634	24	404	20·10	20·13	·0497
5	272	16·49	16·27	·0615	25	402	20·05	20·17	·0496
6	292	17·09	16·79	·0596	26	399	19·97	20·01	·0500
7	310	17·61	17·35	·0576	27	350	18·71	18·80	·0532
8	322	17·94	17·77	0563	28	311	17·64	18·17	·0550
9	333	18·25	18·09	·0553	29	291	17·06	17·35	·0576
10	362	19·03	18·64	·0536	30	280	16·73	16·89	·0592
11	393	19·82	19·43	·0515	31	286	16·91	16·82	·0595
12	413	20·32	20·07	·0498	32	292	17·09	17·00	·0588
13	429	20·71	20·52	·0487	33	260	16·12	16·56	·0604
14	434	20·83	20·77	·0481	34	213	14·59	15·35	·0651
15	436	20·88	20·85	·0480	35	228	15·10	14·85	·0673
16	431	20·76	20·82	·0480	36	210	14·49	14·79	0676
17	426	20·64	20·70	·0483	37	153	12·37	13·43	·0745
18	428	20·69	20·66	·0484	38	129	11·36	11·86	·0843
19	435	20·86	20·77	·0481	39	0	0	5·68	·1761
									2·5065

Hence $$ {}_dT_1 = \frac{2 \times 1760 \times 2{\cdot}5065}{2{\cdot}54 \times \sqrt{32{\cdot}2}}, $$

$$ = 6120'' = 10{\cdot}20'. $$

Therefore ${}_dT_2 = 5{\cdot}10'$.

${}_dT_3 = 3{\cdot}40'$.

Correcting for the map error, we have :—

$$_dT_1 = 10{\cdot}25'.$$
$$_dT_2 = 5{\cdot}13'.$$
$$_dT_3 = 3{\cdot}42'.$$

In this case Du Boys' value is about 12 per cent. in excess for the uninodal period; only about ·6 per cent. in excess for the binodal; while it is about 5 per cent. in defect for the trinodal. The difference between Earn and Treig in this last respect may be due to the fact that at the shallow end of Treig the normal points lie above the representative parabola; whereas the opposite is the case with Earn.

Nodes of Loch Treig.

§ 26. The table of the signs of $\zeta_{A'}$, ζ_0, ζ_A, corresponding to that given for Earn on p. 832, is as follows:—

	$\zeta_{A'}$	ζ_0	ζ_A
Uninodal	−	−	+
Binodal	−		−
Trinodal	−	+	+

The uninode therefore lies north of the deepest normal point. The two binodes are on the same side of the deepest normal point—of course, on the northern side. One trinode lies south of the deepest normal point; and a rough trial easily shews that there is one, and therefore two, trinodes north of the deepest normal point.

Uninode of Treig (North of Deepest Normal Point).

§ 27. We have taken $T_1 = 9{\cdot}14'$, $c = 3{\cdot}8073$.

Coefficients of $L'(3{\cdot}8073, z)$.

n	$1 \backsim \frac{c}{n(n+1)}$	$\log\left(1 \backsim \frac{c}{n(n+1)}\right)$	$\log B_n$	$\log \frac{B_n}{n}$
0	$\log c =$	·58062		
1	·9037	$\bar{1}$·95602	·58062	·58062
2	·3654	$\bar{1}$·56282	·53664	·23561
3	·6827	$\bar{1}$·83424	·09946	$\bar{1}$·62234
4	·8096	$\bar{1}$·90829	$\bar{1}$·93370	$\bar{1}$·33164
5	·8731	$\bar{1}$·94106	$\bar{1}$·84199	$\bar{1}$·14302
6	·9093	$\bar{1}$·95873	$\bar{1}$·78305	$\bar{1}$·00490
7	·9320	$\bar{1}$·96942	$\bar{1}$·74178	$\bar{2}$·89668
8	·9471	$\bar{1}$·97641	$\bar{1}$·71120	$\bar{2}$·80811
9	·9577	$\bar{1}$·98123	$\bar{1}$·68761	$\bar{2}$·73337

Value of L′(3·8073, ·305).

n	$\log\frac{B_n}{n}$	$\log z^n$	$\log T_n$	T_n +	T_n −
0				1·00000	
1	·58062	$\bar{1}$·48429	·06491		1·16121
2	·23561	$\bar{2}$·96859	$\bar{1}$·20420	·16003	
3	$\bar{1}$·62234	$\bar{2}$·45289	$\bar{2}$·07523	·01189	
4	$\bar{1}$·33164	$\bar{3}$·93719	$\bar{3}$·26883	·00186	
5	$\bar{1}$·14302	$\bar{3}$·42149	$\bar{4}$·56451	·00037	
6	$\bar{1}$·00490	$\bar{4}$·90579	$\bar{5}$·91069	·00008	
7	$\bar{2}$·89668	$\bar{4}$·39009	$\bar{5}$·28677	·00001(9)	
8	$\bar{2}$·80811	$\bar{5}$·87439	$\bar{6}$·68250	·00000(48)	
9	$\bar{2}$·73337	$\bar{5}$·35869	$\bar{6}$·09206	·00000(12)	
Series error < ·000002				1·17425	1·16121

Hence L′(3·8073, ·305) = + ·01304.

Calculating in like manner for $z = ·310$, and $z = ·315$, we get the following values :—

z	L′(c, z)	Diff.
·305	+ ·01304	·01301
·310	+ ·00003	·01287
·315	− ·01284	

Interpolating, we find ${}_1z_1 = ·310$; and therefore ${}_1w_1 = ·380$. It follows that

$${}_1r_1 = ·380 \times 62·16 \times 774{,}400 \text{ sq. ft.}$$
$$= 23·61 \times 774{,}400 \text{ sq. ft.};$$

and $${}_1x_1 = 35·6 \times 346·5 \text{ ft.}$$

§ 28. South Binode of Treig (North of Deepest Normal Point).

$$T = 5·099'; \quad c = 12·2275.$$

Coefficients of S′(12·2275, w).

$2n$	$1 \sim \frac{c}{(2n-2)(2n-1)}$	$\log\left(1 \sim \frac{c}{(2n-2)(2n-1)}\right)$	$\log A_{2n}$	$\log\frac{A_{2n}}{2n}$
2	$\log c$ =	1·08734	1·08734	·78631
4	1·0380	·01620	1·10354	·50148

Value of $S'(12{\cdot}2275, {\cdot}0142)$.

$2n$	$\log\frac{A_{2n}}{2n}$	$\log w^{2n}$	$\log T_{2n}$	T_{2n} +	T_{2n} −
0				1·00000	
2	·78631	$\bar{4}$·30458	$\bar{3}$·09089		·00123
4	·50148	$\bar{8}$·60915	$\bar{7}$·11053	·00000(01)	
Series error < ·000001				1·00000	·00123

$S'(12{\cdot}2275, {\cdot}0142) = +{\cdot}99877.$

Coefficients of $C'(12{\cdot}2275, w)$.

$2n+1$	$1\sim\frac{c}{(2n-1)2n}$	$\log\left(1\sim\frac{c}{(2n-1)2n}\right)$	$\log A_{2n+1}$	$\log\frac{A_{2n+1}}{2n+1}$
1	$\log c =$	1·08734	1·08734	1·08734
3	5·1138	·70874	1·79608	1·31896
5	·0190	$\bar{2}$·27875	·07483	$\bar{1}$·37586

Value of $C'(12{\cdot}2275, {\cdot}0142)$.

$2n+1$	$\log\frac{A_{2n+1}}{2n+1}$	$\log w^{2n+1}$	$\log T_{2n+1}$	T_{2n+1} +	T_{2n+1} −
1	1·08734	$\bar{2}$·15229	$\bar{1}$·23963		·17363
3	1·31896	$\bar{6}$·45686	$\bar{5}$·77582	·00006	
5	$\bar{1}$·37586	$\overline{10}$·75144	$\overline{10}$·12730		
Series error < ·000001				·00006	·17363

$C'(12{\cdot}2275, {\cdot}0142) = -{\cdot}17357.$

$\phi({\cdot}0142) = -{\cdot}17464 \times {\cdot}99877 + {\cdot}17357 = -{\cdot}00086.$

In this way we get

w	$\phi(w)$	Diff. for ·00001
·01420	− ·00086	·00012
·01425	− ·00025	·00012
·01429	+ ·00024	

Hence $_2w_1 = {\cdot}01427.$

$_2v_1 = {\cdot}89 \times 774{,}400$ sq. ft.

$_2x_1 = 18{\cdot}7 \times 346{\cdot}5$ ft.

North Binode of Treig (North of Deepest Normal Point).

§29. $T = 5{\cdot}099'$; $c = 12{\cdot}2275$.

Coefficients of $L'(12{\cdot}2275, z)$.

n	$1 \frown \frac{c}{n(n+1)}$	$\log\left(1 \frown \frac{c}{n(n+1)}\right)$	$\log B_n$	$\log \frac{B_n}{n}$
0	$\log c =$	1·08734		
1	5·1138	·70874	1·08734	1·08734
2	1·0380	·01620	1·79608	1·49505
3	·0190	$\bar{2}$·27875	1·81228	1·33516
4	·3886	$\bar{1}$·58950	·09103	·48897
5	·5924	$\bar{1}$·77262	$\bar{1}$·68053	$\bar{2}$·98156
6	·7089	$\bar{1}$·85058	$\bar{1}$·45315	$\bar{2}$·67500

Value of $L'(12{\cdot}2275, {\cdot}1150)$.

n	$\log \frac{B_n}{n}$	$\log z^n$	$\log T_n$	T_n +	T_n −
0				1·00000	
1	1·08734	$\bar{1}$·06070	·14804		1·40618
2	1·49505	$\bar{2}$·12140	$\bar{1}$·61645	·41348	
3	1·33516	$\bar{3}$·18209	$\bar{2}$·51725		·03290
4	·48897	$\bar{4}$·24279	$\bar{4}$·73176	·00054	
5	$\bar{2}$·98156	$\bar{5}$·30349	$\bar{6}$·28505	·00000(19)	
6	$\bar{2}$·67500	$\bar{6}$·36419	$\bar{7}$·03919	·00000(01)	
Series error $< {\cdot}000001$				1·41402	1·43908

Hence $L'(12{\cdot}2275, {\cdot}1150) = -{\cdot}02506$.

Calculating in like manner, we get the following table :—

z	$L'(c, z)$	Diff.
·1100	+ ·00492	·01525
·1125	− ·01033	·01473
·1150	− ·02506	

Hence
$$_2z_2 = {\cdot}1108.$$
$$_2w_2 = {\cdot}7785.$$
$$_2v_2 = 48{\cdot}40 \times 774{,}400 \text{ sq. ft.}$$
$$_2x_2 = 56{\cdot}7 \times 346{\cdot}5 \text{ ft.}$$

§ 30. South Trinode of Treig (South of Deepest Normal Point).

$T_3 = 3{\cdot}587$; $c' = 3{\cdot}8294$.

Coefficients of $L'(3{\cdot}8294, z)$.

n	$1 \backsim \frac{c'}{n(n+1)}$	$\log\left(1 \backsim \frac{c'}{n(n+1)}\right)$	$\log B_n$	$\log \frac{B_n}{n}$
1	log	$c' =$ ·58313	·58313	·58313
2	·9147	$\bar{1}$·96128	·54441	·24338
3	·3618	$\bar{1}$·55847	·10288	$\bar{1}$·62576
4	·6809	$\bar{1}$·83308	$\bar{1}$·93596	$\bar{1}$·33390
5	·8085	$\bar{1}$·90768	$\bar{1}$·84364	$\bar{1}$·14467
6	·8724	$\bar{1}$·94072	$\bar{1}$·78436	$\bar{1}$·00621
7	·9088	$\bar{1}$·95847	$\bar{1}$·74283	$\bar{2}$·89773
8	·9316	·96923	$\bar{1}$·71206	$\bar{2}$·80897

Value of $L'(3{\cdot}8294, {\cdot}3085)$.

n	$\log \frac{B_n}{n}$	$\log z^n$	$\log T_n$	T_n +	T_n −
0				1·00000	
1	·58313	$\bar{1}$·48926	·07239		1·18138
2	·24338	$\bar{2}$·97852	$\bar{1}$·22190	·16669	
3	$\bar{1}$·62576	$\bar{2}$·46778	$\bar{2}$·09354	·01240	
4	$\bar{1}$·33390	$\bar{3}$·95704	$\bar{3}$·29094	·00195	
5	$\bar{1}$·14467	$\bar{3}$·44630	$\bar{4}$·59097	·00039	
6	$\bar{1}$·00621	$\bar{4}$·93556	$\bar{5}$·94177	·00009	
7	$\bar{2}$·89773	$\bar{4}$·42482	$\bar{5}$·32255	·00002	
8	$\bar{2}$·80897	$\bar{5}$·91408	$\bar{6}$·72305	·00000(5)	
Series error $< {\cdot}000008$				1·18154	1·18138

Hence $L'(3{\cdot}8294, {\cdot}3085) = +{\cdot}00016$.

Calculating in like manner, we get the following table :—

z	$L'(c, z)$	Diff.
·3085	+ ·00016	·00130
·3090	− ·00114	·00125
·3095	− ·00239	

Hence
$${}_3z_1 = {\cdot}3086.$$
$${}_3w_1 = {\cdot}3828.$$
$${}_3v_1 = 9{\cdot}367 \times 774{,}400 \text{ sq.}$$
$${}_3x_1 = 12{\cdot}2 \times 346{\cdot}5 \text{ ft.}$$

§ 31. Middle Trinode of Treig (North of Deepest Normal Point).

$T_3 = 3{\cdot}587$; $c = 24{\cdot}711$.

Coefficients of $L'(24{\cdot}711, z)$.

n	$1 \frown \frac{c}{n(n+1)}$	$\log\left(1 \frown \frac{c}{n(n-1)}\right)$	$\log B_n$	$\log \frac{B_n}{n}$
1	log	$c =$ 1·39289	1·39289	1·39289
2	11·3555	1·05521	2·44810	2·14707
3	3·1185	·49395	2·94205	2·46493
4	1·0592	·02498	2·96703	2·36497
5	·2355	$\bar{1}$·37199	2·33902	1·64005
6	·1763	$\bar{1}$·24625	1·58527	·80712
7	·4116	$\bar{1}$·61448	1·19975	·35465
8	·5587	$\bar{1}$·74718	·94693	·04384
9	·6568	$\bar{1}$·81743	·76436	$\bar{1}$·81012
10	·7254	$\bar{1}$·86058	·62494	$\bar{1}$·62494
11	·7754	$\bar{1}$·88953	·51447	$\bar{1}$·47308

Value of $L'(24{\cdot}711, {\cdot}274)$.

n	$\log \frac{B_n}{n}$	$\log z^n$	$\log T_n$	T_n +	T_n −
0				1·00000	
1	1·39289	$\bar{1}$·43775	·83064		6·77080
2	2·14707	$\bar{2}$·87550	1·02257	10·53341	
3	2·46493	$\bar{2}$·31325	·77818		6·00040
4	2·36497	$\bar{3}$·75100	·11597	1·30609	
5	1·64005	$\bar{3}$·18875	$\bar{2}$·82880		·06742
6	·80712	$\bar{4}$·62650	$\bar{3}$·43362		·00271
7	·35465	$\bar{4}$·06425	$\bar{4}$·41890		·00026
8	·04384	$\bar{5}$·50200	$\bar{5}$·54584		·00003
9	$\bar{1}$·81012	$\bar{6}$·93975	$\bar{6}$·74987		·00000
10	$\bar{1}$·62494	$\bar{6}$·37750	$\bar{6}$·00244		·00000
11	$\bar{1}$·47308	$\bar{7}$·81525	$\bar{7}$·28833		·00000
Series error < ·000001				12·83950	12·84163

Hence $L'(24{\cdot}711, {\cdot}274) = -{\cdot}00213$.

And we get the following table :—

z	$L'(c, z)$	Diff.
·273	− ·00637	·00424
·274	− ·00213	·00447
·275	+ ·00234	

Hence
$$_3z_2 = \cdot2745.$$
$$_3w_2 = \cdot4510.$$
$$_3v_2 = 28\cdot03 \times 774{,}400 \text{ sq. ft.}$$
$$_3x_2 = 39\cdot2 \times 346\cdot5 \text{ ft.}$$

§ 32. North Trinode of Treig (North of Deepest Normal Point).

$$T_3 = 3\cdot587\,; \quad c = 24\cdot711.$$

The coefficients of L′(24·711, z) are the same as for the Middle Trinode.

Value of L′(24·711, ·0560).

n	$\log \frac{B_n}{n}$	$\log z^n$	$\log T_n$	T_n +	T_n −
0				1·00000	
1	1·39289	$\bar{2}$·74819	·14108		1·38384
2	2·14707	$\bar{3}$·49638	$\bar{1}$·64345	·44000	
3	2·46493	$\bar{4}$·24457	$\bar{2}$·70950		·05123
4	2·36497	$\bar{6}$·99276	$\bar{3}$·35773	·00228	
5	1·64005	$\bar{7}$·74095	$\bar{5}$·38100		·00002(40)
6	·80712	$\bar{8}$·48914	$\bar{7}$·29626		·00000(02)
Series error < ·000001				1·44229	1·43509

Hence L′(24·711, ·0560) = + ·00719.

In this manner we get the following table :—

z	L′(c, z)	Diff.
·0560	+ ·00719	·00575
·0565	+ ·00144	·00571
·0570	− ·00427	

Therefore
$$_3z_3 = \cdot0566.$$
$$_3w_3 = \cdot8868.$$
$$_3v_3 = 55\cdot12 \times 774{,}400 \text{ sq. ft.}$$
$$_3x_3 = 65\cdot2 \times 346\cdot5 \text{ ft.}$$

Agreement between Theory and Observation.

§ 33. Sufficient observations are not yet available to enable us to test the above theoretical results with the degree of accuracy which we believe to be attainable. We may, however, conclude this memoir by stating what is already known to us—with the reservation that the results alluded to are merely preliminary, and subject to future correction and refinement.

§ 34. In October and November 1904, a series of observations on Treig were made under the superintendence of Mr E. MACLAGAN-WEDDERBURN. Only one limnograph was used, and it was placed at the northern end of the lake. The observations were brought to an untimely conclusion by the partial destruction of the instrument during a storm. The seiches observed were rarely pure for any considerable time; and the depth of the lake varied considerably during the observations, so that the periods do not all belong to the same surface-level. T_1 varied from 9·09′ to 9·45′, the mean of all the determinations attempted being 9·18′. T_2 varied from 5·11′ to 5·22′; mean, 5·15′.

Nothing is known as yet regarding the actual position of the nodes of Treig.

§ 35. In June 1905, observations were begun on Loch Earn by Mr JAMES MURRAY under the superintendence of Professor CHRYSTAL. Two Sarasin limnographs were established—one at the uninode, the other at the binode, as determined by the above calculations. Unfortunately, these instruments proved insufficiently sensitive for the great majority of the delicate seiches which have occurred in the lake during June and July. Mr MURRAY has, however, acquired great skill in using the index limnograph of ENDRÖS; and a considerable number of his charts are already at our disposal. The results here given are merely preliminary, and must not be understood as anticipating the more accurate determinations which Mr MURRAY will doubtless make later on.

Allowance being made for wind denivellations due to the shallow edges of the lake, the traces at the two supposed nodes are nearly pure sinusoids. The calculated positions of the uninode and binode cannot therefore be far out. The values obtained for T_1 vary from 14·35′ to 14·77′, the mean being 14·55′; for T_2, 7·97′ to 8·36′, mean 8·10′. No good determination of T_3 has yet been made.

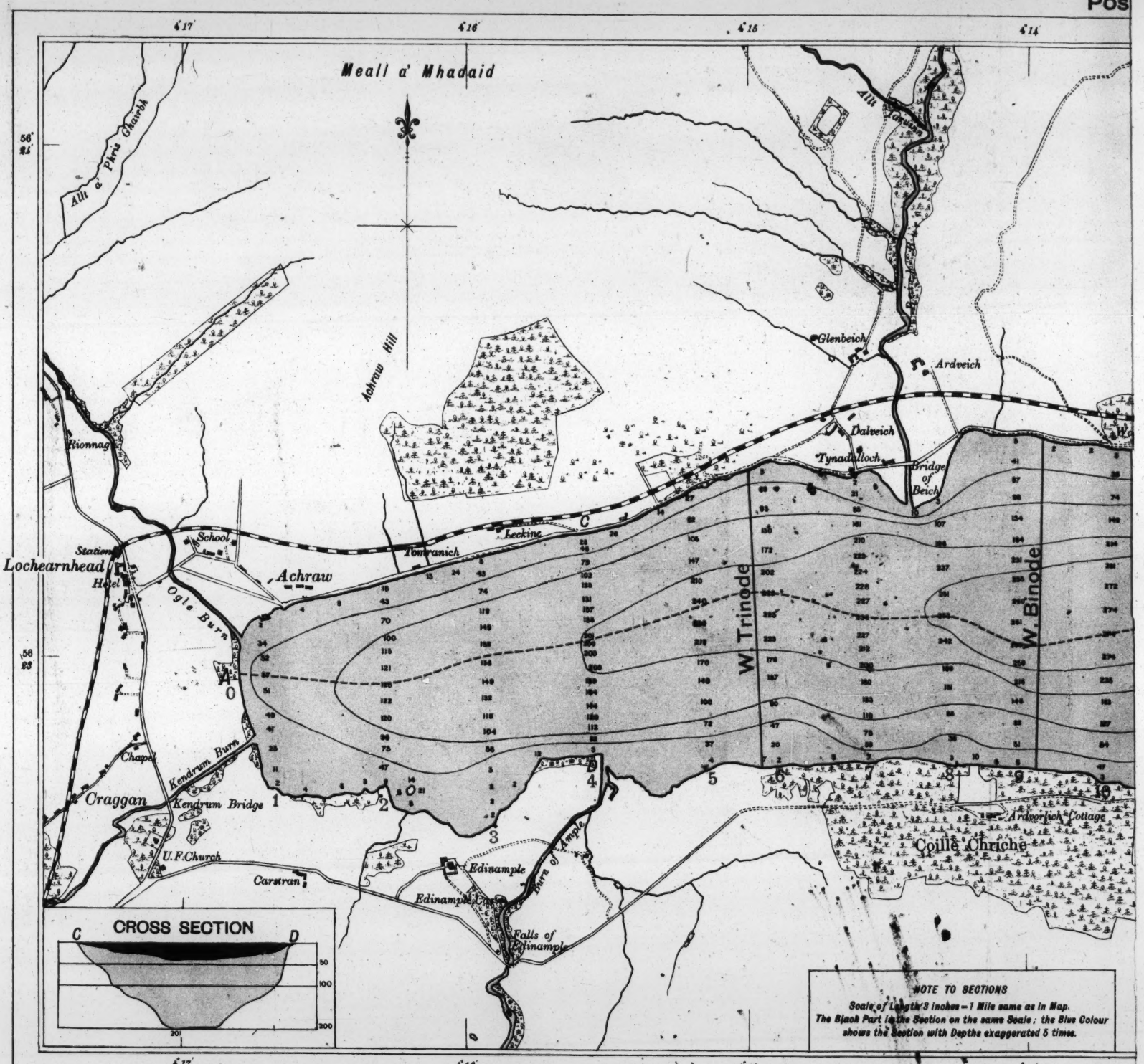
§ 36. The close agreement between the observed and calculated periods may be partly fortuitous. We cannot regard this as finally established until we have additional observations, in which the essential data are more certainly determined.

As regards the agreement between theory and observation to be expected in general, we may point out that more accurate calculation of the periods is to be expected than of the nodes; and that least accuracy of all can be hoped for in the calculation of the amplitudes of the seiches at different parts of the lake. The periods obviously depend more on the whole configuration of the lake, and less upon local irregularities, than do the nodes or the amplitudes. The nodal line would be very seriously displaced by a sudden alteration in the depth or breadth of the lake which might affect the periods very little. Thus, for example, it is easy to see from the position of the dots in fig. 1, with reference to the parabola, that the Western Binode of Earn probably lies some distance west of the position calculated above. The amount of the displacement might be calculated by RAYLEIGH's method if the data from soundings were sufficiently accurate. Also, a gently shelving shallow shore would cause flow across the lake, contrary to the hypothesis of the theory; and the effect of this might be to deform the nodal line, and to alter very considerably the amplitude of the seiche near the shore. These points we propose to discuss in detail in a subsequent communication.

CALCULATIONS OF THE PERIODS AND NODES OF LOCHS EARN AND TREIG. PLATE I.

BAT

Pos



The Edinburgh Geographical Institute

BATHYMETRICAL SURVEY OF THE FRESH-WATER LOCHS

UNDER THE DIRECTION OF SIR JOHN MURRAY, K.C.B., F.R.S., D.Sc., AND LAURENCE PULL

POSITIONS OF NODES AS CALCULATED BY PROFESSOR CHRYSTAL AND E. MACLAGAN-WED

LOCH EARN

(TAY BASIN)

SURVEYED IN 1902 BY

JAMES PARSONS, B.Sc., AND JAMES MURRAY

Height of Surface of Water above Sea Level, 317·2 feet

4'14" 4'13" 4'12" 4'11"

Allt Glenbeich
Glenbeich
Ardveich
Dalveich
Tynadalloch
Bridge of Beich
Creag Bhuidhe
Creag Bheithe
Allt Bheeidh
Allt Chaorabhaidh
Allt na-h-Atha
Allt na Cedar Bosch
Dalkenneth
Ardveich Wood
Woodend Cottage
Derry
Derry Wood
Allt Diseach Lich
Creag Bhuidhe
Meall Re
W. Binode
Unitode
Mid. Trinode
Achadh Ramich
The Haugh
Ardvorlich Cottage
Ardvorlich House
Coille Chriche
Waterfall
Ardvorlich Burn
Allt Dhùnain
Bealach Dhùnain
Balimeanach
Airidh a' Chreagain
Fhaolen Wood
An Dùnan
Druim Chonnaidh

NOTE TO SECTIONS

Scale of Length 3 inches = 1 Mile same as in Map.
The Black Part in the Section on the same Scale; the Blue Colour shows the Section with Depths exaggerated 5 times.

4'14" 4'13" 4'12" 4'11"

LONGITUDINAL SECTION ALONG AXIS OF MAXIMUM DEPTH

W.B. U. M.T.

Scale 1:21,120 3 INCHES TO 1 MILE — Mile 1
0·21 KILOMETRE — 1 CENTIMETRE — Kilometres 1·61

Creag Meagan
Easter Glentarken
Glentarken Burn
Glentarken Wood
Sgiath an Ath
Dun Caoch
St. Fillan's Park
Creag Gharbh
Creag Mhor
Strone Burn
Allt a Phuirt Bhig
Station
Church
Manse
St Fillans
Little Port
The Oaks
River
Choc a Mhadaidh
Wester Dundurn
E. Brinode
E. Trinode
Ardtrostan
Ardtrostan Wood
Finglen
Dùn'n Aon Duine
Creagan na Mointeich
Fuar Chos
Allt Ardtrostan
Sròn Mhor
Mhic Laurain
Bioran Reag
Duirn
Allt Finglen

DEPTHS IN FEET

FEET	METRES
50	15·2
100	30·5
200	61
250	76·2

CROSS SECTION G H
CROSS SECTION E F
B E.T. E.B.

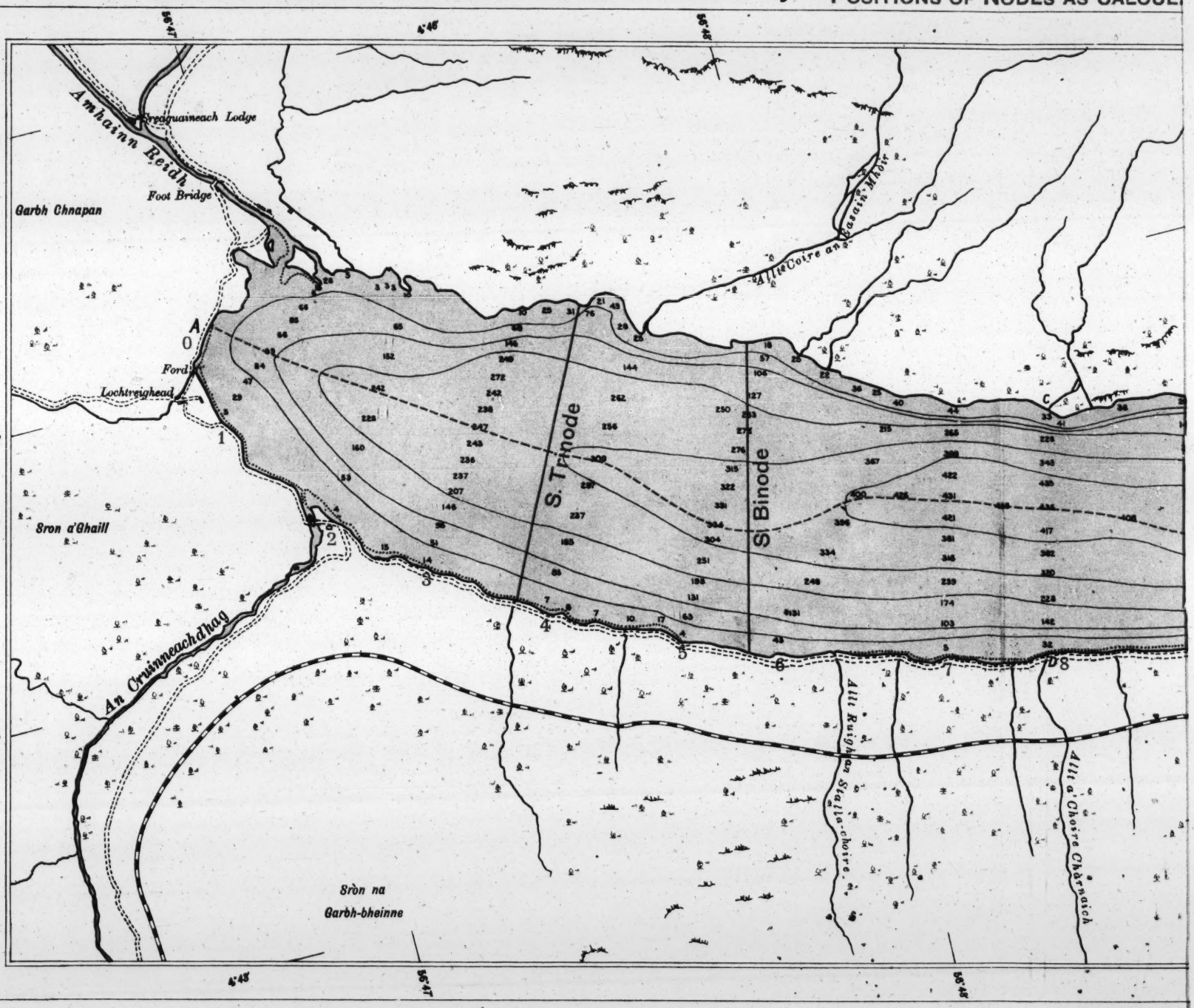
John Bartholomew & Co.

ALCULATIONS OF THE PERIODS AND NODES OF LOCHS EARN AND TREIG. PLATE II.

BATHYMETRICAL SURV

UNDER THE DIRECTION OF SIR

POSITIONS OF NODES AS CALCUL



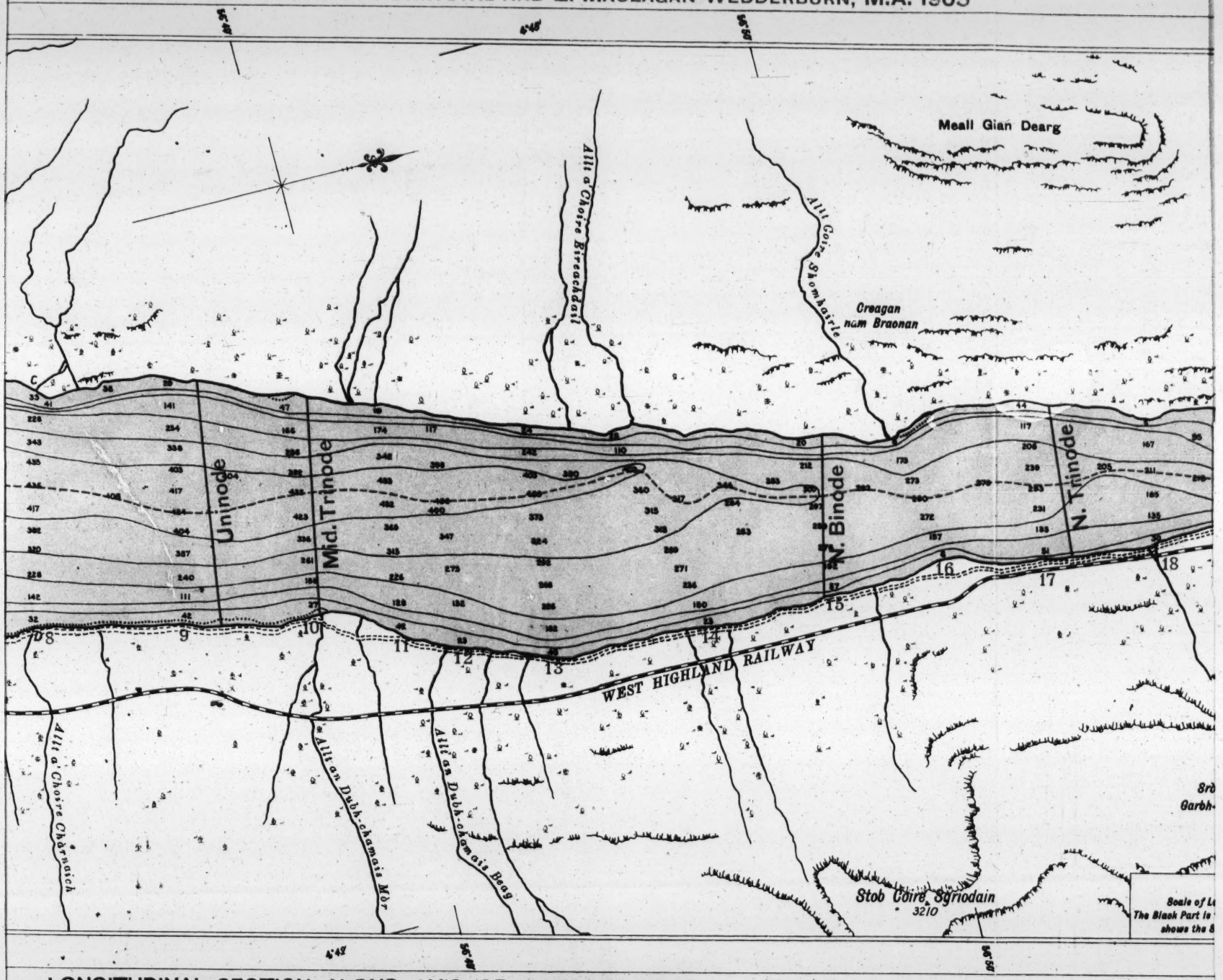
LONGITUDI

dinburgh Geographical Institute

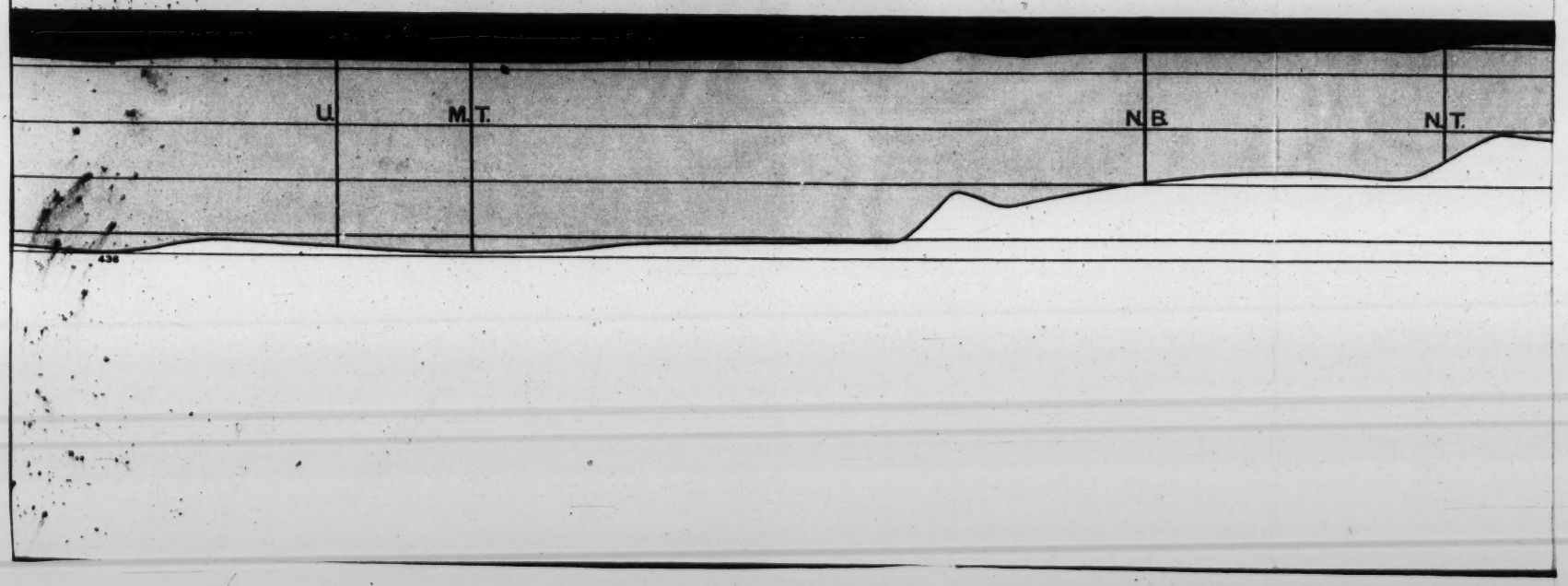
CAL SURVEY OF THE FRESH-WATER LOCHS OF SCOTLAND

DIRECTION OF SIR JOHN MURRAY, K.C.B., F.R.S., D.Sc., AND LAURENCE PULLAR, F.R.S.E.

DES AS CALCULATED BY PROFESSOR CHRYSTAL AND E. MACLAGAN-WEDDERBURN, M.A. 1905



LONGITUDINAL SECTION ALONG AXIS OF MAXIMUM DEPTH



OF SCOTLAND

AR, F.R.S.E.

DERBURN, M.A. 1905

LOCH TREIG

(LOCHY BASIN)

SURVEYED IN 1902 BY

SIR JOHN MURRAY, T. N. JOHNSTON, M.B.C.M.,

JAMES PARSONS, B.Sc., AND JAMES MURRAY

Height of Surface of Water above Sea Level, 787 feet

Scale 1 : 21,120 3 INCHES TO 1 MILE Mile 1

0·21 KILOMETRE — 1 CENTIMETRE Kilometres 1·61

Meall Gian Dearg

Allt Coire Stombairle

Creagan nam Braonan

Lag Odhar

Allt an Luig a Odhair

N. Binnde

N. Trinode

RIVER TREIG

RAILWAY

Creag Dhearg

Sròn na Garbh-bheinne

Stob Coire Sgriodain 3210

DEPTHS IN FEET

FEET	METRES
50	15·2
100	30·5
200	61
300	91·4
400	121·9

CROSS SECTION

C D

50 100 200 300 400

486

NOTE TO SECTIONS

Scale of Length 3 inches = 1 Mile same as in Map.

The Black Part is the Section on the same Scale; the Blue Colour shows the Section with Depths exaggerated 5 times.

B

N.B. N.T.

50 100 200 300 400

John Bartholomew & Co.

XXXIII.—The Alcyonarians of the Scottish National Antarctic Expedition.

By Prof. J. Arthur Thomson, M.A., and Mr James Ritchie, M.A. (With Two Plates.)

(MS. received May 30, 1905. Read July 3, 1905. Issued separately January 18, 1906.)

The Alcyonarians collected by Mr W. S. Bruce on the *Scotia* voyage represent nine species—six of which are new, namely :—

Primnoisis ramosa, n. sp.
Thouarella brucei, n. sp.
Amphilaphis regularis, Wright and Studer.
Primnoella scotiæ, n. sp.
Primnoella magellanica, Studer.
Paramuricea robusta, n. sp.
Gorgonia wrighti, n. sp.
Gorgonia studeri, n. sp.
Umbellula durissima, Kölliker.

Apart from the six new species, the collection is of interest in extending our knowledge of the geographical distribution of previously recorded forms. Thus *Amphilaphis regularis*, Wright and Studer, previously collected off Inaccessible Island, Tristan da Cunha, and off Nightingale Island, was got in abundance off St Helena; *Primnoella magellanica*, Studer, previously collected off Monte Video and in the Magellan Straits, was obtained at Burdwood Bank 54° 25′ S., 57° 32′ W.; while *Umbellula durissima*, Wright and Studer, previously obtained by the *Challenger* from the North Pacific Ocean, south of Yeddo, was found by the *Scotia* at 48° 06′ S., 10° 5′ W.

It may also be noted that the fine specimens of *Umbellula durissima*, Kölliker, give us a better idea of this beautiful species than the single young specimen collected by the *Challenger*. Several of the specimens obtained by Mr Bruce are much larger, older, and of more vigorous growth than that which Kölliker described and named.

With the exception of the much-weathered *Primnoisis ramosa*, n. sp., all the specimens are admirably preserved.

Family ISIDÆ.

Sub-family Mopseinæ.

Primnoisis ramosa, n. sp., Pl. I. fig. 2.

The specimen is much weathered, quite devoid of polyps, and without the basal portion. Although far from complete it attains a height of 230 mm., and a maximum lateral expansion of 45 mm. The bare stem bends frequently, at irregular intervals,

throughout its course, and gives off many branches which are also naked. The branches arise at various acute angles, and some of them, especially towards the lower end, are almost as thick (1·5 mm.) as the main stem (1·75 mm. at the lowest part). Like the latter, they give origin to smaller branches, which may bear minute twigs with a single joint or with two joints. Small branches with only a few joints are much more frequent on the stem than the large branches already mentioned, and they stand off from the stem at greater angles than the large branches—some, indeed, arising perpendicularly.

All the branches spring from the calcareous internodes, and are equally developed on all sides. They vary in number from 3 to 7, or even 8, per joint, 7 perhaps being the most common number. They seem to arise quite irregularly, a frequent interval between two on the same side being 4 mm.; but very occasionally 3 or 4 arise in a whorl.

The axis consists of alternate horny nodes and calcareous internodes, the latter being covered with very fine longitudinal grooves. The internodes are much longer than the nodes, and are themselves longer towards the apex of the colony. The following measurements of successive internodes were taken:—(*a*) from the lowest joint upwards, 5, 6·5, 7, 9 mm.; (*b*) from the topmost joint downwards, 9, 9·5, 10, 9, 10 mm. Near the base the horny nodes are only about 0·5 mm. in length, and gradually decrease towards the apex. The branches never begin with a horny node; in every case a process arises from the originative calcareous node, and on this the first horny node of the branch is based.

This species most closely approaches *P. antarctica*; but the branches arise from all surfaces of the stem and secondary branches, and are equally developed on all sides, whereas in *P. antarctica* the branches arise from only four sides and are unequally developed. Moreover, in the new species the calcareous internodes are much longer than in *P. antarctica*, and may bear 7 or 8 branches, whereas in *P. antarctica* there are only about 4 per joint.

The specimen bears several siliceous sponges, several Polyzoa, a small brown Actinian, and several worm-tubes.

Locality.—Station 411, lat. 74° 1′ S., long. 22° W.; 161 fathoms. Surface temperature 28·9°, March 12, 1904.

Family PRIMNOIDÆ.

Sub-family Primnoinæ.

Thouarella brucei, n. sp., Pl. I. fig. 1; Pl. II. fig. 1.

Several specimens of strong upright branched colonies of a creamy-white colour were found at various stations. The largest specimen is a bushy colony 14 cm. in height by 10·5 cm. in maximum breadth, with an axis 5 mm. in breadth at the base;

but like the others, with one exception, it lacks the basal attachment. The single complete specimen is a graceful bush, 9 cm. in height by 4 cm. in maximum breadth, with an axis 1 mm. in breadth at the base, and an expanded disc of attachment almost 1 cm. across. Of the other specimens the following measurements were taken:—(*a*) 14 cm. in height by 3 in breadth, a single branch, with an axis 4 mm. in breadth; (*b*) 11 cm. in height by 9·5 cm. in maximum breadth, a bushy colony with an axis 3 mm. in breadth at the base; (*c*) 8 cm. in height by 9 cm. in maximum breadth, a bushy colony with an axis 2 mm. in breadth. The colonies bear Comatulids attached by their cirri, encrusting Polyzoa, hydroids, and several sponges.

The branching of the specimens differs from that of the previously described species of Thouarella. A main stem, 1 to 5 mm. in diameter, gives off strong branches almost as thick as itself, and sometimes attaining a length of 12 cm. They arise *in at least three directions* and at irregular intervals. From these branches, as also from the intervals between them on the main stem, slender twigs arise on all sides, and at varying angles. But the strong branches of the first degree may also bear strong branches of the second degree, likewise carrying slender twigs. The larger branches show a tendency to curve inwards towards the main stem.

In all cases the slender, graceful twigs spring *from all sides* at very irregular intervals, and are equally developed all round. As a result of the repeated branching, of the incurving of the larger branches, and of the very numerous close-set twigs, the colony bears a characteristic resemblance to a thickly-growing sturdy bush.

Where the cœnenchyma has been rubbed off in the lower parts of the colonies, the stout, almost inflexible axis is exposed. It is tawny-brown in colour, with in some places a yellowish sheen; but it becomes lighter in colour (honey-yellow), as well as very flexible, towards the tips of the branches and in the twigs. It is composed of horny and calcareous materials, and is *circular* in cross section.

The calices, which are about 1 mm. in height, are borne chiefly on the twigs, but they are occasionally borne by the twig-supporting branches and by the main stem. On the twigs they are closely approximated, arising in all directions and without any definite arrangement. They are pear-shaped, and generally bent inwards to the axis.

The number of transverse rows of scales varies slightly, but five is a very common number. The number of longitudinal rows is about seven. The scales appear to be similar in size and structure on all sides of the polyp, there being none distinctively dorsal or ventral. They have a convex upper edge, frequently assume an almost quadrangular form, and are thickly tuberculated. Fusion of the tubercles occasionally gives rise to very slight ridges running outwards from the nucleus. The embedded edges of all the scales are more ragged than the free edges.

The rows of scales are surmounted by about seven opercular scales, all of which have a ridge projecting for a considerable distance, usually bordered by a narrow leaf-like wing.

This species is marked off from others previously described by the origin of strong

branches in at least three directions, by the origin of twigs on all sides of the axis, by the cylindrical shape of the axis, and by the detailed speculation of the polyps.

Localities.—Burdwood Bank, 56 fathoms, December 1, 1903; Gough Island, 100 fathoms, April 22, 1904; St Helena.

Amphilaphis regularis, Wright and Studer, Pl. II. fig. 5.

Numerous fine specimens of this graceful form were obtained from St Helena. The following measurements of height and lateral expansion were taken in cm.:—40 by 25, 33 by 15, 26 by 15, 20 by 30, 17 by 9, 20 by 19, 20 by 10, 16 by 11; but none of these represent complete specimens. As is the case with *Thouarella brucei*, there are very noticeable differences in the vigour of the various specimens, for some have the polyps much more crowded than others.

The specimens agree closely with the description by Wright and Studer, but it may be noted that the figures of the spicules given in the *Challenger* Report do not show the prominent spines described in the text. We have therefore given a supplementary figure.

We add a few details in reference to the spicules. The scales of the operculum are roughly triangular, usually with an indentation in the base directly opposite the nucleus. A strong ridge, sometimes double, extends from the apex of the triangle towards the nucleus, which, however, it seldom reaches. The calyx scales resemble a rude ellipse, toward the upper edge of which the tubercles have become fused to form prominent ridges, frequently 0·08 to 0·1 mm. in length, radiating from the nucleus and projecting as spines beyond the edge. The arrangement of the ridges resembles that of the teeth in a comb.

The specimens bear numerous small Actinians, clusters of Polyzoa, clambering Ophiuroids, serpuloid worm-tubes, small barnacles, etc.

Locality.—St Helena.

Primnoella scotiæ, n. sp., Pl. II. figs. 3 and 8.

A simple upright colony, 105 mm. in height, of a dirty yellowish-white colour. The basal portion is absent and the axis has disappeared. It looks as if the dredge had dragged the colony from off the axis, for there is a slightly oval central canal, a little over 1 mm. in diameter at the base and narrowing to 0·5 mm. towards the apex.

The stem is closely covered with polyps arranged in whorls of 9 to 11, the most frequent number being ten. The calices are closely apposed to the stem and are pressed against one another laterally, and the whorls themselves overlap, so that the general effect is that of a uniformly thick rod with a diameter of 4 mm.

The calices are from 2·5 to 3 mm. long and 1 mm. broad, but owing to the overlapping at the base less than 2 mm. of the calyx is visible. Under the lens the

verrucæ appear as slightly flattened cylinders covered with fine horizontal striæ, which higher magnification shows to be the smooth edges of regularly arranged broad imbricating scales. These are arranged in two longitudinal parallel rows along the dorsal surface, those in one row interlocking with the alternate scales of the other row. The upper edges of all the dorsal scales are parallel, and the two rows meet in the middle without any distinct angle or keel. Each row has from 21 to 28 scales.

On the ventral side of the calyx there are two small longitudinal rows along the edges, but the rest of the surface is covered with indistinct roundish scales irregularly disposed.

There does not seem to be any special operculum, but several of the uppermost scales bend over so as partly to cover the mouth of the calyx, within which the retracted tentacles of the polyp can usually be seen.

The dorsal calyx-scales are roughly rectangular, very broad and slightly curved to fit the cylindrical polyp body. The upper or projecting margin of each scale is smooth, while the lower or overlapped margin is toothed. The whole of the inside of the scale, except a narrow strip along the upper edge, is covered with numerous small tubercles. On the external surface there are numerous very fine wavy lines running from edge to edge of the scale.

The other scales are irregular in outline, sometimes with toothed margins, sometimes smooth-edged; they may be almost free from tubercles or covered with them.

All the scales are colourless, and show an eccentric darker nucleus from which any slight ridges on the surface run. From these nuclei, as is shown by polarised light, the rest of the scale has been deposited in concentric zones.

Locality.—Burdwood Bank, lat. 54° 25′ S., long. 57° 32′ W.; 52 fathoms. Surface temperature 41·8°, December 1, 1903.

Primnoella magellanica, Studer, Pl. I. fig. 3.

An almost complete specimen of this species, lacking only a small part of the basal region. The stem reaches a height of 148 mm., but towards the lower end the cœnenchyma has disappeared, exposing the brown axis for about 15 mm., while for the next 30 mm. the whorls of polyps are broken and incomplete.

The specimen agrees with the description of *P. magellanica* given in the *Challenger* Report except in the following particulars. In the *Challenger* specimen the number of polyps in a whorl was 8; in the *Scotia* specimen there are 9, 11, 12, 13, 10, 12, 12, 13, in the various whorls counted. In the *Challenger* specimen the opercular scales were in length and breadth 0·48 × 0·2 mm., while the corresponding measurements for the *Scotia* specimen are 0·65 × 0·35, 0·625 × 0·375 mm. Similarly for the calyx scales, the measurements for the *Challenger* specimen were 0·31 × 0·3, 0·36 × 0·37; and for the *Scotia* specimen 0·3 × 0·3, 0·425 × 0·35, 0·3 × 0·25. Thus there are decidedly larger dimensions in the scales of the *Scotia* specimen. The larger and

variable number of polyps in a whorl is of greater importance, but it probably means nothing more than a greater vigour of growth.

The figure in the *Challenger* Report shows the whorls too far apart, as the text points out; we have therefore given a supplementary figure.

Locality.—Burdwood Bank, lat. 54° 25′ S., long. 57° 32′ W.; 52 fathoms. Surface temperature 40·8°, December 1, 1903.

Family MURICEIDÆ.

Paramuricea robusta, n. sp., Pl. I. fig. 6; Pl. II. figs. 2 and 7.

A strong upright colony of a light brown colour, expanded for the most part in one plane, 27·5 cm. in maximum height by 14 cm. in maximum breadth. Not far from the base, which is expanded to 2 cm. and soon narrows to 1 cm., a strong side-branch is given off with a diameter of 6 mm., and this, like the main stem, bears strong offshoots from which smaller, usually simple, branches arise. The branching is very irregular, but anastomosis is rare, being represented in one of the specimens by only two instances, one of which shows the junction of an apparently broken branch of the first degree with the main stem. In another specimen, 18 cm. by 12 cm., there is no anastomosis. Towards the base of the colony the main stem is distinctly flattened, 9·5 by 8 mm., immediately above the basal thickening.

The axis is horny, non-calcareous, fibrous, and of a brownish colour. It narrows from about 6 mm. near the base to 1 mm. near the tips of the branches.

The cœnenchyma is relatively thin (0·5 mm.) and somewhat translucent, allowing the brownish axis to shine faintly through. Its surface is rough, owing to the abundance of large colourless spicules which cover it. Some of these spicules project from the tops of the verrucæ as crowns of spines.

The yellowish verrucæ are cylindrical with a slightly conical summit, 1·5 mm. in height by 1 mm. in diameter, and arise perpendicularly from all sides of the main stem and its branches. They are closely set, without any regular interval between them. Four or five are always grouped at the tip of a branch, giving it a knobbed appearance.

The polyps are wholly retracted, and an operculum of 8 parts, each composed of about 5 spicules resting on the bases of the tentacles. closes over the aperture. Round the top of the verruca a few rows of spicules are arranged horizontally, and on this support the bases of the opercular covering rest.

Various types of spicules characterise the species. Most characteristic are the large tuberculate clubs whose 'handles' form the spiny crowns of the verrucæ, while the much divided root-like 'heads' are embedded in the cœnenchyma. There are also simpler clubs with heads covered with tubercles and spiny processes. Curved spindles are common, some knobbed and thickened, with comparatively large projecting processes and smaller

spines; others are more regular, boomerang-like, with spines and tubercles only; others again are almost smooth with only a few small warts.

The following measurements were taken:—complex clubs, 0·8 to 0·9 mm. in length by 0·45 between the extremes of the spreading heads; broad almost straight spindles, 0·65 × 0·2 mm.; narrow curved spindles, 0·7 × 0·04, 0·5 × 0·05, 0·425 × 0·06 mm.; simple forks with few spines, 0·5 mm. in length by 0·1 at the forked end.

In general the colony presents a remarkably sturdy, rigid appearance, due to the thickening effect of the numerous polyps which arise from the flexible twigs and branches. The various specimens bear numerous epizoic animals, *e.g.* small Actinians, Polyzoa, worm-tubes.

This new species may be distinguished from most of the other representatives of the genus by the absence of any arrangement of the verruca spicules in longitudinal rows. It is separated from all by the characters of its spicules, and in particular by the large tuberculate clubs with expanded divaricate heads. From *P. ramosa*, which it most nearly approaches in appearance, and from *P. laxa*, it may be distinguished, apart from the spicules, by the absence of any intermediate part of the stem or branches free from polyps. The verrucæ are distributed equally on all sides of the stem and branches instead of being disposed, for the most part, on opposite sides. From *P. ramosa* it is also distinguished by the exceedingly rare occurrence of anastomosis. Some of the spicules of Kölliker's *P. spinosa* closely resemble some of those in our species, but in *P. spinosa* the cœnenchyma is very thin, the polyps are rather sparse, and there are many other points of difference.

Localities.—Gough Island, lat. 40° 20′ S., long. 9° 56′ W.; 100 fathoms; surface temperature 55·2°, April 22, 1904. St Helena.

Family GORGONIDÆ.

Gorgonia wrighti, n. sp., Pl. I. figs. 7 and 8; Pl. II. figs. 6 and 9.

A much-branched, flexible, upright white colony with a general height of 22 cm. by about 10 cm. in breadth. The main stem gives off, about 25 mm. above the base, a strong branch which bears long flexible offshoots, and these again bear numerous usually simple branches. There are even some branches of the fourth degree, and with the base of one of these another branch unites—the only instance of anastomosis in the colony. The branches have a fairly uniform thickness of 2 mm., and can hardly be said to taper toward the blunt, rounded, or swollen tip. The larger branches are very slightly flattened towards their base. They all arise at angles rather less than 90°, and the whole system shows a tendency to spread in one plane, though here and there a branch arises at right angles to the rest. The branches of the same degree are markedly parallel when not twisted out of their original direction. There is a tendency in the secondary branching to

preponderate towards the side more remote from the main axis. The first ma branch is 150 mm. in length and 2·5 mm. in breadth.

Towards the base of the colony a portion of the axis is exposed. It is slight flattened, 3 mm. in breadth, horny, non-calcareous, and very flexible. The colo is a rich dark brown, fading into pale brownish yellow towards the tips of t branches. There are very marked 'chambers' or curved transverse septa. cross section shows a central canal filled with whitish material.

The cœnenchyma is thick (0·375 mm.) and has a granular appearance, due to the co plete covering of spicules. On one of the branches there is a calcareous cirripede gall

The polyps occur on all surfaces of the stem and branches, but are more freque along the opposite sides than along the middle. They are completely retractile, a when withdrawn leave small almost circular openings, which are on a level with t surface on the older portions, while in the younger parts their margins are slight raised to form lips, giving a warty appearance to the terminal regions.

The spicules are translucent spindles and scaphoids, almost always curved, and bea ing numerous spines which often equal or exceed the diameter of the spicule prop The spines are generally developed to a much greater extent on the convex side the spicule. They are frequently tubercled or almost branched. Some of the spindl are fairly smooth with only a few tubercles. The following measurements were tak of the length and maximum breadth including the spines:—0·85 × 0·1, 0·8 × 0·1, 0·75 0·06, 0·7 × 0·04, 0·6 × 0·06, 0·5 × 0·03, 0·4 × 0·1, 0·3 × 0·075 mm. As almost eve possible adjective is already preoccupied as the specific name of some *Gorgonia* or s called *Gorgonia*, we have named this new form *G. wrighti* after Prof. E. Percev Wright, joint-author of the *Challenger* Report on Alcyonarians.

Locality.—Station 81; lat. 18° 26′ S., long. 37° 58′ W.; 40 to 50 fathoms.

Gorgonia studeri, n. sp., Pl. I. fig. 4; Pl. II. fig. 4.

A portion of an upright branched white colony, consisting of what may be part the main stem (30 by 2 mm.), bearing on one side two parallel branches from one which a smaller branch arises. The distance between the two parallel branches is mm.; the length of the longer simple branch is 95 mm., of the shorter 70 mm., and its branch 35 mm. There is an indication that still another branch arose from the la so that branching of at least the third degree is present. The branches, which ta almost imperceptibly towards their tips, have a diameter of 2 mm. They lie in one pla leave the axis at an angle of about 70°, and are slightly compressed in their ol portions.

The axis is horny, non-calcareous, and flexible, of a brown colour passing into a ho yellow in the younger portions. It shows transverse 'chambers' or curved septa. diameter at the oldest part is 0·8 mm.

The polyps show a tendency to bilateral arrangement, being more frequent along

two opposite sides of the branches, although by no means confined to these. They are not wholly retracted, but protrude from the surface of the cœnenchyma as small roundish warts surrounded by a gently sloping spicular dome, which rises gradually to form a very slight lip around the polyp aperture.

The spicules, which are whitish and translucent, are of three main types. (*a*) Most abundant are long narrow spindles, *e.g.* 0·75 × 0·06, 0·7 × 0·05 mm., covered with warty tubercles, which are frequently produced into blunt spines. The spines show a marked tendency to unilateral development, being often more prominent and more numerous on one side of the spindle. (*b*) Less abundant are scaphoid forms, *e.g.* 0·7 × 0·12, 0·4 × 0·06 mm. (*c*) There are also some forms which approach the 'club' type and differ greatly in size, *e.g.* 0·45 × 0·12, 0·25 × 0·06 mm. Their heads are covered with long blunt processes, similar to the blunt spines of the spindles, and these are sometimes continued down the 'handle' of the club. Both the 'scaphoids' and the 'clubs' are readily derivable from the spindle type.

We have named this new species *G. studeri* after Prof. Th. Studer, joint-author of the *Challenger* Report on Alcyonarians.

Locality.—Station 81; lat. 18° 26′ S., long. 37° 58′ W.; 40 to 50 fathoms.

Family UMBELLULIDÆ.

Umbellula durissima, Kölliker, Pl. I. fig. 5.

About twenty specimens of this beautiful form were obtained from one locality, from a depth of 1742 fathoms (April 13, 1904). Only one specimen was obtained by the *Challenger* expedition, and that much younger and smaller than the best of the *Scotia* specimens.

The following total length measurements were taken:—50, 45, 42, 37, 34, 32, 22, 20, 18, 17 cm. The heads vary from 2·8 cm. in height and breadth to 1·7 in height by 0·5 in breadth. The stalk is very slender in proportion to the head, and the following breadth measurements were taken:—3·5 mm. almost at the base and 1 mm. near the top of the largest specimen; 1·5 mm. at the base and 0·5 mm. near the top of the smallest specimen.

There is considerable diversity in the number of polyps—thus one head had 9, one had 7, five had 6, one had 5, and four had 3 polyps. The colouring of the polyps is exceptionally beautiful—a milky blue fading basally into white; the tentacles are chocolate brown. Eight vertical rows of rod-like spicules extend up the surface of the polyps and are continued into the tentacles. The largest polyps measure 15 mm. by 8 mm., not including the tentacles, which are 15 mm. in length. The minute siphonozooids are exceedingly numerous, covering the whole ventral surface of the head except a narrow median ridge, and also extending in bands between the bases of the polyps or autozooids. The bluish colour was not noticed in the *Challenger* specimen, and seems to be gradually fading in those under our observation.

The larger spicules are rods with rounded or swollen ends, and have the following dimensions in mm. :—2·5 × 0·25, 2 × 0·2, 2 × 0·15, 1·8 × 0·13, 1·45 × 0·125, 1·4 × 0·1, 1·2 × 0·1. Besides these there are minute rods, 0·14 × 0·023, 0·1 × 0·02.

Locality.—48° 06′ S., 10° 5′ W. Bottom at 1742 fathoms, pebbles and diatom ooze. Surface temperature 40·8° F.

EXPLANATION OF PLATES.

Plate I.

Fig. 1. *Thouarella brucei*, n. sp. A branch with twigs. Nat. size.
Fig. 2. *Primnoisis ramosa*, n. sp. A portion of the axis with branches. Nat. size.
Fig. 3. *Primnoella magellanica*, Studer. Three whorls of polyps. × 9
Fig. 4. *Gorgonia studeri*, n. sp. The whole fragment, natural size; and a portion of the axis with verrucæ, magnified about 10 times.
Fig. 5. *Umbellula durissima*, Kölliker. The largest head, magnified about 2½ times.
Fig. 6. *Paramuricea robusta*, n. sp. A small piece of a branch with verrucæ, magnified about 2 times.
Fig 7. *Gorgonia wrighti*, n. sp. Showing the mode of branching. Nat. size.
Fig. 8. *Gorgonia wrighti*, n. sp. A portion of the axis, showing the chambers, magnified about 10 times.

Plate II.

Fig. 1. *Thouarella brucei*, n. sp.
Fig. 2. *Paramuricea robusta*, n. sp.
Fig. 3. *Primnoella scotiæ*, n. sp.
Fig. 4. *Gorgonia studeri*, n. sp.
Fig. 5. *Amphilaphis regularis*, Wright and Studer.
Fig. 6. *Gorgonia wrighti*, n. sp.
Fig. 7. *Paramuricea robusta*, n. sp. A small portion with two verrucæ. × 10.
Fig. 8. *Primnoella scotiæ*, n. sp. The apex with four whorls of polyps. × 8.
Fig. 9. *Gorgonia wrighti*, n. sp. A small portion of the stem. × 10.

THOMSON AND RITCHIE: ALCYONARIANS.

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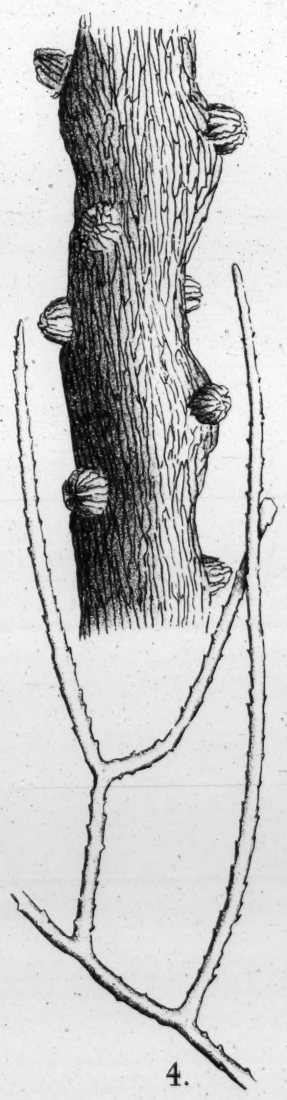

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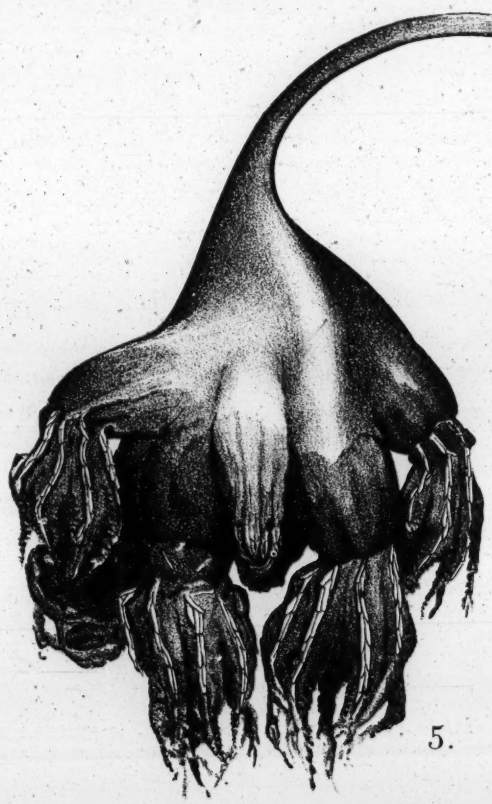

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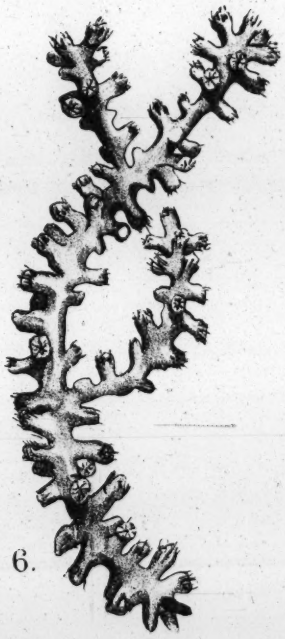

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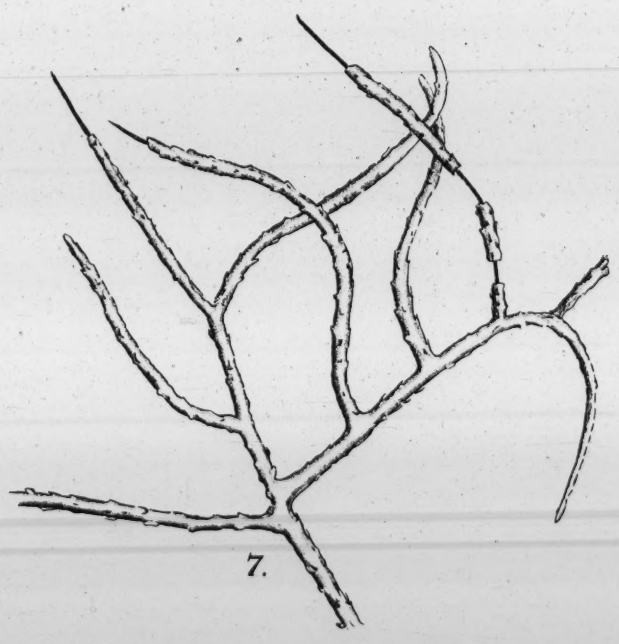

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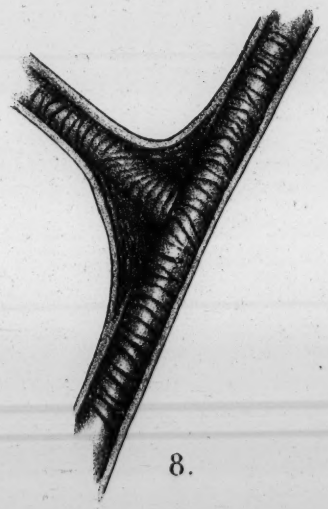

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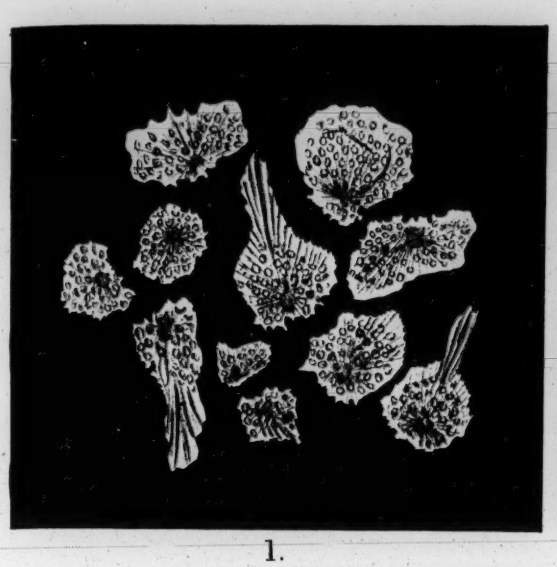
G. Davidson, del. 2-6.
J. Ritchie, del. 1 7, 8.

McFarlane & Erskine, Lith Edin.r

"SCOTIA" ALCYONARIANS. Plate I.



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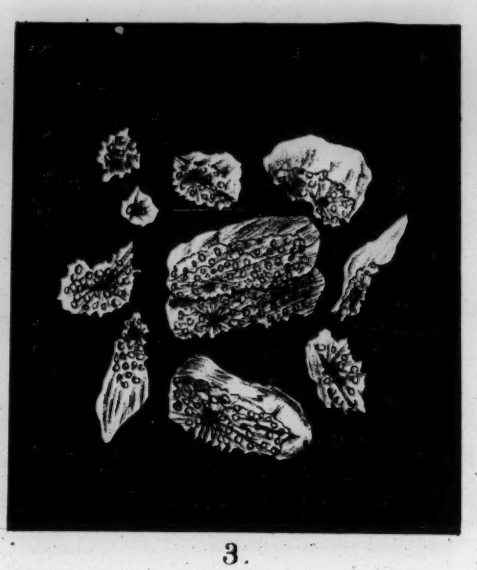

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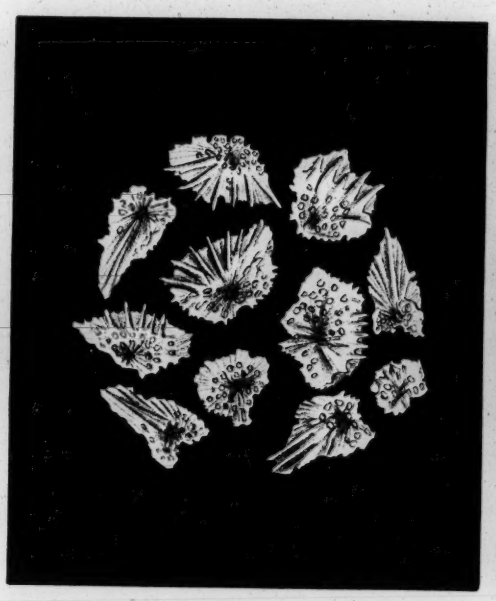

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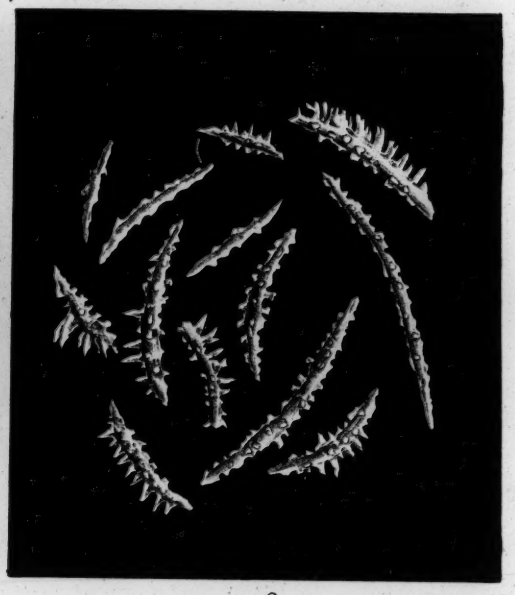

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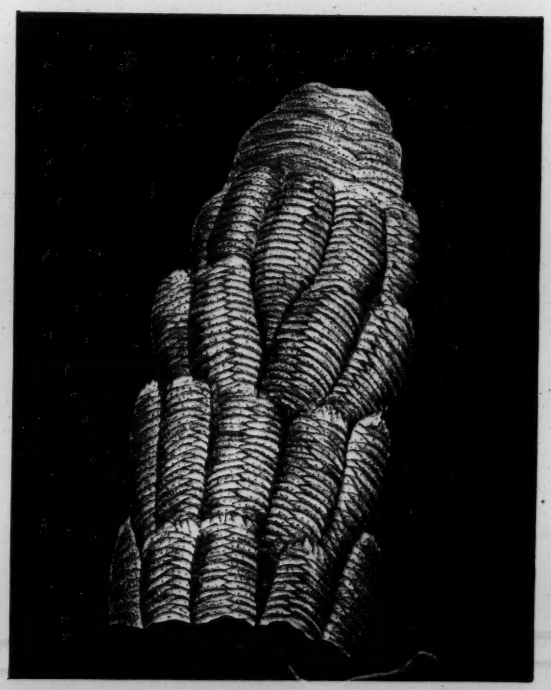

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J. Ritchie del.

M^c^Farlane & Erskine, Lith. Edin^r^

"SCOTIA" ALCYONARIANS. Plate II.

APPENDIX.

TRANSACTIONS

OF THE

ROYAL SOCIETY OF EDINBURGH.

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ROYAL SOCIETY OF EDINBURGH.

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ALPHABETICAL LIST OF ORDINARY FELLOWS,

AND LIST OF HONORARY FELLOWS,

At October 1905.

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OF

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OCTOBER 1905.

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JOHANNES P. KUENEN, Ph.D., Professor of Natural Philosophy in University College, Dundee.

ALPHABETICAL LIST

OF

THE ORDINARY FELLOWS OF THE SOCIETY,

CORRECTED TO OCTOBER 1905.

N.B.—*Those marked * are Annual Contributors.*

B. prefixed to a name indicates that the Fellow has received a Makdougall-Brisbane Medal.
K. ,, ,, ,, Keith Medal.
N. ,, ,, ,, Neill Medal.
V. J. ,, ,, ,, the Gunning Victoria Jubilee Prize.
C. ,, ,, ,, contributed one or more Communications to the Society's Transactions or Proceedings.

Date of Election.		
1898		* Abercromby, The Hon. John, 62 Palmerston Place
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1896		* Affleck, Jas. Ormiston, M.D., F.R.C.P.E., 38 Heriot Row
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1905		* Anderson, William, M.A., Head Science Master, George Watson's College, Edinburgh, 29 Lutton Place
1903		Anderson-Berry, David, M.D., C.M. Edin., F.S.A. Scot., 23 Grosvenor Crescent, St Leonards-on-Sea 15
1905		* Andrew, George, M.A., B.A., H.M.I.S., 3 Mayfield Gardens
1883	C.	Andrews, Thos., Memb. Inst. C.E., F.R.S., F.C.S., Telford Medallist and Prizeman, Inst.C.E., Gold Medallist and Bessemer Prizeman, Soc. Engineers, Metallurgical Testing Laboratory, Wortley, near Sheffield
1881	C.	Anglin, A. H., M.A., LL.D., M.R.I.A., Professor of Mathematics, Queen's College, Cork

Date of Election.		
1867	C.	Annandale, Thomas, M.D., F.R.C.S.E., Professor of Clinical Surgery in the University of Edinburgh, 34 Charlotte Square
1899		Appleyard, James R., Royal Technical Institute, Salford, Manchester
1893		* Archer, Walter E., 17 Sloan Court, London
1883		Archibald, John, M.D., C.M., F.R.C.S.E., Hazleden, Wimborne Road, Bournemouth
1885	C.	* Baildon, H. Bellyse, M.A., Ph.D., F.R.S.L., Lecturer on the English Language and Literature, University College, Dundee
1894		* Bailey, Frederick, Lieut.-Col. (*late*) R.E., 7 Drummond Place
1896		* Baily, Francis Gibson, M.A., Professor of Applied Physics, Heriot-Watt College
1877	C.	Balfour, I. Bayley, M.A., Sc.D., M.D., LL.D., F.R.S., F.L.S., King's Botanist in Scotland, Professor of Botany in the University of Edinburgh and Keeper of the Royal Botanic Garden, Inverleith House
1905		Balfour-Browne, William Alexander Francis, M.A., Barrister-at-Law, Director of the Sutton Broad Biological Laboratory, Catfield, Great Yarmouth
1892		* Ballantyne, J. W., M.D., F.R.C.P.E., 24 Melville Street
1902	C.	Bannerman, W. B., M.D., B.Sc., Lt.-Colonel, Indian Medical Service, Director, Plague Research Laboratory, Bombay, India
1889		* Barbour, A. H. F., M.A., M.D., F.R.C.P.E., 4 Charlotte Square
1886		* Barclay, A. J. Gunion, M.A., 729 Great Western Road, Glasgow
1872		Barclay, George, M.A., 17 Coates Crescent
1883	C.	* Barclay, G. W. W., M.A., 91 Union Street, Aberdeen
1903		Bardswell, Noël Dean, M.D., M.R.C.P. Ed. and Lond., Mundesley, Norfolk
1882	C.	Barnes, Henry, M.D., LL.D., 6 Portland Square, Carlisle
1893		Barnes, R. S. Fancourt, M.D., M.R.C.P.L., Consulting Physician to the Royal Maternity Charity of London, 15 Chester Terrace, Regent's Park, London
1904		Barr, Sir James, M.D., F.R.C.P. Lond., 72 Rodney Street, Liverpool
1874		Barrett, William F., F.R.S., M.R.I.A., Prof. of Physics, Royal College of Science, Dublin
1889		Barry, T. D. Collis, Staff Surgeon, M.R.C.S., F.L.S., Chemical Analyser to the Government of Bombay, and Prof. of Chemistry and Medical Jurisprudence to the Grant Medical College, and of Chemistry, Elphinstone College, Malabar Hill, Bombay
1887		* Bartholomew, J. G., F.R.G.S., The Geographical Institute, Dalkeith Road
1895	C.	Barton, Edwin H., D.Sc., A.M.I.E.E., Memb. Phys. Soc. of London, Senior Lecturer in Physics, University College, Nottingham
1904		* Baxter, William Muirhead, 14 Grange Road
1888		* Beare, Thomas Hudson, B.Sc., Memb. Inst. C.E., Professor of Engineering in the University of Edinburgh
1897	C.	* Beattie, John Carruthers, D.Sc., Professor of Physics, South African College, Cape Town
1892		Beck, J. H. Meining, M.D., M.R.C.P.E., Rondebosch, Cape Town
1893	B. C.	* Becker, Ludwig, Ph.D., Regius Professor of Astronomy in the University of Glasgow, The Observatory, Glasgow
1882	C.	Beddard, Frank E., M.A. Oxon., F.R.S., Prosector to the Zoological Society of London, Zoological Society's Gardens, Regent's Park, London
1887		* Begg, Ferdinand Faithful, Bartholomew House, London
1886		* Bell, A. Beatson, 17 Lansdowne Crescent
1874		Bell, Joseph, M.D., F.R.C.S.E., 2 Melville Crescent
1900		* Bennett, James Bower, Memb. Inst. C.E., 2 Thorburn Road, Colinton
1887		* Bernard, J. Mackay, of Dunsinnan, B.Sc., Dunsinnan, Perth

Date of Election.		
1875		Bernstein, Ludwik, M.D., Lismore, New South Wales
1893	C.	* Berry, George A., M.D., C.M., F.R.C.S., 31 Drumsheugh Gardens
1897	C.	* Berry, Richard J., M.D., F.R.C.S.E., Professor of Anatomy in the University of Melbourne, Victoria
1904		* Beveridge, Erskine, LL.D., St Leonard's Hill, Dunfermline
1880	C.	Birch, De Burgh, M.D., Professor of Physiology in the University of Leeds, 16 De Grey Terrace, Leeds
1900		* Bisset, James, M.A., F.L.S., F.G.S., 9 Greenhill Park
1884		* Black, John S., M.A., LL.D., 6 Oxford Terrace
1850		Blackburn, Hugh, M.A., LL.D., Emeritus Professor of Mathematics in the University of Glasgow, Roshven, Lochailort
1897		* Blaikie, Walter Biggar, 6 Belgrave Crescent
1904	C.	* Bles, Edward J., B.A., B.Sc., Assistant to the Prof. of Natural History, Univ. of Glasgow
1898	C.	* Blyth, Benjamin Hall, M.A., Memb. Inst. C.E., 17 Palmerston Place
1878	C.	Blyth, James, M.A., LL.D., Prof. of Natural Philosophy in Anderson's College, Glasgow
1894		* Bolton, Herbert, Curator of the Bristol Museum, Queen's Road, Bristol
1884		Bond, Francis T., B.A., M.D., M.R.C.S., Gloucester
1872	C.	Bottomley, J. Thomson, M.A., D.Sc., LL.D., F.R.S., F.C.S., Lecturer on Natural Philosophy in the University of Glasgow, 13 University Gardens, Glasgow
1869	C.	Bow, Robert Henry, C.E., 7 South Gray Street
1886		* Bower, Frederick O., M.A., D.Sc., F.R.S., F.L.S., Regius Professor of Botany in the University of Glasgow, 1 St John's Terrace, Hillhead, Glasgow
1884	C.	Bowman, Frederick Hungerford, D.Sc., F.C.S. (Lond. and Berl.), F.I.C., Assoc. Inst. C.E., Assoc. Inst. M.E., M.I.E.E., &c., 4 Albert Square, Manchester
1901		Bradbury, J. B., M.D., Downing Professor of Medicine, University of Cambridge
1903	C.	* Bradley, O. Charnock, M.B., Ch.B., D.Sc., Royal Veterinary College, Edinburgh
1886		* Bramwell, Byrom, M.D., F.R.C.P.E., 23 Drumsheugh Gardens
1895		* Bright, Charles, Assoc. Memb. Inst. C.E., Memb. Inst. E.E., F.R.A.S., F.G.S., 21 Old Queen Street, Westminster, London
1886		Brittle, John Richard, Memb. Inst. C.E., Farad Villa, Vanbrugh Hill, Blackheath, Kent
1877		Broadrick, George, Memb. Inst. C.E., Broughton House, Broughton Road, Ipswich
1893		Brock, G. Sandison, M.D., 2 Via Veneto, Rome, Italy
1892		* Brock, W. J., M.B., D.Sc., 5 Manor Place
1901	C.	* Brodie, W. Brodie, M.B., 28 Hamilton Park Terrace, Hillhead, Glasgow
1887		* Brown, A. B., C.E., Memb. Inst. Mech. E., 19 Douglas Crescent
1864	C. K. B.	Brown, Alex. Crum, M.D., D.Sc., F.R.C.P.E., LL.D., F.R.S. (Vice-President), Professor of Chemistry in the University of Edinburgh, 8 Belgrave Crescent
1898		* Brown, David, F.C.S., F.I.C., Willowbrae House, Midlothian
1883	C.	* Brown, J. J. Graham, M.D., F.R.C.P.E., 3 Chester Street
1885	C.	Brown, J. Macdonald, M.D., F.R.C.S., 2 Frognal, London, N.W.
1883	C.	* Bruce, Alexander, M.A., M.D., F.R.C.P.E., 8 Ainslie Place
1898	C.	* Bryce, T. H., M.A., M.D. (Edin.), 2 Granby Terrace, Glasgow
1888		* Bryson, William A., Electrical Engineer, 16 Charlotte Street, Leith
1869	C. B. V. J.	Buchan, Alexander, M.A., LL.D., F.R.S., Secretary to the Scottish Meteorological Society (Curator of Library and Museum), 2 Dean Terrace
1870	C. K.	Buchanan, John Young, M.A., F.R.S., Christ's College, Cambridge
1902		* Buchanan, Robert M., M.B., F.F.P.S.G., 2 Northbank Terrace, Glasgow

Date of Election.		
1882		* Buchanan, T. R., M.A., 12 South Street, Park Lane, London, W.
1887	C.	* Buist, J. B., M.D., F.R.C.P.E., 1 Clifton Terrace
1905		Bunting, Thomas Lowe, M.D., Scotswood, Newcastle-on-Tyne
1902		* Burgess, A. G., M.A., Mathematical Master, Edinburgh Ladies' College, 2 Craigcrook Terrace, Blackhall
1894	C. K.	* Burgess, James, C.I.E., LL.D., M.R.A.S., M. Soc. Asiatique de Paris, *H.*A.R.I.B.A., 22 Seton Place
1902		* Burn, The Rev. John Henry, B.D., The Parsonage, Ballater
1887		* Burnet, John James, Architect, 18 University Avenue, Hillhead, Glasgow
1888		* Burns, Rev. T., F.S.A. Scot., Minister of Lady Glenorchy's Parish Church, Croston Lodge, Chalmers Crescent
1903		* Butler, Rev. Dugald, M.A., Minister of the Tron Parish, 54 Blacket Place
1896		* Butters, J. W., M.A., B.Sc., Rector of Ardrossan Academy
1887	C.	* Cadell, Henry Moubray, of Grange, B.Sc., Bo'ness
1897		* Caird, Robert, LL.D., Shipbuilder, Greenock
1893	C.	Calderwood, W. L., Inspector of Salmon Fisheries of Scotland, 7 East Castle Road, Merchiston
1894		* Cameron, James Angus, M.D., Medical Officer of Health, Firhall, Nairn
1905	C.	* Cameron, John, M.D., D.Sc., M.R.C.S. Eng., Demonstrator of Anatomy, University of Manchester, Anatomy Department, Owens College, Manchester
1904		* Campbell, Charles Duff, 21 Montague Terrace, Inverleith Row
1878		Campbell, John Archibald, M.D., Gothic Villa, St Aubyn's Road, Jersey
1899	C.	* Carlier, Edmund W. W., M.D., B.Sc., Prof. of Physiology in Mason College, Birmingham
1902		* Carmichael, Sir Thomas D. Gibson, Bart., M.A., Malleny House, Balerno
1905	C.	* Carse, George Alexander, M.A., B.Sc., 120 Lauriston Place
1901		Carslaw, H. S., M.A., D.Sc., Professor of Mathematics in the University of Sydney, New South Wales
1905		Carter, Joseph Henry, F.R.C.V.S., Stone House, Church Street, Burnley, Lancashire
1898		* Carter, Wm. Allan, Memb. Inst. C.E., 32 Great King Street
1898		Carus-Wilson, Cecil, F.R.G.S., F.G.S., Royal Societies Club, St James Street, London
1882		* Cay, W. Dyce, Memb. Inst. C.E., 1 Albyn Place
1890		Charles, John J., M.A., M.D., C.M., Prof. of Anatomy and Physiology, Queen's College, Cork
1899		* Chatham, James, Actuary, 98 Inverleith Place
1874		Chiene, John, C.B., M.D., LL.D., F.R.C.S.E., Professor of Surgery in the University of Edinburgh, 26 Charlotte Square
1880	C. K.	Chrystal, George, M.A., LL.D., Professor of Mathematics in the University of Edinburgh (General Secretary), 5 Belgrave Crescent
1891		* Clark, John B., M.A., Mathematical and Physical Master in Heriot's Hospital School, Garleffin, Craiglea Drive
1903		* Clarke, William Eagle, F.L.S., Natural History Department, Royal Scottish Museum, Edinburgh, 35 Braid Road
1875		Clouston, T. S., M.D., Vice-President of the Royal College of Physicians, Tipperlinn House, Morningside
1892		* Coates, Henry, Pitcullen House, Perth
1887		* Cockburn, John, F.R.A.S., The Abbey, North Berwick
1904	C.	Coker, Ernest George, M.A., D.Sc., Professor of Mechanical Engineering and Applied Mechanics, City and Guilds Technical College, Finsbury, Leonard Street, City Road, London, E.C.

Date of Election.		
1904		Coles, Alfred Charles, M.D., D.Sc., York House, Poole Road, Bournemouth, W.
1888	C.	Collie, John Norman, Ph.D., F.R.S., F.C.S., Professor of Organic Chemistry in the University College, Gower Street, London
1904	C.	* Colquhoun, Walter, M.A., M.B., Muirhead Demonstrator of Physiology, University of Glasgow, 7 Stanley Street, Glasgow, W.
1886		Connan, Daniel M., M.A.
1872		Constable, Archibald, LL.D., 11 Thistle Street 130
1894		Cook, John, M.A., Principal of the Government Central College, Bangalore, India
1891		* Cooper, Charles A., LL.D., 41 Drumsheugh Gardens
1905		* Corrie, David, F.C.S., Nobel's Explosives Company, Polmont Station
1875		Craig, William, M.D., F.R.C.S.E., Lecturer on Materia Medica to the College of Surgeons, 71 Bruntsfield Place
1898		* Crawford, Francis Chalmers, 19 Royal Terrace 135
1903		Crawford, Lawrence, M.A., D.Sc., Professor of Mathematics in the South African College, Cape Town
1887		* Crawford, William Caldwell, 1 Lockharton Gardens, Colinton Road
1870		Crichton-Browne, Sir Jas., M.D., LL.D., F.R.S., Lord Chancellor's Visitor and Vice-President of the Royal Institution of Great Britain, 61 Carlisle Place Mansions, Victoria Street, and Royal Courts of Justice, Strand, London
1886		* Croom, Sir John Halliday, M.D., F.R.C.P.E., Professor of Midwifery in the University of Edinburgh, Vice-President, Royal College of Surgeons, Edinburgh, 25 Charlotte Square
1898		* Cullen, Alexander, F.S.A. Scot., Millburn House, by Hamilton 140
1878	C.	Cunningham, Daniel John, M.D., LL.D., D.C.L., F.R.S., F.Z.S., Professor of Anatomy in the University of Edinburgh (SECRETARY), 18 Grosvenor Crescent
1898		* Currie, James, M.A. Cantab., Larkfield, Golden Acre
1904		* Cuthbertson, John, Secretary, West of Scotland Agricultural College, 4 Charles Street, Kilmarnock
1889		* Dalrymple, James D. G., F.S.A. Lond. and Scot., Meiklewood, Stirling
1885		* Daniell, Alfred, M.A., LL.B., D.Sc., Advocate, c/o Messrs Buchan & Buchan, S.S.C., 37 Great King Street 145
1897		* Davidson, Hugh, of Braedale, Lanark
1884		Davy, R., F.R.C.S. Eng., Surgeon to Westminster Hospital, Burstone House, Bow, North Devon
1894		* Denny, Archibald, Braehead, Dumbarton
1869	C. V. J.	Dewar, Sir James, M.A., LL.D., D.C.L., D.Sc. Dub., F.R.S., F.C.S., Jacksonian Professor of Natural and Experimental Philosophy in the University of Cambridge, and Fullerian Professor of Chemistry at the Royal Institution of Great Britain, London
1905		* Dewar, James Campbell, C.A., 27 Douglas Crescent 150
1904		Dickinson, Walter George Burnett, F.R.C.V.S., Boston, Lincolnshire
1884		* Dickson, The Right Hon. Charles Scott, K.C., Lord-Advocate of Scotland, M.P. for the Bridgeton Division of Glasgow, 22 Moray Place
1888	C.	* Dickson, Henry Newton, M.A., D.Sc., 2 St Margaret's Road, Oxford
1876	C.	Dickson, J. D. Hamilton, M.A., Fellow and Tutor, St Peter's College, Cambridge
1885	C.	Dixon, James Main, M.A., President, Columbia College, Milton, Oregon, United States 151

Date of Election.		
1897		* Dobbie, James Bell, F.Z.S., 2 Hailes Street
1904	C.	* Dobbie, James Johnston, M.A., D.Sc., F.R.S., Director of the Royal Scottish Museum, Edinburgh, 27 Polwarth Terrace
1881	C.	Dobbin, Leonard, Ph.D., Lecturer on Chemistry in the University of Edinburgh, 7 Cobden Road
1902		Dollar, John A. W., M.R.C.V.S., 56 New Bond Street, London
1867	C.	Donaldson, J., M.A., LL.D., Principal of the University of St Andrews, St Andrews 160
1896		* Donaldson, William, M.A., Viewpark House, Spylaw Road
1905		* Donaldson, Rev. William Galloway, Minister of St Paul's Parish, 11 Claremont Crescent
1882	C.	* Dott, D. B., Memb. Pharm. Soc., 29 Spring Gardens
1892		Doyle, Patrick, C.E., M.R.I.A., F.G.S., Editor of *Indian Engineering*, Calcutta
1901		* Douglas, Carstairs Cumming, M.D., B.Sc., Professor of Medical Jurisprudence and Hygiene, Anderson's College, Glasgow, 2 Royal Crescent, Glasgow 165
1866		Douglas, David, 22 Drummond Place
1901		* Drinkwater, Thomas W., L.R.C.P.E., L.R.C.S.E., 25 Blacket Place
1878		Duncanson, J. J. Kirk, M.D., F.R.C.P.E., 22 Drumsheugh Gardens
1904		* Dunlop, William Brown, M.A., 7 Carlton Street
1859	C.	Duns, Rev. Professor, D.D., 5 Greenhill Place 170
1903		* Dunstan, John, M.R.C.V.S., 1 Dean Terrace, Liskeard, Cornwall
1892	C.	Dunstan, M. J. R., M.A., F.I.C., F.C.S., Principal, South-eastern Agricultural College, Wye, Kent
1899		* Duthie, George, M.A., Inspector-General of Education, Salisbury, Rhodesia
1893		Edington, Alexander, M.D., Colonial Bacteriologist, Graham's Town, South Africa
1904		* Edwards, John, 4 Great Western Terrace, Kelvinside, Glasgow 175
1904		* Elder, William, M.D., F.R.C.P.E., 4 John's Place, Leith
1885		Elgar, Francis, Memb. Inst. C.E., LL.D., F.R.S., 18 Cornwall Terrace, Regent's Park, London
1875		Elliot, Daniel G., Curator of Department of Zoology, Field Columbian Museum, Chicago, U.S.
1897	C.	* Erskine-Murray, James Robert, D.Sc., 39 Watcombe Circus, Nottingham
1884		* Evans, William, F.F.A., 38 Morningside Park 180
1879	C. N.	Ewart, James Cossar, M.D., F.R.C.S.E., F.R.S., F.L.S., Professor of Natural History, University of Edinburgh
1902		* Ewen, J. T., B.Sc., Memb. Inst. Mech. E., H.M.I.S., 104 King's Gate, Aberdeen
1878	C.	Ewing, James Alfred, M.A., B.Sc., LL.D., Memb. Inst. C.E., F.R.S., Director of Naval Education, Royal Naval College, Greenwich
1900		Eyre, John W. H., M.D., M.S. (Dunelm), D.P.H. (Camb.), Guy's Hospital (Bacteriological Department), London, 19 Villiers Street, London
1875		Fairley, Thomas, Lecturer on Chemistry, 8 Newton Grove, Leeds 185
1888	C.	* Fawsitt, Charles A., 9 Foremount Terrace, Dowanhill, Glasgow
1859		Fayrer, Sir Joseph, Bart., K.C.S.I., M.D., F.R.C.P.L., F.R.C.S. I. and E., LL.D., F.R.S., Honorary Physician to the Queen, Lamorna, Falmouth
1883	C.	* Felkin, Robert W., M.D., F.R.G.S., Fellow of the Anthropological Society of Berlin, 12 Oxford Gardens, North Kensington, London, W.
1899		* Fergus, Andrew Freeland, M.D., 22 Blythswood Square, Glasgow
1904		* Ferguson, James Haig, M.D., F.R.C.P.E., F.R.C.S.E., 7 Coates Crescent 190
1888		* Ferguson, John, M.A., LL.D., Professor of Chemistry in the University of Glasgow

Date of Election.		
1868	C.	Ferguson, Robert M., Ph.D., LL.D. (Society's Representative on George Heriot's Trust), 5 Douglas Gardens
1898		* Findlay, John R., M.A. Oxon., 27 Drumsheugh Gardens
1899		* Finlay, David W., B.A., M.D., LL.D., F.R.C.P., D.P.H., Professor of Medicine in the University of Aberdeen, 2 Queen's Terrace, Aberdeen
1900	C.N.	* Flett, John S., M.A., D.Sc., Geological Survey Office, 28 Jermyn Street, London 195
1880		Flint, Robert, D.D., Corresponding Member of the Institute of France, Corresponding Member of the Royal Academy of Sciences of Palermo, Emeritus Professor of Divinity in the University of Edinburgh (Vice-President), 1 Mountjoy Terrace, Musselburgh
1872	C.	Forbes, Professor George, M.A., Memb. Inst. C.E., Memb. Inst. E.E., F.R.S., F.R.A.S., 34 Great George Street, Westminster
1904		Forbes, Norman Hay, F.R.C.S.E., Drumminor, Tunbridge Wells, Kent
1892		* Ford, John Simpson, F.C.S., 4 Nile Grove
1858		Fraser, A. Campbell, Fellow of the British Academy, Hon. D.C.L. Oxford, LL.D., Litt.D., Emeritus Professor of Logic and Metaphysics in the University of Edinburgh, Gorton House, Hawthornden 200
1896		* Fraser, John, M.B., F.R.C.P.E., one of H.M. Commissioners in Lunacy for Scotland, 13 Heriot Row
1867	C K. B.	Fraser, Sir Thomas R., M.D., LL.D., F.R.C.P.E., F.R.S., Professor of Materia Medica in the University of Edinburgh, Honorary Physician to the King in Scotland, 13 Drumsheugh Gardens
1891		* Fullarton, J. H., M.A., D.Sc., Brodick, Arran
1891		* Fulton, T. Wemyss, M.D., Scientific Superintendent, Scottish Fishery Board, 417 Great Western Road, Aberdeen
1888	C.	* Galt, Alexander, D.Sc., Keeper of the Technological Department, Royal Scottish Museum, Edinburgh 205
1901		Ganguli, Sanjiban, M.A., Principal, Maharaja's College, and Director of Public Instruction, Jaipur States, Jaipur, India
1899		Gatehouse, T. E., Assoc. Memb. Inst. C.E., Memb. Inst. M.E., Memb. Inst. E.E., Tulse Hill Lodge, 100 Tulse Hill, London
1867		Gayner, Charles, M.D., F.L.S.
1900		Gayton, William, M.D., M.R.C.P.E., 11 Redbourne Avenue, North Finchley, London, N.W.
1889		* Geddes, George H., Mining Engineer, 8 Douglas Crescent 210
1880	C.	Geddes, Patrick, Professor of Botany in University College, Dundee, and Lecturer on Zoology, Ramsay Garden, University Hall, Edinburgh
1861	C. B.	Geikie, Sir Archibald, LL.D. Oxf., D.Sc. Camb. Dub., F.R.S., F.G.S., Foreign Member of the Reale Accad. Lincei, Rome, of the National Acad. of the United States, Corresponding Member of the Institute of France and of the Academies of Berlin, Vienna, Munich, Göttingen, Turin, Belgium, Stockholm, Christiania, Philadelphia, New York, &c., 3 Sloane Court, London
1871	C. B.	Geikie, James, LL.D., D.C.L., F.R.S., F.G.S., Professor of Geology in the University of Edinburgh, Kilmorie, Colinton Road
1881	C.	Gibson, George Alexander, D.Sc., M.D., LL.D., F.R.C.P.E., 3 Drumsheugh Gardens
1890		* Gibson, George A., M.A., LL.D., Professor of Mathematics in the Glasgow and West of Scotland Technical College, 8 Sandyford Place, Glasgow 215

Date of Election.		
1877	C.	Gibson, John, Ph.D., Professor of Chemistry in the Heriot-Watt College, Ringlewood, Colinton, Midlothian
1892		Gifford, Herbert James, Assoc. M. Inst. C.E.
1900		Gilchrist, Douglas A., B.Sc., Professor of Agriculture and Rural Economy, Armstrong College, Newcastle-upon-Tyne
1887		* Gilmour, William, 9 Inverleith Row
1880		Gilruth, George Ritchie, Surgeon, 53 Northumberland Street
1898		* Glaister, John, M.D., F.F.P.S. Glasgow, D.P.H. Camb., Professor of Forensic Medicine in the University of Glasgow, 3 Newton Place, Glasgow
1901		Goodwillie, James, M.A., B.Sc., Liberton, Edinburgh
1899		* Goodwin, Thomas S., F.C.S., Professor of Chemistry, Veterinary College, Glasgow
1897		Gordon-Munn, John Gordon, M.D., 34 Dover Street, London, W.
1891		* Graham, Richard D., 11 Strathearn Road
1898	C.	* Gray, Albert A., M.D., 14 Newton Terrace, Glasgow
1883		* Gray, Andrew, M.A., LL.D., F.R.S., Professor of Natural Philosophy in the University of Glasgow
1880	C.	Gray, Thomas, B.Sc., Professor of Physics, Rose Polytechnic Institute, Terre Haute, Indiana, U.S.
1886		* Greenfield, W. S., M.D., F.R.C.P.E., Professor of General Pathology in the University of Edinburgh, 7 Heriot Row
1897		Greenlees, Thomas Duncan, M.D. Edin., The Residency, Grahamstown, South Africa
1905		* Gregory, John Walter, D.Sc., F.R.S., Professor of Geology in the University of Glasgow, 4 Park Quadrant, Glasgow
1905		* Greig, Robert Blyth, F.Z.S., Fordyce Lecturer in Agriculture, University of Aberdeen, Torloisk, Cults, Aberdeenshire
1899		* Guest, Edward Graham, M.A., B.Sc., 5 Church Hill
1888	C.	Guppy, Henry Brougham, M.B., Rosario, Salcombe, Devon
1905	C.	* Halm, Jacob E., Ph.D., Assistant Astronomer, Royal Observatory, and Lecturer on Astronomy in the University of Edinburgh, Royal Observatory, Blackford Hill, Edinburgh
1899		Hamilton, Allan M'Lane, M.D., 44 East Twenty-ninth Street, New York
1881	C.	Hamilton, D. J., M.B., F.R.C.S.E., Professor of Pathological Anatomy in the University of Aberdeen, 35 Queen's Road, Aberdeen
1876	C.	Hannay, J. Ballantyne, Cove Castle, Loch Long
1902		* Hargreaves, Andrew Fuller, F.C.S., Eskhill House, Roslin
1896		* Harris, David, Fellow of the Statistical Society, Lyncombe Rise, Prior Park Road, Bath
1896	C.	* Harris, David Fraser, B.Sc. (Lond.), M.D., F.S.A. Scot., Lecturer on Physiology in the University of St Andrews
1888		* Hart, D. Berry, M.D., F.R.C.P.E., 29 Charlotte Square
1869		Hartley, Sir Charles A., K.C.M.G., Memb. Inst. C.E., 26 Pall Mall, London
1877	C.	Hartley, W. N., D.Sc., F.R.S., F.I.C., Prof. of Chemistry, Royal College of Science for Ireland, Dublin
1881		Harvie-Brown, J. A., of Quarter, F.Z.S., Dunipace House, Larbert, Stirlingshire
1880	C.	Haycraft, J. Berry, M.D., D.Sc., Professor of Physiology in the University College of South Wales and Monmouthshire, Cardiff
1892	C.	* Heath, Thomas, B.A., Assistant Astronomer, Royal Observatory, Edinburgh

Date of Election.		
1862		Hector, Sir J., K.C.M.G., M.D., F.R.S., Director of the Geological Survey, Colonial Laboratory, Meteorological and Weather Departments, and of the New Zealand Institute, Wellington, New Zealand
1893		Hehir, Patrick, M.D., F.R.C.S.E., M.R.C.S.L., L.R.C.P.E., Surgeon-Captain, Indian Medical Service, Principal Medical Officer, H.H. the Nizam's Army, Hyderabad, Deccan, India
1890	C.	Helme, T. Arthur, M.D., M.R.C.P.L., M.R.C.S., 3 St Peter's Square, Manchester 250
1900		Henderson, John, D.Sc., Assoc. Inst. E.E., Kinnoul, Warwick's Bench Rd., Guildford, Surrey
1890	C.	* Hepburn, David, M.D., Professor of Anatomy in the University College of South Wales and Monmouthshire, Cardiff
1896	C.	* Herbertson, Andrew J., M.A., Ph.D., Reader in Geography, and Curator, School of Geography, University of Oxford, 4 Broad Street, Oxford
1881	C. N.	Herdman, W. A., D.Sc., F.R.S., F.L.S., Prof. of Natural History in University College, Liverpool, Croxteth Lodge, Ullet Road, Liverpool
1894		Hill, Alfred, M.D., M.R.C.S., F.I.C., Valentine Mount, Freshwater Bay, Isle of Wight 255
1902		* Hinxman, Lionel W., B.A., Geological Survey Office, George IV. Bridge
1904		Hobday, Frederick T. G., F.R.C.V.S., 6 Berkeley Gardens, Kensington, London
1885		Hodgkinson, W. R., Ph.D., F.I.C., F.C.S., Prof. of Chem. and Physics at the Royal Military Acad. and Royal Artillery Coll., Woolwich, 18 Glenluce Road, Blackheath, Kent
1881	C. N.	Horne, John, LL.D., F.R.S., F.G.S., Director of the Geological Survey of Scotland, Sheriff-Court Buildings, Edinburgh
1896		Horne, J. Fletcher, M.D., F.R.C.S.E., The Poplars, Barnsley 260
1904		* Horsburgh, Ellice Martin, M.A., B.Sc., Lecturer in Technical Mathematics, University of Edinburgh, 11 Granville Terrace
1897		Houston, Alex. Cruikshanks, M.B., C.M., D.Sc., 14 Upper Addison Gardens, Kensington, London
1893		Howden, Robert, M.A., M.B., C.M., Professor of Anatomy in the University of Durham, 14 Burdon Terrace, Newcastle-on-Tyne
1899		Howie, W. Lamond, F.C.S., Hanover Lodge, West Hill, Harrow
1883	C.	* Hoyle, William Evans, M.A., D.Sc., M.R.C.S., 25 Brunswick Road, Withington, Manchester 265
1872		Hughes-Hunter, Colonel Charles, of Pläs Cöch, Llanfairpwll, Anglesea, and Junior United Service Club, London
1886		Hunt, Rev. H. G. Bonavia, Mus.D. Dub., Mus.B. Oxon., The Vicarage, Burgess Hill, Sussex
1887	C.	* Hunter, James, F.R.C.S.E., F.R.A.S., Rosetta, Liberton, Midlothian
1887	C.	* Hunter, William, M.D., M.R.C.P. L. and E., M.R.C.S., 54 Harley Street, London
1882	C.	* Inglis, J. W., Memb. Inst. C.E., Kenwood, Barnton, Midlothian 270
1904		Innes, R. T. A., Director, Government Observatory, Johannesburg, Transvaal
1904		* Ireland, Alexander Scott, S.S.C., 2 Buckingham Terrace
1875		Jack, William, M.A., LL.D., Professor of Mathematics in the University of Glasgow
1894		Jackson, Sir John, LL.D., 10 Holland Park, London
1889	C.	* James, Alexander, M.D., F.R.C.P.E., 10 Melville Crescent 275
1882		* Jamieson, Prof. A., Memb. Inst. C.E., 16 Rosslyn Terrace, Kelvinside, Glasgow
1901		* Jardine, Robert, M.D., M.R.C.S. Eng., F.F.P. and S. Glas., 20 Royal Crescent, Glasgow
1900		Jee, Sir Bhagvat Sinh, G.C.I.E., M.D., LL.D. Edin., H.H. The Thakore Sahib of Gondal, Gondal, Kathiawar, Bombay

Date of Election.		
1900		* Jerdan, David Smiles, M.A., D.Sc., Ph.D., Temora, Colinton, Midlothian
1895		Johnston, Lieutenant-Colonel Henry Halcro, C.B., R.A.M.S., D.Sc., M.D., F.L.S., Orphir House, Kirkwall, Orkney 280
1903		* Johnston, Thomas Nicol, M.B., C.M., Corstorphine House, Corstorphine
1902		Johnstone, George, Lieut. R.N.R., Marine Superintendent, British India Steam Navigation Co., 16 Strand Road, Calcutta, India
1874		Jones, Francis, M.Sc., Lecturer on Chemistry, Beaufort House, Alexandra Park, Manchester
1888		Jones, John Alfred, Memb. Inst. C.E., Fellow of the Univ. of Madras, Sanitary Engineer to the Government of Madras, c/o Messrs Parry & Co., 70 Gracechurch St., London
1905		Jones, George William, M.A., B.Sc., 28 Roseneath Place 285
1847	C. K. V. J.	Kelvin, The Right Hon. Lord, G.C.V.O., P.C., LL.D., D.C.L., F.R.S. (PRESIDENT), Grand Officer of the Legion of Honour of France, Member of the Prussian Order *Pour le Mérite*, Foreign Associate of the Institute of France, and Emeritus Professor of Natural Philosophy in the University of Glasgow, Netherhall, Largs, Ayrshire, and 15 Eaton Place, London, S.W.
1892		* Kerr, Rev. John, M.A., Manse, Dirleton
1903	C. N.	* Kerr, John Graham, M.A., Professor of Zoology in the University of Glasgow
1891		Kerr, Joshua Law, M.D., Biddenden Hall, Cranbrook, Kent
1886	C. N.	* Kidston, Robert, F.R.S., F.G.S., 12 Clarendon Place, Stirling 290
1877		King, Sir James, of Campsie, Bart., LL.D., 115 Wellington Street, Glasgow
1880		King, W. F., Lonend, Russell Place, Trinity
1883		* Kinnear, The Rt. Hon. Lord, one of the Senators of the College of Justice, 2 Moray Place
1878		Kintore, The Right Hon. the Earl of, M.A. Cantab., LL.D. Cambridge, Aberdeen and Adelaide, Keith Hall, Inverurie, Aberdeenshire
1901		* Knight, The Rev. G. A. Frank, M.A., St Leonard's United Free Church, Perth 295
1880	C. K.	Knott, C. G., D.Sc., Lecturer on Applied Mathematics in the University of Edinburgh (late Prof. of Physics, Imperial University, Japan), (SECRETARY), 42 Upper Gray Street, Edinburgh
1896	C.	* Kuenen, J. P., Ph.D. (Leiden), Prof. of Natural Philosophy in University College, Dundee
1886		* Laing, Rev. George P., 17 Buckingham Terrace
1878	C.	Lang, P. R. Scott, M.A., B.Sc., Professor of Mathematics, University of St Andrews
1885	C.	* Laurie, A. P., M.A., D.Sc., Principal of the Heriot-Watt College, Edinburgh 300
1894	C.	* Laurie, Malcolm, B.A., D.Sc., F.L.S., Royal College of Surgeons, Edinburgh
1870		Laurie, Simon S., M.A., LL.D., Emeritus Professor of Education in the University of Edinburgh, 22 George Square
1905		* Lawson, David, M.A., M.D., L.R.C.P., and S.E., Druimdarroch, Banchory, Kincardineshire
1903		* Leighton, Gerald Rowley, M.D., 51 E. Trinity Road
1874	C. K.	Letts, E. A., Ph.D., F.I.C., F.C.S., Professor of Chemistry, Queen's College, Belfast 305
1905		* Lightbody, Forrest Hay, 56 Queen Street
1889		* Lindsay, Rev. James, D.D., B.Sc., F.G.S., M.R.A.S., Corresponding Member of the Royal Academy of Sciences, Letters and Arts, of Padua, Associate of the Philosophical Society of Louvain, Minister of St Andrew's Parish, Springhill Terrace, Kilmarnock
1870	C. B.	Lister, The Right Hon. Lord, P.C., M.D., F.R.C.S.L., F.R.C.S.E., LL.D., D.C.L., F.R.S., Foreign Associate of the Institute of France, Emeritus-Prof. of Clinical Surgery, King's College, Surgeon Extraordinary to the King, 12 Park Crescent, Portland Pl., London

Date of Election.		
1903		Liston, William Glen, M.D., Captain, Indian Medical Service, c/o Grindlay Groom & Co., Bombay, India
1903		* Littlejohn, Henry Harvey, M.A., M.B., B.Sc., F.R.C.S.E., 1 Atholl Crescent 310
1897	C.	Lloyd, Richard John, M.A., D.Lit., 49A Grove Street, Liverpool
1898		* Lothian, Alexander Veitch, M.A., B.Sc., 16 Clarence Drive, Hyndland, Glasgow
1884		* Low, George M., Actuary, 11 Moray Place
1888		* Lowe, D. F., M.A., LL.D., Head Master of Heriot's Hospital School, Lauriston
1904		* Lowson, Charles Stewart, M.B., C.M., Captain, Indian Medical Service, c/o Messrs Thomas Cook & Son, Bombay, India 315
1900		Lusk, Graham, Ph.D., M.A., Prof. of Physiology, Univ. and Bellevue Medical College, N.Y.
1894		* Mabbott, Walter John, M.A., Rector of County High School, Duns, Berwickshire
1887		M'Aldowie, Alexander M., M.D., 6 Brook Street, Stoke-on-Trent
1891		Macallan, John, F.I.C., 3 Rutland Terrace, Clontarf, Dublin
1888	C.	M'Arthur, John, F.C.S., 196 Trinity Road, Wandsworth Common, London 320
1883		* M'Bride, P., M.D., F.R.C.P.E., 16 Chester Street
1903		* M'Cormick, W. S., M.A., LL.D., 13 Douglas Crescent
1899		* M'Cubbin, James, B.A., Rector of the Burgh Academy, Kilsyth
1905		* Macdonald, Hector Munro, M.A., F.R.S., Professor of Mathematics, University of Aberdeen, 33 College Bounds, Aberdeen
1894		* Macdonald, James, Secretary of the Highland and Agricultural Society of Scotland, 2 Garscube Terrace 325
1897	C.	* Macdonald, James A., M.A., B.Sc., H.M. Inspector of Schools, Glengarry, Dingwall
1904		* Macdonald, J. A., M.A., B.Sc., Olive Lodge, Polwarth Terrace
1886		* Macdonald, The Rt. Hon. Sir J. H. A., K.C.B., K.C., LL.D., F.R.S., M.I.E.E., Lord Justice-Clerk, and Lord President of the Second Division of the Court of Session, 15 Abercromby Place
1904		Macdonald, William, B.Sc., M.Sc., Chief of the Division of Publications under the Department of Agriculture, Pretoria Club, Pretoria, Transvaal
1886		* Macdonald, William J., M.A., Comiston Drive 330
1901	C.	* MacDougal, R. Stewart, M.A., D.Sc., 13 Archibald Place
1888	C.	* M'Fadyean, Sir John, M.B., B.Sc., Principal, and Professor of Comparative Pathology in the Royal Veterinary College, Camden Town, London
1878	C.	Macfarlane, Alexander, M.A., D.Sc., LL.D., Lecturer in Physics in Lehigh University, Pennsylvania, Gowrie Grove, Chatham, Ontario, Canada
1885	C.	* Macfarlane, J. M., D.Sc., Professor of Botany and Director of the Botanic Garden, University of Pennsylvania, Philadelphia, Pennsylvania, U.S.A.
1897		* M'Gillivray, Angus, C.M., M.D., South Tay Street, Dundee 335
1878		M'Gowan, George, F.I.C., Ph.D., 21 Montpelier Road, Ealing, Middlesex
1886		* MacGregor, Rev. James, D.D., 3 Eton Terrace
1880	C.	MacGregor, James Gordon, M.A., D.Sc., LL.D., F.R.S., Prof. of Natural Philosophy in the University of Edinburgh, 24 Dalrymple Crescent
1903		* M'Intosh, D. C., M.A., 37 Warrender Park Terrace
1869	C. N.	M'Intosh, William Carmichael, M.D., LL.D., F.R.S., F.L.S., Professor of Natural History in the University of St Andrews, 2 Abbotsford Crescent, St Andrews 340
1895	C.	* Macintyre, John, M.D., 179 Bath Street, Glasgow
1882		* Mackay, John Sturgeon, M.A., LL.D., late Mathematical Master in the Edinburgh Academy, 69 Northumberland Street

Date of Election.		
1873	C. B.	M'Kendrick, John G., M.D., F.R.C.P.E., LL.D., F.R.S., Professor of Physiology in the University of Glasgow, 2 Buckingham Terrace, Glasgow
1900	C.	* M'Kendrick, John Souttar, M.D., 2 Florentine Gardens, Hillhead, Glasgow
1894		* Mackenzie, Robert, M.D., Napier, Nairn 345
1898		Mackenzie, W. Cossar, D.Sc., Principal of the College of Agriculture, Gheezeh, Egypt
1904		* Mackenzie, W. Leslie, M.A., M.D., D.P.H., Medical Member of the Local Government Board for Scotland, 1 Stirling Road, Trinity
1905		Mackenzie, William Colin, M.D., F.R.C.S., Demonstrator of Anatomy in the University of Melbourne, Elizabeth Street North, Melbourne, Victoria
1904		* Mackintosh, Donald James, M.V.O., M.B., Supt. of the Western Infirmary, Glasgow
1894		* Maclagan, Philip R. D., F.F.A. (TREASURER), St Catherine's, Liberton 350
1869	C.	Maclagan, R. C., M.D., F.R.C.P.E., 5 Coates Crescent
1869	C.	M'Laren, The Hon. Lord, LL.D. Edin. & Glasg., F.R.A.S., one of the Senators of the College of Justice (VICE-PRESIDENT), 46 Moray Place
1899		Maclean, Ewan John, M.D., M.R.C.P. London, 12 Park Place, Cardiff
1888	C.	* Maclean, Magnus, M.A., D.Sc., Memb. Inst. E. E., Prof. of Electrical Engineering in the Glasgow and West of Scotland Technical College, 51 Kerrsland Terrace, Hillhead, Glasgow
1876		Macleod, Very Rev. Norman, D.D., Westwood, Inverness 355
1876		Macmillan, John, M.A., D.Sc., M.B., C.M., F.R.C.P.E., 48 George Square
1893		* M'Murtrie, The Rev. John, M.A., D.D., 13 Inverleith Place
1884		* Macpherson, Rev. J. Gordon, M.A., D.Sc., Ruthven Manse, Meigle
1890		* M'Vail, John C., M.D., 20 Eton Place, Hillhead, Glasgow
1898	C.	Mahalanobis, S. C., B.Sc., Professor of Physiology, Presidency College, Calcutta, India 360
1880	C.	Marsden, R. Sydney, M.B., C.M., D.Sc., F.I.C., F.C.S., Rowallan House, Cearns Road, and Town Hall, Birkenhead
1882	C.	Marshall, D. H., M.A., Professor of Physics in Queen's University and College, Kingston, Ontario, Canada
1901	C.	* Marshall, F. H. A., M.A., D.Sc., Physiological Department, University of Edinburgh
1888	C. K.	* Marshall, Hugh, D.Sc., F.R.S., Lecturer on Chemistry and on Mineralogy and Crystallography in the University of Edinburgh, 12 Lonsdale Terrace
1892		* Martin, Francis John, W.S., 17 Rothesay Place 365
1903		Martin, Nicholas Henry, F.L.S., F.C.S., Ravenswood, Low Fell, Gateshead
1864		Marwick, Sir James David, LL.D., 19 Woodside Terrace, Glasgow
1866		Masson, David, LL.D., Litt.D. Dub., Emeritus-Professor of Rhetoric and English Literature in the Univ. of Edin., H.M. Historiographer for Scotland, 2 Lockharton Gardens
1885	C.	* Masson, Orme, D.Sc., F.R.S., Professor of Chemistry in the University of Melbourne
1898	C. B.	* Masterman, Arthur Thomas, M.A., D.Sc., Inspector of Fisheries, Board of Agriculture, Whitehall, London 370
1890		* Matheson, The Rev. George, M.A., B.D., D.D., LL.D., 19 St Bernard's Crescent
1902		Matthews, Ernest Romney, C.E., F.G.S., Bridlington, Yorkshire
1901		* Menzies, Alan W. C., M.A., B.Sc., F.C.S., Professor of Chemistry in St Mungo's College, Glasgow
1888		* Methven, Cathcart W., Memb. Inst. C.E., F.R.I.B.A., Durban, Natal, S. Africa
1902	C.	Metzler, William H., A.B., Ph.D., Corresponding Fellow of the Royal Society of Canada, Professor of Mathematics, Syracuse University, Syracuse, N.Y. 375
1885	C. B.	* Mill, Hugh Robert, D.Sc., LL.D., 62 Camden Square, London

Date of Election.		
1905		* Miller-Milne, C. H., M.A., Rector, The High School, Arbroath, 8 Dalhousie Place, Arbroath
1905		* Milne, Archibald, M.A., B.Sc., Lecturer on Mathematics and Science, Church of Scotland Training College, 5 Elgin Terrace
1904	C.	* Milne, James Robert, B.Sc., 56 Manor Place
1886		* Milne, William, M.A., B.Sc., 70 Beechgrove Terrace, Aberdeen 380
1899		* Milroy, T. H., M.D., B.Sc., Professor of Physiology in Queen's College, Belfast, 14 Ashley Avenue, Belfast
1866		Mitchell, Sir Arthur, K.C.B., M.A., M.D., LL.D., 34 Drummond Place
1889	C.	Mitchell, A. Crichton, D.Sc., Professor of Pure and Applied Mathematics, and Principal of the Maharajah's College, Trivandrum, Travancore, India
1897		* Mitchell, George Arthur, M.A., 9 Lowther Terrace, Kelvinside, Glasgow
1900		* Mitchell, James, M.A., B.Sc., 7 Bath Street, Nairn 385
1899		* Mitchell-Thomson, Sir Mitchell, Bart., 6 Charlotte Square
1890	C.	Mond, R. L., M.A. Cantab., F.C.S., The Poplars, 20 Avenue Road, Regent's Park, London
1887	C.	Moos, N. A. F., L.C.E., B.Sc., Professor of Physics, Elphinstone College, and Director of the Government Observatory, Colaba, Bombay
1901		* More, James, jun., M. Inst. C.E., 74 George Street
1896		* Morgan, Alexander, M.A., D.Sc., Rector, Church of Scotland Training College, 1 Midmar Gardens 390
1892		Morrison, J. T., M.A., B.Sc., Professor of Physics and Chemistry, Victoria College, Stellenbosch, Cape Colony
1901		Moses, O. St John, M.D., B.Sc., F.R.C.S.E., Captain, Indian Medical Service, 8 Lansdowne Road, Calcutta, India
1892	C.	* Mossman, Robert C., 30 Blacket Place
1874	C. K.	Muir, Thomas, C.M.G., M.A., LL.D., F.R.S., Superintendent-General of Education for Cape Colony, Education Office, Cape Town, and Mowbray Hall, Rosebank, Cape Colony
1888	C.	* Muirhead, George, Commissioner to His Grace the Duke of Richmond and Gordon, K.G., Speybank, Fochabers 395
1887		Mukhopâdhyay, Âsûtosh, M.A., LL.D., F.R.A.S., M.R.I.A., Professor of Mathematics at the Indian Association for the Cultivation of Science, 77 Russa Road North, Bhowanipore, Calcutta
1894		* Munro, J. M. M., Memb. Inst. E.E., 136 Bothwell Street, Glasgow
1891	C.	* Munro, Robert, M.A., M.D., LL.D., Hon. Memb. R.I.A., Hon. Memb. Royal Soc. of Antiquaries of Ireland (Vice-President), 48 Manor Place and Elmbank, Largs, Ayrshire
1896		* Murray, Alfred A., M.A., LL.B., 20 Warriston Crescent
1892	C.	* Murray, George Robert Milne, F.R.S., F.L.S., Keeper of the Botanical Department, British Museum (Natural Hist.), Cromwell Road, London 400
1877	C. B. N.	Murray, Sir John, K.C.B., LL.D., D.C.L., Ph.D., D.Sc., F.R.S., Member of the Prussian Order *Pour le Mérite*, Director of the Challenger Expedition Publications (Vice-President). Office, Villa Medusa, Boswell Road. House, Challenger Lodge, Wardie, and United Service Club
1887		Muter, John, M.A., F.C.S., South London Central Public Laboratory, 325 Kennington Road, London
1902		Mylne, The Rev. R. S., M.A., B.C.L., Oxford, F.S.A. Lond., Great Amwell, Herts
1888		Napier, A. D. Leith, M.D., C.M., M.R.C.P.L., General Hospital, Adelaide, S. Australia
1897		Nash, Alfred George, C.E., B.Sc., Engineer, Department of Public Works, Jamaica, Belretiro, Mandeville, Jamaica, W.I. 405

Date of Election.		
1887		* Nasmyth, T. Goodall, M.D., C.M., D.Sc., Cupar-Fife
1898		Newman, George, M.D., D.P.H. Cambridge, 2 Woburn Square, London
1884		* Nicholson, J. Shield, M.A., D.Sc., Professor of Political Economy in the University of Edinburgh, 3 Belford Park
1880	C.	Nicol, W. W. J., M.A., D.Sc., 15 Blacket Place
1878		Norris, Richard, M.D., M.R.C.S. Eng., 3 Walsall Road, Birchfield, Birmingham 410
1902		Nunn, Joshua Arthur, C.I.E., D.S.O., F.R.C.V.S., Barrister-at-Law, Lincoln's Inn; Veterinary Lieut.-Colonel and Deputy Director-General, Army Veterinary Department, Pretoria, South Africa
1888		* Ogilvie, F. Grant, M.A., B.Sc., Principal Assistant Secretary for Science, Art, and Technology, Board of Education, Whitehall, London
1888		* Oliphant, James, M.A., 12 Murrayfield Road
1886	C.	Oliver, James, M.D., F.L.S., Physician to the London Hospital for Women, 18 Gordon Square, London
1895	C.	Oliver, Thomas, M.D., F.R.C.P., Professor of Physiology in the University of Durham, 7 Ellison Place, Newcastle-upon-Tyne 415
1884	C. K.	* Omond, R. Traill, 3 Church Hill
1905		Pallin, William Alfred, F.R.C.V.S., Captain in the Army Veterinary Department, c/o Messrs Holt & Co., 3 Whitehall Place, London
1892		Parker, Thomas, Memb. Inst. C.E., 1B Chapel Street, Edgeware Road, London
1901		* Paterson, David, F.C.S., Lea Bank, Rosslyn, Midlothian
1886	C.	* Paton, D. Noël, M.D., B.Sc., F.R.C.P.E., 22 Lyndoch Place 420
1889		* Patrick, David, M.A., LL.D., c/o W. & R. Chambers, 339 High Street
1892		* Paulin, David, Actuary, 6 Forres Street
1881	C. N.	Peach, Benjamin N., LL.D., F.R.S., F.G.S., late District Superintendent and Acting Palæontologist of the Geological Survey of Scotland, 72 Grange Loan
1904		* Peck, James Wallace, M.A., Principal Assistant to Executive Officer (Education) of the London County Council, 70 High Street, Hampstead, London
1889		* Peck, William, F.R.A.S., Town's Astronomer, City Observatory, Calton Hill, Edinburgh 425
1863		Peddie, Alexander, M.D., F.R.C.P.E., 15 Rutland Street
1887	C. B.	* Peddie, Wm., D.Sc., Lecturer on Natural Philosophy, Edinburgh University, 14 Ramsay Garden
1900		Penny, John, M.B., C.M., D.Sc., Great Broughton, near Cockermouth, Cumberland
1893		Perkin, Arthur George, F.R.S., 8 Montpellier Terrace, Hyde Park, Leeds
1889		* Philip, R. W., M.A., M.D., F.R.C.P.E., 45 Charlotte Square 430
1905		* Pinkerton, Peter, M.A., Head Mathematical Master, George Watson's College, Edinburgh, 36 Morningside Grove
1886		Pollock, Charles Frederick, M.D., F.R.C.S.E., 1 Buckingham Terrace, Hillhead, Glasgow
1888		Prain, David, Lt.-Col., Indian Medical Service, M.A., M.B., LL.D., F.L.S., F.R.S., Hon. Memb. Soc. Lett. ed Arti d. Zelanti, Acireale; Corr. Memb. Pharm. Soc. Gt. Britain, etc.; Director, Botanical Survey of India, Royal Botanic Gardens, Shibpur, Calcutta
1902		* Preller, Charles Du Riche, M.A., Ph.D., Assoc. Memb. Inst. C.E., 61 Melville Street
1892		* Pressland, Arthur J., M.A. Camb., Edinburgh Academy 435
1875	C.	Prevost, E. W., Ph.D., Weston, Ross, Herefordshire
1885		* Pullar, J. F., Rosebank, Perth
1903		* Pullar, Laurence, The Lea, Bridge of Allan
1880		Pullar, Sir Robert, LL.D., Tayside, Perth

Date of Election.		
1898		* Purves, John Archibald, D.Sc., 53 York Place 440
1897		* Rainy, Harry, M.B., C.M., F.R.C.P. Ed., 16 Gt. Stuart Street
1899		* Ramage, Alexander G., 8 Western Terrace, Murrayfield
1884		Ramsay, E. Peirson, M.R.I.A., F.L.S., C.M.Z.S., F.R.G.S., F.G.S., Fellow of the Imperial and Royal Zoological and Botanical Society of Vienna, Curator of Australian Museum, Sydney, N.S.W.
1891		* Rankine, John, M.A., LL.D., Advocate, Professor of the Law of Scotland in the University of Edinburgh, 23 Ainslie Place
1904		Ratcliffe, Joseph Riley, M.B., C.M., Elmdon, Wake Green Road, Morley, Birmingham 445
1900		Raw, Nathan, M.D., Mill Road Infirmary, Liverpool
1883	C.	* Readman, J. B., D.Sc., F.C.S., Mynde Park, Tram Inn, Hereford
1889		Redwood, Sir Boverton, D.Sc. (Hon.), F.I.C., F.C.S., Assoc. Inst. C.E., Wadham Lodge, Wadham Gardens, London
1902		Rees-Roberts, John Vernon, M.D., D.Sc., D.P.H., Barrister-at-Law, National Liberal Club, Whitehall Place, London
1902		Reid, George Archdall O'Brien, M.B., C.M., 9 Victoria Road South, Southsea, Hants 450
1875		Richardson, Ralph, W.S., 10 Magdala Place
1872		Ricarde-Seaver, Major F. Ignacio, Athenæum Club, Pall Mall, London
1898	C.	Roberts, Alexander William, D.Sc., F.R.A.S., Lovedale, South Africa
1880		Roberts, D. Lloyd, M.D., F.R.C.P.L., 23 St John Street, Manchester
1872		Robertson, D. M. C. L. Argyll, M.D., F.R.C.S.E., LL.D., Surgeon Oculist to the King in Scotland, Mon Plaisir, St Aubins, Jersey 455
1900		* Robertson, Joseph M'Gregor, M.B., C.M., 26 Buckingham Terrace, Glasgow
1896		* Robertson, Robert, M.A., 25 Mansionhouse Road
1902	C.	* Robertson, Robert A., M.A., B.Sc., Lecturer on Botany in the University of St Andrews
1896	C.	* Robertson, W. G. Aitchison, D.Sc., M.D., F.R.C.P.E., 26 Minto Street
1905	C.	* Romanes, George, C.E., Craigknowe, Slateford, Midlothian 460
1881		Rosebery, The Right Hon. the Earl of, K.G., K.T., LL.D., D.C.L., F.R.S., Dalmeny Park, Edinburgh
1880		Rowland, L. L., M.A., M.D., President of the Oregon State Medical Society, and Professor of Physiology and Microscopy in Williamette University, Salem, Oregon
1902	C.	* Russell, James, 11 Argyll Place
1880		Russell, Sir James A., M.A., B.Sc., M.B., F.R.C.P.E., LL.D., Woodville, Canaan Lane
1904		Sachs, Edwin O., Architect, 7 Waterloo Place, Pall Mall, London, S.W. 465
1903		* Samuel, John S., 8 Park Avenue, Glasgow
1897		* Sanderson, William, Talbot House, Ferry Road
1864		Sandford, The Right Rev. Bishop D. F., D.D., LL.D., 4 Coates Crescent
1903		* Sarolea, Charles, Ph.D., D. Litt., Lecturer on French Language, Literature, and Romance Philology, University of Edinburgh, Hermitage, Colinton
1895		Savage, Thomas, M.D., F.R.C.S. England, M.R.C.P. London, Professor of Gynæcology, Mason College, Birmingham, The Ards, Knowle, Warwickshire 470
1891		Sawyer, Sir James, Knt., M.D., F.R.C.P., F.S.A., J.P., Consulting Physician to the Queen's Hospital, 31 Temple Row, Birmingham
1900	C.	* Schäfer, Edward Albert, M.R.C.S., LL.D., F.R.S., Professor of Physiology in the University of Edinburgh
1885	C.	Scott, Alexander, M.A., D.Sc., F.R.S., The Davy-Faraday Research Laboratory of the Royal Institution, London

Date of Election.		
1880		Scott, J. H., M.B., C.M., M.R.C.S., Prof. of Anatomy in the University of Otago, New Zealand
1905		Scougal, A. E., M.A., H.M.C.I.S., 1 Wester Coates Avenue 475
1902		Senn, Nicholas, M.D., LL.D., Professor of Surgery, Rush Medical College, Chicago, U.S.A.
1872	C.	Seton, George, M.A., Advocate, Ayton House, Abernethy, Perthshire
1897		* Shepherd, John William, Carrickarden, Bearsden, Glasgow
1894		* Shield, Wm., Memb. Inst. C.E., 33 Old Queen Street, Westminster, London
1870		Sime, James, M.A., Craigmount House, 10 Grange Road 480
1871		Simpson, A. R., M.D., Emeritus Professor of Midwifery in the University of Edinburgh, 52 Queen Street
1900	C.	* Simpson, James Young, M.A., D.Sc., Professor of Natural Science in the New College, Edinburgh, 52 Queen Street
1903		* Skinner, Robert Taylor, M.A., Governor and Headmaster, Donaldson's Hospital, Edinburgh
1901		* Smart, Edward, B.A., B.Sc., Benview, Craigie, Perth
1891	C.	* Smith, Alex., B.Sc., Ph.D., Prof. of General Chemistry, University of Chicago, Ills., U.S. 485
1882	C.	Smith, C. Michie, B.Sc., F.R.A.S., Director of the Kodaikánal and Madras Observatories, The Observatory, Kodaikánal, South India
1885		* Smith, George, F.C.S., Polmont Station
1871	C.	Smith, John, M.D., F.R.C.S.E., LL.D., 11 Wemyss Place
1904		* Smith, William Charles, K.C., M.A., LL.B., Advocate, 6 Darnaway Street
1880		Smith, William Robert, M.D., D.Sc., Barrister-at-Law, Professor of Forensic Medicine in King's College, 74 Great Russell Street, Bloomsbury Square, London 490
1899		Snell, Ernest Hugh, M.D., B.Sc., D.P.H. Camb., Coventry
1880		Sollas, W. J., M.A., D.Sc., LL.D., F.R.S., late Fellow of St John's College, Cambridge, and Professor of Geology and Palæontology in the University of Oxford
1889	C.	Somerville, Wm., M.A., D.Sc., D.Oec., Assistant Secretary, H.M. Board of Agriculture, 4 Whitehall Place, London
1882		* Sorley, James, F.I.A., C.A., 82 Onslow Gardens, London
1896		* Spence, Frank, M.A., B.Sc., 25 Craiglea Drive 495
1874	C.	Sprague, T. B., M.A., LL.D., Actuary, 29 Buckingham Terrace
1891		* Stanfield, Richard, Professor of Mechanics and Engineering in the Heriot-Watt College
1886	C.	* Stevenson, Charles A., B.Sc., Memb. Inst. C.E., 28 Douglas Crescent
1884		* Stevenson, David Alan, B.Sc., Memb. Inst. C.E., 45 Melville Street
1868		Stevenson, John J., 4 Porchester Gardens, London 500
1888	C.	* Stewart, Charles Hunter, D.Sc., M.B., C.M., Professor of Public Health in the University of Edinburgh, 9 Learmonth Gardens
1868		Stewart, Major-General J. H. M. Shaw, late R.E., Assoc. Inst. C.E., F.R.G.S., 7 Inverness Terrace, London, W.
1904		* Stewart, Thomas W., M.A., B.Sc., Science Master, Edinburgh Ladies' College, 29 Brunts-field Gardens
1873		Stewart, Walter, 3 Queensferry Gardens
1877		Stirling, William, D.Sc., M.D., LL.D., Brackenbury Professor of Physiology and Histology in Owens College and Victoria University, Manchester 505
1902		* Stockdale, Herbert Fitton, Clairinch, Upper Helensburgh, Dumbartonshire
1889	C.	* Stockman, Ralph, M.D., F.R.C.P.E., Professor of Materia Medica and Therapeutics in the University of Glasgow

Date of Election.		
1903		Sutherland, David W., M.D., M.R.C.P. Lond., Captain, Indian Medical Service, Professor of Pathology and Materia Medica, Medical College, Lahore, India
1896		* Sutherland, John Francis, M.D., Dep. Com. in Lunacy for Scotland, Scotsburn Road, Tain, Ross-shire
1905		Swithinbank, Harold William, Denham Court, Denham, Bucks 510
1885	C.	* Symington, Johnson, M.D., F.R.C.S.E., F.R.S., Prof. of Anatomy in Queen's College, Belfast
1904		* Tait, John W., B.Sc., Rector of Leith Academy, 18 Netherby Road, Leith
1898	C.	Tait, William Archer, B.Sc., Memb. Inst. C.E., 38 George Square
1895		Talmage, James Edward, D.Sc., Ph.D., F.R.M.S., F.G.S., Professor of Geology, Univ. of Utah, Salt Lake City, Utah
1890	C.	Tanakadate, Aikitu, Prof. of Nat. Phil. in the Imperial University of Japan, Tokyo, Japan 515
1870		Tatlock, Robert R., F.C.S., City Analyst's Office, 156 Bath Street, Glasgow
1899		* Taylor, James, M.A., Mathematical Master in the Edinburgh Academy, 3 Melgund Terrace
1892		Thackwell, J. B., M.B., C.M.
1885	C.	* Thompson, D'Arcy W., C.B., B.A., F.L.S., Professor of Natural History in University College, Dundee
1905		* Thoms, Alexander, 7 Playfair Terrace, St Andrews 520
1887		* Thomson, Andrew, M.A., D.Sc., F.I.C., Rector, Perth Academy, Ardenlea, Pitcullen, Perth
1896		* Thomson, George Ritchie, M.B., C.M., Cumberland House, Von Brandis Square, Johannesburg, Transvaal
1903		Thomson, George S., F.C.S., Dairy Commissioner for Queensland, Department of Agriculture, Brisbane, Queensland
1887	C.	* Thomson, J. Arthur, M.A., Regius Prof. of Natural History in the Univ. of Aberdeen
1880		Thomson, John Millar, LL.D., F.R.S., Prof. of Chem. in King's College, Lond., 9 Campden Hill Gardens, London 525
1899		* Thomson, R. Tatlock, F.C.S., 156 Bath Street, Glasgow
1870		Thomson, Spencer C., Actuary, 10 Eglinton Crescent
1882		Thomson, Wm., M.A., B.Sc., LL.D., Registrar, University of the Cape of Good Hope, University Buildings, Cape Town
1876		Thomson, William, Royal Institution, Manchester
1874	C. B. N.	Traquair, R. H., M.D., LL.D., F.R.S., F.G.S., Keeper of the Natural History Collections in the Royal Scottish Museum, Edinburgh (VICE-PRESIDENT), The Bush, Colinton 530
1874		Tuke, Sir J. Batty, M.D., D.Sc., LL.D., F.R.C.P.E., M.P. for the Universities of Edinburgh and St Andrews, 20 Charlotte Square
1888		* Turnbull, Andrew H., Actuary, The Elms, Whitehouse Loan
1905		* Turner, Arthur Logan, M.D., F.R.C.S.E., 27 Walker Street
1861	K. N. C.	Turner, Sir William, K.C.B., M.B., F.R.C.S.E., LL.D., D.C.L., D.Sc. Dub., F.R.S., Principal of the University of Edinburgh, 6 Eton Terrace
1895		Turton, Albert H., A.I.M.M., 45 Ribblesdale Road, Streatham Park, London 535
1898	C.	* Tweedie, Charles, M.A., B.Sc., Lecturer on Mathematics in the University of Edinburgh, 12 Nelson Street
1877		Underhill, Charles E., B.A., M.B., F.R.C.P.E., F.R.C.S.E., 8 Coates Crescent
1889		Underhill, T. Edgar, M.D., F.R.C.S.E., Dunedin, Barnt Green, Worcestershire

Date of Election.		
1888		Walker, James, Memb. Inst. C.E., Engineer's Office, Tyne Improvement Commission, Newcastle-on-Tyne
1891	C. B.	* Walker, James, D.Sc., Ph.D., F.R.S., Professor of Chemistry in University College, Dundee, 8 Windsor Terrace, Dundee 540
1873	C.	Walker, Robert, M.A., University, Aberdeen
1902		* Wallace, Alexander G., M.A., 154 Forrest Avenue, Aberdeen
1886	C.	* Wallace, R., F.L.S., Prof. of Agriculture and Rural Economy in the Univ. of Edin.
1898		Wallace, Wm., M.A., Principal, Cockburn Science School, Leeds
1891		* Walmsley, R. Mullineux, D.Sc., Prin. of the Northampton Inst., Clerkenwell, London 545
1901	C.	* Waterston, David, M.A., M.D., F.R.C.S.E., Lecturer on Regional Anatomy in the University of Edinburgh, 23 Colinton Road
1904		* Watson, Charles B. Boog, Huntly Lodge, 1 Napier Road
1866		Watson, Sir Patrick Heron, M.D., F.R.C.S.E., LL.D., Surgeon in Ordinary to the King in Scotland, 16 Charlotte Square
1862	C.	Watson, Rev. Robert Boog, B.A., LL.D., F.L.S., Past President of the Conchological Society, 11 Strathearn Place
1900		* Watson, Thomas P., M.A., B.Sc., Principal, Keighley Institute, Keighley 550
1896		Webster, John Clarence, B.A., M.D., F.R.C.P.E., Professor of Obstetrics and Gynæcology, Rush Medical College, Chicago, 706 Reliance Buildings, 100 State Street, Chicago
1903	C.	* Wedderburn, J. H. Maclagan, M.A., 13 South Charlotte Street
1904		Wedderspoon, William Gibson, M.A., LL.D., Indian Educational Service, Senior Inspector of Schools, Burma, The Education Office, Rangoon, Burma
1896		Wenley, R. M., M.A., D.Sc., D.Phil., LL.D., Prof. of Philosophy in the University of Michigan, U.S.
1896	C.	White, Philip J., M.B., Prof. of Zoology in University College, Bangor, North Wales 555
1890		White, Sir William Henry, K.C.B., Memb. Inst. C.E., LL.D., F.R.S., late Assistant Controller of the Navy, and Director of Naval Construction, Cedarscroft, Putney Heath, London
1881		Whitehead, Walter, F.R.C.S.E., Professor of Clinical Surgery, Owens College and Victoria University, 499 Oxford Road, Manchester
1894		Whymper, Edward, F.R.G.S., 29 Ludgate Hill, London
1879		Will, John Charles Ogilvie, of Newton of Pitfodels, M.D., 17 Bon-Accord Square, Aberdeen
1897		* Williams, W. Owen, F.R.C.V.S., Professor of Veterinary Medicine and Surgery, University of Liverpool, The Veterinary School, The University, Liverpool 560
1900		Wilson, Alfred C., F.C.S., Voewood Croft, Stockton-on-Tees
1879		Wilson, Andrew, Ph.D., F.L.S., Lecturer on Zoology and Comparative Anatomy, 110 Gilmore Place
1902		* Wilson, Charles T. R., M.A., F.R.S., Glencorse House, Peebles, and Sidney Sussex College, Cambridge
1895		Wilson-Barker, David, F.R.G.S., Captain-Superintendent Thames Nautical Training College, H.M.S. "Worcester," Greenhithe, Kent
1882		Wilson, George, M.A., M.D., 7 Avon Place, Warwick 565
1891		* Wilson, John Hardie, D.Sc., University of St Andrews, 39 South Street, St Andrews
1902		Wilson, William Wright, F.R.C.S.E., M.R.C.S. Eng., Cottesbrook House, Acock's Green, Birmingham
1886	C.	* Woodhead, German Sims, M.D., F.R.C.P.E., Prof. of Pathology in the University of Cambridge

Date of Election.		
1884		Woods, G. A., M.R.C.S., Eversleigh, 1 Newstead Road, Lee, Kent
1890		* Wright, Johnstone Christie, Northfield, Colinton 570
1896		* Wright, Robert Patrick, Professor of Agriculture, West of Scotland Agricultural College, 6 Blythswood Square, Glasgow
1882		* Young, Frank W., F.C.S., H.M. Inspector of Science and Art Schools, 32 Buckingham Terrace, Botanic Gardens, Glasgow
1892		Young, George, Ph.D., Lauraville, Bradda, Port Erin, Isle of Man
1896	C.	* Young, James Buchanan, M.B., D.Sc., Dalveen, Braeside, Liberton
1900		* Young, J. M'Lauchlan, F.R.C.V.S., Lecturer on Veterinary Hygiene, University of Aberdeen 575
1904		Young, R. B., M.A., B.Sc., Transvaal Technical Institute, Johannesburg, Transvaal

LIST OF HONORARY FELLOWS

At October 1904.

HIS MOST GRACIOUS MAJESTY THE KING.

FOREIGNERS (LIMITED TO THIRTY-SIX BY LAW X.).

Elected		
1897	Alexander Agassiz,	*Cambridge (Mass.).*
1897	E.-H. Amagat,	*Paris.*
1900	Arthur Auwers,	*Berlin.*
1900	Adolf Ritter von Baeyer,	*Munich.*
1889	Marcellin Pierre Eugène Berthelot,	*Paris.*
1895	Ludwig Boltzmann,	*Vienna.*
1905	Waldemar Chr. Brögger,	*Christiania.*
1897	Stanislao Cannizzaro,	*Rome.*
1905	Moritz Cantor,	*Heidelberg.*
1902	Jean Gaston Darboux,	*Paris.*
1902	Anton Dohrn,	*Naples.*
1905	Paul Ehrlich,	*Frankfurt-a.-M.*
1902	Albert Gaudry,	*Paris.*
1905	Paul Heinrich Groth,	*Munich.*
1888	Ernst Haeckel,	*Jena.*
1883	Julius Hann,	*Graz.*
1879	Jules Janssen,	*Paris.*
1864	Albert von Kölliker,	*Würzburg.*
1902	Samuel Pierpont Langley,	*Washington.*
1897	Gabriel Lippmann,	*Paris.*
1895	Éleuthère-Élie-Nicolas Mascart,	*Paris.*
1888	Demetrius Ivanovich Mendeléef,	*St Petersburg.*
1895	Carl Menger,	*Vienna.*
1897	Fridtjof Nansen,	*Christiania.*
1881	Simon Newcomb,	*Washington.*
1905	Eduard Pflüger,	*Bonn.*
1895	Jules Henri Poincaré,	*Paris.*
1889	Georg Hermann Quincke,	*Heidelberg.*
1897	Giovanni V. Schiaparelli,	*Milan.*
1905	Eduard Suess,	*Vienna.*
1905	Wilhelm Waldeyer,	*Berlin.*
1905	Wilhelm Wundt,	*Leipzig.*
1897	Ferdinand Zirkel,	*Leipzig.*

Total, 33.

BRITISH SUBJECTS (LIMITED TO TWENTY BY LAW X.).

Elected		
1902	Sir Benjamin Baker, K.C.M.G., Mem.Inst.C.E., F.R.S.,	*London.*
1889	Sir Robert Stawell Ball, Kt., LL.D., F.R.S., M.R.I.A., Lowndean Professor of Astronomy in the University of Cambridge,	*Cambridge.*
1900	Edward Caird, LL.D., Master of Balliol College, Oxford,	*Oxford.*
1892	Colonel Alexander Ross Clarke, C.B., R.E., F.R.S.,	*Redhill, Surrey.*
1897	Sir George Howard Darwin, K.C.B., M.A., LL.D., F.R.S., Plumian Professor of Astronomy in the University of Cambridge,	*Cambridge.*
1892	Sir David Gill, K.C.B., LL.D., F.R.S., His Majesty's Astronomer at the Cape of Good Hope,	*Cape of Good Hope.*
1900	David Ferrier, M.D., LL.D., F.R.S., Prof. of Neuro-pathology, King's College, London,	*London.*
1900	Andrew Russell Forsyth, D.Sc., F.R.S., Sadlerian Professor of Pure Mathematics in the University of Cambridge,	*Cambridge.*
1895	Albert C. L. G. Günther, Ph.D., F.R.S.,	*London.*
1883	Sir Joseph Dalton Hooker, K.C.S.I., M.D., LL.D., D.C.L., F.R.S., Corresp. Mem. Inst. of France,	*London.*
1884	Sir William Huggins, K.C.B., LL.D., D.C.L., P.R.S., Corresp. Mem. Inst. of France,	*London.*
1902	Sir Richard C. Jebb, Litt. D., D.C.L., M.P., Regius Professor of Greek in the University of Cambridge,	*Cambridge.*
1900	Archibald Liversidge, LL.D., F.R.S., Professor of Chemistry in the University of Sydney,	*Sydney.*
1905	Afred Newton, M.A., F.R.S., Professor of Zoology and Comparative Anatomy in the University of Cambridge,	*Cambridge.*
1905	Sir William Ramsay, K.C.B., LL.D., F.R.S., Professor of Chemistry in the University College, London,	*London.*
1886	The Lord Rayleigh, D.C.L., LL.D., D.Sc. Dub., F.R.S., Corresp. Mem. Inst. of France,	*London.*
1884	Sir J. S. Burdon Sanderson, Bart., M.D., LL.D., D.Sc. Dub., F.R.S.,	*Oxford.*
1905	Joseph John Thomson, D.Sc., LL.D., F.R.S., Cavendish Professor of Experimental Physics, University of Cambridge,	*Cambridge.*
1900	Thomas Edward Thorpe, D.Sc., LL.D., F.R.S., Principal of the Government Laboratories, London,	*London.*
1895	Sir Charles Todd, K.C.M.G., F.R.S., Government Astronomer, South Australia,	*Adelaide.*

Total, 20.

ORDINARY FELLOWS ELECTED

During Session 1904–1905.

Arranged according to the Date of their Election.

21st November 1904.

William Anderson, F.G.S.
George Alexander Carse, M.A., B.Sc.
Jacob C. Halm, Ph.D.

19th December 1904.

Wm. Alex. Francis Balfour-Browne, M.A.
David Corrie, F.C.S.
David Lawson, M.A., M.D., L.R.C.P. and S.E.

23rd January 1905.

William Anderson, M.A.
John Cameron, M.D., D.Sc.
Rev. Wm. Galloway Donaldson
Prof. Hector Munro Macdonald, M.A., F.R.S.
Archibald Milne, M.A., B.Sc.
C. H. Miller-Milne, M.A.
Peter Pinkerton, M.A.
George Romanes, C.E.
Alexander Thoms.

20th February 1905.

Prof. John Walter Gregory, D.Sc., F.R.S.
Robert Blyth Greig, F.Z.S.
Wm. Colin Mackenzie, M.D., F.R.C.S.
Wm. Alfred Pallin, F.R.C.V.S.
Harold Wm. Swithinbank
Arthur Logan Turner, M.D., F.R.C.S.E.

15th May 1905.

James Campbell Dewar, C.A.
George Wm. Jones, M.A., B.Sc.

19th June 1905.

George Andrew, M.A., B.A.
Thomas Lowe Bunting, M.D.
Forrest Hay Lightbody.

17th July 1905.

Joseph Henry Carter, F.R.C.V.S.
A. E. Scougal, M.A., H.M.C.I.S. (Readmitted).

HONORARY FELLOWS ELECTED

During Session 1904–1905.

BRITISH HONORARY FELLOWS.

Alfred Newton, M.A., F.R.S., Professor of Zoology and Comparative Anatomy in the University of Cambridge.

Joseph John Thomson, D.Sc., LL.D., F.R.S., Cavendish Professor of Experimental Physics University of Cambridge.

Sir William Ramsay, K.C.B., LL.D., F.R.S., Professor of Chemistry in the University College, London.

HONORARY FELLOWS ELECTED—*continued.*

FOREIGN HONORARY FELLOWS.

Moritz Cantor, Hon. Professor of Mathematics, University of Heidelberg.
Wilhelm Wundt, Professor of Philosophy, University of Leipzig.
Wilhelm Waldeyer, Professor of Anatomy, University of Berlin.
Eduard Pflüger, Professor of Physiology, University of Bonn.
Eduard Suess, Em. Professor of Geology, University of Vienna.
Paul Ehrlich, Director of the Institute for Experimental Therapeutics, Frankfurt-a.-M.
Waldemar Chr. Brögger, Professor of Mineralogy and Palæontology, University of Christiania
Paul Heinrich Groth, Professor of Mineralogy in the University of Munich.

ORDINARY FELLOWS DECEASED

During Session 1904–1905.

Walter Berry, of Glenstriven, K.D.
Professor Ralph Copeland, Astronomer-Royal for Scotland.
David Deuchar, F.I.A., F.F.A.
James Duncan.
James Durham, F.G.S.
Patrick Neill Fraser.
Dr A. Lockhart Gillespie.
A. H. Japp, LL.D.
James Napier, M.A.
Dr Charles D. F. Philips.
Eyre Burton Powell, C.S.I., M.A.
Sir John Sibbald, M.D.
Rev. Charles R. Teape, M.A., Ph.D.
Dr Robert Stevenson Thomson.
Charles Wilson Vincent, F.I.C., F.C.S.

HONORARY FELLOWS DECEASED

During Session 1904–1905.

FOREIGN.

Ferdinand von Richthofen.
Otto Wilhelm Struve.
Tobias Robert Thalén.

LAWS

OF THE

ROYAL SOCIETY OF EDINBURGH,

AS REVISED 18TH JULY 1904.

LAWS.

[By the Charter of the Society (printed in the *Transactions*, Vol. VI. p. 5), the Laws cannot be altered, except at a Meeting held one month after that at which the Motion for alteration shall have been proposed.]

I.

THE ROYAL SOCIETY OF EDINBURGH shall consist of Ordinary and Honorary Fellows. Title.

II.

Every Ordinary Fellow, within three months after his election, shall pay Two Guineas as the fee of admission, and Three Guineas as his contribution for the Session in which he has been elected; and annually at the commencement of every Session, Three Guineas into the hands of the Treasurer. This annual contribution shall continue for ten years after his admission, and it shall be limited to Two Guineas for fifteen years thereafter.* Fellows may compound for these contributions on such terms as the Council may from time to time fix. The fees of Ordinary Fellows residing in Scotland.

III.

All Fellows who shall have paid Twenty-five years' annual contribution shall be exempted from further payment. Payment to cease after 25 years.

IV.

The fees of admission of an Ordinary Non-Resident Fellow shall be £26, 5s., payable on his admission; and in case of any Non-Resident Fellow coming to reside at any time in Scotland, he shall, during each year of his residence, pay the usual annual contribution of £3, 3s., payable by each Resident Fellow; but after payment of such annual contribution for eight years, he shall be exempt Fees of Non-Resident Ordinary Fellows.

* A modification of this rule, in certain cases, was agreed to at a Meeting of the Society held on the 3rd January 1831.

At the Meeting of the Society, on the 5th January 1857, when the reduction of the Contributions from £3, 3s. to £2, 2s., from the 11th to the 25th year of membership, was adopted, it was resolved that the existing Members shall share in this reduction, so far as regards their future annual Contributions.

Case of Fellows becoming Non-Resident.

from any further payment. In the case of any Resident Fellow ceasing to reside in Scotland, and wishing to continue a Fellow of the Society, it shall be in the power of the Council to determine on what terms, in the circumstances of each case, the privilege of remaining a Fellow of the Society shall be continued to such Fellow while out of Scotland.

V.

Defaulters.

Members failing to pay their contributions for three successive years (due application having been made to them by the Treasurer) shall be reported to the Council, and, if they see fit, shall be declared from that period to be no longer Fellows, and the legal means for recovering such arrears shall be employed.

VI.

Privileges of Ordinary Fellows.

None but Ordinary Fellows shall bear any office in the Society, or vote in the choice of Fellows or Office-Bearers, or interfere in the patrimonial interests of the Society.

VII.

Numbers Unlimited.

The number of Ordinary Fellows shall be unlimited.

VIII.

Fellows entitled to Transactions.

The Ordinary Fellows, upon producing an order from the TREASURER, shall be entitled to receive from the Publisher, gratis, the Parts of the Society's Transactions which shall be published subsequent to their admission.

IX.

Mode of Recommending Ordinary Fellows.

Candidates for admission as Ordinary Fellows shall make an application in writing, and shall produce along with it a certificate of recommendation to the purport below,* signed by at least *four* Ordinary Fellows, two of whom shall certify their recommendation from personal knowledge. This recommendation shall be delivered to the Secretary, and by him laid before the Council, and shall be exhibited publicly in the Society's Rooms for one month, after which it shall be considered by the Council. If the Candidate be approved by the Council, notice of the day fixed for the election shall be given in the circulars of at least two Ordinary Meetings of the Society.

* "A. B., a gentleman well versed in Science (*or Polite Literature, as the case may be*), being "to our knowledge desirous of becoming a Fellow of the Royal Society of Edinburgh, we hereby "recommend him as deserving of that honour, and as likely to prove a useful and valuable Member."

X.

Honorary Fellows shall not be subject to any contribution. This class shall consist of persons eminently distinguished for science or literature. Its number shall not exceed Fifty-six, of whom Twenty may be British subjects, and Thirty-six may be subjects of foreign states. Honorary Fellows, British and Foreign.

XI.

Personages of Royal Blood may be elected Honorary Fellows, without regard to the limitation of numbers specified in Law X. Royal Personages.

XII.

Honorary Fellows may be proposed by the Council, or by a recommendation (in the form given below*) subscribed by three Ordinary Fellows; and in case the Council shall decline to bring this recommendation before the Society, it shall be competent for the proposers to bring the same before a General Meeting. The election shall be by ballot, after the proposal has been communicated *viva voce* from the Chair at one meeting, and printed in the circulars for two ordinary meetings of the Society, previous to the day of election. Recommendation of Honorary Fellows. Mode of Election.

XIII.

The election of Ordinary Fellows shall take place only at one Afternoon Ordinary Meeting of each month during the Session. The election shall be by ballot, and shall be determined by a majority of at least two-thirds of the votes, provided Twenty-four Fellows be present and vote. Election of Ordinary Fellows.

XIV.

The Ordinary Meetings shall be held on the first and third Mondays of each month from November to March, and from May to July, inclusive; with the exception that when there are five Mondays in January, the Meetings for that month shall be held on its second and fourth Mondays. Regular Minutes shall be kept of the proceedings, and the Secretaries shall do the duty alternately, or according to such agreement as they may find it convenient to make. Ordinary Meetings.

* We hereby recommend _____
for the distinction of being made an Honorary Fellow of this Society, declaring that each of us from our own knowledge of his services to (*Literature or Science, as the case may be*) believe him to be worthy of that honour.

(To be signed by three Ordinary Fellows.)

_____

_____

_____

To the President and Council of the Royal Society of Edinburgh.

XV.

The Transactions.

The Society shall from time to time publish its Transactions and Proceedings. For this purpose the Council shall select and arrange the papers which they shall deem it expedient to publish in the *Transactions* of the Society, and shall superintend the printing of the same.

The Council shall have power to regulate the private business of the Society. At any Meeting of the Council the Chairman shall have a casting as well as a deliberative vote.

XVI.

How Published.

The Transactions shall be published in parts or *Fasciculi* at the close of each Session, and the expense shall be defrayed by the Society.

XVII.

The Council.

That there shall be formed a Council, consisting—First, of such gentlemen as may have filled the office of President; and Secondly, of the following to be annually elected, viz.:—a President, Six Vice-Presidents (two at least of whom shall be resident), Twelve Ordinary Fellows as Councillors, a General Secretary, Two Secretaries to the Ordinary Meetings, a Treasurer, and a Curator of the Museum and Library.

XVIII.

Retiring Councillors.

Four Councillors shall go out annually, to be taken according to the order in which they stand on the list of the Council.

XIX.

Election of Office-Bearers.

An Extraordinary Meeting for the election of Office-Bearers shall be held annually on the fourth Monday of October, or on such other lawful day in October as the Council may fix, and each Session of the Society shall be held to begin at the date of the said Extraordinary Meeting.

XX.

Special Meetings; how called.

Special Meetings of the Society may be called by the Secretary, by direction of the Council; or on a requisition signed by six or more Ordinary Fellows. Notice of not less than two days must be given of such Meetings.

XXI.

Treasurer's Duties.

The Treasurer shall receive and disburse the money belonging to the Society, granting the necessary receipts, and collecting the money when due.

He shall keep regular accounts of all the cash received and expended, which shall be made up and balanced annually; and at the Extraordinary Meeting in October, he shall present the accounts for the preceding year, duly audited.

At this Meeting, the Treasurer shall also lay before the Council a list of all arrears due above two years, and the Council shall thereupon give such directions as they may deem necessary for recovery thereof.

XXII.

Auditor.

At the Extraordinary Meeting in October, a professional accountant shall be chosen to audit the Treasurer's accounts for that year, and to give the necessary discharge of his intromissions.

XXIII.

General Secretary's Duties.

The General Secretary shall keep Minutes of the Extraordinary Meetings of the Society, and of the Meetings of the Council, in two distinct books. He shall, under the direction of the Council, conduct the correspondence of the Society, and superintend its publications. For these purposes he shall, when necessary, employ a clerk, to be paid by the Society.

XXIV.

Secretaries to Ordinary Meetings.

The Secretaries to the Ordinary Meetings shall keep a regular Minute-book, in which a full account of the proceedings of these Meetings shall be entered; they shall specify all the Donations received, and furnish a list of them, and of the Donors' names, to the Curator of the Library and Museum; they shall likewise furnish the Treasurer with notes of all admissions of Ordinary Fellows. They shall assist the General Secretary in superintending the publications, and in his absence shall take his duty.

XXV.

Curator of Museum and Library.

The Curator of the Museum and Library shall have the custody and charge of all the Books, Manuscripts, objects of Natural History, Scientific Productions, and other articles of a similar description belonging to the Society; he shall take an account of these when received, and keep a regular catalogue of the whole, which shall lie in the Hall, for the inspection of the Fellows.

XXVI.

Use of Museum and Library.

All Articles of the above description shall be open to the inspection of the Fellows at the Hall of the Society, at such times and under such regulations as the Council from time to time shall appoint.

XXVII.

Register Book.

A Register shall be kept, in which the names of the Fellows shall be enrolled at their admission, with the date.

XXVIII.

Power of Expulsion.

If, in the opinion of the Council of the Society, the conduct of any Fellow is unbecoming the position of a Member of a learned Society, or is injurious to the character and interests of this Society, the Council may request such Fellow to resign; and, if he fail to do so within one month of such request being addressed to him, the Council shall call a General Meeting of the Fellows of the Society to consider the matter; and, if a majority of the Fellows present at such Meeting agree to the expulsion of such Member, he shall be then and there expelled by the declaration of the Chairman of the said Meeting to that effect; and he shall thereafter cease to be a Fellow of the Society, and his name shall be erased from the Roll of Fellows, and he shall forfeit all right or claim in or to the property of the Society.

THE KEITH, MAKDOUGALL-BRISBANE, NEILL, AND GUNNING VICTORIA JUBILEE PRIZES.

The above Prizes will be awarded by the Council in the following manner:—

I. KEITH PRIZE.

The KEITH PRIZE, consisting of a Gold Medal and from £40 to £50 in Money, will be awarded in the Session 1905–1906 for the "best communication on a scientific subject, communicated, in the first instance, to the Royal Society during the Sessions 1903–04 and 1904–05." Preference will be given to a paper containing a discovery.

II. MAKDOUGALL-BRISBANE PRIZE.

This Prize is to be awarded biennially by the Council of the Royal Society of Edinburgh to such person, for such purposes, for such objects, and in such manner as shall appear to them the most conducive to the promotion of the interests of science; with the *proviso* that the Council shall not be compelled to award the Prize unless there shall be some individual engaged in scientific pursuit, or some paper written on a scientific subject, or some discovery in science made during the biennial period, of sufficient merit or importance in the opinion of the Council to be entitled to the Prize.

1. The Prize, consisting of a Gold Medal and a sum of Money, will be awarded at the commencement of the Session 1906–1907, for an Essay or Paper having reference to any branch of scientific inquiry, whether Material or Mental.

2. Competing Essays to be addressed to the Secretary of the Society, and transmitted not later than 8th July 1906.

3. The Competition is open to all men of science.

4. The Essays may be either anonymous or otherwise. In the former case, they must be distinguished by mottoes, with corresponding sealed billets, superscribed with the same motto, and containing the name of the Author.

5. The Council impose no restriction as to the length of the Essays, which may be, at the discretion of the Council, read at the Ordinary Meetings of the Society. They wish also to leave the property and free disposal of the manuscripts to the Authors; a copy, however, being deposited in the Archives of the Society, unless the paper shall be published in the Transactions.

6. In awarding the Prize, the Council will also take into consideration any scientific papers presented to the Society during the Sessions 1904–05, 1905–06, whether they may have been given in with a view to the prize or not.

III. NEILL PRIZE.

The Council of the Royal Society of Edinburgh having received the bequest of the late Dr PATRICK NEILL of the sum of £500, for the purpose of "the interest thereof being applied in furnishing a Medal or other reward every second or third year to any distinguished Scottish Naturalist, according as such Medal or reward shall be voted by the Council of the said Society," hereby intimate,

1. The NEILL PRIZE, consisting of a Gold Medal and a sum of Money, will be awarded during the Session 1907–1908.

2. The Prize will be given for a Paper of distinguished merit, on a subject of Natural History, by a Scottish Naturalist, which shall have been presented to the Society during the three years preceding the 8th July 1907,—or failing presentation of a paper sufficiently meritorious, it will be awarded for a work or publication by some distinguished Scottish Naturalist, on some branch of Natural History, bearing date within five years of the time of award.

IV. GUNNING VICTORIA JUBILEE PRIZE.

This Prize, founded in the year 1887 by Dr R. H. GUNNING, is to be awarded quadrennially by the Council of the Royal Society of Edinburgh, in recognition of original work in Physics, Chemistry, or Pure or Applied Mathematics.

Evidence of such work may be afforded either by a Paper presented to the Society, or by a Paper on one of the above subjects, or some discovery in them elsewhere communicated or made, which the Council may consider to be deserving of the Prize.

The Prize consists of a sum of money, and is open to men of science resident in or connected with Scotland. The first award was made in the year 1887.

In accordance with the wish of the Donor, the Council of the Society may on fit occasions award the Prize for work of a definite kind to be undertaken during the three succeeding years by a scientific man of recognised ability.

AWARDS OF THE KEITH, MAKDOUGALL-BRISBANE, NEILL, AND GUNNING VICTORIA JUBILEE PRIZES, FROM 1827 TO 1904.

I. KEITH PRIZE.

1st Biennial Period, 1827–29.—Dr Brewster, for his papers "on his Discovery of Two New Immiscible Fluids in the Cavities of certain Minerals," published in the Transactions of the Society.

2nd Biennial Period, 1829–31.—Dr Brewster, for his paper "on a New Analysis of Solar Light," published in the Transactions of the Society.

3rd Biennial Period, 1831–33.—Thomas Graham, Esq., for his paper "on the Law of the Diffusion of Gases," published in the Transactions of the Society.

4th Biennial Period, 1833–35.—Professor J. D. Forbes, for his paper "on the Refraction and Polarization of Heat," published in the Transactions of the Society.

5th Biennial Period, 1835–37.—John Scott Russell, Esq., for his Researches "on Hydrodynamics," published in the Transactions of the Society.

6th Biennial Period, 1837–39.—Mr John Shaw, for his experiments "on the Development and Growth of the Salmon," published in the Transactions of the Society.

7th Biennial Period, 1839–41.—Not awarded.

8th Biennial Period, 1841–43.—Professor James David Forbes, for his papers "on Glaciers," published in the Proceedings of the Society.

9th Biennial Period, 1843–45.—Not awarded.

10th Biennial Period, 1845–47.—General Sir Thomas Brisbane, Bart., for the Makerstoun Observations on Magnetic Phenomena, made at his expense, and published in the Transactions of the Society.

11th Biennial Period, 1847–49.—Not awarded.

12th Biennial Period, 1849–51.—Professor Kelland, for his papers "on General Differentiation, including his more recent Communication on a process of the Differential Calculus, and its application to the solution of certain Differential Equations," published in the Transactions of the Society.

13th Biennial Period, 1851–53.—W. J. Macquorn Rankine, Esq., for his series of papers "on the Mechanical Action of Heat," published in the Transactions of the Society.

14th Biennial Period, 1853–55.—Dr Thomas Anderson, for his papers "on the Crystalline Constituents of Opium, and on the Products of the Destructive Distillation of Animal Substances," published in the Transactions of the Society.

15th Biennial Period, 1855–57.—Professor Boole, for his Memoir "on the Application of the Theory of Probabilities to Questions of the Combination of Testimonies and Judgments," published in the Transactions of the Society.

16TH BIENNIAL PERIOD, 1857–59.—Not awarded.

17TH BIENNIAL PERIOD, 1859–61.—JOHN ALLAN BROUN, Esq., F.R.S., Director of the Trevandrum Observatory, for his papers "on the Horizontal Force of the Earth's Magnetism, on the Correction of the Bifilar Magnetometer, and on Terrestrial Magnetism generally," published in the Transactions of the Society.

18TH BIENNIAL PERIOD, 1861–63.—Professor WILLIAM THOMSON, of the University of Glasgow, for his Communication "on some Kinematical and Dynamical Theorems."

19TH BIENNIAL PERIOD, 1863–65.—Principal FORBES, St Andrews, for his "Experimental Inquiry into the Laws of Conduction of Heat in Iron Bars," published in the Transactions of the Society.

20TH BIENNIAL PERIOD, 1865–67.—Professor C. PIAZZI SMYTH, for his paper "on Recent Measures at the Great Pyramid," published in the Transactions of the Society.

21ST BIENNIAL PERIOD, 1867–69.—Professor P. G. TAIT, for his paper "on the Rotation of a Rigid Body about a Fixed Point," published in the Transactions of the Society.

22ND BIENNIAL PERIOD, 1869–71.—Professor CLERK MAXWELL, for his paper "on Figures, Frames, and Diagrams of Forces," published in the Transactions of the Society.

23RD BIENNIAL PERIOD, 1871–73.—Professor P. G. TAIT, for his paper entitled "First Approximation to a Thermo-electric Diagram," published in the Transactions of the Society.

24TH BIENNIAL PERIOD, 1873–75.—Professor CRUM BROWN, for his Researches "on the Sense of Rotation, and on the Anatomical Relations of the Semicircular Canals of the Internal Ear."

25TH BIENNIAL PERIOD, 1875–77.—Professor M. FORSTER HEDDLE, for his papers "on the Rhombohedral Carbonates," and "on the Felspars of Scotland," published in the Transactions of the Society.

26TH BIENNIAL PERIOD, 1877–79.—Professor H. C. FLEEMING JENKIN, for his paper "on the Application of Graphic Methods to the Determination of the Efficiency of Machinery," published in the Transactions of the Society; Part II. having appeared in the volume for 1877–78.

27TH BIENNIAL PERIOD, 1879–81.—Professor GEORGE CHRYSTAL, for his paper "on the Differential Telephone," published in the Transactions of the Society.

28TH BIENNIAL PERIOD, 1881–83.—THOMAS MUIR, Esq., LL.D., for his "Researches into the Theory of Determinants and Continued Fractions," published in the Proceedings of the Society.

29TH BIENNIAL PERIOD, 1883–85.—JOHN AITKEN, Esq., for his paper "on the Formation of Small Clear Spaces in Dusty Air," and for previous papers on Atmospheric Phenomena, published in the Transactions of the Society.

30TH BIENNIAL PERIOD, 1885–87.—JOHN YOUNG BUCHANAN, Esq., for a series of Communications, extending over several years, on subjects connected with Ocean Circulation, Compressibility of Glass, &c.; two of which, viz., "On Ice and Brines," and "On the Distribution of Temperature in the Antarctic Ocean," have been published in the Proceedings of the Society.

31ST BIENNIAL PERIOD, 1887–89.—Professor E. A. LETTS, for his papers on the Organic Compounds of Phosphorus, published in the Transactions of the Society.

32ND BIENNIAL PERIOD, 1889–91.—R. T. OMOND, Esq., for his Contributions to Meteorological Science, many of which are contained in Vol. XXXIV. of the Society's Transactions.

33RD BIENNIAL PERIOD, 1891–93.—Professor THOMAS R. FRASER, F.R.S., for his papers on *Strophanthus hispidus*, Strophanthin, and Strophanthidin, read to the Society in February and June 1889 and in December 1891, and printed in Vols. XXXV., XXXVI., and XXXVII. of the Society's Transactions.

34TH BIENNIAL PERIOD, 1893–95.—Dr CARGILL G. KNOTT, for his papers on the Strains produced by Magnetism in Iron and in Nickel, which have appeared in the Transactions and Proceedings of the Society.

35TH BIENNIAL PERIOD, 1895–97.—Dr THOMAS MUIR, for his continued Communications on Determinants and Allied Questions.

36TH BIENNIAL PERIOD, 1897–99.—Dr JAMES BURGESS, for his paper "on the Definite Integral $\frac{2}{\sqrt{\pi}}\int_0^t \epsilon^{-t^2} dt$, with extended Tables of Values," printed in Vol. XXXIX. of the Transactions of the Society.

37TH BIENNIAL PERIOD, 1899–1901.—Dr HUGH MARSHALL, for his discovery of the Persulphates, and for his Communications on the Properties and Reactions of these Salts, published in the Proceedings of the Society.

38TH BIENNIAL PERIOD, 1901–03.—Sir WILLIAM TURNER, K.C.B., LL.D., F.R.S., &c., for his memoirs entitled "A Contribution to the Craniology of the People of Scotland," published in the Transactions of the Society, and for his "Contributions to the Craniology of the People of the Empire of India," Parts I., II., likewise published in the Transactions of the Society.

II. MAKDOUGALL-BRISBANE PRIZE.

1ST BIENNIAL PERIOD, 1859.—Sir RODERICK IMPEY MURCHISON, on account of his Contributions to the Geology of Scotland.

2ND BIENNIAL PERIOD, 1860–62.—WILLIAM SELLER, M.D., F.R.C.P.E., for his "Memoir of the Life and Writings of Dr Robert Whytt," published in the Transactions of the Society.

3RD BIENNIAL PERIOD, 1862–64.—JOHN DENIS MACDONALD, Esq., R.N., F.R.S., Surgeon of H.M.S. "Icarus," for his paper "on the Representative Relationships of the Fixed and Free Tunicata, regarded as Two Sub-classes of equivalent value; with some General Remarks on their Morphology," published in the Transactions of the Society.

4TH BIENNIAL PERIOD, 1864–66.—Not awarded.

5TH BIENNIAL PERIOD, 1866–68.—Dr ALEXANDER CRUM BROWN and Dr THOMAS RICHARD FRASER, for their conjoint paper "on the Connection between Chemical Constitution and Physiological Action," published in the Transactions of the Society.

6TH BIENNIAL PERIOD, 1868–70.—Not awarded.

7TH BIENNIAL PERIOD, 1870–72.—GEORGE JAMES ALLMAN, M.D., F.R.S., Emeritus Professor of Natural History, for his paper "on the Homological Relations of the Cœlenterata," published in the Transactions, which forms a leading chapter of his Monograph of Gymnoblastic or Tubularian Hydroids—since published.

8TH BIENNIAL PERIOD, 1872–74.—Professor LISTER, for his paper "on the Germ Theory of Putrefaction and the Fermentive Changes," communicated to the Society, 7th April 1873.

9TH BIENNIAL PERIOD, 1874–76.—ALEXANDER BUCHAN, A.M., for his paper "on the Diurnal Oscillation of the Barometer," published in the Transactions of the Society.

10TH BIENNIAL PERIOD, 1876–78.—Professor ARCHIBALD GEIKIE, for his paper "on the Old Red Sandstone of Western Europe," published in the Transactions of the Society.

11TH BIENNIAL PERIOD, 1878–80.—Professor PIAZZI SMYTH, Astronomer-Royal for Scotland, for his paper "on the Solar Spectrum in 1877–78, with some Practical Idea of its probable Temperature of Origination," published in the Transactions of the Society.

12TH BIENNIAL PERIOD, 1880–82.—Professor JAMES GEIKIE, for his "Contributions to the Geology of the North-West of Europe," including his paper "on the Geology of the Faroes," published in the Transactions of the Society.

13TH BIENNIAL PERIOD, 1882–84.—EDWARD SANG, Esq., LL.D., for his paper "on the Need of Decimal Subdivisions in Astronomy and Navigation, and on Tables requisite therefor," and generally for his Recalculation of Logarithms both of Numbers and Trigonometrical Ratios, —the former communication being published in the Proceedings of the Society.

14TH BIENNIAL PERIOD, 1884–86.—JOHN MURRAY, Esq., LL.D., for his papers "On the Drainage Areas of Continents, and Ocean Deposits," "The Rainfall of the Globe, and Discharge of Rivers," "The Height of the Land and Depth of the Ocean," and "The Distribution of Temperature in the Scottish Lochs as affected by the Wind."

15TH BIENNIAL PERIOD, 1886–88.—ARCHIBALD GEIKIE, Esq., LL.D., for numerous Communications, especially that entitled "History of Volcanic Action during the Tertiary Period in the British Isles," published in the Transactions of the Society.

16TH BIENNIAL PERIOD, 1888–90.—Dr LUDWIG BECKER, for his paper on "The Solar Spectrum at Medium and Low Altitudes," printed in Vol. XXXVI. Part I. of the Society's Transactions.

17TH BIENNIAL PERIOD, 1890–92.—HUGH ROBERT MILL, Esq., D.Sc., for his papers on "The Physical Conditions of the Clyde Sea Area," Part I. being already published in Vol. XXXVI. of the Society's Transactions.

18TH BIENNIAL PERIOD, 1892–94.—Professor JAMES WALKER, D.Sc., Ph.D., for his work on Physical Chemistry, part of which has been published in the Proceedings of the Society, Vol. XX., pp. 255–263. In making this award, the Council took into consideration the work done by Professor Walker along with Professor Crum Brown on the Electrolytic Synthesis of Dibasic Acids, published in the Transactions of the Society.

19TH BIENNIAL PERIOD, 1894–96.—Professor JOHN G. M'KENDRICK, for numerous Physiological papers, especially in connection with Sound; many of which have appeared in the Society's publications.

20TH BIENNIAL PERIOD, 1896–98.—Dr WILLIAM PEDDIE, for his papers on the Torsional Rigidity of Wires.

21ST BIENNIAL PERIOD, 1898–1900.—Dr RAMSAY H. TRAQUAIR, for his paper entitled "Report on Fossil Fishes collected by the Geological Survey in the Upper Silurian Rocks of Scotland," printed in Vol. XXXIX. of the Transactions of the Society.

22ND BIENNIAL PERIOD, 1900–02.—Dr ARTHUR T. MASTERMAN, for his paper entitled "The Early Development of *Cribrella oculata* (Forbes), with remarks on Echinoderm Development," printed in Vol. XL. of the Transactions of the Society.

23RD BIENNIAL PERIOD, 1902–04.—Mr JOHN DOUGALL, M.A., for his paper on "An Analytical Theory of the Equilibrium of an Isotropic Elastic Plate," published in Vol. XLI. of the Transactions of the Society.

III. THE NEILL PRIZE.

1ST TRIENNIAL PERIOD, 1856–59.—Dr W. LAUDER LINDSAY, for his paper "on the Spermogones and Pycnides of Filamentous, Fruticulose, and Foliaceous Lichens," published in the Transactions of the Society.

2ND TRIENNIAL PERIOD, 1859–62.—ROBERT KAYE GREVILLE, LL.D., for his Contributions to Scottish Natural History, more especially in the department of Cryptogamic Botany, including his recent papers on Diatomaceæ.

3rd Triennial Period, 1862–65.—Andrew Crombie Ramsay, F.R.S., Professor of Geology in the Government School of Mines, and Local Director of the Geological Survey of Great Britain, for his various works and memoirs published during the last five years, in which he has applied the large experience acquired by him in the Direction of the arduous work of the Geographical Survey of Great Britain to the elucidation of important questions bearing on Geological Science.

4th Triennial Period, 1865–68.—Dr William Carmichael M'Intosh, for his paper "on the Structure of the British Nemerteans, and on some New British Annelids," published in the Transactions of the Society.

5th Triennial Period, 1868–71.—Professor William Turner, for his papers "on the great Finner Whale; and on the Gravid Uterus, and the Arrangement of the Fœtal Membranes in the Cetacea," published in the Transactions of the Society.

6th Triennial Period, 1871–74.—Charles William Peach, Esq., for his Contributions to Scottish Zoology and Geology, and for his recent contributions to Fossil Botany.

7th Triennial Period, 1874–77.—Dr Ramsay H. Traquair, for his paper "on the Structure and Affinities of *Tristichopterus alatus* (Egerton)," published in the Transactions of the Society, and also for his contributions to the Knowledge of the Structure of Recent and Fossil Fishes.

8th Triennial Period, 1877–80.—John Murray, Esq., for his paper "on the Structure and Origin of Coral Reefs and Islands," published (in abstract) in the Proceedings of the Society.

9th Triennial Period, 1880–83.—Professor Herdman, for his papers "on the Tunicata," published in the Proceedings and Transactions of the Society.

10th Triennial Period, 1883–86.—B. N. Peach, Esq., for his Contributions to the Geology and Palæontology of Scotland, published in the Transactions of the Society.

11th Triennial Period, 1886–89.—Robert Kidston, Esq., for his Researches in Fossil Botany, published in the Transactions of the Society.

12th Triennial Period, 1889–92.—John Horne, Esq., F.G.S., for his Investigations into the Geological Structure and Petrology of the North-West Highlands.

13th Triennial Period, 1892–95.—Robert Irvine, Esq., for his papers on the action of Organisms in the Secretion of Carbonate of Lime and Silica, and on the solution of these substances in Organic Juices. These are printed in the Society's Transactions and Proceedings.

14th Triennial Period, 1895–98.—Professor Cossar Ewart, for his recent Investigations connected with Telegony.

15th Triennial Period, 1898–1901.—Dr John S. Flett, for his papers entitled "The Old Red Sandstone of the Orkneys" and "The Trap Dykes of the Orkneys," printed in Vol. XXXIX. of the Transactions of the Society.

16th Triennial Period, 1901–04.—Professor J. Graham Kerr, M.A., for his Researches on *Lepidosiren paradoxa*, published in the Philosophical Transactions of the Royal Society, London.

IV. GUNNING VICTORIA JUBILEE PRIZE.

1st Triennial Period, 1884–87.—Sir William Thomson, Pres. R.S.E., F.R.S., for a remarkable series of papers "on Hydrokinetics," especially on Waves and Vortices, which have been communicated to the Society.

2nd Triennial Period, 1887–90.—Professor P. G. Tait, Sec. R.S.E., for his work in connection with the "Challenger" Expedition, and his other Researches in Physical Science.

3RD TRIENNIAL PERIOD, 1890–93.—ALEXANDER BUCHAN, Esq., LL.D., for his varied, extensive, and extremely important Contributions to Meteorology, many of which have appeared in the Society's Publications.

4TH TRIENNIAL PERIOD, 1893–96.—JOHN AITKEN, Esq., for his brilliant Investigations in Physics, especially in connection with the Formation and Condensation of Aqueous Vapour.

1ST QUADRENNIAL PERIOD, 1896–1900.—DR T. D. ANDERSON, for his discoveries of New and Variable Stars.

2ND QUADRENNIAL PERIOD, 1900–04.—Sir JAMES DEWAR, LL.D., D.C.L., F.R.S., &c., for his researches on the Liquefaction of Gases, extending over the last quarter of a century, and on the Chemical and Physical Properties of Substances at Low Temperatures: his earliest papers being published in the Transactions and Proceedings of the Society.

PROCEEDINGS

OF THE

STATUTORY GENERAL MEETING,

24TH OCTOBER 1904.

STATUTORY MEETING.

HUNDRED AND TWENTY-SECOND SESSION.

Monday, 24*th October* 1904.

At the Annual Statutory Meeting,

The Hon. Lord M'LAREN in the Chair,

The Minutes of last Annual Statutory Meeting of 26th October 1903 were read, approved, and signed.

On the motion of Dr CRUM BROWN, Dr R. M. FERGUSON and Dr B. N. PEACH were appointed Scrutineers, and the Ballot for the New Council commenced.

The TREASURER submitted his Accounts for the year. These, with the Auditors' Report, were read and approved.

The Scrutineers reported that the following New Council had been duly elected:—

The Right Hon. Lord KELVIN, G.C.V.O., LL.D., D.C.L., F.R.S., President.
Professor JAMES GEIKIE, LL.D., F.R.S.,
The Hon. Lord M'LAREN, LL.D.,
The Rev. Professor FLINT, D.D.,
ROBERT MUNRO, M.A., M.D., LL.D.,
Sir JOHN MURRAY, K.C.B., LL.D., F.R.S.,
RAMSAY H. TRAQUAIR, M.D., LL.D., F.R.S., } Vice-Presidents.
Professor GEORGE CHRYSTAL, LL.D., General Secretary.
Professor CRUM BROWN, F.R.S.,
Professor D. J. CUNNINGHAM, M.D., LL.D., F.R.S., } Secretaries to Ordinary Meetings.
PHILIP R. D. MACLAGAN, F.F.A., Treasurer.
ALEX. BUCHAN, M.A., LL.D., F.R.S., Curator of Library and Museum.

COUNCILLORS.

JOHN HORNE, LL.D., F.R.S.	D. NOËL PATON, M.D., F.R.C.P.E.
CARGILL G. KNOTT, D.Sc.	Professor JOHN CHIENE, C.B., M.D., LL.D.
Professor RALPH STOCKMAN, M.D., F.R.C.P.E.	Professor J. GRAHAM KERR, M.A.
Professor JAMES WALKER, D.Sc., Ph.D., F.R.S.	WILLIAM PEDDIE, D.Sc.
Professor ANDREW GRAY, M.A., LL.D., F.R.S.	LEONARD DOBBIN, Ph.D.
ROBERT KIDSTON, F.R.S., F.G.S.	Professor J. C. EWART, M.D., F.R.S.

On the motion of Professor CRUM BROWN, thanks were voted to the Treasurer.

On the motion of Dr TRAQUAIR, thanks were voted to the Scrutineers.

On the motion of Professor CRUM BROWN, thanks were voted to the Auditors, who were reappointed.

On the motion of Professor CHRYSTAL, thanks were voted to the Chairman.

JOHN M'LAREN, *V.P.*,
Chairman.

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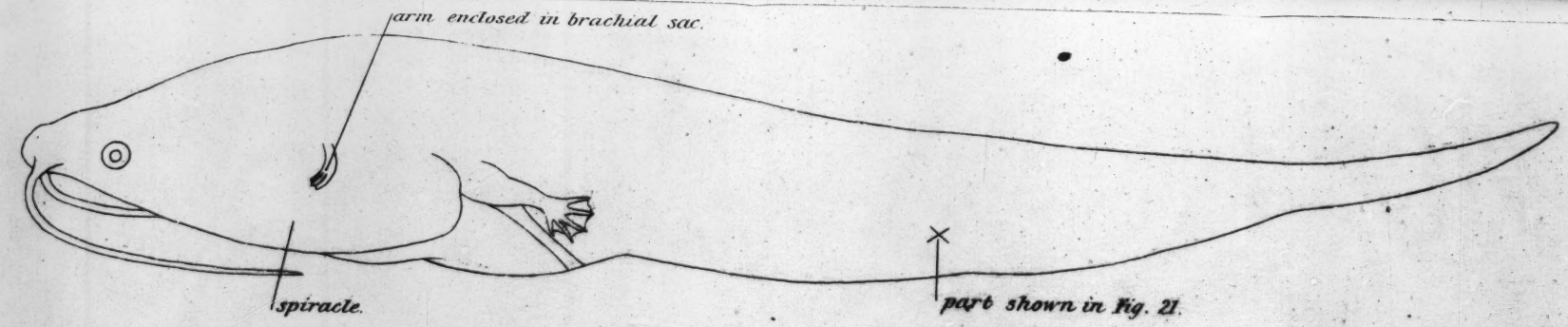

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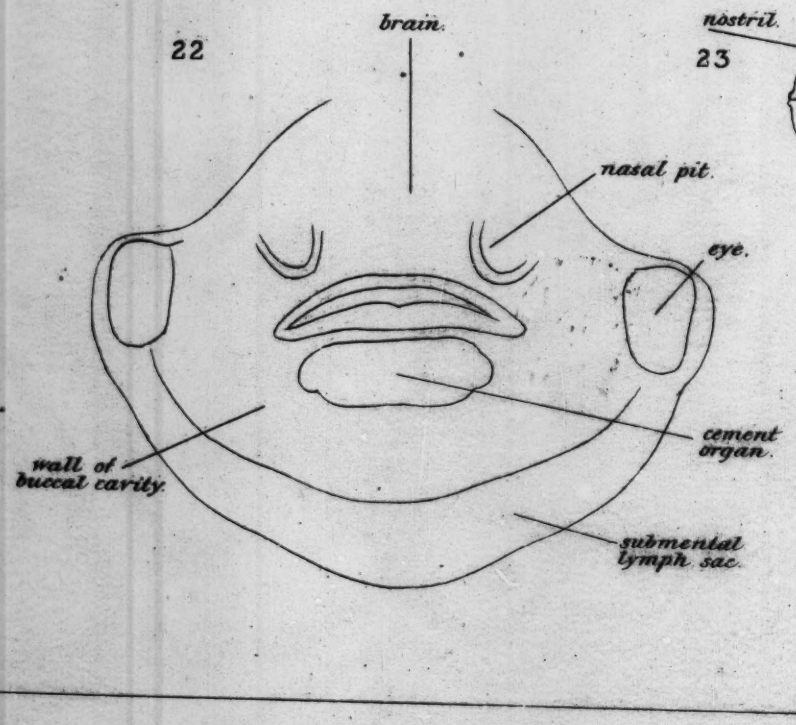

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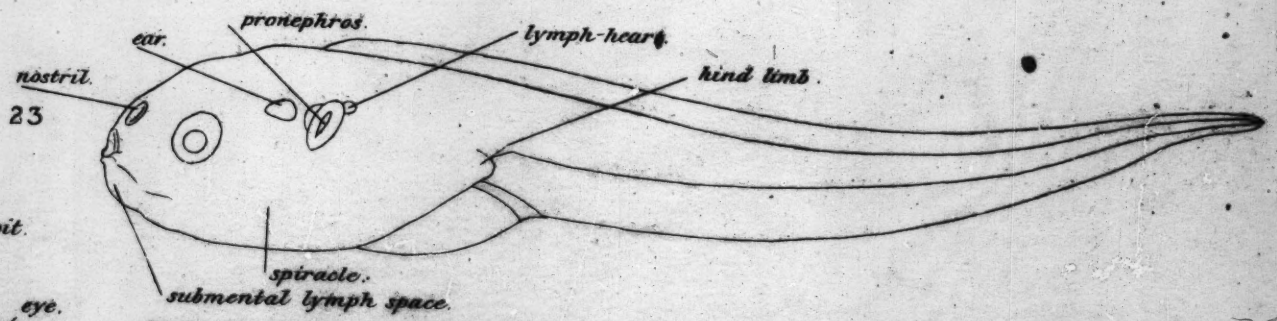

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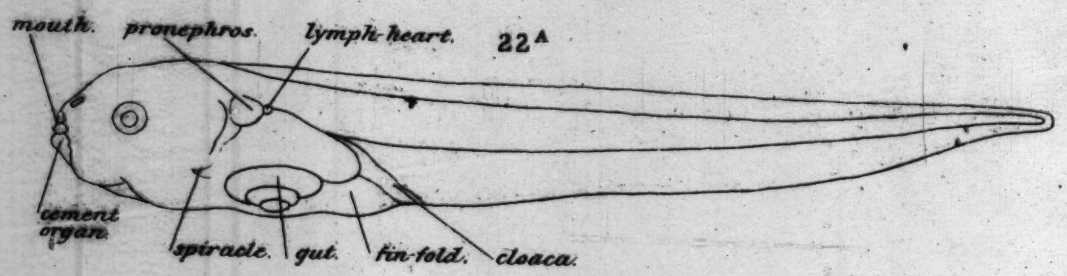

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